EC334 Topics in Financial Economics
Fourth Seminar

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Cheap Talk Model

- Player 1 (or the Agent) has private information and both players' payoffs depend on player 1's private information. Play 2 (or the Principle) makes the decision.
- Player 1's action is a message that has no direct effect on payoffs. (Cheap)
- General way of proceeding
  - Nature selects a type which is the private information that Player 1 has.
  - Player 1 chooses some message to send to Player 2
  - Player 2 observes the message while not knowing the true type of nature
  - Payoffs for both players are realized

A Motivating Example

- Player 1 lives in Coventry area so she knows the true traffic conditions around Warwick Uni, which is one value of 1 (bad traffic), 3 (medium traffic), and 5 (good traffic). *(The type of Nature θ)*
- Player 2 wants to join Warwick and is choosing where to live. He would prefer to live in one of the five locations: 1, 2, 3, 4, 5, where 1 is the closest to the uni but also most boring, and 5 is the furthest but also most lovely place.
- Player 1 is a good friend of Player 2 so she wants Player 2 to live closer to her, and Player 1 is living in 5 currently.
- Their respective payoffs are:
  - $v_1(a_2, \theta) = 5 - (\theta + 1.1 - a_2)^2$ → If *θ=5*, player 1 would want player 2 to choose 5; if *θ = 3*, player 1 would prefer player 2 to choose 5 over 3:
  - $v_2(a_2, \theta) = 5 - (\theta - a_2)^2$

A Motivating Example

Claim 18.1 *There is no perfect Bayesian equilibrium in which player 1 reports the true state of the world.*

Proof Assume in negation that player 1 truthfully reporting $a_1 = \theta$ is part of a perfect Bayesian equilibrium. It therefore must follow that when player 1 sends the message $a_1 \in \{1, 3, 5\}$ player 2 chooses $a_2 = a_1$. We saw that when $\theta = 3$ player 1 prefers $a_2 = 5$ over $a_2 = 3$. But if player 1 believes that player 2 will follow his advice then when $\theta = 3$ player 1's best response is $a_1 = 5$, a contradiction. ■

The intuition behind this result is simple, and easily generalizes to all such information-transmission games in which there is a bias between the sender and receiver with respect to the receiver's optimal choice. If it is indeed the case that the sender is reporting information truthfully, then it is in the receiver's best interest to take the sender's information at face value. But then if the receiver is acting in this way the sender has an incentive to lie.[5] The next observation is also quite straightforward:

Game Theory: An Introduction, by Steven Tadelis

A Motivating Example

Claim 18.2 *There exists a **babbling equilibrium** in which player 1's message reveals no information and player 2 chooses an action to maximize his expected utility given his prior belief.*

Proof To construct the babbling (perfect Bayesian) equilibrium let player 1's strategy be to send a message $a_1 \in \{1, 3, 5\}$ with equal probability of $\frac{1}{3}$ each regardless of θ. This means that the message is completely uninformative: player 2 knows that regardless of the message, $\Pr\{\theta\} = \frac{1}{3}$ for all $\theta \in \{1, 3, 5\}$. This implies that player 2 maximizes his expected payoff,

$$\max_{a_2 \in \{1,2,3,4,5\}} Ev_2(a_2, \theta) = 5 - \tfrac{1}{3}(-(1 - a_2)^2) + \tfrac{1}{3}(-(3 - a_2)^2) + \tfrac{1}{3}(-(5 - a_2)^2),$$

which is maximized when $a_2 = 3$.[6] Because each of player 2's information sets is reached with positive probability, player 2's beliefs are well defined by Bayes' rule everywhere, and player 1 cannot change these beliefs by changing his strategy.[7] Hence player 1 is indifferent between each of the three messages and is therefore playing a best response. ■

Construct and Verify

A Motivating Example — Continuous Case

Claim 18.4 *There exists a babbling perfect Bayesian equilibrium in which player 1's message reveals no information and player 2 chooses an action to maximize his expected utility given his prior belief.*

Proof We construct the equilibrium in a similar way to the finite case. Let player 1's strategy be to send a message $a_1 = a_1^B \in [0, 1]$ regardless of θ. This means that the message is completely uninformative and player 2 believes that θ is distributed uniformly on [0, 1]. This implies that, conditional on receiving the message a_1^B, player 2 maximizes his expected payoff,

$$\max_{a_2 \in \mathbb{R}} Ev_2(a_2, \theta) = \int_0^1 -(\theta - a_2)^2 d\theta = -\tfrac{1}{3} + 2a_2 - a_2^2,$$

which is maximized when $a_2 = \frac{1}{2}$. Let player 2's off-equilibrium-path beliefs be $\Pr\{\theta = \frac{1}{2} | a_1 \neq a_1^B\} = 1$ so that his off-the-equilibrium-path best response to any other message is $a_2 = \frac{1}{2}$ as well. It is easy to see that player 1 is indifferent between any of his messages and hence choosing $a_1 = a_1^B$ is a best response. ■

The probability density function for a uniform distribution taking values in the range a to b is:

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

Cheap Talk: Any Other Way to Improve?

- Two extreme cases: here is no truthful equilibrium and there is always a babbling equilibrium
  - What about something in between, i.e. not completely true but also truthful in terms of **some** information?
- Consider a two-message model: given the continuous type, play 1 can only report one of two message: low or high
- Recall the discrete example:
  - If the true type of the world is somewhere near 5, I would have the incentive to make it looks a bit higher to maximize my payoff
  - However, if the true type of the world is extremely low, I would not actually gain too much payoff by inducing player 2 to choose too high a number — I don't want to be too bad a friend!
- Let $a_2(a_1') < a_2(a_1'')$

Cheap Talk: Two Message Model

Claim 18.5 *In a two-message equilibrium player 1 must use a threshold strategy as follows: if* $0 \le \theta \le \theta^*$ *he chooses* a_1', *whereas if* $\theta^* \le \theta \le 1$ *he chooses* a_1''.

Proof For any θ player 1's payoffs from a_1' and a_1'' are as follows:

$$v_1(a_2(a_1'), \theta) = -(a_2(a_1') + b - \theta)^2$$

$$v_1(a_2(a_1''), \theta) = -(a_2(a_1'') + b - \theta)^2,$$

which implies that the extra gain from choosing a_1'' over a_1' is equal to

$$\Delta v_1(\theta) = -(a_2(a_1'') + b - \theta)^2 + (a_2(a_1') + b - \theta)^2.$$

The derivative of $\Delta v_1(\theta)$ is equal to $2(a_2(a_1'') - a_2(a_1')) > 0$ because $a_2(a_1') < a_2(a_1'')$. This implies that if type θ prefers to send message a_1'' over a_1' then every type $\theta' > \theta$ will also prefer a_1''. Similarly if type θ prefers to send message a_1' over a_1'' then so will every type $\theta' < \theta$. This in turn implies that if two messages are sent in equilibrium then there must be some threshold type θ^* as defined in claim 18.5. It follows that when $\theta = \theta^*$ player 1 must be indifferent between sending the two messages. ■

Cheap Talk: Two Message Model

Now that we know what restrictions apply to player 1's strategy in a two-message equilibrium, we can continue to characterize the strategy of player 2 in a two-message perfect Bayesian equilibrium as follows:

Claim 18.6 *In any two-message perfect Bayesian equilibrium in which player 1 is using a threshold θ^* strategy as described in claim 18.5, player 2's equilibrium best response is $a_2(a_1') = \frac{\theta^*}{2}$ and $a_2(a_1'') = \frac{1-\theta^*}{2}$.*

Proof This follows from player 2's posterior belief and from him playing a best response. In equilibrium player 2's posterior following a message a_1' is that θ is uniformly distributed on the interval $[0, \theta^*]$, and his posterior following a message a_1'' is that θ is uniformly distributed on the interval $[\theta^*, 1]$. Player 2 plays a best response if and only if he sets $a_2(a_1) = E[\theta | a_1]$, which proves the result. ■

Cheap Talk: Two Message Model

Using claims 18.5 and 18.6 we can now characterize the two-message perfect Bayesian equilibrium as follows:

Claim 18.7 *A two-message perfect Bayesian equilibrium exists if and only if* $b < \frac{1}{4}$.

Proof From claim 18.5 we know that when $\theta = \theta^*$ player 1 must be indifferent between his two messages so that

$$v_1(a_2(a_1'), \theta^*) = v_1(a_2(a_1''), \theta^*),$$

which from claim 18.6 and from the fact that $\frac{\theta^*}{2} < \theta^* < \frac{1-\theta^*}{2}$ is equivalent to

$$\theta^* + b - \frac{\theta^*}{2} = -\left(\theta^* + b - \frac{1-\theta^*}{2}\right). \qquad (18.1)$$

The solution to (18.1) is $\theta^* = \frac{1}{4} - b$, which can result in a positive value of θ^* only if $b < \frac{1}{4}$. To complete the specification of off-the-equilibrium-path beliefs, let player 2's beliefs be $\Pr\{\theta = \frac{\theta^*}{2} | a_1 \notin \{a_1', a_1''\}\} = 1$, so that he chooses $a_2 = \frac{\theta^*}{2}$, which causes player 1 to be indifferent between sending the message a_1' and any other message $a_1 \notin \{a_1', a_1''\}$, implying that his threshold strategy is a best response. ■

What About More Messages?

Three Messages: Consider the three-message equilibrium described in Section 18.2. Find the threshold values θ' and θ'' and show that for this to be an equilibrium it must be that $b < \frac{1}{12}$.

To construct a three-message equilibrium we first divide the interval $[0, 1]$ into three segments with a message for each: a_1' for $[0, \theta']$, a_1'' for $[\theta', \theta'']$, and a_1''' for $[\theta'', 1]$. Player 2's best response must be $a_2(a_1') = \frac{\theta'}{2}$, $a_2(a_1'') = \frac{\theta''+\theta'}{2}$, and $a_2(a_1''') = \frac{1+\theta''}{2}$. Finally player 1 must be indifferent between a_1' and a_1'' when $\theta = \theta'$, and indifferent between a_1'' and a_1''' when $\theta = \theta''$. This last condition yields two equations with two unknowns that will determine the equilibrium thresholds θ' and θ''.

You are left to solve this equilibrium in exercise 18.3 and show that it exists if and only if $b < \frac{1}{12}$. This should not be too surprising because the bias is what hampers player 1's ability to transmit more credible information in equilibrium. As the bias drops, we can construct finer and finer partitions of the interval $[0, 1]$ so that there is more meaningful communication between the sender and the receiver. In particular define $b_2 = \frac{1}{4}$ and $b_3 = \frac{1}{12}$. We have shown that if $b > b_2$ we have only a babbling equilibrium, if $b < b_2$ then we can construct a two-message equilibrium, and if $b < b_3$ then we can construct a three-message equilibrium.

Quick Recap on Cheap Talk Models

- First, no matter how small the bias is, as long as $b > 0$ there is no fully truthful equilibrium, which is what we demonstrated in claim 18.1.
- There exists a series $b_2 > b_3 > b_4 > \ldots > b_M$ such that if $b < b_M$, then an equilibrium with M partitions exists.
- In any equilibrium there must be some loss of information that depends on the magnitude of the bias b.
- If $b < b_M$ then there are M different perfect Bayesian equilibria, starting with the babbling equilibrium up until the "most informative" equilibrium with M partitions.

Example from 2021 Exam

A government procurement officer is trying to decide how many doses of a new coronavirus vaccine to order. This decision will depend on the effectiveness of the vaccine, which will be determined by clinical trials conducted by a scientific advisor. You may assume that the effectiveness of the vaccine is given by a random variable ε, uniformly distributed on the interval $[E_0, E_0 + 1]$. The scientific advisor believes the utility of ordering a quantity Q is $U_A(Q|\varepsilon) = 1 + \varepsilon Q - Q^2$; if perfectly-informed about effectiveness, the procurement officer would value Q at $U_G(Q|\varepsilon) = 1 + (\varepsilon + \beta)Q - Q^2$ where β is a non-negative constant. After the trials, the government officer asks the scientific advisor to report on the vaccine's effectiveness and purchases the quantity that maximises his expected utility. The scientific advisor is not paid for his efforts.

a. How would you set up this problem? Can the government advisor be sure of purchasing the optimal quantity (according to his preferences)? If so, how? If not, why not? How does your answer depend on the size of β? (**15 marks**)

b. Suppose that the minimum effectiveness is $E_0 = 25\%$ and that $\beta = 5\%$. Find the 'babbing equilibrium' for this situation – how much will the government order and what expected utilities will the two parties get? (**7 marks**)

c. Now construct a two-part equilibrium – depending on the advice they receive, the government will place either a small order Q^S or a large order Q^L. At what reported level of effectiveness will the government switch its order size, and what are the values of Q^S and Q^L? (**10 marks**)

d. How would you find the most efficient equilibrium (you do not have to compute it explicitly, but should say how it could be identified)? (**10 marks**) a

This is a cheap talk problem; should note that first-best can be achieved only if $\beta = 0$. They should note that there is always an equilibrium in which the government ignores the advisor and purchases the a priori optimal amount $Q_0(E_0, E_0 + 1)$, which they compute in the next part. The optimal strategy is to partition the range of effectiveness into intervals $[x, y]$ and associate to each interval the order that maximises expected utility $Q^*(x,y) = \underset{Q}{\operatorname{argmax}} \int_x^y U_G(Q|\varepsilon)d\varepsilon$. The more intervals, the more efficient is the outcome, but the number (and thus the efficiency) are bounded above by a decreasing function of β. Finally, they should note that for any two adjacent intervals $[x, y]$ and $[y, z]$, the scientific adviser would be indifferent between the purchase levels for both intervals if she was convinced that the true effectiveness was exactly y – in other words $U_A(Q^*(x,y)|y) = U_A(Q^*(y,z)|y)$.

b

In this case, there is only one purchase level regardless of report. If the government believes that the true state is uniformly distributed on $[a, b]$, it's expected utility for purchasing Q is

$$EU_G(Q) = 1 + \left(\frac{\int_a^b \varepsilon d\varepsilon}{b-a} + \beta\right)Q - Q^2 = 1 + \left(\frac{a+b+2\beta}{2}\right)Q - Q^2$$

Optimal Q is $\frac{a+b+2\beta}{4}$. In this case, $Q = \frac{.25+1.25+.1}{4} = 0.4$, $U_G = 1.16$, $U_A = 1.14$.

c

Denote the critical report level by ε^*. The two order sizes are

$$Q^S = \frac{.25 + \varepsilon^* + 2 * .05}{4} = \frac{.35 + \varepsilon^*}{4}$$

$$Q^L = \frac{1.25 + \varepsilon^* + 2 * .05}{4} = \frac{1.35 + \varepsilon^*}{4}$$

ε^* is defined by the condition that the advisor should be indifferent between Q^S and Q^L when the true state is ε^*. Solving $U_A\left(\frac{.35+\varepsilon^*}{4}\middle|\varepsilon^*\right) = U_A\left(\frac{1.35+\varepsilon^*}{4}\middle|\varepsilon^*\right)$ for ε^* gives $\varepsilon =$ (in general, for any level of β), $\varepsilon^* = 0.75 + 2\beta$; in this case,

$$\varepsilon^* = 0.85$$

$$Q^S = 0.3$$

$$Q^L = 0.55$$

d

The most efficient equilibrium is the one with the greatest number of intervals, so they should look for the largest n s.t. there exists a sequence $0.25 = \varepsilon^1, \dots, \varepsilon^n = 1.25$ where

$$Q^i = \frac{\varepsilon^i + \varepsilon^{i+1} + 0.1}{4}$$

And for each $i = 1, \dots, n-1$

$$1 + \varepsilon^{i+1}\left[\frac{\varepsilon^i + \varepsilon^{i+1} + 0.1}{4}\right] - \left[\frac{\varepsilon^i + \varepsilon^{i+1} + 0.1}{4}\right]^2 = 1 + \varepsilon^{i+1}\left[\frac{\varepsilon^{i+1} + \varepsilon^{i+2} + 0.1}{4}\right] - \left[\frac{\varepsilon^{i+1} + \varepsilon^{i+2} + 0.1}{4}\right]^2$$

Example from 2020 Exam

3. A The UK government has embarked on an ambitious plan to build a new high-speed rail network. The work is controlled by a dedicated firm set up to acquire land, commission construction and procure rolling stock. The firm has superior information regarding the actual costs and capabilities of the project.

First, suppose that the firm submits a report to the government detailing the current cost and revenue projections (simplified to $reported\ NPV = V$), along with a check equal to V. The true value may be higher or lower; if it is higher, the firm keeps the surplus. When the contract was signed it was believed that V was uniformly distributed on an interval $[V_0, V_1]$. The government can either accept V or call in an auditor. This will incur an audit cost of C^{audit} (paid by the government). Both the firm and the government are risk neutral. The firm has an outside offer worth W_0; if it does not expect at least this much, it will not sign the contract.

a. If the government must either accept or reject the report, what will its optimal strategy look like (i.e. which reports would it threaten to reject to maximise the expected (net of audit cost) payment by the firm)? Justify your answer. [**10 marks**]

To optimal strategy will be to set a 'trigger' level V, reject any report below this level and retain all NPV in the case of an audit. The reason why this is optimal is standard (see slides); the trigger level is set to balance the losses at higher values of V against the increased cost of audit. The government retains all income in the case of an audit because it could replace any contract that allowed the firm residual profit in the case of audit with an equivalent one that retained all income but audited less frequently. The government's problem is to choose V to maximise

$$U_{gov}(V) = \int_{V_0}^{V} (w - C)\, dw + V \int_{V}^{V_1} dw \;\; s.t.$$

$$U_{firm}(V) = \int_{V}^{V_1} (w - V) dw \geq W_0$$

b. If $V_0 = 10, V_1 = 100, C^{audit} = 30, and\ W_0 = 0$, what are the expected audit cost, expected government receipts (gross) and expected profit for the firm? [**12 marks**]

The interior optimum (where the firm's participation constraint does not bind) is $V = V_1 - C$. The requested quantities are:

$$EC^{audit} = (V_1 - V_0 - C)C = 1800$$

$$EU_{gov} = (V - V_0)V_0 + \frac{(V - V_0)^2}{2} + (V_1 - V)V = 2700$$

$$EU_{firm} = \frac{(V_1 - V)^2}{2} = 450$$

c. What would the trigger value and associated expected net payoff to the government be if $W_0 = 1250$? [**8 marks**]

In this case the government would have to set V at $V = V_1 - \sqrt{2W_0} = 50$ and expected payoff drops to 2500.

d. Would the government prefer a contract that called the auditors in with a probability that decreased with the firm's reported value? (justify your answer) [**5 marks**]

Yes; ideally it would audit with probability 1 at $V = V_0$ and drop to 0 at $V = V_1$; this would economise on audit cost and give the government a greater share of profits above the old trigger value. The audit cost savings mean that both the firm and the government are better off.

Now suppose instead that the firm assesses the state of demand and cost for the project, and reports to the government the amount C^B that the project should cost to build. The government will respond to this report by providing a budget B for the project. The 'real' cost is given by a random variable C that is uniformly distributed on an interval $[C_0, C_1]$. The value to society and to the firm depend on the true cost and the budget as follows

$$V^{soc} = V^* - (B - C)^2$$
$$V^{firm} = W_0 - (B - C - \beta)^2$$

Where β represents the 'inaccuracy' of the government's post-project audits; the more the firm asks for compared to the actual cost, the greater the risk of a demand for repayment and a penalty.

e. Under the optimal contract, how would the government respond to the firm's different possible budget requests? How would the gap between maximum social value (if the government knew C exactly in advance) and the amount achieved by the optimal contract to change if β were to decrease (post-project audits became more accurate)? (You need not provide a mathematical solution, but should clearly justify your answer) [**15 marks**]

This si a standard cheap talk problem (lightly disguised). The discussion should note that the optimum involves a finite set of budget levels the government would approve; each is the midpoint of an interval of possible reports, set so that a firm with a true value at the boundary of the interval is just indifferent between the next highest and next lowest 'acceptable' values. For any value of β there are a set of equilibria with one interval (babbling equilibrium = least efficient) up to a maximum number n*. The lower the value of β (the more accurate the post-project audits), the greater this maximum number and the more efficient the 'best' feasible equilibrium.

Principle-Agent Problem, in General

- Principal pays money
- Agent possesses information and takes action
- Outcomes for both depend on action and type
- Two usual constraints:
  - Incentive-compatibility (i.e. no incentive to deviate)
  - Participation (i.e. taking action should be better than the outside option)
- Most importantly, don't freak out: these problems essentially just consists of several parts:
  - Calculate expected payoff, either using summation (discrete) or integral (continuous)
  - Utility maximization problem solving
  - Check incentive and participation compatibility to make sure it's an equilibrium

2022 Exam Question 3

3. A hedge fund is interested in taking over a formerly-profitable local railroad, hoping to benefit from a post-Covid return to commuting. The railroad's profits are determined by demand (a random parameter θ distributed uniformly on $[0,1]$) and by their effort e. The profit function is θe, the cost of effort is $\frac{e^2}{2}$ and the existing management's outside option $\bar{U} > 0$. The hedge fund wants to specify a contract paying the existing managers bonuses of β – in other words, if the railroad produces profit π the investors get $\pi - \beta$ and the managers get β. The managers must accept or reject the contract before observing θ but choose effort after observing it. All parties are risk neutral.

a. Assume the hedge fund can observe π, but cannot separately observe e or θ. Bonuses are computed as a proportion α of profits: $\beta = \alpha\pi$ where α is a (not necessarily positive) parameter. Find the managers' optimal effort as a function of α, $\bar{U}$ and π and use this to compute the value of α that maximises the hedge funds' expected utility. Also, find the optimised expected value of the railroad as a function of $\bar{U}$. **(20 marks)**

b. How (if at all) would your answer differ if the managers observed θ before accepting or rejecting the contract? **(15 marks)**

c. Now suppose that the hedge fund must pay off the managers before observing actual profits π, but that they have the opportunity to audit (observe) either the random state θ or effort e, but not both. In other words, they can choose a contract that depends on θ ($\beta = \gamma\theta$ for some 'profit share' γ) or one that depends on e ($\beta = \omega e$ for some 'effort wage' ω), but not both. As before, the managers accept or reject the offer before observing θ. Which would the investors prefer to observe? **(15 marks)**

a. Assume the hedge fund can observe π, but cannot separately observe e or θ. Bonuses are computed as a proportion α of profits: $\beta = \alpha\pi$ where α is a (not necessarily positive) parameter. Find the managers' optimal effort as a function of α, $\bar{U}$ and π and use this to compute the value of α that maximises the hedge funds' expected utility. Also, find the optimised expected value of the railroad as a function of $\bar{U}$. (**20 marks**)

Answer: First write the problem facing the principal (hedge fund).

$$W = \max_{\alpha} \int_0^1 \theta e(1-\alpha)d\theta \text{ s.t.}$$

$$V = \int_0^1 \max_{e}\left[\alpha\theta e - \frac{e^2}{2}\right]d\theta \geq \bar{U}$$

From this [**5 marks**],

$$e^*(\theta,\alpha,\bar{U}) = \alpha\theta$$

The managers' expected utility is

$$\frac{\alpha^2}{2}\int_0^1 \theta^2 d\theta = \frac{\alpha^2}{6}$$

So we can write hedge fund's problem as

$$W = \max_{\alpha\geq\sqrt{6\bar{U}}} \alpha(1-\alpha)\int_0^1 \theta^2 d\theta = \max_{\alpha\geq\sqrt{6\bar{U}}} \frac{\alpha(1-\alpha)}{3}$$

This gives optimal contract and expected hedge fund utility:

$$(\alpha^*, W^*) = \begin{cases} \left(\frac{1}{2},\frac{1}{12}\right) \text{ if } \bar{U} \leq \frac{1}{24} \\ \left(\sqrt{6\bar{U}}, \frac{\sqrt{6\bar{U}}-6\bar{U}}{3}\right) \text{ if } \bar{U} \in \left(\frac{1}{24},\frac{1}{6}\right) \\ (0,0) \text{ otherwise} \end{cases}$$

Value of railroad is $\int_0^1 \theta\, e^*(\theta,\alpha,\bar{U})d\theta = \alpha\int_0^1 \theta^2\, d\theta = \begin{cases} \frac{1}{6} \text{ if } \bar{U} \leq \frac{1}{24} \\ \frac{\sqrt{6\bar{U}}}{3} \text{ if } \bar{U} \in \left(\frac{1}{24},\frac{1}{6}\right) \\ 0 \text{ otherwise} \end{cases}$ [**3 marks**]

b. How (if at all) would your answer differ if the managers observed θ before accepting or rejecting the contract? **(15 marks)**

Now we write the hedge fund problem as:

$$W = \max_{\alpha} \int_0^1 \theta e(1-\alpha)d\theta \text{ s.t.}$$

$$V = \max_{e}\left[\alpha\theta e - \frac{e^2}{2}\right] \geq \bar{U}$$

So

$$e^*(\theta,\alpha,\bar{U}) = \begin{cases} \alpha\theta \text{ if } \theta \geq \hat{\theta} = \dfrac{\sqrt{2\bar{U}}}{\alpha} \\ 0 \text{ otherwise} \end{cases}$$

Which lets us write the hedge funds' problem as

$$W = \max_{\alpha} \int_{\hat{\theta}}^1 \theta e(1-\alpha)d\theta = \max_{\alpha} \frac{\alpha(1-\alpha)\left(\alpha^3 - (2\bar{U})^{\frac{3}{2}}\right)}{3\alpha^3}$$

Within this range, the optimal α can be found by solving the following quartic

$$\alpha^3 - 2\alpha^4 - (2\bar{U})^{\frac{3}{2}}(\alpha - 2) = 0$$

I do not expect them to solve this. Very clever ones might note that the optimal α rises and the corresponding W falls as $\bar{U}$ rises from

$$\bar{U} = 0, \text{ where } \alpha = \frac{1}{2} \text{ and } W = \frac{1}{12}$$

up to

$$\bar{U} = \frac{1}{2}, \text{ where } \alpha = 1 \text{ and } W = 0$$

c. Now suppose that the hedge fund must pay off the managers before observing actual profits π, but that they have the opportunity to audit (observe) either the random state θ or effort e, but not both. In other words, they can choose a contract that depends on θ ($\beta = \gamma\theta$ for some 'profit share' γ) or one that depends on e ($\beta = \omega e$ for some 'effort wage' ω), but not both. As before, the managers accept or reject the offer before observing θ. Which would the investors prefer to observe? **(15 marks)**

1. State-dependent contract: the hedge fund solves

$$W = \max_b \int_0^1 \theta(e-b)d\theta \text{ s.t.}$$

$$V = \max_e \int_0^1 \left(b\theta - \frac{e^2}{2}\right) d\theta \geq \bar{U}$$

Obviously, optimal effort is $e^* = 0$. The contract is only accepted if $b \geq 2\bar{U}$ but then hedge fund's payoff would be negative, so this type of contract yields utility 0.

2. Effort-dependent contract. Manager's optimal effort and payoff are independent of θ:

$$\left[\omega e - \frac{e^2}{2}\right] \int_0^1 \theta d\theta = \frac{2\omega e - e^2}{2}$$

Optimal effort is $e^* = \omega$. Manager's expected payoff is $\frac{\omega^2}{2}$; which beats outside option if $\omega^2 \geq 2\bar{U}$. So the hedge fund's problem is:

$$W = \max_{\omega \geq \sqrt{2\bar{U}}} \left\{ \int_0^1 \omega(\theta - \omega)d\theta \right\} = \max_{\omega \geq \sqrt{2\bar{U}}} \left\{ \omega \int_0^1 \theta d\theta - \omega^2 \right\} = \max_{\omega \geq \sqrt{2\bar{U}}} \left\{ \frac{\omega}{2} - \omega^2 \right\}$$

At an interior optimum, $\omega = \frac{1}{4}$ and $W = \frac{1}{16}$. Putting in the corner solutions, the optimal contract and expected return for the hedge fund are

$$(\omega^*, W^*) = \begin{cases} \left(\frac{1}{4}, \frac{1}{16}\right) \text{ if } \bar{U} \leq \frac{1}{32} \\ \left(\sqrt{2\bar{U}}, \sqrt{\frac{\bar{U}}{2}} - 2\bar{U}\right) \text{ if } \bar{U} \in \left(\frac{1}{32}, \frac{1}{8}\right) \\ (0,0) \text{ otherwise} \end{cases}$$

Therefore, they strictly prefer the effort-dependent contract unless $\bar{U} \geq \frac{1}{8}$ when they are indifferent.

Seminar Problem 3 Question 2

A firm is being run by a manager. A risky project requires an investment of £200,000. The return is given by R(ω) = £500,000 * ω, where ω is uniformly distributed on [0, 1]. The manager is supposed to report the state to the investors and pay them the amount h(ω). The investor(s) can audit the firm for a cost of £100,000. The manager has initial wealth of £100,000, and a reservation payoff of £200,000. A contract for the entrepreneur specifies his initial stake (the amount s of his own money he should invest in the project), the repayment scheme h(ω) and the set of states E where the auditors will be called in.

a. Write the participation (IR) and incentive compatibility (IC) constraints, the expected payoff for the manager and the shareholders, and the shareholders contract design problem.

b. What should σ be?

c. In the optimal contract, how much should the manager pay the investors in each state?

d. In which states (if any), will the shareholders call in the auditors?

Comment: to solve this, we wrote the expected payoffs an used the result from class that the optimal contract involved full equity participation by the manager and a 'debt contract' (pay a fixed amount if you can afford it, and all revenue otherwise). To prevent cheating, the firm must audit whenever the manager pays less than h*. Finally, the optimal h* is completely determined by the IR constraint – the shareholders want h* to be as big as possible for two reasons: it maximises their income in states where h* can be afforded and it shrinks the set of states where they pay audit costs. They will push h* up until the manager is indifferent between accepting and rejecting the job.

First things first: identify who is who

- Manager: Agent
- Investor: Principal
- Manager pays *s* into the project, decides the payment scheme to investors, gets whatever is left from the project, and risks auditors being called to investigate whether the state is reported correctly

a. Write the participation (IR) and incentive compatibility (IC) constraints, the expected payoff for the manager and the shareholders, and the shareholders contract design problem.

$$\text{IR}: \int_0^1 [500000\omega - h(\omega)]d\omega + 100000 - s \geq 200000$$

$$\text{IC}: \forall \omega' \neq \omega, \text{either} \begin{cases} \omega' \notin \text{E and } 500000\omega - h(\omega) \geq 500000\omega' - h(\omega') \\ \omega' \in \text{E and } 500000\omega \geq h(\omega) \end{cases}$$

$$\text{Manager gets}: \int_0^1 [500000\omega - h(\omega)]d\omega + 100000 - s$$

$$\text{Shareholders get}: \text{V} = \int_0^1 h(\omega)d\omega - 200000 + s - 100000 \int_{\omega \in \text{E}} d\omega$$

$$\text{Problem}: \text{maximise V s.t. IR, IC, } s \in [0,100000], h(\omega) \leq 500000\omega$$

Solutions

b. What should s be?

WLOG, can take s = £100,000

c. In the optimal contract, how much should the manager pay the investors in each state?

Optimal contract specifies a fixed payment h*; shareholders get min{500000ω, h*}-100000-audit cost; manager gets max{0,500000ω-h*}. To find h*, need to determine E – also, h* should give manager exactly £200000 (next step). Note also that for any h*, the manager can just make the payment at the state ω* = h*/500000).

d. In which states (if any), will the shareholders call in the auditors?

Let $\alpha = 500000, \beta = 100000\ and\ \gamma = 200000$. The principal gets $V := \frac{2(\alpha-\beta)h-h^2}{2\alpha}$; the agent gets $\frac{(\alpha-h)^2}{2\alpha}$.

The agent's payoff exceeds its outside offer if $h \notin \left[\alpha - \sqrt{2\alpha\gamma}, \alpha + \sqrt{2\alpha\gamma}\right]$ i.e. $h \leq 52786.4$ or $h \geq 947213.6$. These boundaries give payoffs to the principal of 39443 and -139443, resp.

Assuming an interior solution gives $h = \alpha - \beta = 400000$, but that would be rejected by the agent. The best acceptable offer from the perspective of the principal is the lower boundary, 52786.4

$$\int_{\frac{h^*}{\alpha}}^{1} (\alpha w - h^*)\,dw = \frac{\alpha}{2} w^2 - h^* \cdot w \Big|_{\frac{h^*}{\alpha}}^{1}$$

$$= \frac{\alpha}{2} - h^* - \left(\frac{\alpha}{2} \cdot \frac{h^{*2}}{\alpha^2} - \frac{h^{*2}}{\alpha}\right)$$

$$= \frac{\alpha}{2} - h^* - \frac{h^{*2}}{2\alpha} + \frac{h^{*2}}{\alpha}$$

$$= \frac{\alpha^2 - 2\alpha h^* + h^2}{2\alpha}$$

$$= \frac{(\alpha - h)^2}{2\alpha}$$

Let $\frac{(\alpha-h)^2}{2\alpha} \geq r$ Solve h.

$$\frac{h^2 - 2\alpha h + \alpha^2 - 2\alpha r}{2\alpha} \geq 0$$

$$h^2 - 2\alpha h + (\alpha^2 - 2\alpha r) \geq 0$$

$$h^* = \frac{2\alpha \pm \sqrt{4\alpha^2 - 4\alpha^2 + 8\alpha r}}{2} = \alpha \pm \sqrt{2\alpha r}$$

$$V = \int_0^1 h(w)\,dw - \beta - \beta \int_0^{\frac{h^*}{\alpha}} dw$$

$$= \int_0^{\frac{h^*}{\alpha}} \alpha \cdot w\,dw + h^* \int_{\frac{h^*}{\alpha}}^{1} dw - \beta - \beta \int_0^{\frac{h^*}{\alpha}} dw$$

$$= \frac{\alpha}{2} w^2 \Big|_0^{\frac{h^*}{\alpha}} + h^* \cdot w \Big|_{\frac{h^*}{\alpha}}^{1} - \beta - \beta \cdot w \Big|_0^{\frac{h^*}{\alpha}}$$

$$= \frac{\alpha}{2} \cdot \frac{h^{*2}}{\alpha^2} + h^* - \frac{h^{*2}}{\alpha} - \beta - \beta \cdot \frac{h^*}{\alpha}$$

$$= \frac{h^2 + 2\alpha h - 2h^2 - 2\alpha\beta - 2\beta h}{2\alpha}$$

$$= \frac{2(\alpha - \beta)h - h^2 - 2\alpha\beta}{2\alpha}$$