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Microeconomic Theory (501b)

Problem Set 9. Auctions
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This problem set is due on Tuesday, 4/15/14.

1. **All Pay Auction. Complete Information.**The rules of the all pay auction are: (*i*) the highest bid receives the object, (*ii*) each bidder pays his bid, independent of whether he wins or loses the object. If two bidders offer exactly the same bid, then each bidder receives the object with equal probability

   (a) Consider the complete information version of the all pay auction. There are two bidders, and each bidder values the object at $v > 0$.

      i. Argue first that there cannot be a pure strategy equilibrium in this game.
      ii. Argue next that there cannot be a mixed strategy equilibrium where any specific bid receives a strictly positive probability.
      iii. Argue next that the support of the equilibrium bidding strategy must form an interval, that is, it is convex and does not display any gaps.
      iv. Now compute the mixed strategy Nash equilibrium of the all pay auction and argue that the expected net utility has to be equal to zero.

   (b) Consider the complete information version of the all pay auction. There are two bidders, but now each bidder values the object differently, namely $v_1 > v_2 > 0$. Compute the mixed strategy Nash equilibrium of the all pay auction.

      i. Verify that the losing bidder may now place an atom, a positive probability on the lower bound, of the support of his strategy.
      ii. Now compute the mixed strategy Nash equilibrium of the all pay auction and identify the expected net utility of each bidder.
      iii. Is the resulting equilibrium leading to an efficient allocation of the object.

   [Solution]

(a) (i) It is easy to see that bidding above v is always strictly dominated, so it is easy to see that it will never be played. On the other hand, note that all players bidding 0 is also not an equilibrium, as one player could get the object a price ε (were ε is always considered a "small enough" number). By the same argument there must be at least two players bidding more than 0 in equilibrium.

Consider now pure strategies. If one player is submitting a strictly positive bid that does not win the object for sure there are two possibilities. Either he never wins the object, in which case he has a best response that is bidding 0. Otherwise there is a tie at the maximum bid, but in this case one of the players can increase his own bid by ε and win it for sure, making it a profitable deviation.

(ii) Consider a bid that receives a strictly positive probability, which we call b_i. There are two possibilities. Either at least one other player is playing a mixed strategies and bids ε smaller than b_i receive a positive probability density. In this case the other players can increase the bid by ε, and increase their probability of winning in a positive amount (as b_i receives a mass of probability). If there is no bid in an interval $[b_i - \varepsilon, b_i]$, then player i that is submitting b_i could decrease b_i and keep exactly the same probability of winning. Thus, this cannot be an equilibrium either.

(iii) Suppose there exists an interval $[b, b']$ in which on bid is submitted, but some bids greater than b' are submitted, which we know must have continuous density. Then, for some bid close enough to b' a player can decrease the bid all the way down to b, keeping the probability of winning ε small, but paying $b - b'$ less. For ε small enough this is a profitable deviation. Note that we have proved that there are no gaps in the submitted bids by all players. If we restrict attention to symmetric strategies this implies that no players plays a strategy that has a gap.

(iv) We look for a symmetric equilibrium in which players on average get a payoff of 0, which we will denote σ. Note that σ represents a probability density in $[0, v]$, and we denote by Σ the cumulative probability distribution of σ. The probability of winning when submitting bid b is given by $\Sigma(b)^{n-1}$. Thus, since a player gets a payoff of 0 for each bid he submits, we have that,

$$v \cdot \Sigma^{n-1}(b) = b \quad \forall b \in [0, v] \text{ such that } \Sigma(b) < 1.$$

This implies that the cumulative distribution is given by:

$$\Sigma(b) = (\frac{b}{v})^{n-1}$$

Thus, the strategy of players is given by:

$$\sigma(b) = (\frac{b}{v})^{n-2}(n-1) \text{ for all } b \in [0, v].$$

Note that all players get on expectation 0 from their bids.

(b) (i) This will become clear from the construction of the equilibrium.

(ii) We look for an equilibrium in which the player with lower valuation gets 0 expected profits. Thus, we must have that for all bids that agent 2 submits he gets 0 profits. Thus,

$$v_2 \cdot \Sigma_1(b) = b \quad \forall b \in [0, v_2] \text{ such that } \Sigma_2(b) < 1.$$

where we denote the strategy of high and low valuation player by subindices 1 and 2. This implies that:

$$\Sigma_1(b) = \frac{b}{v_2} \quad \forall b \in [0, v_2] \text{ such that } \Sigma_2(b) < 1.$$

Note that we are looking for an equilibrium in which player 2 gets 0 expected profits, so he cannot bid v_2 and win the object for sure. Thus, the support of σ_1 must be $[0, v_2]$. Thus, the support of σ_2 must also be $[0, v_2]$. Thus, we have that:

$$\sigma_1(b) = \frac{1}{v_2} \text{ for all } b \in [0, v_2]$$

We know that for all bids that player 1 submits, he must get the same expected profits. Thus, there exists a c such that (since player 1 gets positive expected $c \neq 0$).,

$$v_1 \cdot \Sigma_2(b) - b = c \quad \forall b \in [0, v_1] \text{ such that } \Sigma_1(b) < 1.$$

We know that player 1 can bid v_2 and get the object for sure. Thus, his expected rents must be $v_1 - v_2$. Thus, we have that:

$$c = v_1 - v_2$$

Whenever player 1 bids 0, his expected profits must be $v_1 - v_2$, so he must win the object with probability $(v_1 - v_2)/v_1$ Thus, we have that the strategy of player 2 will be

$$\Sigma_2(b) = \frac{(v_1 - v_2)}{v_1} + (\frac{v_2}{v_1})(\frac{b}{v_2}) \text{ for all } b \in [0, v_2]$$

This implies that player 2 plays 0 with probability $(v_1 - v_2)/v_1$ and a uniform $[0, v_2]$ with the complementary probability. Finally, note that the tie breaking rule is key to keep player 2 indifferent between bidding 0 and some number ε above 0, as he always wins against v_l if he is v_h.

(iii) Obviously not as high valuation player. Note that the revenue equivalence does not hold as bids are not strictly increasing in agents type, and thus the low valuation agent gets the object with positive probability

2. **All Pay Auction. Incomplete Information.** The rules of the all pay auction are: (i) the highest bid receives the object, (ii) each bidder pays his bid, independent of whether he wins or loses the object.

   (a) Characterize the equilibrium of the all-pay auction in the symmetric environment with a *uniform distribution on the unit interval* for $I = 2$ bidders. (Hint: Guess that the equilibrium bidding function is an increasing and quadratic function.)

   (b) Characterize the equilibrium of the all-pay auction in the symmetric environment with a *continuously differentiable distribution function.* (You may either proceed similarly to the method we proceeded in class and/or directly the assume the validity of the revenue equivalence theorem.)

   [SOLUTION]

   (a) Let there be N bidders with uniformly distributed valuations. Remember that in this incomplete information game, a player's type is her valuation and a strategy is a function mapping valuations into bids.

   Look for a symmetric equilibrium in which players use a strictly increasing (identical) bidding function $b_i(v_i) = b(v)\ \ \forall i$. If all players $j \neq i$ follow a strategy $\beta_j(v_j)$, then player i's payoff is given by:

$$\begin{aligned}
u_i\left(v_i, b_i, \beta_j\right) &= v_i \prod_{j\neq i} \Pr\left(b_i > \beta_j\right) - b_i \\
&= v_i \prod_{j\neq i} \Pr\left(\beta_j^{-1}(b_i) > v_j\right) - b_i \\
&= v_i F^{N-1}\left(\beta_j^{-1}(b_i)\right) - b_i \\
&= v_i \left[\beta_j^{-1}(b_i)\right]^{N-1} - b_i.
\end{aligned}$$

   Differentiating with respect to b_i one obtains

$$v_i (N-1) \left[\beta_j^{-1}(b_i)\right]^{N-2} \frac{1}{\beta_j'\left(\beta_j^{-1}(b_i)\right)} - 1 = 0$$

Imposing symmetry $\left(\beta \equiv b, \beta^{-1}(b) = v\right)$:

$$\begin{aligned} v(N-1)\,v^{N-2}\frac{1}{b'(v)} - 1 &= 0 \\ b'(v) &= (N-1)\,v^{N-1} \\ b(v) &= \frac{N-1}{N}v^{N} \end{aligned}$$

From this solution you can verify that for $N = 2$ the bidding function is quadratic.

(b) Follow the previous steps, suppose a symmetric equilibrium in which players use a strictly increasing (identical) bidding function $b_i(v_i) = b(v)\ \forall i$. If all players $j \neq i$ follow a strategy $\beta_j(v_j)$, then player i's payoff is given by:

$$\begin{aligned} u_i\left(v_i, b_i, \beta_j\right) &= v_i \prod_{j \neq i} \Pr\left(b_i > \beta_j\right) - b_i \\ &= v_i \prod_{j \neq i} \Pr\left(\beta_j^{-1}(b_i) > v_j\right) - b_i \\ &= v_i F^{N-1}\left(\beta_j^{-1}(b_i)\right) - b_i \end{aligned}$$

Differentiating with respect to b_i one obtains

$$v_i(N-1)\left[F(\beta_j^{-1}(b_i))\right]^{N-2}\frac{f(\beta_j^{-1}(b_i))}{\beta_j'\left(\beta_j^{-1}(b_i)\right)} - 1 = 0$$

Imposing symmetry $\left(\beta \equiv b, \beta^{-1}(b) = v\right)$:

$$\begin{aligned} v(N-1)\,F(v)^{N-2}\frac{f(v)}{b'(v)} - 1 &= 0 \\ b'(v) &= v(N-1)\,F(v)^{N-2}f(v) \\ b(v) &= (N-1)\int_0^v yF(y)^{N-2}f(y)dy \end{aligned}$$

Notice that if $F(y) \sim Uniform[0,1]$, we obtain the result of part (a).

3. **First Price Auction.** Consider the first price auction in a symmetric environment with binary valuations, i.e. the value of bidder i is given by $v_i \in \{v_l, v_h\}$ with $0 \leq v_l < v_h < \infty$. It is sufficient to consider the case of $i = 1, 2$. (You may assume an efficient tie-breaking rule; i.e. if the there are two bidders, then the bidder with the higher value receives the object, if they have the same value, then the probability of receiving the object is the same.)

   (a) The prior probability is given by $\Pr(v_i = v_h) = \alpha$ for all i. Characterize the equilibrium in the first price auction. (Hint: Can you find a pure strategy Bayesian Nash equilibrium?)

(b) The prior probability is now given by $\Pr(v_i = v_h) = \alpha_i$ with $0 < \alpha_1 < \alpha_2 < 1$. Characterize the equilibrium in the first price auction. (Hint: Can you find a pure strategy Bayesian Nash equilibrium?)

(c) Does the revenue equivalence result between the first and the second price auction still hold with the binary payoff types.

[SOLUTION]

(a) Consider only the case with 2 bidders and focus on symmetric interim BNE. First of all, note that for any type $\{v_l, v_h\}$, it is irrational to bid above her own valuation, ie, $b(v_i) \leq v_i$, for all i. Given this, any bidder of the high type, v_h, will always bid above v_l, since there is no point in bidding below v_l and incur the risk of loosing against a low type when she could easily win and make positive profit. So, in any BNE it must be the case that $v_l \leq b(v_h) \leq v_h$. Now let's focus on each type separetely.

- *Low Type:* in any BNE, we must have $b(v_l) = v_l$, because for any strategy such that $b(v_l) < v_l$, the low type bidder has an incentive to deviate and bid slightly more than b, win with positive probability (when the other bidder is a low type) and, so, obtain a positive expected payoff. (Of course she always get a payoff of zero when playing against a high type).
- *High Type:* once that $b(v_l) = v_l$ and $v_l \leq b(v_h) \leq v_h$ in any BNE, we see that the high type prefers to bid as close as possible to v_l when playing against a low type (since then she wins the auction and pays the lowest price possible) but as large as v_h when against a high type (since for any given strategy profile such that $b(v_h) < v_h$, she would have an incentive to deviate and bid slightly above $b(v_h)$, and obtain positive payoff for sure). In any case, if (a) $b_i < b_j < v_h$, then i would have a profitable deviation; if (b) $v_l \leq b_j < b_i < v_h$, then again i would have a profitable deviation; and if (c) $b_j \leq b_i = v_h$, once more i would have a profitable deviation since this action garantees a zero payoff while biding less than v_h gives a positive expected payoff (since high type always win against low type and she would pay less than her own valuation). Because of this discontinuity in the expected payoff, there is no pure strategy best reply to strategies such that $b(v_l) = v_l$ and $v_l \leq b(v_h) \leq v_h$. Ie, there is no pure strategy BNE here.
  So, let's look for a mixed strategy BNE. From the discussion above, the only type that is willing to mix is the high type. So let's propose a BNE such that $b(v_l) = v_l$ and the high type mixes with the distribution $F(b)$ over some support $[\underline{b}, \bar{b}]$.
  Of course, from above, we must have $v_l \leq \underline{b} \leq \bar{b} \leq v_h$. Moreover, the lower bound must be $\underline{b} = v_l$, because if $\underline{b}$ was strictly greater than v_l then bids in the interval $(v_l, \underline{b})$ would be preferred to $\underline{b}$. A

subtlety here is that although $\underline{b} = v_l$, we must have the support to be open at the lower bound, ie, $(v_l, \bar{b}]$. The reason is that when the high type bids v_l there is a tie whenever the opponent is a low type and they get the good with probability 1/2, by the tie-breaking rule. This situation gives a smaller payoff for the high type than the expected payoff obtained when bidding slightly above v_l. However, if the mixed strategy $F(b)$ puts mass zero at this left-end point, then this event occurs with zero probability and we do not need to care about it. In this case we could use the support as $[v_l, \bar{b}]$.[1]
Actually, the distribution $F(b)$ has to be continuous because if it is not (at some point b), the other player could put more mass right above b and get higher expected payoff. So from now on, suppose $F(b)$ is continuous and the support is $[v_l, \bar{b}]$. Now we have to find both $\bar{b}$, $F(b)$ and we have to show that there is no incentive for the high type to deviate from this strategy.

- *Find $\bar{b}$:* when $b = v_l$, the expected payoff of the high type is $(1-\alpha)(v_h - v_l)$. At any other b in the support, the expected payoff is $(1-\alpha)(v_h - b) + \alpha(v_h - b)F(b)$. Therefore, $\bar{b}$must be such that $(1-\alpha)(v_h - v_l) = (1-\alpha)(v_h - \bar{b}) + \alpha(v_h - \bar{b})$, since $F(\bar{b}) = 1$. Implying $\bar{b} = \alpha v_h + (1-\alpha)v_l$. Note, as (should be) expected, that $\bar{b} < v_h$.
- *Find $F(b)$:* from the previous reasoning it is easy to deduce that for all $b \in [v_l, \alpha v_h + (1-\alpha)v_l]$, we have that

$$(1-\alpha)(v_h - v_l) = (1-\alpha)(v_h - b) + \alpha(v_h - b)F(b)$$

$$\Rightarrow F(b) = \frac{1-\alpha}{\alpha}\left[\frac{(v_h - v_l)}{(v_h - b)} - 1\right]$$

- *Check this is a BNE:* the high type is indifferent, by construction, over the support $[v_l, \alpha v_h + (1-\alpha)v_l]$. Moreover, she does not want to deviate since (i) any $b < v_l$ gives a payoff of zero, which is smaller than the payoff $(1-\alpha)(v_h - v_l)$ obtained when using the mixed strategy; and (ii) any $b \in (\alpha v_h + (1-\alpha)v_l, v_h]$ gives a payoff smaller than $v_h - \{\alpha v_h + (1-\alpha)v_l\} = (1-\alpha)(v_h - v_l)$. Hence, bids within the support are better than bids outside the support. Consequently, the high type does not deviate from the proposed mixed strategy. Since the low type gives a best response $b(v_l) = v_l$, we found a symmetric BNE of this game.

(b) The strategies calculated in the previous part where meant to leave the other agent indifferent when he is the high type. Yet, for an agent conditional on being the high type, these strategies will still work for

[1] Another solution would be to impose a tie-breaking rule that gives the object to the highest type whenever a tie ocurs.

all bids in the support of the distribution. That is, we just index the strategy by the agents own prior distribution as follows:

$$F_i(b) = \frac{1-\alpha_i}{\alpha_i}\left[\frac{(v_h - v_l)}{(v_h - b)} - 1\right]$$

Once again, as these strategies left the other agent indifferent in the previous part, they will still keep the agent indifferent independent on what was the prior. Yet, this is only true for $b \in [v_l, \alpha_i v_h + (1-\alpha_i)v_l]$. Thus, if we assume each player plays according to F_i ,then for player two there will be bids in the support of his strategy that are not in the support of the strategy of player 1. This cannot be optimal, as he could reduce the bids keeping the probability of winning constant.

Thus, we assume that player 1 plays the same strategy as before, but we modify the strategy of player 2. We do this by assuming he will play a mass of probability at v_l when his type is v_h. The mass of probability is such that it induces the same expected distribution over bids as the one that player 1 has. Since player 2 is high type with higher probability, he can play v_l with a mass probability when he is v_h and induce the same distribution over bids as player 1.

Summing up, the equilibrium bidding strategies are given by (these are the distributions conditional on being v_h, conditional on being v_l both players bid v_l):

$$F_1(b) = \frac{1-\alpha_1}{\alpha_1}\left[\frac{(v_h - v_l)}{(v_h - b)} - 1\right]$$

$$F_2(b) = (1 - \frac{\alpha_2 - \alpha_1}{\alpha_2})\frac{1-\alpha_1}{\alpha_1}\left[\frac{(v_h - v_l)}{(v_h - b)} - 1\right] + \frac{\alpha_2 - \alpha_1}{\alpha_2}.$$

Just to repeat the argument one more time. Both players play v_l with probability $1-\alpha_1$. Player 1 plays v_l when he is type v_l only. On the other hand, player 2 plays v_l when he is v_l, but when he is v_h he also plays v_l with probability $(\alpha_2 - \alpha_1)/\alpha_2$. Note that,

$$1 - \alpha_1 = 1 - \alpha_2 + \alpha_2 \cdot \frac{(\alpha_2 - \alpha_1)}{\alpha_2}.$$

Since the strategy of both player plays all other bids with the same probability, the final (unconditional) distribution over bids is the same for player 1 and player 2 (even when their strategies are different!). Since we already showed that under these distributions players were indifferent between all bids, this is still an equilibrium (as we have not changed the distribution over bids).

(c) Consider the Second Price Auction first. The equilibrium in dominant strategies is, as usual, $b(v_i) = v_i$ for both types. The expected revenue therefore is $R_{(SPA)} = (1-\alpha)^2 v_l + 2\alpha(1-\alpha)v_l + \alpha^2 v_h$, where the first term is the probability of having two bidders with low type times the corresponding payment; the second term reflects the cases where one bidder is low type and the other is a high type and so on. In the First Price Auction, the expected revenue is $R_{(FPA)} = (1-\alpha)^2 v_l + 2\alpha(1-\alpha)E(b_h) + \alpha^2 E(b_{(1)})$, where $E(b_h)$ is the expected bid of the high type when she mixes (since that is the expected payment when one bidder is a low type and the other is a high type); and $E(b_{(1)}) = E(\max\{b_1, b_2\})$ (since that is the expected payment when the two bidders are of a high type).

Now we have to solve some tedious algebra. First note that $R_{(SPA)} = R_{(FPA)}$ if and only if the last two terms of each coincide, i.e., iff $2\alpha(1-\alpha)v_l + \alpha^2 v_h = 2\alpha(1-\alpha)E(b_h) + \alpha^2 E(b_{(1)})$.

Now, notice that, by integration by parts,

$$E(b_h) = \int_{v_l}^{\overline{b}} b.dF(b) = \overline{b} - \int_{v_l}^{\overline{b}} \frac{1-\alpha}{\alpha}\left[\frac{(v_h - v_l)}{(v_h - b)} - 1\right] db$$

therefore,

$$2\alpha(1-\alpha)E(b_h) = 2\alpha(1-\alpha)\overline{b} - 2(1-\alpha)^2 \int_{v_l}^{\overline{b}} \left[\frac{(v_h - v_l)}{(v_h - b)} - 1\right] db \quad (1)$$

Also, by integration by parts again and noticing that $b_{(1)} \sim [F(b)]^2$

$$E(b_{(1)}) = \overline{b} - \int_{v_l}^{\overline{b}} \left(\frac{1-\alpha}{\alpha}\right)^2 \left[\frac{(v_h - v_l)}{(v_h - b)} - 1\right]^2 db$$

implying

$$\alpha^2 E(b_{(1)}) = \alpha^2\overline{b} - (1-\alpha)^2 \int_{v_l}^{\overline{b}} \left[\frac{(v_h - v_l)}{(v_h - b)} - 1\right]^2 db$$

$$= \alpha^2\overline{b} - (1-\alpha)^2 \int_{v_l}^{\overline{b}} \left(\frac{v_h - v_l}{v_h - b}\right)^2 db + 2(1-\alpha)^2 \int_{v_l}^{\overline{b}} \left(\frac{v_h - v_l}{v_h - b}\right) db - (1-\alpha)^2 \int_{v_l}^{\overline{b}} 1.db \quad (2)$$

Notice the last term of $2\alpha(1-\alpha)E(b_h)$, in (1), and the two last terms of $\alpha^2 E(b_{(1)})$, in (2). Therefore, we have that

$$[2\alpha(1-\alpha)E(b_h)] + [\alpha^2 E(b_{(1)})] =$$

$$= \left[2\alpha(1-\alpha)\overline{b} + (1-\alpha)^2 \int_{v_l}^{\overline{b}} 1.db\right] + \left[\alpha^2\overline{b} - (1-\alpha)^2 \int_{v_l}^{\overline{b}} \left(\frac{v_h - v_l}{v_h - b}\right)^2 db\right]$$

$$= \left[2\alpha(1-\alpha)\overline{b} + (1-\alpha)^2(\overline{b} - v_l)\right] + \left[\alpha^2\overline{b} - (1-\alpha)^2(v_h - v_l)^2 \left[\frac{1}{v_h - b}\right]_{v_l}^{\overline{b}}\right]$$

$$= \left[2\alpha(1-\alpha)\bar{b} + (1-\alpha)^2(\bar{b} - v_l)\right] + \\ + \left[\alpha^2\bar{b} - (1-\alpha)^2(v_h - v_l)^2\left[\frac{1}{(1-\alpha)(v_h - v_l)} - \frac{1}{(v_h - v_l)}\right]\right]$$

$$= 2\alpha(1-\alpha)\bar{b} + (1-\alpha)^2(\bar{b} - v_l) + \alpha^2\bar{b} - (1-\alpha)(v_h - v_l) + (1-\alpha)^2(v_h - v_l)$$

After some more algebra and using $\bar{b} = \alpha v_h + (1-\alpha)v_l$, we obtain

$$= 2\alpha(1-\alpha)v_l + \alpha^2 v_h$$

Hence $[2\alpha(1-\alpha)E(b_h)] + [\alpha^2 E(b_{(1)})] = 2\alpha(1-\alpha)v_l + \alpha^2 v_h$, and, so, the revenue equivalence holds here.

4. **Bilateral Trading.** Suppose there is a *continuum* of buyers and sellers (with quasilinear preferences). Each seller initially has one unit of indivisible good and each buyer initially has none. A seller's valuation for consumption of the good is $\theta_1 \in [\underline{\theta}_1, \overline{\theta}_1]$, which is independently and identically drawn from distribution $\Phi_1(\cdot)$ with associated strictly positive density $\phi_1(\cdot)$. A buyer's valuation from consumption of the good is $\theta_2 \in [\underline{\theta}_2, \overline{\theta}_2]$, which is independently and identically drawn from distribution $\Phi_2(\cdot)$ with association strictly positive density $\phi_2(\cdot)$.

   (a) Characterize the trading rule in an ex post efficient social choice function. Which buyers and sellers end up with a unit of the good?

   (b) Exhibit a social choice function that has the trading rule you identified in **(a)**, is Bayesian incentive compatible, and is the individually rational. [Hint: Think of a "competitive" mechanism.] Conclude that the inefficiency identified in the Myerson-Satterthwaite theorem goes away as the number of buyers and sellers grows large.

   [SOLUTION]

   (a) The ex post efficient trading rule will have all the highest valuation agents owning a unit of the good. That is, we may have only buyers, only sellers or a mixture of both ending up with the good, as long as there is some $\tilde{\theta}$ such that all agents with $\theta \geq \tilde{\theta}$ have one unit of the good and all others do not. The total amount of good is given by the continuum $[\underline{\theta}_1, \overline{\theta}_1]$.

   (b) Define the following "competitive" social choice function as follows: let q_S and q_D denote market supply and demand, and be defined as:

$$q_D = \begin{cases} \overline{\theta}_2 - \underline{\theta}_2 & \text{for } p \leq \underline{\theta}_2 \\ (\overline{\theta}_2 - \underline{\theta}_2)\int_p^{\overline{\theta}_2} d\Phi_2(\theta) & \text{for } \underline{\theta}_2 \leq p \leq \overline{\theta}_2 \\ 0 & \text{for } p \geq \overline{\theta}_2 \end{cases} \qquad q_S = \begin{cases} 0 & \text{for } p \leq \underline{\theta}_1 \\ (\overline{\theta}_1 - \underline{\theta}_1)\int_{\underline{\theta}_1}^{p} d\Phi_1(\theta) & \text{for } \underline{\theta}_1 \leq p \leq \overline{\theta}_1 \\ \overline{\theta}_1 - \underline{\theta}_1 & \text{for } p \geq \overline{\theta}_1 \end{cases}$$

   The market equilibrium price $p^* \equiv \min\{p : q_D(p) \leq q_S(p)\}$, which is a well defined object, will cause efficient trade, and will lead to the efficient outcome described in (a) above. Trivially, incentive compatibility holds since there is no need for announcements. It is individually rational since a buyer will buy if and only if $p \leq \theta_2$ and a seller will sell if and only if $p \geq \theta_1$. This example shows that for a continuum of buyers and sellers the Myerson-Satterthwaite theorem (nonexistence of an individual rational, incentive compatible and ex post efficient direct mechanism) no holds.

5. **Single Unit Auction.** Suppose the valuation of agent $i = 1, 2$, and $j \neq i$, for the object is given by

$$u_i(\theta_i, \theta_j) = \theta_i + \gamma\theta_j$$

with $0 < \gamma < 1$. The type θ_i is private information of agent i and as the valuation of the object by agent i also depends on the type of his competitor j, we are in a world of interdependent rather than private values.

(a) Find a transfer rule t^* such that truthtelling is an ex post equilibrium in the direct revelation game and such that the efficient allocation is realized and such that the transfer of each agent only depends on the announcement of the other agent and the allocation decision, but not on the announcement of agent i.

(b) Given the transfer rule, is truthtelling also an equilibrium in dominant strategies?

[SOLUTION]

(a) Let $\{q(\theta), t(\theta)\}$ be a direct mechanism where $q_i(\theta)$ is the probability that i gets the object, and $t_i(\theta)$ the transfer from i to the seller.

Efficiency requires that the object goes to the player with the highest valuation (note that $u_i \geq u_j \Leftrightarrow \theta_i \geq \theta_j$ since $\gamma \in (0, 1)$). Therefore, an efficient mechanism[2] should have for all $i \neq j$

$$q_i(\theta) = \begin{cases} 1 & \text{if } \theta_i > \theta_j,\, \theta_i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

With quasilinear utilities we have that player i's payoff is then

$$U_i(\theta) = q_i(\theta_i + \gamma\theta_{-i}) - t_i$$

Consider the following transfer rule

$$t_i(\theta) = (1 + \gamma)\theta_j q_i(\theta)$$

where q_i is the efficient allocation rule defined above.

[2] Assuming the prior distributions are such that ties are zero probability events.

Let's check in this direct mechanism truthtelling is an ex post equilibrium and there is efficient allocation of the good. Truthtelling is an ex post equilibrium iff $\forall i$, $\forall \theta$

$$q_i(\theta)[(\theta_i + \gamma\theta_j) - (1+\gamma)\theta_j] \geq q_i(\theta_i', \theta_j)[(\theta_i + \gamma\theta_j) - (1+\gamma)\theta_j] \quad \forall \theta_i' \tag{3}$$

For arbitrary θ_i and θ_j, from (3) we need to check $\forall \theta_i'$

$$1\{\theta_i > \theta_j\}[\theta_i - \theta_j] \geq 1\{\theta_i' > \theta_j\}[\theta_i - \theta_j] \tag{4}$$

If $\theta_i > \theta_j$ (4) implies

$$1 \geq 1\{\theta_i' > \theta_j\}$$

which holds $\forall \theta_i'$, and if $\theta_i < \theta_j$ (4) implies

$$0 \leq 1\{\theta_i' > \theta_j\}$$

which again holds $\forall \theta_i'$.

(b) Truthtelling is an equilibrium in (weakly) dominant strategies iff[3] $\forall i$, $\forall \theta$, $\forall \theta_i'$

$$q_i(\theta_i, \theta_j')[(\theta_i + \gamma\theta_j) - (1+\gamma)\theta_j'] \geq q_i(\theta_i', \theta_j')[(\theta_i + \gamma\theta_j) - (1+\gamma)\theta_j'] \quad \forall \theta_j' \tag{5}$$

and

$$q_i(\theta_i, \theta_j')[(\theta_i + \gamma\theta_j) - (1+\gamma)\theta_j'] > q_i(\theta_i', \theta_j')[(\theta_i + \gamma\theta_j) - (1+\gamma)\theta_j'] \quad \text{for some } \theta_j'. \tag{6}$$

To show that truthtelling is not an equilibrium in dominant strategies it suffices an example that violates (5) and (6). Say j has valuation θ_j but reports θ_j' with $\theta_j' > \theta_i > \theta_j$. If i reports his true valuation, θ_i, he looses and gets a payoff of 0. On the other hand if he reports $\theta_i' > \theta_j'$, he wins and gets a payoff of $\theta_i - \theta_j > 0$.

Reading MWG: 23, S (=Salanie) 2 and 3

[3] A similar definition can be given in an interim version without changing the results.