The following are the requirements for a Perfect Bayesian Equilibirum, you should know what this means as well as what an information set is.

Requirement 1

At each information set, the player with the move must have a belief about which node in the information set has been reached by the play of the game. For a nonsingleton information set, a belief is a probability distribution over the nodes in the information set; for a singleton information set, the player's belief puts probability one on the single decision node.

Requirement 2

Given their beliefs, the players' strategies must be **sequentially rational**. *That is, at each information set the action taken by the player with the move (and the player's subsequent strategy) must be optimal given the player's belief at that information set and the other players' subsequent strategies (where a "subsequent strategy" is a complete plan of action covering every contingency that might arise after the given information set has been reached).*

Requirement 3

At information sets on the equilibrium path, beliefs are determined by Bayes' rule and the players' equilibrium strategies.

In questions (like in this problem set) on (pure strategy) separating and pooling Perfect Bayesian Equilibrium.

- Player 1 usually observes the state of nature, thus he can have a strategy for each realisation.
- Player 2 usually does not observe the state of nature (or the type of Player 1 which it determines). He however can condition his strategy on the observed action of Player 1
- If we have 2 states of nature, S1 and S2, then for each action of Player 1 he observes, Player 2 is unable to distinguish exactly which state of nature he is in. In the extensive form diagram, this is represented by the dotted lines connecting the nodes following (S1,a) and (S2,a) where a is the action of Player 1: the nodes are said to be in the same information set.
- Player 2 however has beliefs about which state of nature he is in for each observed action of Player 1 he observes.
    a. The strategies chosen by Player 1 affects Player 2's beliefs about the true state of nature in each information set. This then affects Player 2's conditional strategies.
- There are usually 2 possible pooling equilibria and 2 possible separating equilibria (because the first player is the one to observe nature's move and he can have)
- Pooling equilibria are those in which Player 1 has similar strategies for the different states of nature. Separating equilibria are those in which Player 1 has different strategies in the different states of nature.

The steps I would follow to solve such questions are:

1. Make a hypothesis about Player 1's equilibrium strategies (this will correspond to one of the 4 types of equilibria above)
2. Given these Player 1 strategies, identify the beliefs of Player 2 about the true state of nature at every information set that is on the "equilibrium[1]" path.
3. Given these beliefs, compute Player 2's best response at the information set that is on the equilibrium path.
4. Verify that Player 1's equilibrium strategies which we started from are indeed his best reply given the state of nature which he observes.
5. This will impose restrictions on the receiver's beliefs that are at information sets that are off the equilibrium path.
    a. In particular, we need to specify beliefs on information sets that are off the equilibrium path which lead to best responses of Player 2 which are consistent with (4).

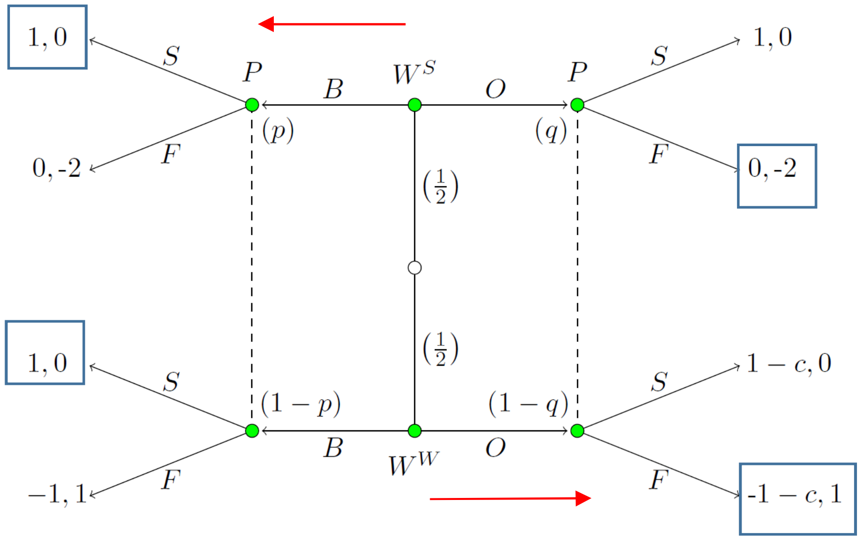
Sometimes, there may be shortcuts to rule out certain equilibria, but if you follow the above method it should not go wrong.

I hope this procedure is clear enough… It should be easier to understand if we look at it in the context of Question 2 of Problem set 5b).

Also, the idea behind the above steps should also be applicable to more general questions which involve solving for PBE.

[1] I use inverted commas here because we do not know yet whether it is an equilibrium. It is just an assumption at the moment.

Q2.



1. Suppose B when strong and O when weak is an equilibrium strategy of Wesley. (and thus a candidate for a separating equilibrium)

Both the left and the right information sets lie on the equilibrium path. We thus compute the beliefs p and q via Bayes rule. This should lead to $p = 1, q = 0$.

There are no information sets off the equilibrium path.

Given the beliefs $p = 1, q = 0$, the best response for the Prince is to play (S when he observes B, F when he observes O). This can be seen by just solving for the Prince's best response in the top left and bottom right nodes respectively.

Given this best response, we need to check that our hypothesised Wesley's strategy: (B when strong, O when weak) is a best response:

So, given that the Prince plays the strategy (S when he observes B, F when he observes O),

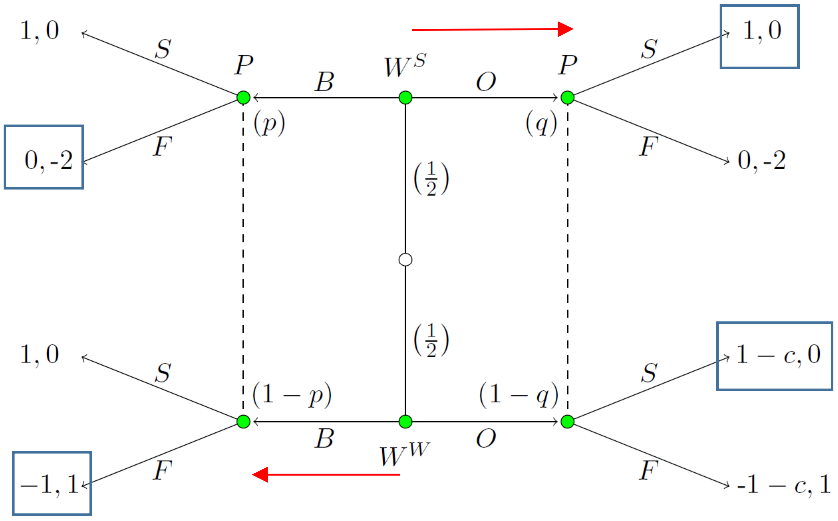
<u>When Wesley is strong:</u>

Playing B gets him 1 while playing O gets him 0. So, Playing B when strong is indeed a best response.

<u>When Wesley is weak:</u>

Playing B gets him 1 while playing O gets him $-1 - c$. So, playing O when weak is not a best response.

There is thus no separating Perfect Bayesian Equilibrium (PBE) where Wesley plays (B when strong, O when weak).



2. Suppose O when strong and B when weak is an equilibrium strategy of Wesley. (and thus a candidate for a separating equilibrium)

Both the left and the right information sets lie on the equilibrium path. We thus compute the beliefs p and q via Bayes rule. This should lead to $p = 0, q = 1$.

There are no information sets off the equilibrium path.

Given the beliefs $p = 0, q = 1$, the best response for the Prince is to play (F when he observes B, S when he observes O). This can be seen by just solving for the Prince's best response in the bottom left and top right nodes respectively.

Given this best response, we need to check that our hypothesised Wesley's strategy: (O when strong, B when weak) is a best response:

So, given that the Prince plays the strategy (F when he observes B, S when he observes O),

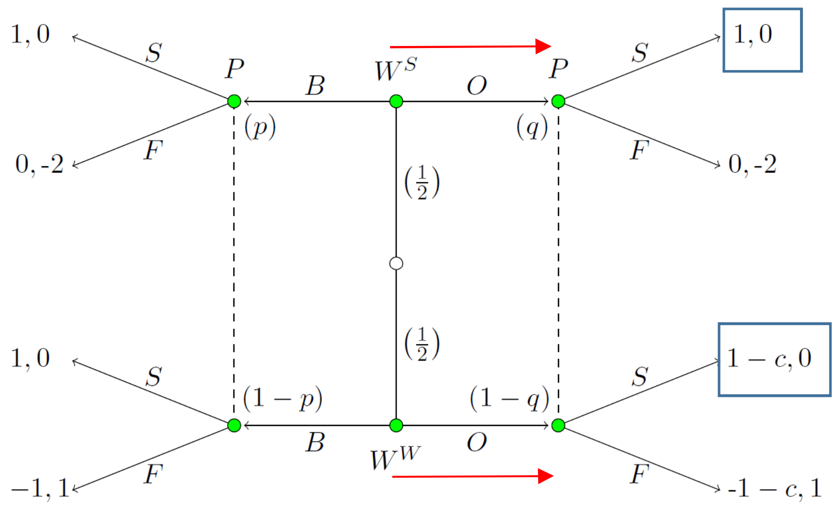
<u>When Wesley is strong:</u>

Playing B gets him 0 while playing O gets him 1. So, Playing O when strong is indeed a best response.

<u>When Wesley is weak:</u>

Playing B gets him -1 while playing O gets him $1 - c$. Playing B is only a best response if $-1 \geq 1 - c$ or $c \geq 2$.

There thus exists a separating Perfect Bayesian Equilibrium (PBE) where Wesley plays (O when strong, B when weak) when $c \geq 2$. The Prince plays (F when he observes B, S when he observes O) and his beliefs are $p = 0, q = 1$.



3. Suppose O when strong and O when weak is an equilibrium strategy of Wesley. (and thus a candidate for a separating equilibrium)

Only the right information set lies on the equilibrium path. We thus compute the beliefs q via Bayes rule which leads to $q = 0.5$.

Given the beliefs $q = 0.5$,

<u>If the Prince plays S,</u>

Expected payoffs are: $0.5 \times 0 + 0.5 \times 0 = 0$.

<u>If the Prince plays F,</u>

Expected payoffs are: $0.5 \times -2 + 0.5 \times 1 = -0.5$.

Thus, it is a best response for the Prince to play S when he observes O.

Now, the left information set is off the equilibrium path.

We need to specify beliefs p such that the Prince choses an action when he observes B which does not lead Wesley to want to deviate. Let us check if such an action exists.

<u>If the Prince plays S when he observes B</u>; combined with our computed best response when he observes O.

When Wesley is weak:

Playing B gets him 1, while playing O gets him $1 - c$. Thus, Playing O when strong is not a best response here.

We thus cannot have that <u>the Prince plays S when he observes B</u> in such a pooling equilibrium.

If the Prince plays F when he observes B; combined with our computed best response when he observes O.

When Wesley is strong:

Playing O gets him 1, while playing B gets him 0. Thus, Playing O when strong is a best response here.

When Wesley is weak:

Playing O gets him $1 - c$, while playing B gets him -1. Thus, Playing O when weak is only a best response when $1 - c \geq -1$ or $c \leq 2$.

All that is left is specifying the beliefs p which "support" Prince plays F when he observes B.

If the Prince plays F when he observes B,

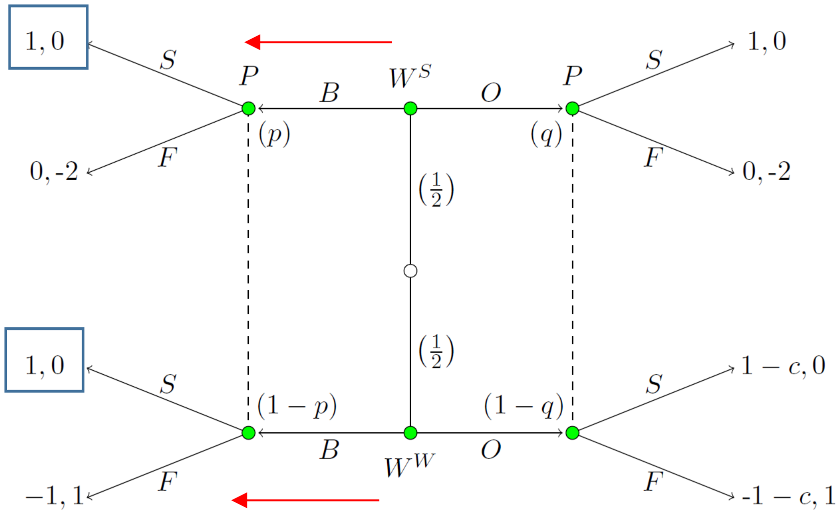
His expected payoff is $-2p + (1 - p)$

If the Prince plays S when he observes B,

His expected payoff is 0.

Thus, the above is preferred when $-3p + 1 \geq 0$ or $p \leq 1/3$.

So, for $c \leq 2$, there is a pooling equilibrium with Wesley playing (O when strong, O when weak); Prince playing (F when he observes B, S when he observes O) and beliefs $q = 0.5, p \leq 1/3$.



4. Suppose B when strong and B when weak is an equilibrium strategy of Wesley. (and thus a candidate for a separating equilibrium)

Only the left information set lies on the equilibrium path. We thus compute the beliefs p via Bayes rule which leads to $p = 0.5$.

Given the beliefs $p = 0.5$,

<u>If the Prince plays S,</u>

Expected payoffs are: $0.5\times0 + 0.5\times0 = 0$.

<u>If the Prince plays F,</u>

Expected payoffs are: $0.5\times-2 + 0.5\times1 = -0.5$.

Thus, it is a best response for the Prince to play S when he observes B.

Now, the right information set is off the equilibrium path.

We need to specify beliefs q such that the Prince choses an action when he observes O which does not lead Wesley to want to deviate.

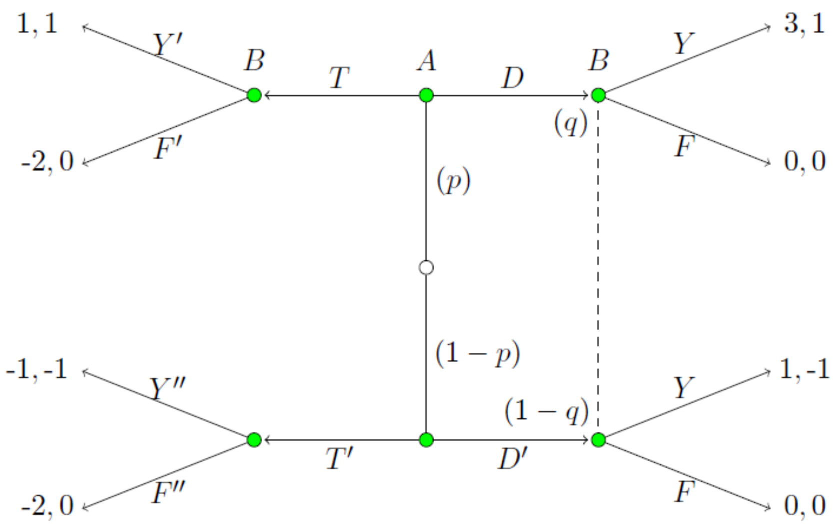
Notice that since B is the non-costly action, and that the Prince will choose S when he observes B, which is the beneficial action for (both) Wesleys, none of them will strictly want to deviate for any action the Prince chooses when he observes O.

Thus, any beliefs q will be alright and we just need to the Prince to choose an action when he observes O which is in line with such (off the equilibrium path) beliefs.

In particular, there is a pooling equilibrium with Wesley playing (B when strong, B when weak); Prince playing (S when he observes B, S when he observes O) and beliefs $p = 0.5, q \geq 1/3$.

And another pooling equilibrium with Wesley playing (B when strong, B when weak); Prince playing (S when he observes B, F when he observes O) and beliefs $p = 0.5, q \leq 1/3$.

Question 1i)



In this question, it is relatively simpler because it is assumed that when Amy takes the newspaper, Brenda will discover the true state of nature (SALE or NO SALE). Thus, there is no non-singleton information set following such an action.

Notice that there are now three strategies $(Not\ brought\ newspaper, Brought\ newspaper\ and\ Sale, Brought\ newspaper\ and\ No\ sale)$ and 1 set of beliefs of Brenda (q) which we need to specify (in contrast to 2 strategies and 2 beliefs in the previous question)

If Brenda is brought the newspaper when there is a sale (T), the best response in that sub-game would be for her to pick Y’.

If Brenda is brought the newspaper when there is no sale (T’), the best response in that sub-game would be for her to pick F’’.

Given these choices, notice that if there is no sale, it will not be a best response for Amy to bring the newspaper to Brenda. This is because she will get a negative payoff in contrast to the lowest payoff if she does not bring the newspaper (0).

ii) Separating Bayesian Equilibria

From the observation above, the only potential separating equilibria is (T, D').

Given this, by Bayes rule, $q = 0$.

Thus, when Brenda is not brought the newspaper, her beliefs is that there is no sale with probability 1. Her best response is thus F.

We then need to check that (T,D') is a best response for Amy given these strategies of Brenda above.

When there is no sale, we know from above that she will choose D'

When there is a sale, given that Brenda will choose F if she does not bring the newspaper, the payoffs are 0 if Amy chooses D, but 1 if she chooses T.

This shows that (T,D'); (F,Y', F'') together with $q = 0$ forms a perfect Bayesian equilibrium which is separating.

iii) Pooling Bayesian Equilibria

Remember that from above, T' is never a best response, so the only potential pooling equilibria is where Amy chooses (D, D') ; i.e. never bring the newspaper to Brenda.

Given this assumed strategy, by Bayes rule, when not brought the newspaper, Brenda believes that $q = p$ by Bayes rule.

Brenda's expected payoff if she chooses Y is then $p + (1-p)\times(-1) = 2p - 1$. If she chooses F, her expected payoff is 0. Y is a best response if $p \geq \frac{1}{2}$.

In order for Amy to want to choose D, we need Brenda to choose Y when not brought the newspaper (Amy gets 3 if Y and 0 if F), otherwise it would be better for Amy to bring her the newspaper (Amy gets 1).

Thus, a Pooling Bayesian Equilibria is only possible for $p \geq 1/2$.

Then, (D,D'); (Y,Y',F'') together with $q = p$ forms a perfect Bayesian equilibrium which is pooling.