Q1. Solving for Mixed Strategy Nash Equilibria

Key concepts:

1. *Normal Form*
2. *Expected Utility: probability weighted sum of utilities/payoffs in each scenario.*
3. *Best responses*
4. *Nash equilibria (mixed + pure)*

1.1

		Kate	
		Rush	Swerve
Jane	Rush	$-5, -5$	$2, -1$
	Swerve	$-1, 2$	$0, 0$

1.2

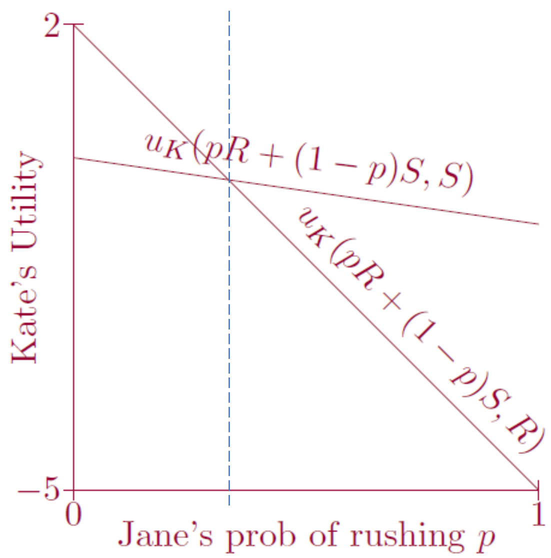
Here, we want to graph Kate's expected utility for each pure strategy as a function of Jane's strategy.

Let p be the probability that Jane plays Rush, $1 - p$ the probability that Jane plays Swerve.

If Kate plays Rush, she gets $pU_k(R, R) + (1 - p)U_k(S, R) = p(-5) + (1 - p)2 = 2 - 7p$

If Kate plays Swerve, she gets $pU_k(R, S) + (1 - p)U_k(S, S) = p(-1) + (1 - p)0 = -p$

Plotting this, we get:



1.3.

From the graph, we can see what action is the best response given Kate's chosen strategy p. (Alternatively, can just compare the expected utilities of each action without the diagram)

First, we calculate the intersection point of the two lines:

$$2 - 7p = -p$$

$$p = 1/3$$

Hence Rush is the best response when $p < \frac{1}{3}$. Indifferent between Rush and Swerve when $p = 1/3$. (Any mixed strategy) Swerve is the best response when $p > 1/3$.

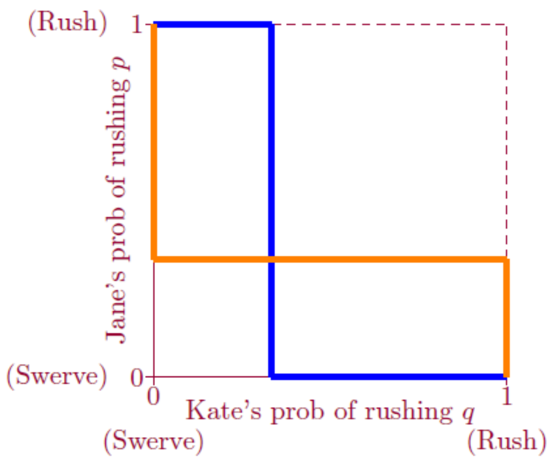
Let us call q the probability that Kate plays Rush. Then we can write out the best response for Kate given p.

$$BR(p): q = \begin{cases} 1 & if\ p < \frac{1}{3} \\ [0,1] & if\ p = \frac{1}{3} \\ 0 & if\ p > \frac{1}{3} \end{cases}$$

Likewise, by symmetry, we have that the following best response for Jane given q.

$$BR(q): p = \begin{cases} 1 & if\ q < \frac{1}{3} \\ [0,1] & if\ q = \frac{1}{3} \\ 0 & if\ q > \frac{1}{3} \end{cases}$$

We can graph these correspondences (Jane's BR in Blue, Kate's in Orange):



1.4.

The Nash equilibria are the intersections of the best responses in the diagram.

(Remember that the definition of Nash equilibria is that each strategy is a best response to other's strategies: best responses to best responses)

From this we can identify 2 Pure strategy NE and 1 Mixed strategy NE.

These are: $(R; S)$, $(S; R)$ and $(\frac{1}{3}R, \frac{2}{3}S; \frac{1}{3}R, \frac{2}{3}S)$

1.5.

If Jane has insurance, this means that her payoff is $x - 5$ when both choose to rush.

		Kate: Rush	Kate: Swerve
Jane	Rush	$x-5, -5$	$2, -1$
	Swerve	$-1, 2$	$0, 0$

For any positive value of x, $(R; S)$ is still an equilibrium. Hence, we need to find values of x for which other equilibria do not exist. This means, that we need Jane to not want to play Swerve for any strategy q of Kate.

If Jane plays Rush, she gets $q(x - 5) + (1 - q)2 = 2 - 7q + qx$.

If Jane plays Swerve, she gets $q(-1) + (1 - q)0 = -q$.

Rushing is strictly preferred if $x > 6 - \frac{2}{q}$.

The RHS is increasing with q, and takes its maximum at $q = 1$. Thus if $x > 4$, she will never choose to Swerve and we will have a unique equilibrium. Intuitively, insurance has the greatest potential payoff for Jane if Kate definitely chooses to Rush, hence that becomes the deciding factor.

For $0 \leq x \leq 4$, there always exists another NE $(S; R)$. This is because insurance is not enough to encourage Jane to Rush if she knows Kate is definitely going to Rush.

(When x=4, there is a continuum of mixed strategy NE!)

Notice that the indifference point for Jane is $q = \frac{2}{6-x}$ which rises with x, so raising x graphically shifts the vertical blue line to the right. In the figure there will only be one intersection point when the blue vertical line occurs at $q > 1$.

Q2. Auctions

Note that an important assumption here is complete information. In future, you will look at cases where they have uncertainty about other's valuation.

In contrast to question 1, we have continuous instead of discrete action sets.

(Food for thought: try solving it when they have different valuations)

Key concepts:

1. *Being able to translate actions into utility given the rules of the game: In particular, for auctions, we can usually separate these into several simple cases.*
2. *Subsequently constructing best responses.*

2.1.

Rules: A player wins if he has a higher bid and he pays his bid if he wins. A draw in bids means that the winner is chosen at random. This threshold for winning leads to a natural partition where we consider cases where Player 1's bid is higher, equal or lower than Player 2's bid.

Therefore, Player 1's utility function is

$$u_1(x_1, x_2) = \begin{cases} 500 - x_1, & x_1 > x_2 \\ \frac{1}{2}(500 - x_1), & x_1 = x_2 \\ 0, & x_1 < x_2 \end{cases}$$

Similarly, player 2's utility function is

$$u_2(x_1, x_2) = \begin{cases} 500 - x_2, & x_2 > x_1 \\ \frac{1}{2}(500 - x_2), & x_2 = x_1 \\ 0, & x_2 < x_1 \end{cases}$$

2.2.

<u>For best responses of Player 1,</u> let us consider 3 cases for the strategy of Player 2: $x_2 < 500, x_2 = 500, x_2 > 500$. We do so because 500 is Player 1's valuation of the item and thus is a critical threshold for whether he wants to win over Player 2 or not.

If $x_2 < 500$,

There is the possibility of having positive utility and he will want to win by bidding an infinitely small amount greater than Player 2's bid: $x_2 + \epsilon$. However, technically there is no defined best response here because of the infinite divisibility of bids.

If $x_2 = 500$,

There is no possibility of having positive utility, he is indifferent between all bids between 0 and 500 which give 0 utility and will not want to "strictly" win by bidding more than 500 as it gives negative utility.

If $x_2 > 500$,

Winning will only give him negative utility, hence he will want to lose (and get 0 utility). Thus, any bid strictly under x_2 will suffice and be a best response.

So, we have

$$BR_1(x_2) = \begin{cases} \varnothing, & x_2 < 500 \\ [0, 500], & x_2 = 500 \\ [0, x_2), & x_2 > 500 \end{cases}$$

And similarly, for Player 2, we have

$$BR_2(x_1) = \begin{cases} \varnothing, & x_1 < 500 \\ [0, 500], & x_1 = 500 \\ [0, x_1), & x_1 > 500 \end{cases}$$

2.3.

From the best response functions above, one can see that only (500,500) satisfies this.

Intuitively, it is a unique equilibrium for the following reasons:

- None of the bids can be above 500. If the bid above 500 wins, he gets negative utility, so lowering the bid will be a better response. (If a bid above 500 does not win, then the other bid must be above 500 and won, so the same reasoning holds).
- None of the bids can be below 500. Otherwise, the higher of the two bids can stand to gain by lowering one's bid to an infinitely smaller amount than the other. If both bids are the same and below 500, then by raising one's bid to an infinitely smaller amount than the other, payoff increases.

2.4.

We will show this by contradiction. Suppose, there is a symmetric mixed strategy described by the pdf f.

Then, player 1 wins if the others' bid is less than x_1: the probability of winning with a bid x_1 is hence $F(x_1)$, where F is the CDF.

$$EU(x_1) = F(x_1)(500 - x_1)$$

Let $\underline{x}$ be the lowest bid in the support of f. We can assume that this is < 500, otherwise it would automatically not be an equilibrium.

Then, $F(\underline{x}) = 0, EU = 0$

But, in order to mix between bids in the support, one must be indifferent between them: This is not true here as a profitable deviation can be made by shifting some probability mass to the bids which give positive expected utility (such bids always exist by the assumption that $\underline{x} < 500$.) Hence there is no symmetric mixed strategy pdf f.

2.5.

Rules: Now we have an all pay auction where everyone pays their bids, with the highest winner winning.

$$u_1(x_1, x_2) = \begin{cases} 500 - x_1, & x_1 > x_2 \\ 250 - x_1, & x_1 = x_2 \\ -x_1, & x_1 < x_2 \end{cases}$$

$$u_2(x_1, x_2) = \begin{cases} 500 - x_2, & x_2 > x_1 \\ 250 - x_2, & x_2 = x_1 \\ -x_2, & x_2 < x_1 \end{cases}$$

2.6.

For the best responses of player 1, we have 2 cases: $x_2 < 500, x_2 \geq 500$.

If $x_2 < 500$, the best response would be to bid slightly higher than x_2 and to get positive utility. However, technically there is no defined best response here because of the infinite divisibility of bids.

If $x_2 \geq 500$, the best response would be not to win and get 0 utility by bidding 0. Any positive bid gives negative utility whether one wins or loses.

The reason why the $x_2 = 500$ case is different from before is because even at $x_2 = 500$, trying to get a draw will get -250 in utility.

So,

$$BR_1(x_2) = \begin{cases} \varnothing, & x_2 < 500 \\ \{0\}, & x_2 \geq 500 \end{cases}$$

And similarly, for Player 2, we have

$$BR_2(x_1) = \begin{cases} \varnothing, & x_1 < 500 \\ \{0\}, & x_1 \geq 500 \end{cases}$$

2.7.

There are no mutual best responses: hence no pure strategy NE. Intuitively,

- There cannot be any bids below 500 for the same reason as before.
- There cannot be any bids above 500 for the same reason as before.
- (500,500) is not an equilibrium due to the all pay rule.

2.8.

We construct an equilibrium as follows:

Suppose (f, f) is a mixed strategy NE with support $[0{,}500]$. Then expected utility of a bid x_1 is

$$F(x_1) \times 500 - x_1$$

Since $F(0) = 0$ and all strategies in the support of a mixed strategy need to have equal payoffs, this means that

$$F(x_1) \times 500 - x_1 = 0$$

$$F(x_1) = \frac{x_1}{500}$$

This implies that F is the CDF of a uniform distribution over $[0{,}500]$.

2.9.

If there are n other players, then the probability of winning with a bid of $x_1$changes to $F(x_1)^n$;

i.e. all other players must have a bid less than x_1.

The mixed strategy nash equilibrium with support $[0{,}500]$ is the pdf which has a CDF which satisfies

$$F(x_1) = (\frac{x_1}{500})^{\frac{1}{n}}$$

Q3. Cournot Competition

Key concepts:

- *How to solve for a Cournot Nash equilibrium (prices/quantities)*
- *What is an iterated elimination equilibrium?*

3.1

Firm utility is profit which is $(p - c)q$. Since marginal costs are 0, and substituting the demand function, we have

$$\pi_1(q_1, q_2) = q_1(1 - q_1 - q_2)$$

$$\pi_2(q_1, q_2) = q_2(1 - q_1 - q_2)$$

3.2.

Let us suppose that Firm 2 has a mixed strategy f for q_2.

Firm 1 maximises

$$\int_0^1 q_1(1-q_1-q_2)f(q_2)dq_2$$

$$= q_1 - q_1^2 - q_1\int_0^1 q_2 f(q_2)dq_2$$

Taking the F.O.C wrt to q_1, we get

$$1-2q_1 = \int_0^1 q_2 f(q_2)dq_2$$

$$1-2q_1 = E(q_2)$$

$$q_1 = (1-E(q_2))/2$$

By symmetry,

$$q_2 = (1-E(q_1))/2$$

3.3.

Since the best responses are pure strategies, NE have to be in pure strategies. I.e. we can get rid of the expectations notation above.

$$q_1 = (1-q_2)/2$$

$$q_2 = (1-q_1)/2$$

We solve for the intersection of the two best response functions. Since the firms are symmetric, a short cut would be that $q_1 = q_2$ and so the unique NE is

$$q_1 = q_2 = \frac{1}{3}$$

3.4

Firstly, I show that any strategies above $\frac{1}{2}$ are dominated by the strategy $\frac{1}{2}$.

$$u_1(q_1,q_2) - u\left(\frac{1}{2},q_2\right) = q_1(1-q_1-q_2) - \frac{1}{2}\left(1-\frac{1}{2}-q_2\right)$$

$$= q_1 - q_1^2 - \frac{1}{4} - q_2\left(q_1 - \frac{1}{2}\right)$$

$$= -\left(q_1 - \frac{1}{2}\right)^2 - q_2\left(q_1 - \frac{1}{2}\right)$$

This is negative for any $q_1 > \frac{1}{2}$ and any $q_2 > 0$. This means we can restrict attention to $[0,\frac{1}{2}]$

In general, we can show that any strategy which is not a best response to an opponent's set of remaining strategies will be strictly dominated by some strategy.

We can iteratively eliminate strategies which do not appear in the best response function we got earlier (If they are never best responses, they are strictly dominated by some strategy.).

Given that $q_2 \in [0,1]$, using the best response for firm 1, $q_1 \in [0, \frac{1}{2}]$.

Given that $q_1 \in [0, \frac{1}{2}]$, using the best response for firm 2, $q_2 \in [\frac{1}{4}, \frac{1}{2}]$

Given that $q_2 \in [\frac{1}{4}, \frac{1}{2}]$, using the best response for firm 1, $q_1 \in [\frac{1}{4}, \frac{3}{8}]$

Given that $q_1 \in [\frac{1}{4}, \frac{3}{8}]$, using the best response for firm 2, $q_2 \in [\frac{5}{16}, \frac{3}{8}]$

Round	q_2	q_1
0	$[0, 1]$	$[0, \frac{1}{2}]$
1	$[\frac{1}{4}, \frac{1}{2}]$	$[\frac{1}{4}, \frac{3}{8}]$
2	$[\frac{5}{16}, \frac{3}{8}]$	....
4	...	...

We have the following pattern after iterative elimination:

The right hand bound of q_2 determines the left hand bound of q_1 while the left hand bound determines the right hand bound of q_1. Likewise, for how $q_1's$ bounds determines $q_2's$ bounds.

Let the bounds for q_2 after n rounds of elimination be $[a_n, b_n]$.

In the next step, we obtain the bounds for q_1 using the best response function.

This will be $\left[\frac{1-b_n}{2}, \frac{1-a_n}{2}\right]$.

This implies that the bounds for q_2 after n rounds of elimination are $\left[0.5(1 - \frac{1-a_n}{2}), 0.5(1 - \frac{1-b_n}{2})\right]$.

I.e. $a_{n+1=}0.25 + 0.25a_n$

And $b_{n+1} = 0.25 + 0.25b_n$

Since the coefficient on a_n is less than 1, we have a convergent sequence which has a steady state value of 1/3 (let $a_{n+1} = a_n = a$ and solve). Likewise for b_n.

In particular, if you solve the difference equation,

$a_n = -\frac{1}{3}(0.25)^n + \frac{1}{3}$ and $b_n = \frac{2}{3}(0.25)^n + 1/3$.

As $n \to inf$, we get exactly 1/3.

Note that since the bounds of q_2 converge to 1/3, that of q_1 should also as well.

Proof:

Suppose player 2 has a set of remaining strategies $[a_2, b_2]$, Let us assume that Player 1 has a wider interval $[a_1, b_1]$ which we are trying to reduce in size via elimination.

For any strategy $q_1 < \frac{1-b_2}{2}$, they will be strictly dominated by the strategy $z = \frac{1-b_2}{2}$.

$$\pi_1(q_1, q_2) - \pi_1(z, q_2) = q_1 - q_1^2 - q_1 q_2 - z + z^2 + z q_2$$

Since $q_1 < z$

The above function is increasing in q_2, given the upper bound of q_2: b_2 , we have

$$q_1 - q_1^2 - q_1 q_2 - z + z^2 + z q_2 < q_1 - q_1^2 - q_1 b_2 - z + z^2 + z b_2$$

$$= q_1 - q_1^2 - q_1 b_2 - \frac{1-b_2}{2} + \left(\frac{1-b_2}{2}\right)^2 + \left(\frac{1-b_2}{2}\right) b_2$$

$$= -q_1^2 + (1-b_2) q_1 - \frac{1-b_2}{2} + \left(\frac{1-b_2}{2}\right)^2 + \left(\frac{1-b_2}{2}\right) b_2$$

$$= -\left(q_1 - \frac{1-b_2}{2}\right)^2 + \left(\frac{1-b_2}{2}\right)^2 - \frac{1-b_2}{2} + \left(\frac{1-b_2}{2}\right)^2 + \left(\frac{1-b_2}{2}\right) b_2$$

$$= -\left(q_1 - \frac{1-b_2}{2}\right)^2 + 2\left(\frac{1-b_2}{2}\right)^2 - \left(\frac{1-b_2}{2}\right)(1-b_2)$$

$$= -\left(q_1 - \frac{1-b_2}{2}\right)^2$$

This is less than 0 for any value of $q_1 < z$.

Likewise, we can show that for any strategy $q_2 > \frac{1-a_2}{2}$, they will be strictly dominated by the strategy $z = \frac{1-a_2}{2}$.

Notice that the remaining strategies for Player 1 is then $[\frac{1-b_2}{2}, \frac{1-a_2}{2}]$. These are the set of strategies which can be obtained from just observing the best responses to the opponent's strategy set.