**Main things to learn**

- Distinguish between the different equilibria and apply the solution concepts:

The main difference is that in BNE, off the equilibrium path information set beliefs (and actions) "do not matter" as they do not factor into payoffs: hence they might turn out to be non-credible threats. In PBE, we need to specify off the equilibrium path beliefs (sometimes free to specify) and the resultant actions must also be sequentially rational at the information set given these beliefs.

Somewhat similar to how SPNE compares to NE (off equilibrium path sub-games).

However, PBE differs from SPNE in that SPNE considers sequential rationality in each subgame while PBE considers sequential rationality at each information set.

See http://www.eecs.harvard.edu/cs286r/courses/fall12/presentations/lecture3.pdf for a summary.

**1) Adverse selection.**

**1.1**

Insuree with type $x$ gets $x$ from insurance. Cost of providing is $x/2$.

Since benefits ≥ costs, social planner (someone who cares about maximizing total utility) will always want to provide insurance.

Knowing $x$ is not essential since the type does not matter for the social planners decision.

**1.2**

Insurance costs price $p$. Those with $x \geq p$ will purchase it.

**1.3**

Suppose one's type $x$ is common information. **The potential insuree's best response is to accept if $p \leq x$ and reject if $p > x$ (in each subgame).**

It is always possible for an insurer to get strictly positive profits if $(cost\ of\ providing)\frac{x}{2} < p \leq x(Maximum\ acceptable\ price)$. Any other price will give ≤ 0 profits.

This is true for any $x \in (0,1]$. Hence all types in this range will be served.

**The profit maximizing strategy is thus to choose the highest price within this range which is $x = p$.**

$x = 0$ is a trivial case since getting or not getting insurance is the same.

**1.4**

By sequential rationality and since the potential insuree knows his own type, **he/she accepts if $p \leq x$ and rejects if $p > x$.**

Given the insuree's strategy, the profits of the insurer are:

$$u_I(p,x) = \begin{cases} 0, & x < p \\ p - x/2, & x \geq p \end{cases}$$

Since the insurer does not know $x$, he maximises the expected profits given his priors about $x$:

$$E[u_I(p,x)] = \int_0^1 u_I(p,x)dx = \int_0^p 0dx + \int_p^1 (p - \frac{x}{2})dx$$

$$= \left[px - \frac{x^2}{4}\right]_p^1 = -\frac{3}{4}p^2 + p - \frac{1}{4}$$

Taking the first derivate and setting it to 0, we get

$$-\frac{3}{2}p + 1 = 0$$

$$\boldsymbol{p = 2/3}$$

**The bolded actions give their actions in the PBE. However, we also need the insurer's beliefs for the PBE. Since the insurer moves first, he does not have any additional information, so his beliefs in equilibrium are just his priors.**

**1.5**

Types with $x < 2/3$ will not be served by the market.

**Duopoly case**

**1.6**

Insurer $i's$ profits are:

$$u_i(p_i, p_j, x) = \begin{cases} 0, & p_i > x \text{ or } p_i > p_j \\ p_i - x/2, & p_i \leq x \text{ and } p_i < p_j \\ \frac{p_i - x/2}{2}, & p_i = p_j \leq x \end{cases}$$

(Somewhat similar to our auction examples)

**1.7**

Here, we can use the condition that in Bertrand price competition, Insurers should earn 0 profits and price identically.

1) 0 profits because otherwise undercutting (the positive profit firm) in price can strictly increase profits.
2) Equal prices because otherwise, there is an incentive from the firm with 0 profits (and the market share) to raise prices and earn positive profits.

Thus, we are in the third case above:

$$\int_p^1 \frac{1}{2}\left[p - \frac{x}{2}\right] dx = 0$$

$$\left[px - \frac{x^2}{4}\right]_p^1 = 0$$

$$-\frac{3p^2}{4} + p - \frac{1}{4} = 0$$

Setting this to 0, we get $p = 1\ or\ 1/3$.

Profits are positive between $\frac{1}{3}\ and\ 1$, so undercutting will lead to an equilibrium price of $\frac{1}{3}$.

Beliefs are the same as before: equal to the prior.

**1.8**

As we can see, more consumers obtain insurance, but having more insurers will not help since everyone is already earning zero profits.

1.9 With all customers purchasing insurance, profits for the lower-pricing insurer are

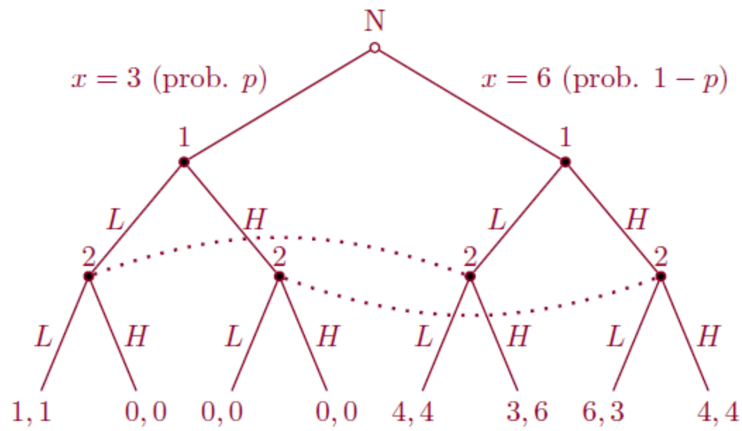
$$\int_0^1 \left[p - \frac{x}{2}\right] dx$$
$$p - \frac{1}{4}$$

Once again, price competition forces profits to zero, so $p_i = p_j = \frac{1}{4}$.

Now, to find $k$: as all consumers now find it optimal to purchase insurance, it has to be the case that for all consumers $x - p \geq -k$, which implies $k + x \geq p$; as the lowest-type consumer is $x = 0$, we have that $k \geq p$ has to be satisfied. The lowest $k$ satisfying this is $k = p$; so the lowest fine that leads to all consumers purchasing insurance is $k = \frac{1}{4}$. Notice that this is less than the price before the legislation was implemented!

This is because lower risks are in the pool, so they can now charge less than before.

2.1




The dotted lines indicate that player 2 when observing L does not know whether he is facing a situation with low price or high prices: this is an information set.

**2.2**

For player 1, his strategies are given by his choices in two scenarios $(x = 3, x = 6)$

For player 2, his strategies are given by his choices in two scenarios $(observe\ L, observe\ H)$

The normal form given all possible strategies can be calculated as follows:

*Note: the reason why we can use this normal form (assuming a common prior for all payoffs) to calculate Pure- BNE is nuanced: you can show that solving this would get best responses equivalent to if you used posteriors updated by Bayes rule on the equilibrium path.*

| Player 1 \ Player 2 | $LL$ | $LH$ | $HL$ | $HH$ |
|---|---|---|---|---|
| $LL$ | $3, 3$ | $\underline{3}, 3$ | $2, \underline{4}$ | $2, \underline{4}$ |
| $LH$ | $\underline{13/3}, 7/3$ | $\underline{3}, \underline{3}$ | $\underline{4}, 2$ | $\underline{8/3}, 8/3$ |
| $HL$ | $8/3, 8/3$ | $8/3, 8/3$ | $2, \underline{4}$ | $2, \underline{4}$ |
| $HH$ | $4, 2$ | $8/3, \underline{8/3}$ | $\underline{4}, 2$ | $\underline{8/3}, \underline{8/3}$ |

Figure 1: Payoffs in Normal Form using the prior $p = \frac{1}{3}$. Strategies for player 1 given as $s_1(3)s_1(6)$; for player 2 given as $s_2(L)s_2(H)$. That is, player 1's strategies are functions of $x$, whereas player 2's strategies are functions of player 1's actions.

Example (LH,LL)

Expected payoffs for player 1:

$$p(x = 3)U_1(L, L, 3) + p(x = 6)U_1(H, L, 6)$$

$$= \frac{1}{3}.1 + \frac{2}{3}.6$$

$$= \frac{13}{3}$$

Expected payoffs for player 2:

$$p(x=3)U_2(L,L,3)+p(x=6)U_2(H,L,6)$$
$$=\frac{1}{3}.1+\frac{2}{3}.3$$
$$=\frac{7}{3}$$

Using the underline method, we get the pure strategy BNE (LH,LH) and (HH,HH).

Now we check whether they are PBE as well by checking whether there are any beliefs of Player 2 which sustain their actions. (remember PBE are refinements of BNE)

In the case of $(LH, LH)$ all of 2's information sets are reached, so denoting by $\mu(a)$ the belief that 2 puts on $x = 3$ given player 1 played $a$, the supporting system of beliefs corresponds to

$$\mu(L) = \frac{P(L|x=3)\cdot P(x=3)}{P(L|x=3)\cdot P(x=3)+P(L|x=6)\cdot P(x=6)} = \frac{1\cdot 1/3}{1\cdot 1/3+0\cdot 2/3} = 1$$

and $\mu(H) = 0$ is found similarly. Therefore $((LH, LH), 1, 0)$ is a PBE. (Here, the first belief is $\mu(L)$ and the second $\mu(H)$, the beliefs at 2's first and second information set, respectively)

(This is like a separating equilibrium since player 2 knows that in equilibrium Player 1 will only play L if $x = 3$ and vice versa).

For $(HH, HH)$ the belief $\mu(H) = \frac{1}{3}$ is given by Bayes' Rule; but $\mu(L)$ is not. $L$ by player 1 has to induce player 2 to play $H$; this is a requirement that

$$\mu(L)u_2(L, \boxed{H,} 3) + (1-\mu(L))u_2(L, \boxed{H,} 6) \geq \mu(L)u_2(L, \boxed{L,} 3) + (1-\mu(L))u_2(L, \boxed{L,} 6)$$
$$\mu(L)0 + (1-\mu(L))6 \geq \mu(L)1 + (1-\mu(L))4$$

which reduces to simply

$$\mu(L) \leq \frac{2}{3}$$

so that for any $\mu(L) \in [0, \frac{2}{3}]$, we have that $((HH, HH), \mu(L), \frac{1}{3})$ is a PBE.

Here, $\mu(L)$ is off the equilibrium path, so we are free to specify it as above. Observing H does not update any beliefs since Player 1 chooses it in both cases (like a pooling equilibrium).

**3: Solving signalling games**

In questions (like in this problem set) on (pure strategy) separating and pooling Perfect Bayesian Equilibrium.

- Player 1 usually observes the state of nature, thus he can have a strategy for each realisation.
- Player 2 usually does not observe the state of nature (or the type of Player 1 which it determines). He however can condition his strategy on the observed action of Player 1
- If we have 2 states of nature, S1 and S2, then for each action of Player 1 he observes, Player 2 is unable to distinguish exactly which state of nature he is in. In the extensive form diagram, this is represented by the dotted lines connecting the nodes following (S1,a) and (S2,a) where a is the action of Player 1: the nodes are said to be in the same information set.
- Player 2 however has beliefs about which state of nature he is in for each observed action of Player 1 he observes.
    a. The strategies chosen by Player 1 affects Player 2's beliefs about the true state of nature in each information set. This then affects Player 2's conditional strategies.
- There are usually 2 possible pooling equilibria and 2 possible separating equilibria (because the first player is the one to observe nature's move and he can have)
- Pooling equilibria are those in which Player 1 has similar strategies for the different states of nature. Separating equilibria are those in which Player 1 has different strategies in the different states of nature.

The steps I would follow to solve such questions are:

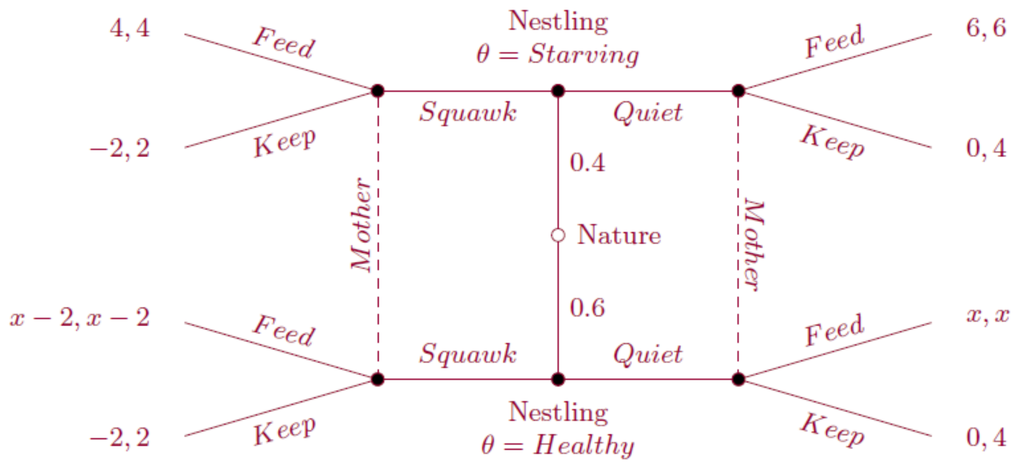
1. Make a hypothesis about Player 1's equilibrium strategies (this will correspond to one of the 4 types of equilibria above)
2. Given these Player 1 strategies, identify the beliefs of Player 2 about the true state of nature at every information set that is on the "equilibrium[1]" path.
3. Given these beliefs, compute Player 2's best response at the information set that is on the equilibrium path.
4. Verify that Player 1's equilibrium strategies which we started from are indeed his best reply given the state of nature which he observes.
5. This will impose restrictions on the receiver's beliefs that are at information sets that are off the equilibrium path.
    a. In particular, we need to specify beliefs on information sets that are off the equilibrium path which lead to best responses of Player 2 which are consistent with (4).

Sometimes, there may be shortcuts to rule out certain equilibria, but if you follow the above method it should not go wrong.

**Note: Problem set 4b from last year's material has another example of this kind of question.**

[1] I use inverted commas here because we do not know yet whether it is an equilibrium. It is just an assumption at the moment.

**3.1**




**3.2** $\boldsymbol{x = 1}$

For healthy types, we can see that being quiet is the dominant strategy:

Squawking gets a healthy type at most $-1$ (if fed) which is less than 0 if it is quiet and is not fed.

Hence, we can focus on equilibria where either both types are quiet, or healthy types are quiet, while starving types are healthy.

In an equilibrium where the two types pool on Quiet, the mother computes that it is preferable to keep the worm (4 utility) than feed it to the nestling ($0.4 * 6 + 0.6 * 1 = 3$ utility). For such an equilibrium, we need to check the Starving type must not wish to deviate to Squawk. We're only looking for pure-strategy equilibria, so there are two cases: either the mother feeds after Squawking, or not. If she doesn't, the deviation to Squawking by the Starving type is clearly a bad idea. If she does, the Starving nestling would get 4 by deviating to Squawking, versus 0 in the equilibrium; so she must not want to feed a squawking nestling. For her to not want to feed a squawking nestling, we need beliefs that a Squawking bird is Starving, $\mu_S$, to be such that

$$\mu_S u_M(S, F, Starving) + (1-\mu_S) u_M(S, F, Healthy) \le \mu_S u_M(S, K, Starving) + (1-\mu_S) u_M(S, K, Healthy)$$

$$\mu_S 4 + (1 - \mu_S)(-1) \le 2$$

$$5\mu_S \le 3$$

$$\mu_S \le \frac{3}{5}$$

Therefore, for any $\mu_S \in [0, 3/5]$ we have that $((QQ, KK), \mu_S, .4)$ is a PBE. A simple check also shows that a separating equilibrium also exists here, where only the Starving type Squawks; the PBE is $((SQ, FK), 1, 0)$.

### 3.3 $x = 3$

Now we have that healthy types might want to squawk (not dominated anymore).

However, we have that a separating equilibrium is impossible.

In a separating equilibrium, the mother will be able to identify each type. She will thus not want to feed the healthy type because $3 - 2 < 2$ (if it squawks) or $3 < 4$ (if it does not squawk). She will always feed the starving type.

Because of this, the healthy type will always want to imitate the starving type: imitating a squawking starver gives $3 - 2 > 0$ and imitating a quiet starver gives $3 > -2$.

If the types pool, the mother feeds because $0.4 \times 4 + 0.6 \times 1 > 2$ $(both\ squawk)$

Or $0.4 \times 6 + 0.6 \times 3 > 4$ $(both\ quiet)$

For PBE, we will need to specify the off-equilibrium path beliefs and the corresponding best response for the mother in that case.

- $((QQ, FF), \mu_S, .4)$ with any $\mu_S \in [\frac{1}{3}, 1]$
- $((QQ, KF), \mu_S, .4)$ with any $\mu_S \in [0, \frac{1}{3}]$
- $((SS, FK), .4, \mu_Q)$ with any $\mu_Q \in [0, \frac{1}{3}]$.

Notice that there is no (SS, FF): why?

If the mother also fed when they kept quiet, the nestlings would both want to deviate to keeping quiet.

3.4 Here, Feeding is a dominant strategy for the Mother. Therefore, in any equilibrium, she plays $FF$; therefore, all nestlings will be Quiet. For any $\mu_S \in [0, 1]$, we have that $((QQ, FF), \mu_S, .4)$ is a PBE.

3.5 The cost of squawking is sufficiently large that, when $x = 1$, the starving nestling can signal the mother it's type and a healthy nestling would not want to follow suit. The signaling value of Squawking is larger for a starving nestling than for a healthy one, and this allows it to credibly signal its type. When $x$ is 3 or 5, however, the healthy nestling has a strong enough incentive to appear Starving, however, and is willing to Squawk to accomplish that, so that separation becomes impossible.

Note cost of squawking relative to benefits falls as $x$ rises.