Things to know

1. What are (strictly) dominant strategies?
2. Definition of Nash equilibrium + Conditions
    a. Non-cooperative (No joint actions)
    b. Complete information
    c. Solution Concept: Best response to others best response
3. Definition of Correlated equilibrium
    a. Models some form of communication
    b. Sort of allows probabilities to differ depending on what the other plays: Hence the term "Correlated".
    c. Public Signal which gives recommendations of strategies to play
    d. No observation of recommendation other receives
    e. Solution Concept: Following recommendation is best response for everyone
4. All Nash equilibria can be formulated as Correlated equilibria as well. Correlated equilibria are more general.
5. What is Cournot Competition, Betrand Competition?

Problem Set 1b

1a) Game of Battle

Normal Form

(1,2)	F	O
F	5,1	0,0
O	4,4	1,5

1b) Solving for Nash equilibria

Note that Nash equilibria in these kinds of questions usually implies both *pure strategy* and *mixed strategy* Nash equilibria.

Step 1: Our aim is to get the best responses of each Player given an assumed fixed strategy of the other.

(1,2)	F (σ)	O $(1-\sigma)$
F (ρ)	5,1	0,0
O$(1-\rho)$	4,4	1,5

Step 1a) Assume Player 2 plays F with probability σ. Then,

$$\text{Payoff of Player 1} = \begin{cases} 5\sigma & if\ plays\ F \\ 4\sigma + (1-\sigma) & if\ plays\ O \end{cases}$$

Player 1's best response is thus

Play F, if $\sigma > \frac{1}{2}$

Play any strategy which mixes both F and O, if $\sigma = \frac{1}{2}$

Play O, if $\sigma < \frac{1}{2}$

Step 1b) Assume Player 1 plays F with probability ρ. Then,

$$\text{Payoff of Player 2} = \begin{cases} \rho + 4(1-\rho) & \textit{if plays F} \\ 5(1-\rho) & \textit{if plays O} \end{cases}$$

Player 2's best response is thus

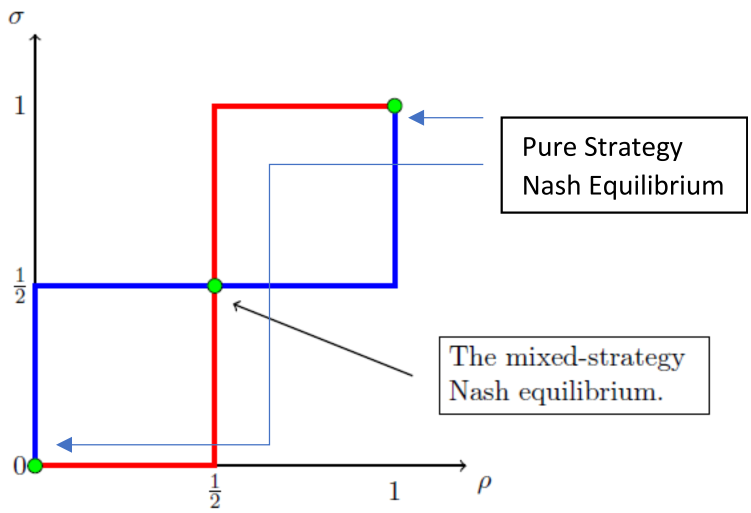
Play F, if $\rho > \frac{1}{2}$

Play any strategy which mixes both F and O, if $\rho = \frac{1}{2}$

Play O, if $\rho < \frac{1}{2}$

Step 2: Draw out the best responses in a diagram to obtain all the Nash equilibria.

These are the points of intersection between the two best-response curves (why?).



So, in total there are 3 different Nash Equilibria (ρ, σ)[1]:

1) $(1,1)$: Both Player 1 and Player 2 play F.
2) $(0,0)$: Both Player 1 and Player 2 play O.
3) $(\frac{1}{2}, \frac{1}{2})$: Both Player 1 and Player 2 play a mixed strategy where there play F with probability $\frac{1}{2}$ (and implicitly O with probability ½ since there are only 2 actions).

Note 1: This should usually be summarized at the end of solving the question.

Note 2: This technique is meant for cases where each Player has 2 discrete actions and drawing out the figure will give you <u>all</u> NE. However, there is an easier way to solve for <u>Pure Strategy</u> by just examining the payoff table (carefully).

1c) Payoffs of each player in each Nash equilibrium.

Just calculate (expected) payoffs given the strategies you have:

1. (5,1)
2. (1,5)
3. (5/2, 5/2)

For 3, as an example Player 1's payoff is given by

$$\left(\frac{1}{4} \times 5\right) [F,F] + \left(\frac{1}{4}\times 0\right) [F,O] + \left(\frac{1}{4}\times 4\right) [O,F] + \left(\frac{1}{4}\times 1\right) [O,O].$$

1d) Show that prescribed random strategies form a correlated equilibria.

Probabilities of signals

(1,2)	F	O
F	1/3	0
O	1/3	1/3

We need to show that when a player is given a signal to play X, he will follow the recommendation assuming that the other player also always follows the recommendation. (Remember that the player does not know the other's recommendation.)

<u>Step 1: Calculate the conditional probabilities of the other player's signal, given own signal.</u>

Player 1's Signal	**Probability that other is told to play F**	**Probability that other is told to play O**
Play F	$1/3 \div 1/3 = 1$	0
Play O	$1/3 \div (1/3 + 1/3) = 1/2$	$1/2$

[1] The answer given in the solutions has a different way of writing it, but as long as you define your notation clearly, it should be fine.

Player 2's Signal	Probability that other is told to play F	Probability that other is told to play O
Play F	${}^{1}/_{3} \div ({}^{1}/_{3} + {}^{1}/_{3}) = {}^{1}/_{2}$	${}^{1}/_{2}$
Play O	0	${}^{1}/_{3} \div {}^{1}/_{3} = 1$

Step 2: Check that they following each recommendation is the best response, assuming that the other follows the recommendation.

Expected payoffs are calculated by using the above conditional probabilities given the recommendation.

If Player 1 is told to play F, her expected payoff is 5 if she plays F, but 4 if she plays O.

If Player 1 is told to play O, her expected payoff is $\frac{1}{2}(4) + \frac{1}{2}(1) = \frac{5}{2}$ if she plays O, but $\frac{1}{2}(5) + \frac{1}{2}(0) = \frac{5}{2}$ if she plays F.

Thus, Player 1 will not want to strictly deviate from either recommendation.

If Player 2 is told to play F, her expected payoff is $\frac{1}{2}(4) + \frac{1}{2}(1) = \frac{5}{2}$ if she plays F, but $\frac{1}{2}(5) + \frac{1}{2}(0)$ if she plays O.

If Player 2 is told to play O, her expected payoff 5 if she plays O, but 4 if she plays F.

Thus, Player 2 will not want to strictly deviate from either recommendation.

1e) Compare the expected payoffs in the correlated equilibrium with the mixed strategy NE.

Payoffs in the correlated equilibrium are:

$$\left(\frac{1}{3}(5) + \frac{1}{3}(4) + \frac{1}{3}(1), \left(\frac{1}{3}(1) + \frac{1}{3}(4) + \frac{1}{3}(5),\right)\right) = \left(\frac{10}{3}, \frac{10}{3}\right) > \left(\frac{5}{2}, \frac{5}{2}\right)$$

2a) Cournot Quantity Competition

Step 1: Get each firm's profits in quantity terms

$$\pi_1 = (p_1 - c)q_1 = (a - q_1 - \gamma q_2)q_1$$

$$\pi_2 = (p_2 - c)q_2 = (a - q_2 - \gamma q_1)q_2$$

Step 2: Maximise each firm's profits with respect to their own quantity, holding the quantity of the other firm fixed to get reaction functions.

For Firm 1: $\max_{q_1}(a - q_1 - \gamma q_2)q_1$ gives

$q_1 = \frac{a-\gamma q_2 - c}{2}$

For Firm 2: $\max_{q_2}(a - q_2 - \gamma q_1)q_2$ gives

$q_2 = \frac{a-\gamma q_1 - c}{2}$

Step 3: Solve the two simultaneous equations

We thus get

$q_1 = \frac{a-c}{2+\gamma}$, $q_2 = \frac{a-c}{2+\gamma}$

Step 4: Substitute q_1, q_2 into the inverse demand functions

We then get

$$p_1 = \frac{a + c + \gamma c}{2 + \gamma}, p_2 = \frac{(a + c + \gamma c)}{2 + \gamma}$$

Note: As you can see, both prices and quantities in equilibrium are exactly the same! The reason for this is that the two firms are symmetric (they have exactly the same inverse demand functions and cost). Thus, this should be expected. In fact, we can use a shortcut at Steps 2-3 by setting $q_1 = q_2$ without having to use the other firms' best response function. If you use this method, you should explain why you can use this shortcut.

2b) Bertrand Price Competition. For simplicity, let us assume that $0 < \gamma < 1$ (When $\gamma = 0$, they are separate monopolies; when $\gamma = 1$ they are like perfect substitutes. For a more mathematical approach when $\gamma = 1$, refer to last years' Extra Stuff for PS1B document.)

Step 1: Get each firm's profit in terms of only prices.

This involves first solving for indirect demand functions (i.e. quantities in terms of prices).

We have 2 linear equations:

$$p_1 = a - q_1 - \gamma q_2$$

$$p_2 = a - q_2 - \gamma q_1$$

From the 2nd equation, $q_2 = a - p_2 - \gamma q_1$.

Substituting into the 1st equation, we get $p_1 = a - q_1 - \gamma(a - p_2 - \gamma q_1)$.

This then simplifies to $q_1 = \frac{a(1-\gamma)-p_1+\gamma p_2}{1-\gamma^2}$.

Substituting it back into the converted second equation, we get $q_2 = \frac{a(1-\gamma)-p_2+\gamma p_1}{1-\gamma^2}$.

We can then substitute these q back into the profit function which will then be in terms of p only.

<u>Step 2: Maximise the profit functions to get reaction functions.</u>

For Firm 1: $\max_{p_1} \left(\frac{a(1-\gamma)-p_1+\gamma p_2}{1-\gamma^2}\right)(p_1 - c)$ gives

$p_1 = \frac{c+\gamma p_2+a(1-\gamma)}{2}$

For Firm 2: $\max_{p_2} \left(\frac{a(1-\gamma)-p_2+\gamma p_1}{1-\gamma^2}\right)(p_2 - c)$ gives

$p_1 = \frac{c+\gamma p_1+a(1-\gamma)}{2}$

<u>Step 3: Solve the two simultaneous equations to get p_1, p_2</u>

Here, we get

$$p_1 = \frac{a(1-\gamma)+c}{2-\gamma}, p_2 = \frac{a(1-\gamma)+c}{2-\gamma}$$

<u>Step 4: Substitute into the indirect demand functions obtained earlier.</u>

This gives:

$$q_1 = \frac{a-c}{(2-\gamma)(1+\gamma)}, q_2 = \frac{a-c}{(2-\gamma)(1+\gamma)}$$

Note that the comment about symmetric firms can be applied here at steps 2-3 again.

2c) Compare welfare consequences for consumers and producers under both kinds of competition.

$$p_{Bertrand} = \frac{a(1-\gamma)+c}{2-\gamma} \;, q_{Bertrand} = \frac{a-c}{(2-\gamma)(1+\gamma)}$$

$$p_{cournot} = \frac{a+c+\gamma c}{2+\gamma} \;, q_{cournot} = \frac{a-c}{2+\gamma}$$

Let us compare quantities for simplicity:

In particular, let us check under what conditions $q_{Betrand} > q_{cournot}$:

$$\frac{a-c}{(2-\gamma)(1+\gamma)} > \frac{a-c}{2+\gamma}$$

$$(2+\gamma) > 2+\gamma-\gamma-\gamma^2$$

$$\gamma+\gamma^2 > 0$$

So, this always occurs for any $0 < \gamma < 1$.

This implies that $p_{Bertrand} < p_{cournot.}$

So, consumers are better off under Bertrand competition because they face lower prices and consume more as well.

Looking at producers,

$$\pi_{Bertrand} = \left(\frac{a(1-\gamma)+c}{2-\gamma} - c\right)\left(\frac{a-c}{(2-\gamma)(1+\gamma)}\right)$$

$$\pi_{cournot} = \left(\frac{a+c+\gamma c}{2+\gamma} - c\right)\left(\frac{a-c}{2+\gamma}\right)$$

We can show that the condition for $\pi_{cournot} > \pi_{betrand}$ is $2\gamma^3 > 0$. So, it always holds for $0 < \gamma < 1$.

Overall, deadweight welfare loss should rise when prices are further away from MC and since Cournot prices are further away from MC, there should be higher deadweight welfare losses and lower social welfare.

Note that if prices are lower than marginal cost, it would be better not to produce, so to compare both scenarios, it would be better to restrict $\gamma \leq 1$ such that $p_{Bertrand} \geq c$. This implies that we have $c < p_{Bertrand} < p_{cournot}$ and that Bertrand prices are closer to marginal costs.