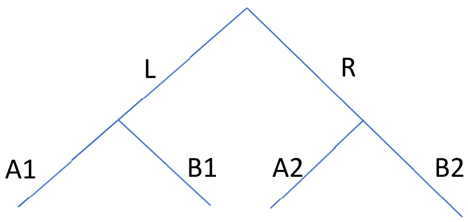
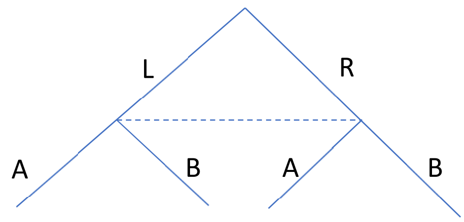
Things to know:

1. Extensive form (game tree) and Normal form (table) representation
2. Simultaneous vs Sequential moves (Information available to each player and thus possibility of conditioning)
3. Perfect vs Imperfect Information
    a. Whether one knows the action choices of others from an earlier stage





Dotted line in right figure indicates that player 2 doesn't know whether L or R is played.

4. Repeated games
5. What is a sub-game perfect Nash equilibria, what is a subgame (in the perfect and complete information setting we have here)
    a. Starts with single node, contains all nodes that are successors and their successors as well.
6. What are max-min strategies?
7. Strictly competitive games in pure and mixed strategies.
    a. $u_1(a_1, a_2) \geq u_1(b_1, b_2) \Leftrightarrow u_2(b_1, b_2) \geq u_2(a_1, a_2)$ where a,b are PS.
    b. $EU_1(p_1, p_2) \geq EU_1(q_1, q_2) \Leftrightarrow EU_2(q_1, q_2) \geq EU_2(p_1, p_2)$ where p, q are MS.

    Note that this implies $u_1(a_1, a_2) = u_1(b_1, b_2) \Leftrightarrow u_2(b_1, b_2) = u_2(a_1, a_2)$ *and likewise in b) as well.*
8. Relation between max-min strategies and mixed strategy NE in strictly competitive games.

1i) Solve for the Nash equilibria:

1,2	L	M	R
U	8,8	0,9	0,0
C	9,0	0,0	3,1
D	0,0	1,3	3,3

3 Pure Strategy NE: (C,R), (D,M) and (D,R)

1ii) Describe a SPNE where players select (U,L) in the first period.

In repeated games, it is possible to construct SPNE where the strategies in later periods depend on outcomes in earlier periods; this is because players are assumed to observe what happens then. This means that earlier outcomes in the stage game in a SPNE need not be NE in the stage game by itself; for example (U,L) here.

The idea here is to construct some kind of punishment which utilises Nash Equilibrium outcomes if they deviate from the desired strategy: if Player 1 deviates, select the NE which gives him a lower payoff; If player 2 deviates, select the NE which gives him a lower payoff.

A necessary condition for this is that we have multiple Nash equilibria which each prefers over the other. Using NE outcomes as punishment means that if there is a deviation, the strategies chosen will be NE in their respective subgames starting after the deviation.

The strategies are defined as follows:

Observation of Period 1	**Player 1's Strategy**	**Player 2's Strategy**
$(not\ U, L)$	D	M
$(U, not\ L)$	C	R
$(not\ U, not\ L)$	D	R
(U, L)	D	R

Seeing that this is a subgame perfect equilibrium:

In the second period, we have 4 different subgames which depend on the history observed:

1. $(not\ U, L)$
2. $(U, not\ L)$
3. $(not\ U, not\ L)$
4. (U, L)

Since we have set actions of each player such that they will play a NE strategy of the stage game in each case, it will always be a NE in each sub-game.

In the first period, we check that following the period 1 strategies is a NE given the strategies for period 2.

If player 1 deviates, he gets at most 9+1. If he follows the strategy, he gets 8+3=11.

Exactly the same for player 2. Thus, the first period strategies are a NE of the overall subgame.

Hence, we have a SPNE because the strategies constitute a NE in every subgame of the 2 period repeated game.

2i)

1,2	X	Y
A	5,6	0,0
B	8,2	2,2

There are 2 pure strategy NE: (B,X) and (B,Y).

2ii) Construct an equilibrium where the players select (A,X) in the first period.

Observation of Period 1	**Player 1's Strategy**	**Player 2's Strategy**
$(not\ A, X)$	B	Y
$(A, not\ X)$	B	Y (X is also ok)
$(not\ A, not\ X)$	B	Y
(A, X)	B	X

Gist of equilibrium: select NE (B,Y) if outcome is not (A,X) in period 1, otherwise select NE (B,Y).

In period 1,

If Player 1 deviates and plays B, will get 8+2. If plays A, will get 5+8.

If Player 2 deviates and plays Y, will get 0+2. If plays X, will get 6+2.

Thus, neither will want to deviate and we have that strategies constitute a NE of the overall game.

Note: during the seminar, there was a question about whether SPNE which support any payoff profile exist. In general, not all payoffs can be supported: for example, in question 1 above, we will not be able to support (9,0) in Period 1. Intuitively, this is because one of the payoffs (0) is below all of the NE payoffs, which means that it will not be possible to punish Player 2 for deviating. Another required factor would be sufficient patience. In particular, for your interest there is the following theorem which applies to infinitely repeated games:

Theorem (Nash Threat Folk Theorem.)

Let α^* be a NE of the stage game G with payoff vector e^*. Then for any π such that $\pi_i \geq e_i^*$ for all $i \in N$ there exists a $\underline{\delta}_\pi$ such that for every $\underline{\delta}_\pi < \delta_\pi < 1$ there exists a Subgame Perfect equilibrium of $G^\infty(\delta)$ with payoff vector π.

3) To find out which is strictly competitive in pure strategies, arrange outcomes in ascending order of Player 1's preferences and check that Player 2's preferences are descending.

(a)	(b)	(c)	(d)
2,9	2,8	0,2	1,1
3,8	3,6	1,1	2,2
4,7	4,5	1,1	3,3
5,6	6,4	2,0	4,4
6,5	7,2		
7,4	7,2		
7,3	8,1		
8,2	8,1		
9,1	9,0		
N	Y	Y	N

Hint: constant sum games are always strictly competitive in pure and mixed strategies.

Note:

Strictly competitive in mixed strategies ⟺ there is a positive affine utility transformation V_1, V_2 *of payoffs for each player such that* $V_1(x) = -V_2(x)$ *for any outcome* x.

Proof:
Let U_1, U_2 represent the preferences of each player (here these are just the material payoffs). Then any positive affine transformation of each represents the same VNM preferences.

Suppose $\overline{a}$ is the most preferred while $\underline{a}$ is the least preferred outcome for player 1.
Let $V_1(x) = \frac{U_1(x)-U_1(\underline{a})}{U_1(\overline{a})-U_1(\underline{a})}$ and $V_2(x) = \frac{U_2(x)-U_2(\underline{a})}{U_2(\underline{a})-U_2(\overline{a})}$.
Then $V_1(\overline{a}) = 1, V_1(\underline{a}) = 0, V_2(\overline{a}) = -1, V_2(\underline{a}) = 0$.
These are positive affine transformations and thus represent the same preferences.

Let $p = V_1(x)$.
Notice that $V_1(x) = pV_1(\overline{a}) + (1-p)\,V_1(\underline{a})$. I.e. this means that Player 1s is indifferent between x and the mixture on the right with reduced form probabilities p[1].

This means that Player 2 has to be indifferent as well by the assumption that the game is strictly competitive in mixed strategies.
So, $V_2(x) = pV_2(\overline{a}) + (1-p)V_2(\underline{a}) = -p$.

Thus, we have shown that for any x, we have given a utility representation V_1, V_2 for each player such that $V_1(x) = -V_2(x)$.

[1] Note that the proof from the textbook here sort of assumes that they can play correlated equilibria because ¯a and ¯a might not be in the same column or row as the outcome x. However it should be possible to show that that if this holds for any x in a particular column, we can repeat the same process for each row which has a common item with the initial column and obtain a similar function V.

Note that *strictly competitive in pure strategies* does not imply *strictly competitive in mixed strategies*, although the converse holds.

Example :

1,2	L	R
T	2,1	0,5
B	1,3	5,0

5	0
2	1
1	3
0	5

This game is strictly competitive in pure strategies.

Player 1 is indifferent between (T,L) and 3/5*(T,R) +2/5*(B,R) [i.e. Player 2 plays pure strategy R while Player 1 plays T with probability 3/5] because both give him 2. However, Player 2 prefers 3/5*(T,R) +2/5*(B,R) because she gets 3 instead of 1.

Checking whether we can get a positive affine transformation of U_2 such that the sum of the transformed payoffs is 0.

From (T,R), we want $a + b * 0 = -5 \rightarrow a = -5$. And from (B,R), we want $a + 5b = 0 \rightarrow b = -\frac{1}{5a} = 1$. However, in (T,L), $-5 + 2 * 1 \neq 0$. So, there does not exist such a transformation.

Question 4.

1,2	L	R
U	2,8	7,3
D	9,1	3,7

4a) Strictly competitive?

Constant sum (all boxes add up to 10), so it is strictly competitive in pure (and mixed) strategies.

4b) Calculate Max-min Payoffs in pure strategies for Player 1.

If player 1 plays U, worst she can get is 2. If player 1 plays D, worst she can get is 3. So maxmin strategy is D with maxmin payoff 3.

4c) Calculate Max-min Payoffs in mixed strategies

Notice that one's worst outcome will always occur when the other is playing a pure strategy (because the payoffs when other is playing a mixed strategy is a linear combination of multiple pure strategies, one of which must give a worst outcome). So, we need only restrict attention to the other's pure strategies.

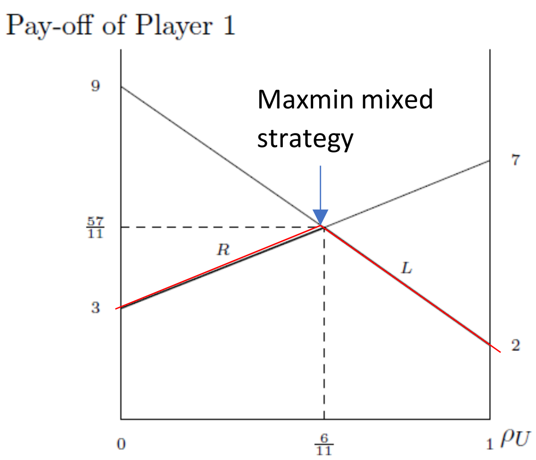
Player 1:

Suppose player 1 plays U with probability p_u.

If player 2 plays L for sure, Expected payoff of player 1: $9 - 7p_u$.

If player 2 plays R for sure, Expected payoff of player 1: $3 + 4p_u$.

Red line indicates the minimum payoffs at each p_u; we want to choose p_u to maximise it.



Player 2:

Try this yourself!

Note that if we solve for the 2 max-min strategies, you can check that they are exactly equal to the mixed strategy NE (try to solve this as well!).

This is according to the theorem in your slides which says that for strictly competitive games, a pair of mixed strategies is a NE iff it is each player's maxmin strategy/