Problem set 2

Key concepts

- Information sets, Sequential rationality and relation to subgame perfect equilibria.
- Difference between NE and SPNE.

1.1

Now we have a sequential first price auction.

Player 2 observes Player 1's choice.

We are looking for all Nash equilibria. The main difference of NE from SPNE is that in information sets off the equilibrium path, we do not require choices there to be best responses.

Suppose we have a candidate NE where they choose $x_{1,}x_2$,

- The information sets on the equilibrium path consist of Player 2's choice after $b_1 = x_1$.
- The information sets off the equilibrium path consist of Player 2's choice after $b_1 \neq x_1$.

For this to be a NE:

- $b_1 = x_1$ must be a best response given Player 2's choices on and off the equilibrium path.
- $b_2 = x_2$ must be a best response given that Player 1 is choosing $b_1 = x_1$.

First, we show that x_1 is always 500 in equilibrium.

Suppose $x_1 > 500$ in equilibrium, by the second condition, Player 2 will choose to lose and Player 1 will get a negative payoff. Player 1 can improve on this by bidding 0.

Suppose $x_1 < 500$ in equilibrium, there is no defined best response by Player 2.

So $x_1 = 500$ in equilibrium. And Player 2 best responds (condition 1) by choosing any bid in $[0, 500]$.

Given this, we just need to construct choices of Player 2 in off the equilibrium path information sets such that Player 1 will want to bid 500. Basically, we just need any choices such that he gets ≤ 0 in utility.

These are given by:

- At information sets where $b_1 < 500$, Player 2 bids (x_1, ∞)
- At information sets where $b_1 > 500$, Player 2 bids anything.

In total we have that a NE is any strategies satisfying

- Player 1 bids 500
- At the information set where $b_1 = 500$, Player 2 bids anything in $[0,500]$
- At information sets where $b_1 < 500$, Player 2 bids (x_1, ∞)
- At information sets where $b_1 > 500$, Player 2 bids anything.

As can be seen, by bidding below 500, Player 1 will get 0 because he loses given Player 2's action. By bidding above 500, Player 1 will get ≤ 0 because he either loses and gets 0, or he wins and gets < 0.

Notice that in the last 2 conditions, not all of Player 2's strategies are sequentially rational.

1.2

For a SPNE, we need Player 2's strategies to be sequentially rational in information sets off the equilibrium path. However, this is not possible when $b_1 < 500$. So, there is no SPNE.

1.3

Now we have that bids can only be in integers. We are looking for NE in a simultaneous game.

Let us examine Player 1's best responses:

For $b_2 > 500$, winning gives negative utility, so he will want to lose by bidding anything strictly less than that in integers.

For $b_2 = 500$, winning (with 500) or losing is the same. So, he will want to bid anything ≤ 500

For $b_2 < 500$, he will not want to lose: but will he choose to win strictly, or draw?

Bidding b_2 gives $(500 - b_2)/2$. Bidding $b_2 + 1$ gives $(500 - b_2 - 1)$.

$$\frac{(500 - b_2)}{2} \geq (500 - b_2 - 1)$$

$$500 - b_2 \geq 998 - 2b_2$$

$$b_2 \geq 498$$

So, if $b_2 < 498$, he will always choose to bid 1 higher than the others bid.

If $b_2 = 498$, he is indifferent between drawing (bidding 498) and winning strictly (bidding 499)

If $b_2 = 499$, he only wants to draw.

The best response function can be summarized by

$$BR_1(x_2) = \begin{cases} \{x_2+1\}, & x_2 < 498 \\ \{498, 499\}, & x_2 = 498 \\ \{499\}, & x_2 = 499 \\ \{0, 1, .., 500\}, & x_2 = 500 \\ \{0, 1, ..., x_2 - 1\}, & x_2 > 500 \end{cases}$$

Likewise, for Player 2 by symmetry.

As can be seen the NE are (499,499) (498,498) (500,500)

1.4.

Now Player 2 moves after Player 1.

In a SPNE, Player 2 must be best responding in every subgame (following P1's bid).

Player 1 must then be choosing the optimal responses given Player 2's best responses to his strategies.

Notice that by bidding 499, Player 1 can get at least ½ in utility (Player 2's best response is 499). So, we can restrict attention to cases where she can possibly get $\geq \frac{1}{2}$.

These are not possible when $b_1 \neq 498{,}499$. She always gets 0 in those cases.

Notice that when Player 1 bids 498, Player 2 has 2 choices: these will determine what kinds of SPNE we get.

Player 2 can either choose 498 or 499 to get 1 in that subgame.

- If he chooses 499, Player 1 gets 0.
- If he chooses 498, Player 1 gets 1.

Let us start with a candidate equilibria where Player 1 bids 499. He gets $\frac{1}{2}$ in such a case. For this to be an equilibria, he must not want to deviate to 498.

In order for Player 1 not to want to deviate from 499, Player 2 has to choose 499 in the subgame where Player 1 chooses 498.

Next, let us find the candidate equilibria where Player 1 bids 498. His payoff here must be higher than ½ when he chooses 499 (and Player 2 best responds).

So, Player 2 must choose 498.

So, the set of SPNE are the set of strategies which satisfy

Either

- $b_1 = 498$
- $b_2 = 498$ when $b_1 = 498$
- Any best response for $b_1 \neq 498$

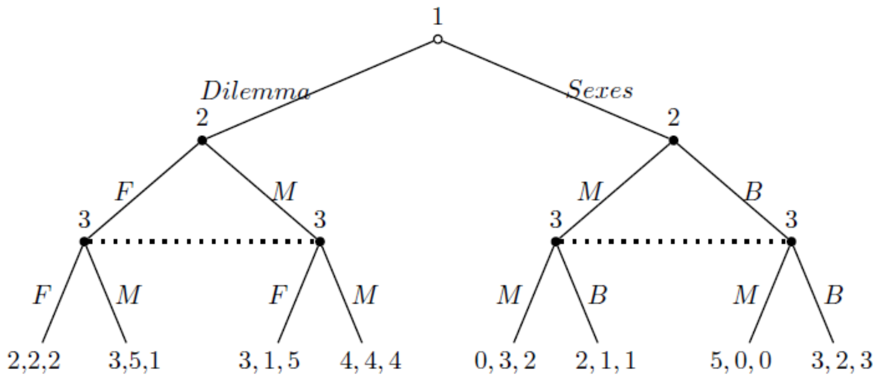
Or

- $b_1 = 499$
- $b_2 = 499$ when $b_1 = 498$
- Any best response for $b_1 \neq 498$

2.1

Key concepts:

- Solving for NE in extensive form games
- Solving for SPNE



Remember that since each of players 2 and 3 observe player 1's action, their strategies have 2 dimensions (choice if Dilemma, choice if Sexes).

Normal form

P2\P3	FM	FB	MM	MB
FM	2,2,2	2,2,2	3,5,1	3,5,1
FB	2,2,2	2,2,2	3,5,1	3,5,1
MM	3,1,5	3,1,5	4,4,4	4,4,4
MB	3,1,5	3,1,5	4,4,4	4,4,4

Player 1 chooses Dilemma

P2\P3	FM	FB	MM	MB
FM	0,3,2	2,1,1	0,3,2	2,1,1
FB	5,0,0	3,2,3	5,0,0	3,2,3
MM	0,3,2	2,1,1	0,3,2	2,1,1
MB	5,0,0	3,2,3	5,0,0	3,2,3

Player 1 chooses Sexes

To solve for PSNE, we can use the underline method on the normal form tables. The best responses are highlighted in red while NE are shaded grey.

Alternatively, we can consider candidate equilibria and construct them accordingly.

Suppose in Equilbrium, P1 chooses the Prisoner's Dilemma,

Then the choices of P2 and P3 need to form NE in the PD subgame. This is given by actions (F,F). In this scenario, P1 gets a payoff of 2. In order for P1 not to want to deviate, the actions of P2 and P3 when he choose the battle of sexes game cannot be such that he gets a higher payoff. These actions are (M,M) , (M,B).

These result in the NE (D,FM, FM) (D,FM,FB).

Suppose in Equilbrium, P1 chooses the battle of sexes game,

Then the choices of P2 and P3 need to form NE in the PD subgame. These are given by actions (M,M) or (B,B).

In the (M,M) scenario, P1 gets a payoff of 0. Choosing D will always get P1 a higher payoff. So this is not a plausible case.

In the (B,B) scenario, P1 gets a payoff of 3. In order for P1 not to want to deviate, the actions of P2 and P3 when he choose the PD game cannot be such that he gets a higher payoff. These actions are (F,F), (F,M) , (M,F).

These result in the NE (S,FB, FB) (S,FB,MB) (S,MB,FB).

2.2

Remember that in a SPNE, we must have a NE in each subgame. There are 3 subgames: the overall game, the PD game and BOS game.

- In the Prisoner's Dilemma, (F, F) is the only NE, and
- In the Battle of the Sexes, (B, B), (M, M) and $(\frac{3}{4}M + \frac{1}{4}B, \frac{1}{4}M + \frac{3}{4}B)$ are the NE.

The payoffs of Player 1 in each of these cases is respectively

- (F,F)=2
- (B,B)= 3 , (M,M)= 0 , (Mixed)= $2 = \frac{3}{4}.\frac{3}{4}.2 + \frac{3}{4}.\frac{1}{4}.0 + \frac{3}{4}.\frac{1}{4}.3 + \frac{1}{4}.\frac{1}{4}.5$

There are 3 (classes of) SPNE

- (D,FM,FM) because the PD game gives $2 > 0$ in (M,M) for the BOS game.
- (S,FB,FB) because the BOS game (B,B) gives 3$>$ 0 in the PD game.
- $(p, F\frac{3}{4}M, F\frac{1}{4}M)$ where $p \in [0,1]$ is the probability of choosing the PD game. This arises because of the indifference by P1 between the two games when players arrive at the Mixed strategy NE in the BOS game.

3

Key concepts:

- Calculating discounted utility. Note that in Costas' solutions, he multiplies all payoffs by $(1-\delta)$ to give the future discounted payoffs.
- Constructing Nash Reversion strategies to sustain some equilibrium.
  - The idea behind this is that for a SPNE, both on (no deviation) and off the equilibrium path (with deviation) actions need to form a NE in each subgame.
  - One way to do so is to directly set actions when a deviation is detected to be a NE of the stage game (which should make them worse off than if they didn't deviate). This is called a Nash Punishment/ Nash Reversion strategy.
- Note that from the literature, it is sufficient to consider 1 step deviations by each player: only the strategy of 1 period changes while keeping the strategy in other periods the same.

3.1

Payoff if x other shepherds choose to gorge:

- Graze: $1-2X$
- Gorge: $N-2X$

Since $N>1$, Gorging is the dominant strategy and everyone gorging is a unique NE.

3.2

If the game is finitely repeated, we can use backward induction.

In the final subgame, the unique NE is (gorge, gorge, gorge).

In the penultimate stage, they know that no matter what they do, the outcome in the next round will be (gorge, gorge, gorge). Hence, they all gorge.

Iterating this backwards, they always gorge in every stage.

3.3

Since we have a unique NE in the stage game, our Nash reversion strategy would involve gorging if we observe anyone transgressing.

$$s_i(h) = \begin{cases} Graze, & \text{if } h \text{ does not contain any } Gorge \\ Gorge, & \text{otherwise.} \end{cases}$$

There are two kinds of subgames:

- one where no one has transgressed
- one where at least 1 person has transgressed

For a SPNE, we need to see whether they want to deviate from $s_i(h)$ in each subgame.

In the latter subgame, since we have defined a NE outcome, no one will deviate from the specified action (Gorge).

Let us consider whether anyone will consider to choose Gorge instead of Graze in a period where everyone has not yet transgressed. (it is sufficient to consider 1 step deviations in strategies)

If he chooses to Graze, he gets $1 + \delta + \delta^2 + \cdots = 1 + \frac{\delta}{1-\delta}$

This is because they will always get 1 in each period.

If he chooses to Gorge, he gets $3 - \delta - \delta^2 - \cdots = 3 - \frac{\delta}{1-\delta}$

This is because he gets 3 in the period he gorges, but everyone chooses Gorge after which gives a payoff of -1 .

Grazing is preferred to Gorging if

$$1 + \frac{\delta}{1-\delta} \geq 3 - \frac{\delta}{1-\delta}$$

$$\frac{2\delta}{1-\delta} \geq 2$$

$$\delta \geq 1 - \delta$$

$$\delta \geq \frac{1}{2}$$

3.4

Now we repeat it with N players instead.

Payoffs if everyone gorges are $N - 2(N-1) = -(N-2)$

If he chooses to Gorge, he gets $N - (N-2)\delta - (N-2)\delta^2 - \cdots = N - (N-2)\frac{\delta}{1-\delta}$

Grazing is preferred to Gorging if

$$1 + \frac{\delta}{1-\delta} \geq N - (N-2)\frac{\delta}{1-\delta}$$

$$(N-1)\frac{\delta}{1-\delta} \geq (N-1)$$

$$\delta \geq 1 - \delta$$

$$\delta \geq \frac{1}{2}$$

So, it leads to exactly the same conditions.

3.5

In order to get a payoff of at least 2, shepherd 1 needs to gorge in some cases, but not so often such that others will want to gorge all the time.

The general idea is to construct a SPNE equilibria with Nash reversion strategies where Shepherd 2 and 3 allow Shepherd 1 to Gorge, say in even periods. Shepherd 1 however does not allow them to gorge at all. Observations of any deviations from this are again punished by gorging there after (which by construction is a NE).

I.e. Shepherd 1 is being treated more leniently.

We can then find conditions on the discount factor such that each will prefer not to deviate from the strategy. (Note let starting period be period 0)

Shepherd 1's strategy is:

$$s_1(h) = \begin{cases} Graze, & \text{if odd period and } h \text{ contains no } Gorges \text{ other} \\ & \text{than by shepherd 1 on even period} \\ Gorge, & \text{otherwise} \end{cases}$$

and the strategy of any shepherd $i \neq 1$ be

$$s_i(h) = \begin{cases} Graze, & \text{if } h \text{ contains no } Gorge \text{ other than by shepherd 1 on even period} \\ Gorge, & \text{otherwise} \end{cases}$$

Firstly, we show that if everyone follows this strategy, then shepherd 1 gets at least 2.

His future discounted payoff is $(1-\delta)(3+\delta+3\delta^2+\delta^3+3\delta^4+\cdots) = (1-\delta)(\frac{3}{1-\delta^2} + \frac{\delta}{1-\delta^2}) = \frac{3+\delta}{1+\delta} > 2$ for any $\delta < 1$.

Then we need to show that no one wants to deviate from this strategy. In particular, we only need to check this in information sets where no deviation has been observed because strategies after a deviation have been observed are automatically Nash Equilibria by construction.

For Shepherd 1:

In even periods, he will definitely not want to deviate from gorging.

In odd periods, he earns 1 in odd periods and 3 in even periods for a total payoff:

$$1 + 3\delta + \delta^2 + 3\delta^3 + \cdots = \frac{1}{1-\delta^2} + \frac{3\delta}{1-\delta^2}$$

$$= \frac{1+3\delta}{1-\delta^2}$$

If he deviates to gorge in the odd period, he gets 3 in the current period, but -1 thereafter:

$$3 - \delta - \delta^2 - \cdots = 3 - \frac{\delta}{1-\delta}$$

$$= \frac{3-4\delta}{1-\delta}$$

He will not want to strictly deviate if

$$\frac{1+3\delta}{1-\delta^2} \geq \frac{3-4\delta}{1-\delta}$$

Simplifying this, we get:

$$\delta \geq \frac{\sqrt{3}-1}{2}$$

For Shepherds 2 and 3:

We need to check whether they want to deviate in both odd and even periods.

In odd periods (Shepherd 1 is grazing),

If they do not deviate, their payoffs are -1 in even periods and 1 in odd periods:

$$1 - \delta + \delta^2 - \delta^3 + \cdots = \frac{1-\delta}{1-\delta^2}$$

If they deviate, their payoffs are 3 in the current period, but -1 thereafter:

$$3 - \delta - \delta^2 - \delta^3 - \cdots = \frac{3-4\delta}{1-\delta}$$

The condition for not wanting to deviate is:

$$\frac{1-\delta}{1-\delta^2} \geq \frac{3-4\delta}{1-\delta}$$

$$\delta \geq \frac{1}{\sqrt{2}}$$

In even periods (Shepherd 1 is gorging),

If they do not deviate, their payoffs are

$$-1 + \delta - \delta^2 + \delta^3 + \cdots = \frac{\delta - 1}{1-\delta^2}$$

If they deviate, their payoffs are 1 in the current period, but -1 thereafter:

$$1 - \delta - \delta^2 - \delta^3 + \cdots = \frac{1-2\delta}{1-\delta}$$

The condition for not wanting to deviate is:

$$\frac{\delta - 1}{1 - \delta^2} \geq \frac{1 - 2\delta}{1 - \delta}$$

$$\delta \geq \frac{\sqrt{5} - 1}{2}$$

So overall, we have 3 conditions:

$$\delta \geq \frac{\sqrt{5} - 1}{2}$$

$$\delta \geq \frac{1}{\sqrt{2}}$$

$$\delta \geq \frac{\sqrt{3} - 1}{2}$$

The strictest condition is:

$$\delta \geq \frac{1}{\sqrt{2}}$$

So, for all δ satisfying this, our defined strategies are a SPNE!