Optimal equity auctions with two-dimensional types*

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Abstract

We analyze the design and performance of equity auctions when bidder's valuations and opportunity costs are private information, distributed according to an arbitrary joint density that can differ across bidders. We identify, for any incentive compatible mechanism, an equivalent single-dimensional representation for uncertainty. We then characterize the revenue-maximizing and surplus-maximizing equity mechanisms, and compare revenues in optimal equity and cash auctions. Unlike in cash auctions, the adverse selection arising from bidders' two-dimensional types in equity auctions can lead to a global violation of the regularity condition, which represents a maximal mismatch between incentive compatibility and maximization of revenue or surplus. Such mismatch can lead a seller to exclude bidders and demand a bidder-specific stake from a non-excluded bidder; or separate bidders into groups, and sell to them sequentially.

Keywords: Global violation of regularity; Adverse selection; Dimensionality reduction

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1 Introduction

A central premise underlying the use of auctions is that bidders have private information that affects both their payoffs and a seller's revenue. In practice, bidder information is often multi-dimensional. Concretely, a bidder may have private information about both the gross revenues that it can generate with an auctioned asset and its opportunity cost. For instance, an acquiring firm in a corporate takeover may be privately informed about its synergy with the target and its own standalone value; and in project-rights auctions bidders are often privately informed about the opportunity costs associated with alternative projects to pursue.[1]

In auctions where bidders pay with cash, if the asset for sale is indivisible, the sole determinant of a bidder's strategy and seller revenues is the bidder's net valuation, i.e., the difference between gross revenues and costs.[2] However, when bidders pay with securities that tie payments to the revenues generated (Hansen 1985; DeMarzo, Kremer and Skrzypacz, DKS 2005), because the winner retains only a share of revenues but incurs all opportunity costs, the revenue-cost composition affects bidding strategies. Moreover, bidders with different revenue-cost compositions may select the same bid, even though the bid's value to a seller can vary. Thus, the multi-dimensional informational structure affects seller revenue in security-bid auctions in non-trivial ways.

Illustrating the potential consequences, Che and Kim (2010) show that when a bidder's opportunity cost rises deterministically with the expected total cash flow sufficiently quickly, an extreme form of adverse selection arises in standard second-price security-bid auctions: bidders with higher NPVs bid less, resulting in low seller revenues. This suggests that securities auctions should be designed with care when opportunity costs are private information.

We derive the optimal equity auction design when bidders' values and opportunity costs are private information.[3] We allow for an arbitrary joint density over the expected cash flows

[1]Examples include auctions to sell oil or timber leases (e.g., Hendricks and Porter 1988, Paarsch 1997), auctions for highway building contracts; DeMarzo, Kremer and Skrzypacz (2005) and Skrzypacz (2013) provide other examples.

[2]With divisible assets, if a bidder's opportunity cost of acquiring a quantity is not proportional to its gross valuation of the quantity, the two dimensions of uncertainty need not collapse to one dimension.

[3]Equity auctions are the most common form of security auctions: Andrade et al. (2001) report that 58% of mergers and acquisitions are paid entirely in equity. Faccio and Masulis (2005) and Eckbo et al. (2015) document reasons underlying the exclusive use of equity.

v_i and opportunity cost x_i of a bidder i, requiring only independence across bidders, mild continuity conditions on the density, and a compact, connected support. Thus, the extent of adverse selection is arbitrary, and can vary across bidders.

We first find a transformation of a bidder's objective that simplifies the resulting envelope condition.[4] We show that in any incentive-compatible mechanism, an agent's preferences are determined only by $r_i \equiv \frac{x_i}{v_i}$, the ratio of his opportunity cost to expected project revenue. This ratio represents the equity stake the agent needs to break even. While an agent's preferences are determined only by this one-dimensional summary of his two-dimensional type, it does not guarantee that two type pairs with the same r_i are pooled in equilibrium. Nonetheless, an application of the envelope theorem reduces the design problem to a one-dimensional problem. It follows that almost all types of a bidder with the same r_i have the same equilibrium probability of winning and the same expected equity share retained—even if project revenues net of opportunity costs are very different for these types.

These results yield a general characterization of any incentive-compatible equity mechanism that we use to analyze revenue-maximizing and surplus-maximizing mechanisms. When a seller seeks to maximize revenues, we use this one-dimensional representation to express a bidder's virtual valuation as a function of r_i only, denoted $\phi(r_i)$. Different bidder types with the same r_i have the same virtual valuation, which depends on the aggregate properties of all types with that r_i. Expected seller revenues decompose into the sum of a component that reflects the virtual valuation of the winning bidder and a component that reflects the rents obtained by a bidder type that needs the highest equity share to break even.

These results yield a general characterization of equity mechanisms, where incentive-compatibility requires the winning probability of all bidders i to weakly decrease in r_i. We use this characterization to analyze revenue-maximizing and surplus-maximizing mechanisms. When a seller seeks to maximize revenues, we show a bidder's virtual valuation—which represents the rents the seller can extract from the bidder—is a function of r_i only, denoted $\phi(r_i)$. Different bidder types (x_i, v_i) with the same r_i have the same virtual valuation, which

[4]The original, un-transformed objective contains a quantity that is the product of a term that depends on a bidder's private type, and two terms that are endogenously determined in equilibrium—the equity share q_i and the probability of winning G_i. The product of these three terms complicates the envelope condition. We transform the objective function so that the term that depends on a bidder's private type is multiplied by only one endogenously-determined term, resulting in a tractable envelope condition.

depends on the aggregate properties of all types with that r_i.

Under the regularity condition that ϕ_i decreases in the break-even equity stake r_i, incentive compatibility and virtual valuations move in the same direction. We show that the resulting optimal selling mechanism has the features that (1) a bidder with the highest virtual valuation wins whenever his virtual valuation exceeds the seller's valuation, and (2) a bidder type with the highest r_i extracts no rents. When the design problem is not regular, a concavification argument similar to that in Myerson (1981) is used to solve for the optimal mechanism. In the optimal design, the winning bidder has the highest adjusted virtual valuation, and a bidder type with the highest r_i still earns no rents.

With two-dimensional uncertainty, the regularity condition in equity auctions becomes demanding. Virtual valuations measure available rents, and thus tend to grow with a bidder's net valuation $v_i - x_i$. In cash auctions, the relevant bidder type *is* this net valuation, so the regularity condition, which mandates that virtual valuations increase with bidder type, holds under mild assumptions—a monotone hazard condition on the distribution of types suffices. In contrast, in equity auctions with two-dimensional types, the relevant bidder type is r_i. What determines regularity is the monotonicity of virtual valuation with respect to r_i, but a lower r_i need not imply a higher net valuation. This is true even when x_i depends deterministically on v_i, and more so, due to aggregation, when x_i and v_i are distributed on a two-dimensional space. Indeed, the regularity condition can be violated at *every* r_i: virtual valuations can *increase* in r_i, going in the opposite direction of incentive compatibility over the entire domain. Global violations imply a maximal mismatch between revenue maximization and incentive compatibility. Revenue maximization requires the asset to be allocated to bidders with higher virtual valuations—who have higher r_i when the regularity condition is globally violated—but incentive compatibility demands that bidder types with lower r_i be weakly more likely to win.

If the regularity condition for a bidder is globally violated, the optimal design pools all types so that the winning probability does not vary with type. If such violation occurs for all bidders, it is optimal to identify the bidders with the highest (constant) adjusted virtual valuation and sell to one of them when that adjusted virtual valuation exceeds the asset's value to the seller, demanding the highest share that this bidder would cede regardless of his type.

Hence, if bidders are ex-ante identical and the regularity condition is globally violated, then even with multiple bidders, the optimal mechanism is a take-it-or-leave-it offer—expected seller revenue in the optimal design is unaffected by the number of bidders.[5]

This feature contrasts with optimal cash auctions. Consistent with this design feature, Boone and Mulherin (2007) find that mergers and acquisitions involving equity are twice as likely as pure cash acquisitions to have a single bidder.[6] If, alternatively, the regularity condition for some, but not all, bidders is globally violated, one can implement the optimal mechanism by first conducting an auction among bidders for whom the regularity condition is not globally violated, setting a reserve price that reflects the value of selling to bidders for whom the regularity condition is globally violated. A seller only turns to bidders exhibiting extreme adverse selection when the virtual valuations of the other bidders are not high enough.

Our analysis provides insights into the forces affecting revenue comparisons of optimal equity and optimal cash auctions. When uncertainty solely concerns valuations, Hansen (1985) shows that equity auctions generate higher revenues because equity bids tie payments to bidder types. When bidders have two-dimensional private information, additional forces come into play. First, with severe adverse selection, where the regularity condition is largely violated, revenues in optimal equity auctions rise minimally with the number of bidders. This favors cash if there are enough bidders. Second, the distribution of net valuations in the one-dimensional representation for equity mechanisms second-order stochastically dominates that for the original two-dimensional distribution (and hence for cash auctions). With few bidders, this favors equity because it lets a seller set the reserve price more efficiently, reducing the risk of no sale. In contrast, with many bidders, what matters primarily is the upper tail of the distribution, especially with substantial two-dimensional uncertainty. Reflecting the combined effects, optimal equity auctions generate more revenues than cash auctions if there are few bidders or moderate levels of two-dimensional uncertainty, while cash auctions do better with extensive two-dimensional uncertainty and many bidders.

Our dimensionality reduction result extends to mechanisms that maximize expected social

[5]In contrast, in standard auction formats expected revenues *fall* with the number of bidders: the greater is competition, the lower is the winner's r_i; and if regularity is globally violated, this lower r_i corresponds to a *lower* virtual valuation, and hence lower revenue.

[6]Computed from data in their Table IV.

surplus, where we identify a 'surplus valuation' that drives the optimal design. This analysis reveals the inefficiencies of equity auctions relative to cash auctions when bidders have two-dimensional types. With cash, the Vickrey-Clarke-Groves mechanism ensures an efficient allocation, and any pooling in net valuations is suboptimal. In contrast, in equity auctions, the VCG mechanism breaks down due to the constraint that payments are non-cash whose value depends on bidder type. Our findings show that this constraint binds in the sense that sometimes surplus-maximizing mechanisms can feature pooling, failing to achieve the first-best welfare. Indeed, when adverse selection is severe, disregarding all bids and selling to any bidder with the highest adjusted surplus valuation maximizes surplus. This contrast with cash auctions reinforces how in equity auctions with severe adverse selection, incentive compatibility requires "inferior" types (in terms of either virtual or surplus valuations) to win with a weakly higher probability. Thus, a mechanism designer can do no better than randomize.

Literature. The study of equity auctions begins with Hansen (1985). Like Hansen (1985), subsequent papers (e.g., DKS, Che and Kim (2010), Deb and Mishra (2014) or Liu (2016)) assume one-dimensional information for bidders.[7] DKS analyze optimal selling mechanisms in standard auction formats when bidders have the same opportunity cost and select bids from a class of ordered securities. They show that steeper securities yield higher seller revenues. Che and Kim (2010) introduce bidders whose opportunity costs rise deterministically with their valuations and show that severe adverse selection can arise that causes steeper securities to generate less revenues in standard auctions. They also uncover how the monotonicity of break-even stakes affects the extent of adverse selection and hence equity auction designs.

In contrast, we consider two-dimensional private information for bidders. We identify a way to reduce the dimensionality that we use to characterize incentive-compatible equity mechanisms, and to identify revenue- and surplus-maximizing mechanisms. We can recover Che and Kim's (2010) framework when the distribution of types has a positive mass only on a one-dimensional subset of the two-dimensional space. However, one cannot just myopically use a one-dimensional formulation: not all two-dimensional distributions reduce to a setting

[7]For other work on security-bid auctions, see Crémer (1987), Samuelson (1987), Zheng (2001), Board (2007), Povel and Singh (2010), Kogan and Morgan (2010), Gorbenko and Malenko (2011), Liu (2012), Sogo et al. (2016), Sogo (2017), Sun et al. (2018); Skrzypacz (2013) surveys the literature. See Hansen (1987), Laffont et al. (1987), Fishman (1988; 1989), Dasgupta and Tsui (2004), Biais et al. (2007), or Burkart and Lee (2015), for analyses of security design and rent extraction.

in which x_i is a deterministic function of v_i; and even if x_i is deterministically and monotonically related to v_i, our dimensionality reduction approach is still needed when the relevant bidder type r_i evolves non-monotonically. Moreover, for a given two-dimensional distribution, equity auctions typically reduce to a different one-dimensional distribution than that for cash auctions. We derive a stochastic dominance relationship between the two distributions, and show how the implications for revenue comparisons of optimal equity and cash auctions depend on both the extent of two-dimensional uncertainty and the number of bidders.

Ekmekci, Kos and Vohra (2016) consider the problem of selling a firm to a *single* buyer who is privately informed about post-sale cash flows and the benefits of control. The offer consists of a menu of tuples of cash-equity mixes, and the bidder must obtain a minimum 50% stake to gain control. They provide sufficient conditions for the optimal mechanism to take the form of a take-it or leave-it offer for either 50% of the firm, or for all shares. The minimum stake introduces a discontinuity in buyer payoffs, which Ekmekci et al. address by characterizing incentive-compatible mechanisms via the exclusion boundary, separating types who obtain controlling stakes from those who do not. In contrast, we examine auctions in which bidders offer equities for full ownership and characterize the optimal auction for multiple buyers.

The study of mechanism design problems when agents have type-dependent opportunity costs (outside options) was initiated in Jullien (2000) and Rochet and Stole (2002), who show that an interior type can have zero surplus. In their models, the value of the opportunity cost is independent of the allocated quantity. In our single unit auction setting, however, this value depends on the allocated quantity: a bidder wins and incurs an opportunity cost, or loses and incurs no cost; and if allocations are probabilistic, the expected opportunity cost is proportional to the expected allocation. More fundamentally, equity complicates design, because the monetary value of payments depends on a bidder's private information about valuations, and private information about opportunity costs introduces further complications.

Lastly, Deneckere and Severinov (2017) consider an agent whose gross utility $u(q, \alpha, \theta)$ depends on quantity q and privately-known type (α, θ). His net utility in a direct mechanism is

$$\pi = u(q(\hat{\alpha}, \hat{\theta}), \alpha, \theta) - t(\hat{\alpha}, \hat{\theta}), \tag{1}$$

where $(\hat{\alpha}, \hat{\theta})$ is the reported type, and $t(\hat{\alpha}, \hat{\theta})$ is the principal's revenue. They identify conditions on preferences such that if $q_1 > 0$ is optimal for a type (α_1, θ_1), and

$$u_q (q_1, \alpha_2, \theta_2) = u_q (q_1, \alpha_1, \theta_1) \tag{2}$$

for any (α_2, θ_2) with $\alpha_2 < \alpha_1$, then (α_2, θ_2)'s optimal choice is also q_1, reducing the dimensionality. Their set-up relates to a cash-only version of our setting. To see why, let (α, θ) be the bidder's gross valuation and opportunity cost, let q be the expected probability of winning, and let t be the expected cash payment (unconditional on winning). Then $u(q, \alpha, \theta) = q(\alpha - \theta)$, and (2) becomes $\alpha_2 - \theta_2 = \alpha_1 - \theta_1$. That is, the relevant bidder type is the net valuation. Equity payments materially alter the framework: net utility becomes

$$\pi = q(\hat{\alpha}, \hat{\theta})e(\hat{\alpha}, \hat{\theta})\alpha - q(\hat{\alpha}, \hat{\theta})\theta, \tag{3}$$

and seller revenue is $q(\hat{\alpha}, \hat{\theta})(1 - e(\hat{\alpha}, \hat{\theta}))\alpha$, where $q(\hat{\alpha}, \hat{\theta})$ and $e(\hat{\alpha}, \hat{\theta})$ are the winning probability and the bidder's equity share. Equation (3) cannot be cast in the form of (1), and the form of seller revenue differs, hinging on both the bidder's reported and true (private) type.

These differences highlight how mechanism design involving equity payments with two-dimensional types differ from existing studies of mechanism design with cash transfers, underscoring that different approaches for dimensionality reduction are needed.[8]

2 The Model

$n \geq 1$ risk-neutral bidders compete to acquire an asset/project, over which control rights are indivisible. Bidder i $(i = 1, ..., n)$ is privately informed about both the expected value $v_i > 0$ of the cash flows that the asset would generate if he wins, and his opportunity cost $x_i \in (0, v_i)$ of acquiring the asset. The seller has a publicly-known reservation value v_s of retaining the asset. Thus, the expected social surplus if i wins is $v_i - x_i - v_s$. For each bidder i, v_i and x_i are jointly distributed according to a strictly positive, continuous pdf $f_i(v_i, x_i)$ over a compact and connected set S_i. The densities f_i can differ across bidders, and v_i and s_i are independently distributed across bidders. We assume $\max_{i,(v_i,x_i)\in S_i} v_i - x_i - v_s > 0$, i.e., there

[8]In a one-dimensional setting where x_i is a deterministic function of v_i, the methodology in Liu (2016) can be used to derive the virtual valuation and construct the optimal design. However, when x_i and v_i are distributed over a two-dimensional space, approaches such as the ones developed in this paper are needed.

are potential gains to trade. We use $t_i \equiv (x_i, v_i)$ to denote bidder i's type, $f(\mathbf{t}) \equiv \Pi_{i=1}^{n} f_i(t_i)$ to denote the joint density of $\mathbf{t} \equiv (t_1, t_2, ..., t_n)$, and $f_{-i}(\mathbf{t}_{-i}) \equiv \Pi_{k \neq i} f_k(t_k)$ to denote the joint density of the types of bidders other than i, where $\mathbf{t}_{-i} \equiv (t_1, ..., t_{i-1}, t_{i+1}, ..., t_n)$.

We consider direct-revelation mechanisms in which the winner pays with equities. Let $Q_i(t_i', \mathbf{t}_{-i}) \in [0, 1]$ be the equity share of the project's cash flows that bidder i retains when he reports being type t_i' and other bidders report $\mathbf{t}_{-i}$. Let $W_i(t_i', \mathbf{t}_{-i})$ be the probability that bidder i wins, and let $W_0(t_i', \mathbf{t}_{-i})$ be the probability that no bidder wins (i.e., the seller retains the asset). We require $\Sigma_{j=0}^{n} W_j(\mathbf{t}) = 1$ for all $\mathbf{t}$. We define $G_i(t_i')$ to be the probability bidder i wins when i reports t_i' and all other bidders report truthfully:

$$G_i(t_i') = \int W_i(t_i', \mathbf{t}_{-i}) f_{-i}(\mathbf{t}_{-i}) \, d\mathbf{t}_{-i}. \tag{4}$$

We define $q_i(t_i')$ to be the expected equity share that bidder i retains conditional on winning by reporting type t_i' when all others report truthfully,

$$q_i(t_i') G_i(t_i') = \int Q_i(t_i', \mathbf{t}_{-i}) W_i(t_i', \mathbf{t}_{-i}) f_{-i}(\mathbf{t}_{-i}) \, d\mathbf{t}_{-i}. \tag{5}$$

If bidder i has type t_i but reports t_i', and other bidders report truthfully, his expected profit is:

$$h_i(t_i, t_i') = (v_i q_i(t_i') - x_i) G_i(t_i'). \tag{6}$$

Equilibrium requires that the incentive compatibility condition hold,

$$t_i = \arg\max_{t_i'} h_i(t_i, t_i'). \tag{7}$$

The seller's expected revenue is the sum of the winner's payments plus the reservation value when the seller retains the asset:

$$\pi_s = \int \Big(\sum_{i=1}^{n} W_i(\mathbf{t}) (1 - Q_i(\mathbf{t})) v_i + W_0(\mathbf{t}) v_s \Big) f(\mathbf{t}) \, d\mathbf{t}. \tag{8}$$

Our objective is to identify the mechanism that maximizes expected revenue (8) subject to incentive compatibility (7) and individual rationality, i.e., that $h_i(t_i, t_i) \geq 0$ for all i and t_i.

We start by examining the properties of any incentive-compatible mechanism. Observe that bidder i's objective function (6) contains a term $v_i q_i(t_i') G_i(t_i')$, where v_i depends on the bidder's private type, and both $q_i(\cdot)$ and $G_i(\cdot)$ are endogenously determined in equilibrium. This complicates the envelope condition vis à vis cash auctions, where the term in a bidder's objective function that depends on the bidder's private type is multiplied only by one endogenous term, $G_i(\cdot)$. That is, $q_i = 1$ in cash auctions.

Our first step is to simplify the envelope condition by transforming a bidder's objective function. We make a simple but fundamental observation: in equation (7), bidder i's gross expected valuation v_i is in his information set, so it can be treated as a constant in his optimization problem. This lets us rescale (6), bidder i's expected profit when he has type t_i, but reports t_i' and all other bidders report truthfully. We define

$$
\begin{aligned}
m_i(t_i, t_i') &\equiv \frac{h_i(t_i, t_i')}{v_i} && (9)\\
&= (q_i(t_i') - r_i)\, G_i(t_i'), && (10)
\end{aligned}
$$

where

$$r_i \equiv \frac{x_i}{v_i} \tag{11}$$

is the fraction of equity a bidder must retain to break even. Equivalently, r_i is the fractional cost bidder i incurs to generate a unit cash flow. We can express incentive compatibility (7) as

$$m_i(t_i, t_i) = \max_{t_i'} (q_i(t_i') - r_i)\, G_i(t_i'). \tag{12}$$

Equation (12) reveals the advantage of the transformation: in the argument for optimization, the term that depends on a bidder's private information is multiplied by only G_i. This simplifies the resulting envelope condition relative to working directly with (7).

Our second step is to reduce the dimensionality to a single dimension. The right-hand side of equation (12) only depends on a bidder's type according to $r_i = \frac{x_i}{v_i}$. This means that $m_i(t_i, t_i)$ must be a function of r_i only. Hence, any two type pairs $t_i' = (x_i', v_i')$ and $t_i'' = (x_i'', v_i'')$ such that $\frac{x_i'}{v_i'} = \frac{x_i''}{v_i''} = r_i'$ should have the same m_i:

$$m_i(t_i', t_i') = m_i(t_i'', t_i''), \tag{13}$$

i.e.,

$$(q_i(t_i') - r_i')\, G_i(t_i') = (q_i''(t_i) - r_i')\, G_i''(t_i). \tag{14}$$

Equation (14) relates the winning and allocation rules, G_i and q_i, of the two types. However, it does not imply that

$$G_i(t_i') = G_i(t_i'') \quad \text{and} \quad q_i(t_i') = q_i(t_i''), \tag{15}$$

because $G_i''(t_i)$ and $q_i(t_i'')$ can differ from $G_i'(t_i)$ and $q_i(t_i')$, and yet (14) still holds. That is, due to the two-dimensional nature of the set S_i, an infinite number of type pairs (x_i, v_i) typically correspond to any given r_i, and (15) need not hold for all such pairs. We next derive the stronger result that (15) holds in "almost all cases".

To proceed, we project every point of S_i onto r_i, and define $\underline{r}_i$ and $\bar{r}_i$ to be respectively the minimum and maximum values of r_i. To facilitate the analysis, we add the mild structure that there is no atom at any value of $x_i/v_i = r_i$.[9] Thus, $\underline{r}_i < \bar{r}_i$. Define $\widetilde{f}_i(r_i)$ to be the associated probability density function over r_i, which is strictly positive for $r_i \in (\underline{r}_i, \bar{r}_i)$.

By (13), without loss of generality we define

$$M_i(r_i) \equiv m_i(t_i, t_i). \tag{16}$$

By (12), $M_i(r_i)$ is the maximum of a family of affine functions and hence is convex. Because $M_i(r_i)$ is also bounded and hence absolutely continuous, it is differentiable almost everywhere in the interior of its domain. Define R_i to be the set of all $r_i \in (\underline{r}_i, \bar{r}_i)$ at which $M_i(r_i)$ is differentiable, which includes all points in $(\underline{r}_i, \bar{r}_i)$ except those of measure zero. In light of the no-atom condition, we ignore the measure zero set of points at which $M_i(r_i)$ is non-differentiable in any analysis that involves integration. By (12) and the envelope theorem, at every point $t_i = (v_i, x_i) \in S_i$ with $x_i/v_i \in R_i$, we obtain the simple relation:

$$\frac{dM_i(r_i)}{dr_i} = -G_i(t_i). \tag{17}$$

Thus, if $\frac{x_i'}{v_i'} = \frac{x_i''}{v_i''} \in R_i$, then because $G_i(t_i')$ and $G_i(t_i'')$ both equal $-\frac{dM_i(r)}{dr}|_{r=r_i'}$, we have:

Lemma 1 *In any incentive-compatible equity mechanism, (15) holds if* $\frac{x_i'}{v_i'} = \frac{x_i''}{v_i''} \in R_i$.

We now derive the necessary and sufficient conditions for incentive compatibility and individual rationality.

[9] This ensures that bidder types where $M_i(r_i)$ (see equation (16)) is non-differentiable have zero mass.

Theorem 1 ***(necessary and sufficient conditions for IC and IR)*** *An equity mechanism is incentive compatible and individually rational if and only if, for all i:*

(i) If $t_i' = (v_i', x_i')$ and $t_i'' = (v_i'', x_i'')$ satisfy $\frac{x_i'}{v_i'} = \frac{x_i''}{v_i''}$, then $m_i(t_i', t_i') = m_i(t_i'', t_i'')$. That is, $m_i(t_i, t_i)$ is a function of r_i only.

(ii) The function $M_i(r_i)$ defined in (16) is weakly convex over $r_i \in [\underline{r}_i, \bar{r}_i]$.

(iii) If $x_i/v_i \in R_i$, then (17) holds. Further, (15) holds for any type pairs t_i' and t_i'' such that $\frac{x_i'}{v_i'} = \frac{x_i''}{v_i''} \in R_i$.

(iv) The winning probability is non-increasing in r_i for all i: for any type pairs $t_i' = (x_i', v_i')$ and $t_i'' = (x_i'', v_i'')$ if $\frac{x_i'}{v_i'} > \frac{x_i''}{v_i''}$, then $G_i(t_i') \leq G_i(t_i'')$.

(v) $M_i(\bar{r}_i) \geq 0$.

Part (iv) follows from the incentive compatibility condition. It reflects the intuition that r_i represents the fractional cost of generating one unit of revenue, and that a bidder with a smaller fractional cost is more competitive in equity auctions. Part (iii) shows that (15) holds if $x_i/v_i \in R_i$. However, for type pairs t_i' and t_i'' such that $\frac{x_i'}{v_i'} = \frac{x_i''}{v_i''} \notin R_i$, (15) need not hold. That is, at r_i where $M_i(r_i)$ is not differentiable, $G_i(t_i')$ and $G_i(t_i'')$ can differ. Part (iv) bounds the amount by which they can differ: any $G_i(t_i')$ must lie between $\lim_{r \to r_i^-} \frac{dM_i(r)}{dr}$ and $\lim_{r \to r_i^+} \frac{dM_i(r)}{dr}$.

We next derive an expression for expected revenue in any incentive compatible mechanism. Abusing notation slightly, for all $t_i \in S_i$ such that $r_i \in R$, we use (17) to write $G_i(t_i)$ as $G_i(r_i)$. Because an absolutely continuous function is the definite integral of its derivative,

$$M_i(r_i) = M_i(\bar{r}_i) + \int_{r_i}^{\bar{r}_i} G_i(r)\, dr. \tag{18}$$

By (18), $M_i(r_i)$ is non-increasing in $r_i = \frac{x_i}{v_i}$, as is $G_i(r_i)$.

To facilitate the analysis, for a given r, we define

$$a_i(r) \equiv \lim_{\epsilon \to 0} E\left[v_i | r \leq x_i/v_i \leq r + \epsilon\right], \tag{19}$$

where the expectation is taken over S_i. Here, $a_i(r)$ is the expected value of v_i conditional on x_i/v_i being in an "ϵ" neighborhood of r. Note that this expectation is *not* the same as $E[v_i|x_i/v_i = r]$ because the two expressions imply different weightings in calculating the conditional expectations.[10] The lemma below shows how to do this weighting in the two-dimensional case where the probability density $f_i(x_i, v_i)$ is finite, and Example 1 provides an illustration. Example 2 will show how to do the weighting in the limiting case where the two-dimensional distribution degenerates to a single dimension, so that $f_i(x_i, v_i)$ is infinite.

Lemma 2 *For each $R \geq 0$, define $\rho_i(R) \equiv f_i\Big(x_i = \frac{r}{\sqrt{(1+r^2)}}R, v_i = \frac{1}{\sqrt{(1+r^2)}}R\Big)$, where $f_i(x_i, v_i)$ is the probability density for $(x_i, v_i) \in S_i$, and $f_i(v_i, x_i) = 0$ for $(v_i, x_i) \notin S_i$. Then*

$$a_i(r) = \frac{1}{\sqrt{(1+r^2)}} \frac{\int_0^\infty R^2 \rho(R)\, dR}{\int_0^\infty R\rho(R)\, dR}.$$

We now use standard mechanism-design techniques to decompose expected seller revenue.

Definition 1 *Bidder i's virtual valuation at $r_i \in [\underline{r}_i, \bar{r}_i]$ in equity auctions is*

$$\phi_i(r_i) \equiv (1 - r_i)\, a_i(r_i) - \frac{\int_{\underline{r}_i}^{r_i} a_i(r)\, \widetilde{f}_i(r)\, dr}{\widetilde{f}_i(r_i)}. \tag{20}$$

Theorem 2 ***(Revenue decomposition)*** *In any incentive-compatible mechanism of equity auctions, the seller's expected revenue (8) decomposes:*

$$\pi_s = \pi_{s,a} + \pi_{s,b}, \tag{21}$$

where

$$\pi_{s,a} \equiv -\sum_{i=1}^{n} M_i(\bar{r}_i) \int_{\underline{r}_i}^{\bar{r}_i} a_i(r)\, \widetilde{f}_i(r)\, dr \tag{22}$$

and

$$\pi_{s,b} \equiv \int_{S_1 \times \ldots \times S_n} \Big(\sum_{i=1}^{n} W_i(\mathbf{t})\, \phi_i(r_i) + W_0(\mathbf{t})\, v_s\Big) f(\mathbf{t})\, dt_1 \ldots dt_n. \tag{23}$$

$\pi_{s,a}$ is bounded by bidders' rationality constraints: because $M_i(\bar{r}_i) \geq 0$, its maximum possible value is zero, which obtains if $M_i(\bar{r}_i) = 0$ for all i. $\pi_{s,b}$ is maximized by allocating the asset to the bidder with maximal virtual valuation $\phi_i(x_i)$ if that $\phi_i(x_i)$ exceeds the seller's valuation. It is instructive to use (19) and $r_i = v_i/x_i$ to rewrite the first term of $\phi_i(x_i)$ in (20) as

$$(1 - r_i)\, a_i(r_i) \equiv \lim_{\epsilon \to 0} E\left[v_i - x_i \;\middle|\; r \leq x_i/v_i \leq r + \epsilon\right]. \tag{24}$$

[10]A geometric perspective provides insight into the proper weighting for (19): the two-dimensional space between two line segments emanating from the origin (i.e., the two lines with slopes r and $r + \epsilon$ respectively) is a fan-shaped sector, which has the property that the length of the arc at radius R is $Rd\theta$ (when $d\theta$ is small). Thus, the total mass between R and dR is proportional to R.

Hence, this term is the bidder's expected net valuation (conditional on x_i/v_i being in an "ϵ" neighborhood of r), similar to the first term in the virtual valuation for a cash auction. The second term of the virtual valuation reflects the difference that the relevant bidder type is r_i in equity auctions whereas it is the net valuation in cash auctions.

Theorem 2 implies a dimensionality reduction result: a seller's revenue from any given mechanism for any two-dimensional distribution over (x_i, v_i) equals that for a one-dimensional distribution where x_i and v_i are parameterized in terms of r_i:

Corollary 1 ***(One-dimensional representation)*** *A seller's revenue in a given mechanism for any two-dimensional distribution of (x_i, v_i) is the same as that for the one-dimensional distribution in which for each bidder i, all points (x_i, v_i) associated with a given r_i are replaced by the single point $(r_i a_i(r_i), a_i(r_i))$, and r_i is distributed according to $\widetilde{f}_i(r_i)$.*

Corollary 1 contains subtle implications. First, in our one-dimensional representation, only one type pair $(x_i(r_i), v_i(r_i))$ corresponds to any r_i. However, this does not imply that v_i is a deterministic function of x_i (Example 1 will illustrate this). Thus, it is not without loss of generality to consider only settings in which x_i and v_i are deterministically related. Second, if we start with a one-dimensional problem, our one-dimensional representation result is still needed if multiple type pairs correspond to a given r_i (Example 2 illustrates this). Moreover, dimensionality reduction in cash auctions typically results in a different one-dimensional distribution as bidder types aggregate via net valuation, so if one wishes to compare equity and cash auctions, one needs to begin with the higher dimensional primitives.

The following proposition highlights this difference in the dimensionality reduction.

Proposition 1 *The distribution of net valuations $v_i - x_i$ for the one-dimensional distribution in Corollary 1 second-order stochastically dominates the distribution of net valuations for the original two-dimensional distribution.*

Proposition 1 has implications for revenue comparisons of equity and cash auctions that we later explore. We next illustrate how to use the one-dimensional representation.

Example 1 (Skrzypacz (2013)). *n ex-ante identical bidders bid with equity. Bidder i expects revenue $v_i = x_i + y_i$, where the net valuation, y_i, and opportunity cost, x_i, are each independently and uniformly distributed on $[0, 1]$.*

We have $\underline{r}_i = 0$ and $\bar{r}_i = 1$. In the appendix, we obtain the form of $a_i(r_i)$ and show that a seller's expected revenue from any mechanism given the two-dimensional distribution of the example is the same as it would be were x_i and v_i parameterized (via r_i) as

$$(x_i, v_i) = \Big(\frac{2r_i}{3\max\{r_i, 1-r_i\}}, \frac{2}{3\max\{r_i, 1-r_i\}}\Big), \tag{25}$$

where r_i is distributed according to

$$\widetilde{f}_i(r_i) = \frac{1}{2}\Big(1 + \big(\min\{\frac{1-r_i}{r_i}, \frac{r_i}{1-r_i}\}\big)^2\Big) / \left(r_i^2 + (1-r_i)^2\right). \tag{26}$$

In equation (25), the relationship between x_i and v_i is $\supset$-shaped: there are two x_i corresponding to each $v_i \in \left(\frac{2}{3}, \frac{4}{3}\right)$. Nonetheless, there is only one type pair associated with each r_i, which is what our one-dimensional representation result details.

3 Optimal Equity Mechanisms

An *optimal equity mechanism* maximizes a seller's expected revenue subject to the incentive compatibility and individual rationality conditions. We first derive optimal equity mechanisms given the regularity condition in Assumption 1:

Assumption 1 *The design problem is regular: $\phi_i(r_i)$ strictly decreases in r_i over $[\underline{r}_i, \bar{r}_i]$ for all i.*

Proposition 2 *Under Assumption 1, a direct-revelation mechanism is optimal if and only if*

(i) the winning rule $W_i(t_i, \mathbf{t}_{-i})$ depends on t_i only through r_i, satisfying

$$W_i(\mathbf{t}) = \begin{cases} 1 & \text{if} \quad \phi_i(r_i) > \max\{\max_{j\neq i}\{\phi_j(r_j)\}, v_s\} \\ 0 & \text{if} \quad \phi_i(r_i) < \max\{\max_{j\neq i}\{\phi_j(r_j)\}, v_s\} \end{cases}, \tag{27}$$

where $\phi_i(r_i)$ is the virtual valuation in (20)*, for all i and $\mathbf{t}$, ties are broken arbitrarily,*

(ii) the equity-retention rule $q_i(t_i)$ depends on t_i only through r_i, via

$$(q_i(r_i) - r_i)\, G_i(r_i) = \int_{r_i}^{\bar{r}_i} G_i(r)\, dr, \quad \text{for all } i \text{ and } r_i. \tag{28}$$

Corollary 2 *Under Assumption 1, an optimal equity mechanism is given by: the winning rule in* (i) *of Proposition 2, and the following equity-retention rule for the winning bidder,*

$$Q_i(\mathbf{t}) = \phi_i^{-1}\left(\max\left\{\max_{j\neq i}\left\{\phi_j(r_j)\right\}, v_s\right\}\right). \tag{29}$$

$Q_i(\mathbf{t})$ *is the maximum value of the break-even* r_i *corresponding to a virtual valuation that exceeds* v_s *and allows* i *to win against* $\mathbf{t}_{-i}$*, where* $\phi_i^{-1}(\cdot)$ *denotes a bounded inverse of* $\phi_i(\cdot)$*:*

$$\phi_i^{-1}(x) \equiv \begin{cases} \bar{r}_i & if \quad x < \phi_i(\bar{r}_i) \\ y \in [\underline{r}_i, \bar{r}_i] \;\; s.t. \;\; \phi_i(y) = x & if \quad x \in [\phi_i(\bar{r}_i), \phi_i(\underline{r}_i)] \end{cases}. \tag{30}$$

The mechanism in Corollary 2 is simply the second-price auction with an optimal reserve when bidders are ex-ante identical. Thus, the optimal design is exactly the same as in cash auctions (Myerson 1981) or equity auctions where bidders' opportunity costs are common knowledge (Liu 2016). This correspondence reflects properties that hold generally for both cash and equity auctions: (1) the regularity condition requires virtual valuations to strictly increase in the direction where incentive compatibility requires the winning probability to weakly increase, and (2) standard formats select in this "reinforced" direction of incentive compatibility. To see what these two properties mean, consider equity auctions: (1) incentive compatibility requires the winning probability to *weakly* decrease in r_i, whereas regularity requires the virtual valuation to *strictly* decrease in r_i, and (2) the winning probability in standard formats *strictly* decreases in r_i. Thus, when the regularity condition holds, this "reinforced" direction is in line with revenue maximization. It follows that with ex-ante identical bidders, standard formats are optimal when the regularity condition holds.

We now solve for the optimal mechanism in a general setting in which Assumption 1 need not hold. We follow the approach of Myerson (1981), making adjustments for the fact that incentive compatibility in equity auctions requires that the winning probability G_i decrease in r_i, whereas in cash auctions, the winning probability must increase in bidder type. We then show that in equity auctions the features of the optimal mechanism differ dramatically according to whether the regularity condition is or is not violated. These differences are more striking than the analogous differences that emerge in cash auctions.

Let $\widetilde{F}_i(r_i)$ be the cdf of $\widetilde{f}_i(r_i)$. For any $y \in [0,1]$, define

$$b_i(y) = \phi_i\left(\widetilde{F}_i^{-1}(y)\right) \quad \text{and} \quad B_i(y) = \int_0^y b_i(z)\,dz. \tag{31}$$

Define $C_i(\cdot)$ to be the concave hull of $B_i(\cdot)$. Define

$$\bar{\phi}_i(r_i) = C_i'(\widetilde{F}_i(r_i)) \tag{32}$$

for $r_i \in [\underline{r}_i, \bar{r}_i]$. For any vector of bidder types $\mathbf{t}$, let $N(\mathbf{t})$ be the set of bidders for whom $\bar{\phi}_i(r_i)$ is both maximal and higher than v_s

$$N(\mathbf{t}) = \left\{ i | \bar{\phi}_i(r_i) \geq \max\left\{ \max_j \left\{ \bar{\phi}_j(r_j) \right\}, v_s \right\} \right\}.$$

Proposition 3 *The following constitutes an optimal equity mechanism:*

(i) *the winning rule is*

$$\bar{W}_i(\mathbf{t}) = \begin{cases} 1/|N(\mathbf{t})| & \text{if } \; i \in N(\mathbf{t}) \\ 0 & \text{if } \; i \notin N(\mathbf{t}) \end{cases}, \tag{33}$$

for all i and $\mathbf{t}$, and

(ii) *the equity-retention rule for the winning bidder is*

$$\bar{Q}_i(\mathbf{t}) = \bar{\phi}_i^{-1}\left(\max\left\{ \max_{j \neq i} \left\{ \bar{\phi}_j(r_j) \right\}, v_s \right\} \right), \tag{34}$$

which is the maximum value of the break-even r_i that corresponds to a virtual valuation exceeding v_s that allows i to win against $\mathbf{t}_{-i}$, and

$$\bar{\phi}_i^{-1}(x) \equiv \begin{cases} \bar{r}_i & \text{if } \quad x < \bar{\phi}_i(\bar{r}_i) \\ \max\left\{ y | y \in [\underline{r}_i, \bar{r}_i] \;\; \text{and} \;\; \phi_i(y) = x \right\} & \text{if } \;\; x \in \left[\bar{\phi}_i(\bar{r}_i), \bar{\phi}_i(\underline{r}_i) \right] \end{cases}. \tag{35}$$

When the design problem is not regular, virtual valuations increase in r_i on some range. Because incentive compatibility requires the allocation probability to be weakly decreasing in r_i, always allocating the asset to a bidder with the highest virtual valuation is not incentive compatible. Proposition 3 concavifies with an adjusted virtual valuation that pools types on the increasing range. The adjusted virtual valuation is weakly decreasing in r_i. Allocating the asset to a bidder with the highest adjusted virtual valuation maximizes a seller's revenue subject to the incentive compatibility requirement.

Regardless of whether the regularity condition holds for a bidder, in the optimal design, the type of that bidder with the highest break-even equity stake extracts no rents. This implication is less direct than it may seem: in a non-optimal, but incentive-compatible mechanism, the type with the lowest rent of a given bidder varies across mechanisms, and need not be the one with the highest break-even equity stake for that bidder.[11] Nonetheless, a key insight is that while the bidder's rent $h_i(t_i, t_i)$ can be non-monotone, h_i/v_i must be non-increasing by (9) and (17). By individual rationality, it follows that if $h_i(t_i, t_i)$ is zero for any type t_i, then h_i/v_i and hence h_i for a type with the highest r_i must also be zero. Thus, in the optimal design, for all bidders, any type with the highest break-even equity stake earns zero rent.

3.1 Illustrations of virtual valuations in a one-dimensional setting

We illustrate virtual valuations in the setting of Che and Kim (2010) where x_i is a deterministic function of v_i with $\frac{dx_i}{dv_i} < 1$ so that both gross valuation and net valuation increase in v_i. Abusing notation slightly, we use $f_i(v_i)$ to denote the pdf of v_i, and let $F_i(v_i)$ denote the cdf.

$\mathbf{r_i = \frac{x_i(v_i)}{v_i}}$ **strictly monotone in** v_i**.** Then

$$\widetilde{f}_i(r_i) = f_i(v_i)\,|\frac{dv_i}{dr_i}|, \tag{36}$$

and the expected value of v_i given r_i in (19) becomes $a_i(r_i) = v_i$. The form of $\int_{\underline{r}_i}^{r_i} a_i(r)\,\widetilde{f}_i(r)\,dr$ in (20), however, will depend on whether $\frac{x_i(v_i)}{v_i}$ is decreasing or increasing in v_i. If $\frac{x_i(v_i)}{v_i}$ strictly increases in v_i, then

$$\int_{\underline{r}_i}^{r_i} a_i(r)\,\widetilde{f}_i(r)\,dr = \int_{\underline{v}_i}^{v_i} v f_i(v)\,dv. \tag{37}$$

If, instead, $\frac{x_i(v_i)}{v_i}$ strictly declines in v_i, then

$$\int_{\underline{r}_i}^{r_i} a_i(r)\,\widetilde{f}_i(r)\,dr = -\int_{\bar{v}_i}^{v_i} v f_i(v)\,dv = \int_{v_i}^{\bar{v}_i} v f_i(v)\,dv. \tag{38}$$

The corresponding virtual valuations follow directly from (20). Importantly, the virtual valuation takes a different form depending on whether $\frac{x_i(v_i)}{v_i}$ increases or decreases in v_i, reflecting the differences between (37) and (38). Next, we examine a more complicated setting where $\frac{x_i(v_i)}{v_i}$ is not monotone in v_i, and show that virtual valuations take yet further different forms.

Example 2 (hybrid case): $\mathbf{r_i = \frac{x_i(v_i)}{v_i}}$ **nonmonotone in** $\mathbf{v_i}$ **(**$\cap$**-shaped in** $\mathbf{v_i}$**).**

[11] As an extreme illustration, suppose the seller faces a single bidder i and (suboptimally) lets the bidder win for free. Then the lowest-rent bidder type has the lowest net valuation $v_i - x_i$, which, as we show in section 3.1, need not correspond to the bidder with the highest r_i.

Let v_i be uniformly distributed on $[5,15]$ with $x_i = 0.6v_i - 1$ for $v_i \in [5,10]$ and $x_i = 0.2v_i + 3$ for $v_i \in [10,15]$, and let $v_s = 0$. Then $x_i = 2$ at $v_i = 5$, $x_i = 5$ at $v_i = 10$, and $x_i = 6$ at $v_i = 15$, and $\frac{x_i}{v_i} = 0.6 - \frac{1}{v_i}$ for $v_i \in [5,10]$, and $\frac{x_i}{v_i} = 0.2 + \frac{3}{v_i}$ for $v_i \in [10,15]$.

Even though x_i is deterministic and strictly increasing in v_i, because multiple points correspond to the same r_i, aggregation is still needed. We have $\underline{r}_i = 0.4$ and $\bar{r}_i = 0.5$. Each $r_i \in (0.4, 0.5)$ corresponds to two points:

$$0.6 - \frac{1}{v_i} = r_i \Rightarrow v_i = \frac{1}{0.6 - r_i} \quad \text{and} \quad 0.2 + \frac{3}{v_i} = r_i \Rightarrow v_i = \frac{3}{r_i - 0.2}. \tag{39}$$

To calculate the associated density $\widetilde{f}_i(r_i)$ and to find the proper weighting in calculating $a_i(r_i)$, observe that the absolute value of the derivative of v_i over r_i at these two points are

$$|\frac{dv_i}{dr_i}| = \frac{1}{(0.6 - r_i)^2} \quad \text{and} \quad |\frac{dv_i}{dr_i}| = \frac{3}{(r_i - 0.2)^2}.$$

Because v_i is uniformly distributed over an interval of length 10, the density over v_i is 0.1. Hence, the density over r_i is $0.1|\frac{dv_i}{dr_i}|$ summed over the two points, yielding[12]

$$\widetilde{f}_i(r_i) = \frac{0.1}{(0.6 - r)^2} + \frac{0.3}{(r - 0.2)^2}.$$

To calculate $a_i(r_i)$, refer to equation (19). Noting that the weight of mass at each of these two points is $0.1|\frac{dv_i}{dr_i}|/\widetilde{f}_i(r_i)$, and that v_i relates to r_i via (39), we have

$$a_i(r_i) = \frac{\frac{0.1}{(0.6-r)^2}}{\widetilde{f}_i(r_i)}\left(\frac{1}{0.6 - r_i}\right) + \frac{\frac{0.3}{(r_i-0.2)^2}}{\widetilde{f}_i(r_i)}\left(\frac{3}{r - 0.2}\right). \tag{40}$$

Substituting for these components into (20) yields the virtual valuation.

Discussion. These examples share the salient feature that $v_i - x_i$ increases in v_i so that a high type is unambiguously defined. In cash auctions, the relevant bidder type is the net valuation $v_i - x_i$, and in the optimal design the zero-rent type has the lowest net valuation. In contrast, in equity auctions, the relevant bidder type is r_i, and in the optimal design the zero-rent type has the highest r_i. Thus, when $\frac{x_i(v_i)}{v_i}$ first increases and then decreases in v_i,

[12] In this example the derivative of r_i is not continuous at $\bar{r}_i$. Were r_i continuous and differentiable in v_i, then, from the first-order condition, $\widetilde{f}(r_i)$ would be infinite at $\bar{r}_i$. This would not violate the premise of no atom at any x_i/v_i, because the integral of $\widetilde{f}_i(r_i)$ goes to zero as the width of the interval vanishes.

the zero-rent-type has an intermediate net valuation. Moreover, when $r_i = \frac{x_i(v_i)}{v_i}$ increases in v_i, the zero-rent-type in the optimal design has the highest net valuation, which is the opposite of what arises in cash auctions.

Our one-dimensional representation result does more than reduce the dimensionality of analysis. In a one-dimensional setting like Example 2, although the relationship between x_i and v_i is monotone, it is not the final form of our representation result because r_i evolves non-monotonically such that multiple v_i's correspond to the same r_i. In such settings, the representation result helps to derive the virtual valuation and identify the lowest rent type. Moreover, the representation result shows r_i is the relevant bidder type: the virtual valuation (20) takes a simple and unique form as a function of r_i, whereas if one uses v_i or $v_i - x_i$ as the bidder type, the functional form of the virtual valuation will differ depending on the monotonicity of r_i.[13]

4 Global Violation of Regularity Condition

Assumption 1 requires virtual valuations to decrease in r_i, i.e., to move in line with incentive compatibility. The virtual valuation measures available rents and thus tends to increase with a bidder's net valuation $v_i - x_i$. In cash auctions, the net valuation *is* the relevant bidder type, and hence, the regularity condition holds under mild assumptions—a monotone hazard condition on the distribution suffices. In contrast, in equity auctions with two-dimensional private information, the relevant bidder type is given by r_i, and a lower r_i need not correspond to a higher net valuation. This is true even when x_i depends deterministically on v_i, and more so, due to aggregation, when x_i and v_i are distributed on a two-dimensional space.

Definition 2 *The regularity condition is* ***globally violated*** *for bidder i if for any types t_1 and t_2 such that $\phi_i(t_1) > \phi_i(t_2)$, incentive compatibility requires $G_i(t_1) \leq G_i(t_2)$.*

Global violations imply that virtual valuations go in the opposite direction of the regularity requirement over the entire domain: From Theorem 1 part (iv), the regularity condition is globally violated for bidder i if $\phi_i(r_i)$ increases in r_i over $[\underline{r}_i, \bar{r}_i]$.

[13]Further, the relevant regularity condition is the monotonicity of the virtual valuation with respect to r_i, not to v_i or $v_i - x_i$.

Proposition 4 *Suppose that in the one-dimensional representation, x_i is deterministically related to v_i. Then, for any $f_i(v_i)$ strictly positive over $[\underline{v}_i, \bar{v}_i]$ and $x_i(\underline{v}_i) < \underline{v}_i$, there exist $\epsilon_1, \epsilon_2 > 0$ such that the regularity condition is globally violated for bidder i if $\frac{d \ln x_i}{d \ln v_i} \in (1, 1+\epsilon_1)$ and $|\frac{d}{dv}\left(\frac{d \ln x_i}{d \ln v_i}\right)| < \epsilon_2$ at all $v_i \in [\underline{v}_i, \bar{v}_i]$.*

Proposition 4 shows that global violations occur when $x_i(v_i)$ is slightly more than unit elastic in the reduced dimensionality space of Corollary 1. This condition imposes no restrictions on the *scale* of opportunity costs relative to v_i. This means that global violations can arise in "vanilla" economies in which opportunity costs are tiny, making the "efficiency cost" of not allocating the asset to a bidder with the highest valuation high.

This feature that virtual valuations go against the regularity requirement over the entire domain can never arise in cash auctions or in equity auctions with constant opportunity costs. In cash auctions, the virtual valuation is

$$\psi_i(y_i) = y_i - \frac{1 - D_i(y_i)}{d_i(y_i)}, \tag{41}$$

where $y_i \equiv v_i - x_i$ is bidder i's net valuation and $D(\cdot)$ and $d(\cdot)$ are the cdf and pdf of y_i. Thus, $\psi_i(y_i) < y_i < \bar{y}_i = \psi_i(\bar{y}_i)$ for $y_i < \bar{y}_i$. Similarly, $\phi_i(y_i) < y_i < \bar{y}_i = \phi_i(\bar{y}_i)$ for $y_i < \bar{y}_i$ for virtual valuations in equity auctions with constant opportunity costs (Liu 2016). Thus, even though the virtual valuation may decline over some range of v_i in violation of the regularity condition, it cannot decrease over the entire range.

The global violation of the regularity condition represents the greatest possible mismatch between revenue maximization and incentive compatibility. Revenue maximization requires a seller to allocate the asset to bidders with higher virtual valuations as much as possible, but incentive compatibility demands the opposite. This mismatch has novel implications:

Lemma 3 *If the regularity condition is globally violated for bidder i, then $\bar{\phi}_i$ in (32) is the same for all types of bidder i.*

Corollary 3 *If the regularity condition is globally violated for bidder i, then in the optimal design, bidder i's expected winning probability is independent of his type.*

Corollary 4 *If the regularity condition is globally violated for all bidders, then it is optimal for the seller to disregard all bids and sell to any bidder $i \in \arg\max_j \bar{\phi}_j$ (if $\bar{\phi}_i > v_s$), asking for fraction $1 - \bar{r}_i$ of equity. The bidder always accepts the offer.*

Boone and Mulherin (2007) find that in corporate takeovers that use equities as payments, selling firms typically only ask a subset of potential bidders, often a single bidder, to participate. Consistent with this, we show that if the regularity condition is globally violated for all bidders, then it is optimal for a seller first to identify those bidders whose adjusted virtual valuation is maximal, and to exclude bidders with lower adjusted virtual valuations. The seller selects one non-excluded bidder and sells to that bidder when the maximal adjusted virtual valuation exceeds v_s. The seller demands the highest equity share that this selected bidder i would be willing to cede regardless of his type, hence demanding share $1 - \bar{r}_i$. Note that when bidders are ex-ante heterogeneous, multiple bidders can have the highest adjusted virtual valuation but their values of $\bar{r}_i$ can differ. When this is so, the seller can choose any of these bidders, but the seller must tailor its equity demand to the selected bidder.

Corollary 5 *Suppose bidders are ex-ante identical and the regularity condition is globally violated. Then expected seller revenue in the optimal equity auction does not vary with the number of bidders n. In contrast, in a standard first- or second-price format, revenue falls with n.*

When the regularity condition is globally violated, not only are standard (first- or second-price) equity auction designs not optimal, even with optimal reserve prices, but expected seller revenue falls as competition rises. The greater is n, the lower is the winner's break-even equity stake r_i, and this corresponds to a *lower* virtual valuation ϕ_i when the regularity condition is globally-violated. By (23), seller revenue falls. Indeed, as n goes to infinity, the winner's break-even stake approaches the $\underline{r}_i$ associated with the lowest virtual valuation. This contrasts with cash auctions (or equity auctions with constant opportunity costs), in which with ex-ante identical bidders, revenues in standard auction formats strictly rise with n (when the highest net valuation exceeds v_s), and a seller extracts full rents as n goes to infinity.

The result that competition reduces seller revenues when the regularity condition is globally violated reflects the intuition highlighted earlier that standard auction formats select in a "reinforced" direction of incentive compatibility. Thus, with global violations of regularity, the lowest virtual-valuation bidder is selected; and increased competition reduces this "lowest" virtual valuation. In contrast, in cash auctions (or equity auctions with constant opportunity costs), the regularity condition can be locally violated, but it cannot be violated at sufficiently high net-valuations, and as n grows large so does the probability of a high valuation.

If opportunity costs are common knowledge, optimal equity auctions generate higher revenues than cash auctions, even with ex-ante heterogeneous bidders (Hansen 1985; Liu 2016). Corollary 5 shows that if adverse selection is so severe that the regularity condition is globally violated, optimal equity auctions generate less revenues than cash auctions when n is large, consistent with Che and Kim (2010). In fact, this holds as long as virtual valuations for equity auctions are distorted at high types (i.e., if $\phi(\underline{r})$ is less than the maximum net valuation).[14]

We now give a simple way to implement the optimal mechanism if the regularity condition is globally violated for some bidders, showing how to distinguish between bidder populations.

Corollary 6 *Suppose the regularity condition is globally violated for bidders 1 through $\hat{n}$, but not bidders $\hat{n}+1...n$, where $\hat{n} < n$. Then it is optimal to sell sequentially in two stages: the seller first sells to bidders $\{\hat{n}+1, \ldots n\}$ via the modified mechanism in Proposition 3 that replaces v_s with $max\{v_s, \max_{j=1\ldots\hat{n}} \bar{\phi}_j\}$. If the asset is not sold in the first stage, the seller sells to the first $\hat{n}$ bidders, using the mechanism in Corollary 4.*

Of note, Corollaries 3 through 6 hold as long as regularity is largely, even if not globally, violated: the corollaries follow as long as $\bar{\phi}_i$ (see Lemma 3) is the same for all types of bidder i, for which global violation of regularity is a sufficient, but not necessary, condition.

Revenue Comparisons. Our analysis provides insights into the forces that affect revenue comparisons of optimal equity and cash auctions. If uncertainty solely concerns valuations, then optimal equity auctions generate higher revenues, reflecting that equity bids tie payments to bidder type (Hansen 1985). When bidders have two-dimensional private information, additional forces come into play. First, if the regularity condition is largely violated, it favors cash over equity, a difference that is enhanced by more bidders because the adverse

[14] A sufficient condition for such distortion is that for a positive measure of types that correspond to the lowest break-even stake, the associated net valuation is strictly less than the maximum net valuation.

selection causes equity revenues to rise slowly at best with the number of bidders. This effect tends to rise with increased two-dimensional uncertainty that magnifies the adverse selection.

Second, as Proposition 1 shows, the distribution of net valuations in the one-dimensional representation for equity mechanisms second-order stochastically dominates the distribution of net valuations for the original two-dimensional distribution (and hence for cash auctions). The effects of this stochastic dominance are subtle. With few bidders, second-order stochastic dominance lets a seller set the reserve more efficiently, reducing the probability of no sale, and raising revenues. In contrast, with many bidders, the probability of no sale is small and what matters primarily is the upper-tail of the distribution, so that second-order stochastic dominance reduces revenues. Thus, this feature favors equity when there are few bidders, but it favors cash when n is large. Both of these forces are stronger when the extent of two-dimensional uncertainty increases, which increases the differences in the distributions.

Reflecting the combined forces [is "forces" here repetitive with "Both of these forces are stronger" above. check], optimal equity auctions yield more revenues than cash auctions if there are few bidders or moderate two-dimensional uncertainty, but cash auctions do better with many bidders and high two-dimensional uncertainty. Example 3 illustrates.

Example 3. n ex-ante identical bidders bid with equity or cash. Bidder i's expected valuation is $v_i = 2x_i + y_i$, where y_i is independently and uniformly distributed on $[1, 3]$. Bidder i's opportunity cost x_i is uniformly distributed over $[4 - d, 4 + d]$, where $d \in [0, 4]$.

Implicitly, d measures the extent of two-dimensional uncertainty. In cash auctions, the regularity condition holds for all d. In contrast, in equity auctions, the regularity condition is violated unless d is very close to zero. For a given d, we use the fraction of all type pairs (v_i, x_i) for which $\frac{d}{dr}\phi(r) > 0$ at their corresponding r to measure the extent of violations. Figure 1 shows how violations of the regularity condition in equity auctions rise with d, leading the extent of pooling in the optimal mechanism to rise.

Figure 2 plots how the expected revenue differences between optimal equity and cash auctions varies with d for different numbers of bidders.[15] The non-monotone curvature ("rip-

[15]The regularity condition for cash auctions is satisfied for all d in this example. Thus, a second-price auction with optimal reserve is always optimal. In contrast, with equity auctions, the mechanism in Proposition 3 becomes the optimal mechanism when the regularity condition is violated.

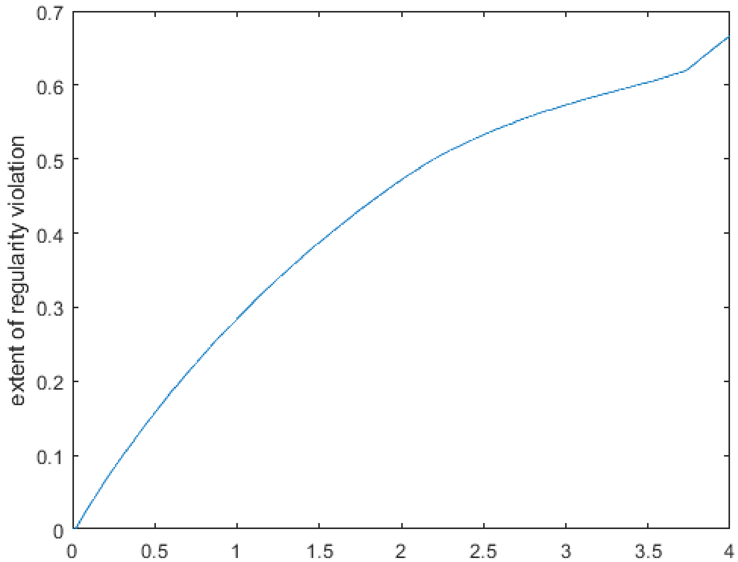

Figure 1: **Violations.** d is on the horizontal axis. The vertical axis is the fraction of probability mass for which the regularity condition is violated.

ples") in the evolution of the expected revenue difference with d illustrates the possible complicated interactions of the different forces. With one or two bidders, equity auctions always generate higher revenues, even with extensive two-dimensional uncertainty. With more than two bidders, cash auctions generate higher expected revenues as long as there is sufficient two-dimensional uncertainty, a difference that rises with n. Once d is sufficiently large, further increases in d amplify revenue differences between the two auction designs for all n. Thus, with substantial two-dimensional uncertainty, optimal equity auctions greatly outperform optimal cash auctions with one bidder, but the opposite holds once there are enough bidders.

5 Surplus-maximizing mechanisms

Lemma 1 holds for any incentive-compatible mechanism. As a result, our one-dimensional representation can be used to analyze mechanisms that maximize expected social surplus:

$$\eta = \int \Big(\sum_{i=1}^{n} W_i\left(\mathbf{t}\right)\left(v_i - x_i\right) + W_0\left(\mathbf{t}\right) v_s \Big) f\left(\mathbf{t}\right) d\mathbf{t}. \tag{42}$$

To proceed, we identify a 'surplus valuation', which is the analogue of a virtual valuation:

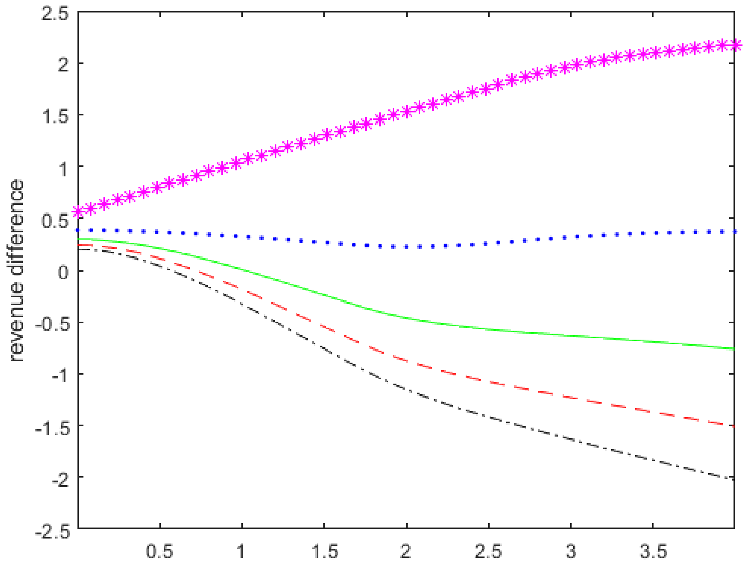

Figure 2: **Revenue comparison.** Number of bidders: 1 (magenta, star), 2 (blue dot) 3 (green solid line), 4 (red, dash), 5 (black, dash dot). The horizontal axis measures d. The vertical axis is revenues for optimal equity minus revenues for optimal cash auctions.

Definition 3 *Bidder i's surplus valuation at $r_i \in [\underline{r}_i, \bar{r}_i]$ in equity auctions is*

$$\kappa_i(r_i) \equiv (1 - r_i)\, a_i(r_i)\,. \tag{43}$$

Theorem 3 *In any incentive-compatible mechanism of equity auctions, expected surplus is:*

$$\eta = \int_{S_1 \times \ldots \times S_n} \Big(\sum_{i=1}^{n} W_i(\mathbf{t})\, \kappa_i(r_i) + W_0(\mathbf{t})\, v_s \Big) f(\mathbf{t})\, dt_1 \ldots dt_n. \tag{44}$$

By (24), the surplus valuation $\kappa_i(r_i)$ is the expected net valuation at r_i. Comparing (44) with (23) suggests that the role $\kappa_i(r_i)$ plays in surplus maximizing is analogous to that played by the virtual valuation in revenue maximizing. Indeed, analysis of surplus-maximizing mechanisms follows that of revenue-maximizing mechanisms, simply by replacing 'virtual valuation' with 'surplus valuation'. Thus, the regularity condition for surplus maximization requires $\kappa_i(r_i)$ rather than $\phi_i(r_i)$ to strictly decrease over $[\underline{r}_i, \bar{r}_i]$. When this regularity condition holds for all i, then in the surplus-maximizing mechanism: (1) a bidder with the highest surplus valuation wins whenever this surplus valuation exceeds the seller's valuation, and (2) the equity share r retained by the winning bidder i solves $\kappa_i(r) = \max_{j \neq i} \{\kappa_j(r_j), v_s\}$. When the regularity condition is violated for one or more bidders, a surplus-maximizing mechanism can be constructed analogously via concavification. We provide details on these results in the appendix.

The properties of cash and equity auctions for surplus-maximizing mechanisms differ. With cash, a second-price auction (with optimal reserve) always maximizes surplus, and achieves first-best social welfare. With equity, however, not only can a second-price auction fail to maximize surplus, but the surplus-maximizing mechanism may not achieve first-best social welfare. This difference reflects that in cash auctions, a bidder's "surplus valuation" is given by his net valuation, and the regularity condition for surplus maximization—which mandates that the surplus valuation increase in net valuation—always holds: the net valuation is trivially an increasing function of itself. In contrast, in equity auctions, the relevant bidder type is r_i, and the regularity condition requires $\kappa_i(r_i)$ to increase in r_i. This regularity condition can be violated, in which case (i) the second-price auction is no longer surplus-maximizing, and (ii) the surplus-maximizing mechanism fails to achieve the first best.[16]

In particular, when the regularity condition for surplus maximization is violated everywhere,[17] disregarding all bids and selling to any bidder with the highest adjusted surplus-valuation maximizes expected surplus (when the adjusted surplus valuation exceeds v_s). Thus, in equity auctions, pooling can be surplus-maximizing, just as it can be revenue-maximizing. This feature reinforces how with severe adverse selection, incentive compatibility requires "inferior" types (in terms of either virtual or surplus valuations) to win with a weakly higher probability. As a result, a mechanism designer can do no better than to make the winning probability independent of bidder type.

Hence, in equity auctions, when adverse selection leads to a conflict between incentive compatibility and surplus maximization, there is a surplus gap between the first best and what the surplus-maximizing mechanism can obtain. In addition, there is a second layer of efficiency loss if the seller seeks to maximize his own payoff at the possible expense of surplus. Indeed, when the regularity condition is sometimes but not always violated, the support over

[16]In the appendix we identify another important difference between cash and equity auctions: in equity auctions, the surplus valuation function $\kappa_i(\cdot)$ differs across heterogeneous bidders, but in cash auctions bidders always have the same surplus valuation function.

[17]Lemma 4 in the appendix provides a condition for the regularity condition to be locally violated. This condition implies that global violations occur when the extreme adverse selection (decreasing bidding strategy) condition identified by Che and Kim (2010) holds.

which adjusted surplus valuations pool typically differs from that over which adjusted virtual valuations pool, implying that revenue-maximizing mechanisms are not efficient. Figure 3 reveals that in the canonical uniform-uniform setting of Example 3, violations are *greater* for surplus maximization than revenue maximization. Of note, this feature differs from cash auctions in which the opposite holds. In fact, in cash auctions, while the regularity condition for revenue maximization can be violated, it cannot be violated with surplus maximization.

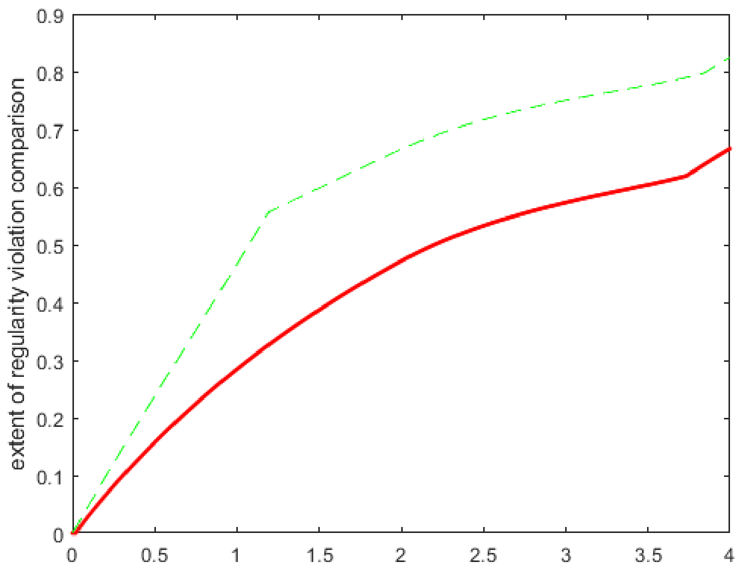

Figure 3: **Violations: Surplus-maximizing vs. Revenue-maximizing mechanisms.** The horizontal axis measures d. The vertical axis is fraction of probability mass with violations of regularity condition. Surplus-maximizing: green line. Revenue-maximizing: red line.

To gain further intuition for the difference, observe that the virtual valuation (20) differs from the surplus valuation (43) by the second term in (20). This term is zero at $r_i = \underline{r}_i$, and it is negative (it is subtracted) for all $r_i > \underline{r}_i$. For the most part, this term decreases in r_i, causing regularity to be violated to a lesser extent for revenue maximization. So, too, in cash auctions, the virtual valuation (41) differs from the surplus valuation by the second term. This second term is zero for the highest net-valuation and it is strictly negative elsewhere, generally (but not always) trending upward in net valuation—in the direction mandated by regularity. What differs from equity auctions is that, without this second term, the regularity condition in cash auctions *always* holds for surplus maximization: adding this second term can only lead to violations of regularity, but not reduce them. Thus, from an efficiency perspective, revenue-maximizing mechanisms can only over-pool in cash auctions, whereas they tend to under-pool in equity auctions.

6 Conclusion

In equity auctions when bidders' values and opportunity costs are private information, adverse selection can arise that results in conflict between revenue maximization, which requires a seller to allocate the asset to bidders with higher virtual valuations; and incentive compatibility, which demands that bidder types with lower break-even equity requirements be weakly more likely to win. Unlike in cash auctions, virtual valuations and incentive compatibility may go globally in opposite directions. We derive the implications for the optimal design. For example, if adverse selection is extreme for all bidders, a seller does best to identify bidders with the highest adjusted virtual valuation, exclude all others, select one non-excluded bidder, and demand the highest equity share that this bidder would cede regardless of his type. Thus, our work provides guidance on standard formats, identifying when they cease to be an appropriate selling mechanism; and when they still perform well given simple modifications. These insights also hold for equity mechanisms that maximize expected social surplus.

The general principles identified should extend to auctions in which bidders offer ordered securities other than equities. For example, the optimality of simple ways to sell when adverse selection is severe for some or all bidders should hold for a given class of ordered securities. Going beyond such analyses, Liu and Bernhardt (2018) investigate settings in which a seller can combine different classes of ordered securities, showing that a seller can then do better when adverse selection obtains.

7 Bibliography

Andrade, G., M. Mitchell and E. Stafford. 2001. "New Evidence and Perspectives on Mergers", *Journal of Economic Perspectives* 15: 103-120.

Boone, A. and H. Mulherin. 2007. "How Are Firms Sold?" *Journal of Finance* 62: 847-875.

Biais, B., T. Mariotti, G. Plantin, and J-C. Rochet. 2007. "Dynamic Security Design: Convergence to Continuous Time and Asset Pricing Implications", *Review of Economic Studies* 74: 345-390.

Board, S. 2007. "Bidding into the Red: A Model of Post-Auction Bankruptcy", *Journal of Finance* 62: 2695-2723.

Burkart, M. and S. Lee. 2015. "Signalling to dispersed shareholders and corporate control." *Review of Economic Studies* 82: 922-962.

Che, Y-K. and J. Kim. 2010. "Bidding with securities: comment", *American Economic Review* 100: 1929-1935.

Crémer, J. 1987. "Auctions with Contingent Payments: Comment," *American Economic Review* 77: 746.

Dasgupta, S. and K. Tsui, 2004. "Auctions with Cross-Shareholding", *Economic Theory* 24: 163-194.

Deneckere, R. and S. Severinov. 2017. "A Solution to a Class of Multi-Dimensional Screening Problems: Isoquants and Clustering", *mimeo.*

DeMarzo, P., I. Kremer and A. Skrzypacz. 2005. "Bidding with Securities: Auctions and Security Design", *American Economic Review* 95: 936-959.

Ekmekci, M., N. Kos, and R. Vohra. 2016. "Just Enough or All: Selling a Firm", *American Economic Journal: Microeconomics* 8: 223-56.

Fishman, M., 1988, "A Theory of Pre-emptive Takeover Bidding," Rand Journal of Economics 19: 88-101.

Fishman, M. 1989. "Preemptive bidding and the role of the medium of exchange in acquisitions." *Journal of Finance* 44: 41-57.

Gorbenko, A. and A. Malenko. 2011. “Competition among Sellers in Securities Auctions”, *American Economic Review* 101: 1806-1841.

Hansen, R. 1985. “Auctions with Contingent Payments.” *American Economic Review* 75: 862-65.

Hansen, R. 1987. “A Theory for the Choice of Exchange Medium in the Market for Corporate Control,” *Journal of Business* 60: 75-95.

Hendricks, K. and R. Porter. 1988. “An Empirical Study of an Auction with Asymmetric Information”, *American Economic Review* 78: 865-883.

Jullien, B. 2000. “Participation Constraints in Adverse Selection Models”, *Journal of Economic Theory* 93: 1-47.

Kogan, S. and J. Morgan. 2010. “Securities under Moral Hazard”, *Review of Finance* 14: 477-520.

Laffont, J-J., E. Maskin, and J-C. Rochet. 1987. “Optimal Nonlinear Pricing with Two-Dimensional Characteristics”, *Information, Incentives, and Economics Mechanisms: Essays in Honor of Leonid Hurwicz.* T. Groves, R. Radner, and S. Reiter (eds.), University of Minnesota Press, chapter 8: 256-266.

Liu, T., 2012, “Takeover Bidding with Signaling Incentives”, *Review of Financial Studies* 2: 522-556.

Liu, T., 2016, “Optimal Equity Auctions with Heterogeneous Bidders”, *Journal of Economic Theory* 166: 94-123.

Liu, T. and D. Bernhardt. 2018. “Rent extraction with securities-plus-cash”, *mimeo.*

Myerson, R. 1981. “Optimal Auction Design”, *Mathematics of Operations Research*: 58-83.

Paarsch, H. 1997. “Deriving an estimate of the optimal reserve price: An application to British Columbian timber sales”, *Journal of Econometrics* 78: 333-357.

Povel, P. and R. Singh. 2010. “Stapled Finance”, *Journal of Finance*, 65: 927-953.

Rochet, J-C. and L. Stole. 2002. “Nonlinear Pricing with Random Participation,” *Review of Economic Studies* 69: 277-311.

Samuelson, W. 1987. “Auctions with Contingent Payments: Comment”, *American Eco-*

nomic Review, 77: 740-745.

Skrzypacz, A. 2013. "Auctions with contingent payments: An overview", *International Journal of Industrial Organization*, 31: 666-675.

Sogo, T. 2017. "Effects of Seller's Information Disclosure in Equity Auctions Requiring Post-Auction Investment", *International Journal of Industrial Organization*, 55: 166-181.

Sogo, T., D. Bernhardt and T. Liu. 2016. "Endogenous Entry to Security-Bid Auctions", *American Economic Review*, 106: 3577-3589.

Sun, W., D. Wang and Y. Zhang. 2018. "Optimal profit sharing mechanisms with type-dependent outside options", *Journal of Mathematical Economics*, 75: 57-66.

8 *Appendix: Proofs*

Proof of Lemma 2: Consider a plane where x_i is on the x-axis and v_i is on the y-axis. Then, R is the distance from point $\left(x_i = \frac{rR}{\sqrt{(1+r^2)}}, v_i = \frac{R}{\sqrt{(1+r^2)}}\right)$ to the origin. Now consider the fan-shaped sector in the first quadrant defined by lines $x_i/v_i = r$ and $x_i/v_i = r+\epsilon$. Denote the opening angle between the two lines by $d\theta$. From the property of a fan-shaped sector that the length of the arc at radius R is $Rd\theta$, the total mass between R and dR is proportional to R. $\square$

Proof of Theorem 1: We first prove the "only if". Conditions (i) – (iii) follow from the text. To prove condition (iv), note that if it is not profitable for t_i to mimic t_i', and vice versa, then

$$(q_i(t_i') - r_i) G_i(t_i') \leq (q_i(t_i) - r_i) G_i(t_i)$$

and

$$(q_i(t_i) - r_i') G_i(t_i) \leq (q_i'(t_i) - r_i') G_i'(t_i).$$

Adding these two equations yields $(r_i' - r_i)(G_i(t_i') - G_i(t_i)) \leq 0$, which yields condition (iv). By (9) and individual rationality, $M_i(r_i) \geq 0$ for all r_i, establishing condition (v).

To prove the "if" part, assume conditions (i) through (v) hold. We show it is not profitable for a type t_i to mimic a type t_i', i.e., $m_i(t_i, t_i') - m_i(t_i, t_i) \leq 0$. By conditions (i) and (ii),

$$\begin{aligned} m_i(t_i, t_i') - m_i(t_i, t_i) &= m_i(t_i, t_i') - m_i(t_i', t_i') + M_i(r_i') - M_i(r_i) \\ &= (r_i' - r_i) G_i(t_i') + M_i(r_i') - M_i(r_i). \end{aligned} \tag{45}$$

If $r_i' = r_i$, then $m_i(t_i, t_i') - m_i(t_i, t_i) = 0$, hence deviation is not profitable. Now consider a report $t_i' \neq t_i$ for which $r_i' \neq r_i$. By conditions (i) – (iii) and the fact that an absolutely continuous function is the definite integral of its derivative, (18) holds and (45) yields

$$m_i(t_i, t_i') - m_i(t_i, t_i) = (r_i' - r_i) G_i(t_i') + \int_{r_i'}^{r_i} G_i(r)\, dr. \tag{46}$$

By condition (iv), the right-hand side of (46) is non-positive regardless of whether $r_i' > r_i$ or $r_i' < r_i$. Thus, $m_i(t_i, t_i') - m_i(t_i, t_i) \leq 0$, i.e., deviation is not profitable. This establishes incentive compatibility. By condition (v) of the theorem, (18), and $G_i \geq 0$, we have $M_i(r_i) \geq 0$ for all r_i. Therefore, by (9), individual rationality holds. ∎

Proof of Theorem 2. Let $\pi_{s,i}$ be bidder i's contribution to the seller's expected profit:

$$\pi_{s,i} = \int_{S_i} G_i(t_i)(v_i - x_i - v_s) f_i(t_i)\, dt_i - \int_{S_i} h_i(t_i, t_i) f_i(t_i)\, dt_i. \tag{47}$$

Adding v_s, the seller's expected revenue becomes:

$$\pi_s = \sum_{i=1}^{n} \pi_{s,i} + v_s. \tag{48}$$

To aid analysis, define

$$A_i(r_i) = \int_{\underline{r}_i}^{r_i} a_i(r) \widetilde{f}_i(r)\, dr \tag{49}$$

and

$$l_i(r) = \lim_{\epsilon \to 0} E\left[v_i - x_i - v_s | r \leq x_i/v_i \leq r + \epsilon\right], \tag{50}$$

where the expectation is taken over S_i. We next rewrite π_i in terms of r_i. Integrate the first term on the RHS of (47) using (15):

$$\int_{S_i} G_i(t_i)(v_i - x_i - v_s) f_i(t_i)\, dt_i = \int_{\underline{r}_i}^{\bar{r}_i} G_i(r_i) l_i(r_i) \widetilde{f}_i(r_i)\, dr_i, \tag{51}$$

Similarly, by (9), the second term on the RHS of (47) becomes

$$\int_{S_i} h_i(t_i, t_i) f_i(t_i)\, dt_i = \int_{S_i} M_i(r_i) v_i f_i(t_i)\, dt_i = \int_{\underline{r}_i}^{\bar{r}_i} M_i(r_i) a_i(r_i) \widetilde{f}_i(r_i)\, dr_i. \tag{52}$$

Substituting (51) and (52) into (47) yields

$$\pi_{s,i} = \int_{\underline{r}_i}^{\bar{r}_i} G_i(r_i) l_i(r_i) \widetilde{f}_i(r_i)\, dr_i - \int_{\underline{r}_i}^{\bar{r}_i} M_i(r_i) a_i(r_i) \widetilde{f}_i(r_i)\, dr_i. \tag{53}$$

Substituting (18) for $M_i(r_i)$ and using (49), we rewrite the second integral in (53) as

$$\int_{\underline{r}_i}^{\bar{r}_i} \Big(M_i(\bar{r}_i) + \int_{r_i}^{\bar{r}_i} G_i\left(r\right) dr \Big) dA_i\left(r_i\right) = M_i\left(\bar{r}_i\right) A_i\left(\bar{r}_i\right) + \int_{\underline{r}_i}^{\bar{r}_i} A_i\left(r_i\right) G_i\left(r_i\right) dr_i. \tag{54}$$

Next, substitute (54) into (53) to obtain

$$\pi_{s,i} = \int_{\underline{r}_i}^{\bar{r}_i} G_i(r_i) \Big(l_i\left(r_i\right) - \frac{A_i\left(r_i\right)}{\widetilde{f}_i\left(r_i\right)} \Big) \widetilde{f}_i\left(r_i\right) dr_i - M_i\left(\bar{r}_i\right) A_i\left(\bar{r}_i\right). \tag{55}$$

We now return to the original two-dimensional framework and write $\pi_{s,i}$ as

$$\begin{aligned} \pi_{s,i} &= \int_{S_i} G_i\left(t_i\right) \left(l_i\left(r_i\right) - \frac{A_i\left(r_i\right)}{\widetilde{f}_i\left(r_i\right)} \right) f_i\left(t_i\right) dt_i - M_i\left(\bar{r}_i\right) A_i\left(\bar{r}_i\right) \\ &= \int_{S_1 \times \ldots \times S_n} W_i\left(\mathbf{t}\right) \left(l_i\left(r_i\right) - \frac{A_i\left(r_i\right)}{\widetilde{f}_i\left(r_i\right)} \right) f\left(\mathbf{t}\right) dt_1 \ldots dt_n - M_i\left(\bar{r}_i\right) A_i\left(\bar{r}_i\right). \end{aligned}$$

Noting $l_i\left(r\right) = \left(1 - r\right) a_i\left(r\right) - v_s$ by $x_i = (x_i/v_i)v_i$ and (50), plugging in (48) and substituting for the virtual valuation, the theorem follows. ■

Derivations for Example 1. We use the uniform-uniform distribution of (x_i, y_i) rather than that of (x_i, v_i). For $\frac{x_i}{v_i} = r_i$, we have $\frac{x_i}{x_i+y_i} = r_i$, yielding

$$y_i = \left(\frac{1}{r_i} - 1 \right) x_i. \tag{56}$$

For a given $r_i \in (0, 1)$, (56) is the straight line through (0,0) with slope $k\left(r_i\right) \equiv \frac{1}{r_i} - 1 = \frac{1-r_i}{r_i}$. If $k\left(r_i\right) \leq 1$, then this line forms a right-angle triangle with the x_i axis and line $x_i = 1$. The length of the segment inside the unit square is therefore $\left(1 + k\left(r_i\right)^2\right)^{0.5}$. Similarly, if $k\left(r_i\right) > 1$, the length of the line segment inside the square is $\left(1 + \frac{1}{k(r_i)^2}\right)^{0.5}$. Combining, for all r_i, we write the length of the line segment inside the square as

$$\left\{ \left(1 + \left(\min\left\{ k\left(r_i\right), \frac{1}{k\left(r_i\right)} \right\} \right)^2 \right\}^{.5} = \left\{ \left(1 + \left(\min\left\{ \frac{1-r_i}{r_i}, \frac{r_i}{1-r_i} \right\} \right)^2 \right\}^{.5}.$$

The angle corresponding to this line is $\theta\left(r_i\right) \equiv \arctan\left(\frac{1}{r_i} - 1\right)$, which has derivative $\frac{d\theta}{dr_i} = \frac{-1}{r_i^2 + (1-r_i)^2}$. Thus, the mass in the differential fan-shaped sector with slopes between $k\left(r_i\right)$ and $k\left(r_i + dr_i\right)$ is $\frac{1}{2}\left(1 + \min\left\{\frac{1-r_i}{r_i}, \frac{r_i}{1-r_i}\right\}^2\right) / \left(r_i^2 + \left(1 - r_i\right)^2\right) dr_i$, reflecting that the area of a fan-shaped sector is 1/2 times the radius squared times the opening angle. This yields (26).

We now calculate $a_i\left(r_i\right)$ in (19). If $r_i \geq 0.5$, then $k\left(r_i\right) \leq 1$ and line (56) leaves the unit square at point $(x_i = 1, y_i = 1/r_i - 1)$ where $v_i = x_i + y_i = 1/r_i$. Consider the fan-shaped sector inside the square bounded by lines with angles θ and $\theta+d\theta$. Taking the mass-weighted expectation of v_i in this sector yields $a_i(r_i) = \frac{2}{3r_i}$, where the factor 2/3 comes from the property of a fan-shaped sector that the length of the arc at radius R is $Rd\theta$—and hence the total mass between R and dR is proportional to R, and that the value of v_i at radius R is proportional to R. Including the other scenario in which $r_i < 0.5$, we have

$$a_i(r_i) = \frac{2}{3}\begin{cases} 1/r_i & \text{if} \quad r_i \geq 0.5 \\ 1 + 1/k(r_i) = 1/(1-r_i) & \text{if} \quad r_i < 0.5 \end{cases}. \tag{57}$$

That is, $a_i(r_i) = \frac{2}{3\max\{r_i, 1-r_i\}}$. By Corollary 1, (25) follows. ■

Proof of Proposition 1: In the one-dimensional distribution prescribed by Corollary 1, for any r, conditional on $r_i \in [r, r+\epsilon]$ where ϵ is infinitesimal, i's net valuation is $a_i(r)(1-r)$. By (19),

$$a_i(r)(1-r) \equiv \lim_{\epsilon\to 0} E\left[v_i(1-r) \mid r \leq x_i/v_i \leq r+\epsilon\right] \tag{58}$$

$$= \lim_{\epsilon\to 0} E\left[v_i - x_i \mid r \leq x_i/v_i \leq r+\epsilon\right]. \tag{59}$$

Thus, the net valuation distribution for the original two-dimensional distribution is a mean-preserving spread of that for the one-dimensional representation of Corollary 1. □

Proofs of Proposition 2 and Corollary 2:

Step 1: To prove Corollary 2, we first show that truth-telling is an equilibrium. Compare the profits of bidder i with type r_i from truthful reporting and under-reporting (i.e., reporting a type pair that corresponds to a lower break-even equity stake than r_i). (1) If truthful and under-reporting both result in winning, then by (29), profits are the same. (2) If truthful and under-reporting both result in losing, profits are also the same (zero). (3) If truthful reporting results in losing, but under-reporting results in winning, under-reporting is strictly unprofitable because $\max\{\max_{j\neq i}\{\phi_j(r_j)\}, v_s\} > \phi_i(r_i)$ and $Q_i < r_i$ by (29). Summarizing, truthful reporting weakly dominates under-reporting. A similar argument shows that truthful reporting weakly dominates over-reporting. Further, individual rationality is satisfied because a bidder can always ensure a zero profit by reporting a sufficiently high r_i.

Next, we show $M_i(\bar{r}_i) = 0$ for all i. By (10), it suffices to consider $G_i(\bar{r}_i) > 0$. By (27) and (29), when bidder $\bar{r}_i$ wins in equilibrium, $Q_i = \bar{r}_i$ and hence $M_i(\bar{r}_i, \bar{r}_i) = 0$. Thus, $\pi_{s,a}$ (equation (22)) is zero, which is its maximum possible value. Further, because a bidder with the maximum virtual valuation $\phi_i(x_i)$ wins when $\phi_i(x_i) > v_s$, $\pi_{s,b}$ obtains its maximum possible value. Because this mechanism maximizes $\pi_{s,a}$ and $\pi_{s,b}$ simultaneously, it is optimal.

Step 2: We show the "if" part of Proposition 2 by assuming (27) and (28). By (10), (27), and (28), it is without loss to denote $m_i(t_i, t_i')$ by $m_i(r_i, r_i')$. Equation (10) yields

$$\begin{aligned} m_i(r_i, r_i) - m_i(r_i, r_i') &= (q_i(r_i) - r_i) G_i(r_i) - (q_i(r_i') - r_i) G_i(r_i') \\ &= (q_i(r_i) - r_i) G_i(r_i) - (q_i(r_i') - r_i') G_i(r_i') + (r_i - r_i') G_i(r_i'), \end{aligned}$$

which, by (28), yields

$$m_i(r_i, r_i) - m_i(r_i, r_i') = \int_{r_i}^{r_i'} G_i(r)\, dr - (r_i' - r_i) G_i(r_i') \geq 0,$$

where the inequality follows because $W_i(r_i, \mathbf{t}_{-i})$ is weakly decreasing in r_i and hence so is $G_i(r_i)$. This establishes that truth-telling is an equilibrium. Further, by (28) and (10), $m_i(\bar{r}_i, \bar{r}_i) = 0$. By arguments similar to those in Step 1, the mechanism is optimal.

Step 3: We prove the "only if" part of Proposition 2. As Corollary 2 shows, a mechanism exists that simultaneously maximizes $\pi_{s,a}$ and $\pi_{s,b}$. An optimal mechanism cannot do worse, so it must also maximize both $\pi_{s,a}$ and $\pi_{s,b}$. Thus, we have (27). Similarly, we have $m_i(\bar{r}_i, \bar{r}_i) = 0$ for all i. By (4) and (27), G_i is a function of r_i only and is continuous in r_i, and $M_i(r_i)$ ((18)) is differentiable everywhere. By Theorem 1, q_i is a function of r_i only. Thus (28) follows upon equating the right-hand sides of (10) and (18). ■

Proof of Proposition 3: As $C_i(\cdot)$ is the concave hull of $B_i(\cdot)$, we have

$$C_i(y) = \max_{\omega, z_1, z_2} \{\omega B_i(z_1) + (1-\omega) B_i(z_2)\}$$

$$s.t. \{\omega, z_1, z_2\} \in [0,1]^3 \text{ and } \omega z_1 + (1-\omega) z_2 = y.$$

Thus, $C_i(\cdot)$ is the lowest concave function on $[0,1]$ such that $C_i(y) \geq B_i(y)$ for all y.

It is straightforward to show that truth telling is an equilibrium. We next establish optimality. Given any allocation rule $W_i(\cdot)$ and associated $G_i(\cdot)$ as defined in (4), we have

$$\begin{aligned}
& \int_{\underline{r}_i}^{\bar{r}_i} G_i(r_i)\left(C_i'\left(\widetilde{F}_i(r_i)\right)-b_i\left(\widetilde{F}_i(r_i)\right)\right)\widetilde{f}_i(r_i)\,dr_i \\
= \;& G_i(r_i)\left(C_i\left(\widetilde{F}_i(r_i)\right)-B_i\left(\widetilde{F}_i(r_i)\right)\right)\Big|_{\underline{r}_i}^{\bar{r}_i}+\int_{\underline{r}_i}^{\bar{r}_i}\left(B_i\left(\widetilde{F}_i(r_i)\right)-C_i\left(\widetilde{F}_i(r_i)\right)\right)dG_i(r_i) \\
= \;& \int_{\underline{r}_i}^{\bar{r}_i}\left(B_i\left(\widetilde{F}_i(r_i)\right)-C_i\left(\widetilde{F}_i(r_i)\right)\right)dG_i(r_i)\geq 0, \qquad (60)
\end{aligned}$$

where the inequality in (60) follows because $B_i(\widetilde{F}_i(r_i)) \leq C_i(\widetilde{F}_i(r_i))$ and $G_i(\cdot)$ is non-increasing in r_i by incentive compatibility.

Let $\bar{G}_i(r_i)$ be the distribution corresponding to allocation rule $\bar{W}(\cdot)$ in (33). Then (60) holds as an equality,

$$\int_{\underline{r}_i}^{\bar{r}_i} \bar{G}_i(r_i)\left(C_i'\left(\widetilde{F}_i(r_i)\right)-b_i\left(\widetilde{F}_i(r_i)\right)\right)\widetilde{f}_i(r_i)\,dr_i = 0, \qquad (61)$$

because when $B_i(\widetilde{F}_i(r_i)) < C_i(\widetilde{F}_i(r_i))$, the concave hull $C_i(\cdot)$ is a straight line locally, implying the derivative $C_i'(\cdot)=\bar{\phi}_i(\cdot)$ is constant. Thus $G_i(\cdot)$ is constant and $dG_i(\cdot)=0$. Further,

$$\begin{aligned}
\int\Big(\sum_{i=1}^n \bar{W}_i(\mathbf{t})(\bar{\phi}_i(t_i)-v_s)\Big)f(\mathbf{t})dt_1..dt_n \;&\geq\; \int\Big(\sum_{i=1}^n W_i(\mathbf{t})(\bar{\phi}_i(t_i)-v_s)\Big)f(\mathbf{t})dt_1...dt_n \\
&=\; \sum_{i=1}^n\int_{\underline{r}_i}^{\bar{r}_i} G_i(r_i)\left(C_i'\left(\widetilde{F}_i(r_i)\right)-v_s\right)\widetilde{f}_i(r_i)\,dr_i \\
&\geq\; \sum_{i=1}^n\int_{\underline{r}_i}^{\bar{r}_i} G_i(r_i)\left(b_i\left(\widetilde{F}_i(r_i)\right)-v_s\right)\widetilde{f}_i(r_i)\,dr_i \\
&=\; \int\Big(\sum_{i=1}^n W_i(\mathbf{t})(\phi_i(t_i)-v_s)\Big)f(\mathbf{t})\,dt_1..dt_n, \qquad (62)
\end{aligned}$$

where the second inequality follows from (60). Moreover, from (61), we have

$$\int\Big(\sum_{i=1}^n \bar{W}_i(\mathbf{t})\left(\bar{\phi}_i(t_i)-v_s\right)\Big)f(\mathbf{t})dt_1...dt_n = \int\Big(\sum_{i=1}^n \bar{W}_i(\mathbf{t})\left(\phi_i(t_i)-v_s\right)\Big)f(\mathbf{t})\,dt_1...dt_n. \qquad (63)$$

Substituting (63) into the left-hand-side of (62) yields

$$\int\Big(\sum_{i=1}^n \bar{W}_i(\mathbf{t})\left(\phi_i(t_i)-v_s\right)\Big)f(\mathbf{t})\,dt_1...dt_n \geq \int\Big(\sum_{i=1}^n W_i(\mathbf{t})\left(\phi_i(t_i)-v_s\right)\Big)f(\mathbf{t})dt_1...dt_n.$$

Thus, $\bar{W}_i(\cdot)$ maximizes the term $\pi_{s,a}$ in (21). The mechanism in the proposition also maximizes $\pi_{s,b}$ by having $M_i(\bar{r}_i)=0$ for all i. Optimality of the mechanism follows. ■

Proof of Proposition 4: Observe that if, for some $d_1 > 0$,

$$\frac{d\ln x_i}{d\ln v_i} \in (1, 1+d_1) \tag{64}$$

at all $v_i \in [\underline{v}_i, \bar{v}_i]$ then

$$\frac{\underline{x}_i}{\underline{v}_i} v_i < x_i < \underline{x}_i \left(\frac{v_i}{\underline{v}_i}\right)^{1+d_1}. \tag{65}$$

This observation means that for any $r^* \in \left(\frac{\underline{x}_i}{\underline{v}_i}, 1\right)$ (the value of r^* is unimportant), there exists a $d_1^* > 0$, such that for any $d_1 \in (0, d_1^*)$, we have $\frac{x_i}{v_i} < r^*$ for $\forall v_i \in [\underline{v}_i, \bar{v}_i]$. By (64),

$$\frac{d\ln x_i}{d\ln v_i} = \frac{dx_i}{dv_i}\frac{v_i}{x_i} \leq 1 + d_1 \Leftrightarrow \frac{dx_i}{dv_i} \leq \frac{x_i}{v_i}(1+d_1). \tag{66}$$

Define $d_1^{**} \equiv \min\left\{d_1^*, \frac{1}{r^*} - 1\right\}$. Then, for all $d_1 \in (0, d_1^{**})$, $\frac{x_i}{v_i} < r^*$ and (66) yield $\frac{dx_i}{dv_i} < 1$.

To proceed further, in (64) let $d_1 \in (0, d_1^{**})$. We have

$$\begin{aligned}\frac{d}{dv_i}\phi_i(v_i) &= 1 - \frac{dx_i}{dv_i} - \left(\frac{d}{dv_i}\frac{\int_{\underline{v}_i}^{v_i} vdF_i(v)}{f_i(v_i)}\right)\frac{x_i(v_i)}{v_i^2}\left(\frac{d\ln x_i(v_i)}{d\ln v_i} - 1\right) \\ &- \frac{\int_{\underline{v}_i}^{v_i} vdF_i(v)}{f_i(v_i)}\Big(\frac{d}{dv_i}\frac{x_i(v_i)}{v_i^2}\Big)\Big(\frac{d\ln x_i(v_i)}{d\ln v_i} - 1\Big) - \frac{\int_{\underline{v}_i}^{v_i} vdF_i(v)}{f_i(v_i)}\frac{x_i(v_i)}{v_i^2}\Big(\frac{d}{dv_i}\frac{d\ln x_i(v_i)}{d\ln v_i}\Big).\end{aligned}$$

We first show that the absolute values of $\frac{\int_{\underline{v}_i}^{v_i} vdF_i(v)}{f_i(v_i)}$ and its derivative are both bounded:

$$|\frac{\int_{\underline{v}_i}^{v_i} vdF_i(v)}{f_i(v_i)}| \leq \frac{\bar{v}_i}{\min_{v_i}\{f_i(v_i)\}} \equiv B_1$$

and

$$|\frac{d}{dv_i}\frac{\int_{\underline{v}_i}^{v_i} vdF_i(v)}{f_i(v_i)}| = |v_i - \int_{\underline{v}_i}^{v_i} vdF_i(v)\frac{f_i'(v_i)}{f_i^2(v_i)}| \leq \bar{v}_i\left(1 + \max_{v_i}\left\{|\frac{f_i'(v_i)}{f_i^2(v_i)}|\right\}\right) \equiv B_2.$$

Further, the absolute values of $\frac{x_i}{v_i^2}$ and its derivative are both bounded:

$$|\frac{x_i}{v_i^2}| < \frac{\bar{x}_i}{\underline{v}_i^2} \equiv B_3 \quad \text{and} \quad \frac{d}{dv_i}\frac{x_i}{v_i^2} = \frac{1}{v_i^2}\frac{dx_i}{dv_i} - \frac{2x_i}{v_i^3}$$

and thus

$$|\frac{d}{dv_i}\frac{x_i}{v_i^2}| < \frac{1}{\underline{v}_i^2} + \frac{2\bar{x}_i}{\underline{v}_i^3} \equiv B_4.$$

Thus,

$$\frac{d}{dv_i}\phi_i(v_i) > 1 - \frac{dx_i}{dv_i} - (B_2B_3 + B_1B_4)\left(\frac{d\ln x_i(v_i)}{d\ln v_i} - 1\right) - B_1B_3\left(\frac{d}{dv_i}\frac{d\ln x_i(v_i)}{d\ln v_i}\right). \tag{67}$$

Let $\epsilon_1 \in (0, d_1^{**})$. When $\frac{d \ln x_i}{d \ln v_i} \in (1, 1+\epsilon_1)$, $\frac{dx_i}{dv_i} < 1$. In addition, by (67), there exists $\epsilon_2 > 0$ such that when $|\frac{d}{dv}\left(\frac{d \ln x_i}{d \ln v_i}\right)| < \epsilon_2$, $\frac{d}{dv_i}\phi_i(v_i) > 0$ at all $v_i \in [\underline{v}_i, \bar{v}_i]$. Because r_i strictly increases in v_i by $\frac{d \ln x_i}{d \ln v_i} > 1$, ϕ_i strictly increases in r_i, completing the proof. ■

Proof of Lemma 3: If the regularity condition is globally violated for bidder i, then $B_i(\cdot)$ in (31) is convex. Therefore, the concave hull $C_i(\cdot)$ is a straight line on $[0, 1]$. ■

Sketch of Analysis of Surplus-maximizing mechanisms. The analysis directly follows that for revenue-maximizing mechanism. Here we define the key notation; substituting into the analogous arguments for revenue-maximization yields the results.

Let η_i be bidder i's contribution to expected social surplus (over the value when the seller retains the asset):

$$\eta_i = \int_{S_i} G_i(t_i)(v_i - x_i - v_s) f_i(t_i)\, dt_i.$$

Adding the surplus of v_s when the seller retains the asset, total expected surplus is

$$\eta = \sum_{i=1}^{n} \eta_i + v_s. \tag{68}$$

We next rewrite η_i in terms of r_i using (15) and (19):

$$\eta_i = \int_{\underline{r}_i}^{\bar{r}_i} G_i(r_i)(\kappa_i(r_i) - v_s)\, \widetilde{f}_i(r_i)\, dr_i. \tag{69}$$

Returning to the original two-dimensional framework, we have

$$\eta_i = \int_{S_1 \times \ldots \times S_n} \Big(W_i(\mathbf{t})(\kappa_i(r_i) - v_s) \Big) f(\mathbf{t})\, dt_1 \ldots dt_n$$

Substituting yields the formulation in (44).

Definition 4 *The design problem for surplus maximization is regular if $\kappa_i(r_i)$ strictly decreases over $[\underline{r}_i, \bar{r}_i]$ for all i.*

Proposition 5 *Suppose that the design problem for surplus maximization is regular. Then the following constitutes a surplus-maximizing equity mechanism:*

(i) the winning rule is

$$W_i(\mathbf{t}) = \begin{cases} 1 & \text{if} \quad \kappa_i(r_i) > \max\{\max_{j \neq i}\{\kappa_j(r_j)\}, v_s\} \\ 0 & \text{if} \quad \kappa_i(r_i) < \max\{\max_{j \neq i}\{\kappa_j(r_j)\}, v_s\} \end{cases}, \tag{70}$$

for all i *and* $\mathbf{t}$.

(ii) the equity-retention rule for the winning bidder is

$$Q_i(\mathbf{t}) = \kappa_i^{-1}\left(\max\left\{\max_{j\neq i}\{\kappa_j(r_j)\}, v_s\right\}\right), \tag{71}$$

where $\kappa_i^{-1}(\cdot)$ *denotes a bounded inverse of* $\kappa_i(\cdot)$*:*

$$\kappa_i^{-1}(x) \equiv \begin{cases} \bar{r}_i & \text{if} \quad x < \kappa_i(\bar{r}_i) \\ y \in [\underline{r}_i, \bar{r}_i] \ \text{s.t.}\ \kappa_i(y) = x & \text{if} \quad x \in [\kappa_i(\bar{r}_i), \kappa_i(\underline{r}_i)] \end{cases}. \tag{72}$$

In the special case when bidders have publicly-known but heterogeneous opportunity costs, regularity holds and the surplus-maximizing equity mechanism in 5 reduces to the efficient mechanism identified by Hansen (1985).

Note that in cash auctions, a second-price auction (with optimal reserves) always maximizes surplus, even if bidders are ex-ante heterogeneus. In equity auctions, however, a second-price auction is typically not surplus maximizing with heterogeneous bidders, even if the corresponding regularity condition holds. This difference partially reflects that in cash auctions, bidders have the same "surplus valuation" function—the surplus valuation is given by a bidder's net valuation, even when bidders are ex-ante heterogeneous—whereas in equity auctions, the surplus valuation function $\kappa_i(\cdot)$ differs across heterogeneous bidders. More importantly, in contrast to cash auctions in which the regularity condition always holds, in equity auctions the regularity condition that $\kappa_i(r_i)$ decrease in r_i need not always hold. Indeed, in one-dimensional deterministic setting of Che and Kim (2010), we have:

Lemma 4 *If* x_i *is deterministic of* v_i *and* $\frac{dx_i}{dv_i} \in (0,1)$*, then the regularity condition holds for bidder* i *if and only if* r_i *is strictly decreasing in* v_i*.*

Proof: By $\frac{dx_i}{dv_i} \in (0,1)$, bidder i's net valuation strictly increases with v_i. Thus net valuation strictly decreases in r_i if and only if r_i strictly decreases in v_i. The lemma follows. ■

When the regularity condition is violated for one or more bidders, a surplus-maximizing mechanism can again be constructed along the lines for the revenue-maximizing mechanism, defining an adjusted surplus valuation $\bar{\kappa}_i(r_i)$ that is constructed via $\kappa_i(r_i)$ in a similar way as how $\bar{\phi}_i(r_i)$ is constructed via $\phi_i(r_i)$. For any vector of bidder types $\mathbf{t}$, let $N(\mathbf{t})$ be the set of bidders for whom $\bar{\kappa}_i(r_i)$ is both maximal and higher than v_s.

Proposition 6 *The following constitutes a surplus-maximizing equity mechanism:*

(i) *the winning rule is*

$$\bar{W}_i(\mathbf{t}) = \begin{cases} 1/|N(\mathbf{t})| & \text{if } \ i \in N(\mathbf{t}) \\ 0 & \text{if } \ i \notin N(\mathbf{t}) \end{cases} \tag{73}$$

for all i *and* $\mathbf{t}$*, and*

(ii) *the equity-retention rule for the winning bidder is the same as that in part (ii) of Proposition 5 with* $\kappa_i(r_i)$ *replaced by* $\bar{\kappa}_i(r_i)$*.*

If the regularity condition for surplus maximization is globally violated for all bidders, then as with revenue-maximizing mechanisms, pooling can be optimal. That is, the best a mechanism designer can do is to randomize:

Corollary 7 *If the regularity condition for surplus maximization is globally violated for all bidders, then it maximizes expected surplus to disregard all bids and sell to any bidder* $i \in \arg\max_j \bar{\kappa}_j$ *(if* $\bar{\kappa}_i > v_s$*), asking for equity share* $1-\bar{r}_i$*. The bidder always accepts the offer.*