Simplifying Auction Designs via Market Feedback*

Tingjun Liu
Faculty of Business and Economics, University of Hong Kong
tjliu@hku.hk

Dan Bernhardt
University of Illinois, danber@illinois.edu
University of Warwick

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Abstract

Implementation of optimal or near-optimal mechanisms with heterogeneous bidders is informationally demanding for auctioneers, and such mechanisms invariably employ discriminatory winning and payment rules—creating legal and moral hazard concerns. We show how an auctioneer can alleviate informational burdens and avoid discrimination by exploiting information aggregation by capital markets, linking auction outcomes to post-auction market prices to obtain high revenues even when arbitrarily heterogeneous bidders pay with different securities. Our insight is that the market will collect information and respond to details ex post when pricing the winner, so the selling mechanism can be *detail-free ex ante and informationally un-demanding* for the auctioneer.

Keywords: mechanism robustness; information aggregation; security bidding and security design; financial markets; bidder heterogeneities

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1 Introduction

A common, fundamental critique of theoretically-derived optimal or near-optimal mechanism designs with heterogeneous agents is that they are invariably complicated and informationally demanding for auctioneers.[1] When bidders in an auction are ex-ante heterogeneous—as is the case in practice—the optimal design is sensitive to the specific characteristics of all agents, requiring an auctioneer to know all features of the environment in order to determine the winner and payments. Moreover, even if an auctioneer knows everything, optimal auction rules are discriminatory, requiring different rules for different bidders and favoring some bidders at the expense of others. Such discriminatory designs give rise to legal and moral hazard concerns.[2]

Heterogeneous bidders impose even heavier demands for information and discrimination in security-bid auctions, where the winner pays the seller with securities rather than cash. Unlike cash bids, the monetary values of security bids depend on the fine details of the underlying cash flow distributions, and since bidder cash flows are private information, a security bid's monetary value is not transparent. Without intricate discriminatory adjustments for bidder heterogeneities and the security used, the "wrong" (lower valuation) winning bid is likely to be selected, and both the probability and costs of such selections rise when a security's value is more sensitive to cash flows.

Our paper develops a simple and nondiscriminatory mechanism for security-bid auctions that imposes minimal information burdens on the seller and treats all bidders identically, but yet—regardless of how bidders differ ex ante—it always generates high revenues. Our insight is that a seller can *alleviate informational burdens* and *avoid discrimination* by exploiting information aggregation by capital markets, linking auction outcomes to prices set by the market post-auction.

Our "floating-parameter" securities auction consists of two stages. In the first stage, bidders who are privately informed about their synergies—which add value to the auctioned asset—submit cash bids in a second-price auction, and the winning price is publicly announced. In the second stage, in lieu of cash, the winner pays with securities (claims to future cash flows) priced by a competitive capital market that incorporates all public information. The security paid is from a possibly bidder-

[1] See e.g., Hurwicz 1972, Wilson 1985, Dasgupta and Maskin 2000, Bergemann and Morris 2005.

[2] As a consequence, regulators may place restrictions on discriminatory designs. For example, in takeovers, fiduciary duties may require a target to accept the "highest" bid.

specific set of securities ordered by a parameter (e.g., the set of equities ordered by share). This set is either pre-specified by a seller or exogenously determined. The parameter paid is such that the security's post-auction market value—its expected payoff conditional on the second-highest cash bid (the auction's winning price) and the winner's identity—equals that second-highest cash bid. That is, the parameter paid *floats* after the first stage of the auction so as to deliver a guaranteed cash value.

We allow for arbitrary heterogeneity across bidders in (1) standalone values, (2) distributions of expected synergies from acquisition, (3) conditional distributions of future payoffs given expected synergies, and (4) *the types of ordered securities used by different bidders*. Despite the bidder heterogeneity, we prove that with our nondiscriminatory floating-parameter design, steeper securities—securities whose payoffs are more sensitive to underlying cash flows—always generate greater seller revenues than flatter securities, and flatter securities generate greater revenues than cash auctions, regardless of how bidders differ. Specializing to equity securities, we show that the revenues raised are close to those with the optimal design.

We contrast outcomes in our "floating-parameter" auction design with those in the "fixed-parameter" securities auctions studied by the literature where the security bids submitted in the auction determine the security parameter paid. To start, we consider the benchmark setting of existing studies—ex-ante identical bidders who use the same ordered set of securities. In such settings, DeMarzo, Kremer, and Skrzypacz (2005, DKS) show that in standard (nondiscriminatory) fixed-parameter security auctions, expected seller revenue is higher when the securities used for payment are steeper, so that payments are tied more tightly to the winner's valuation.

In this homogeneous bidder setting, we derive a revenue equivalence result for nondiscriminatory fixed- and floating-parameter designs. In our floating-parameter auction, a bidder breaks even when it barely wins, i.e., when the first and second highest bids are equal so that the winner pays its own bid. When this happens, the market does not know that the winner is just about to lose, inferring instead that the winner's synergies are at least as high as those associated with the bid, and likely higher. As a result, at the break-even point, the market overestimates the winner's synergies, so the actual value of the security paid by the winner is *less* than the cash bid. This provides bidders in floating-parameter auctions incentives to submit more aggressive cash bids than in cash auctions, and we show this results in the floating- and fixed-parameter designs delivering exactly the same expected revenue to the seller.

Revenue equivalence breaks down once there is heterogeneity. As a first step, we consider a setting with bidders who are ex-ante identical in all regards except for stand-alone values, and stand-alone values are small relative to synergies. We prove that when the stand-alone values of different bidders go to zero at different rates, floating-parameter auctions always achieve first-best revenues (full surplus extraction), whereas nondiscriminatory fixed-parameter security auctions earn strictly less—and, of course, so do cash auctions.

We then consider the general case with heterogeneous bidders whose standalone values need not be small. We prove that given any fixed set of reserves (including no reserves), seller revenues in floating-parameter auctions are always higher with steeper securities than with flatter securities, which, in turn, exceed revenues from cash auctions, regardless of how bidders differ ex ante. This result holds even when different bidders use different types of securities—for example, if some bidders use equity and the others use debt. In particular, expected revenue is higher if some bidder switches to steeper securities, as long as all other bidders use securities that are at least as steep as those they had used before. Relatedly, we prove that revenues in floating-parameter auctions always rise when the standalone value of any *single* bidder is reduced. These results reflect that bidders are more concerned with the market's inference about their synergies when they use steeper securities—whose values are more sensitive to bidder synergy types—or have lower standalone-values, so a bidder would pay out a larger fraction of its total cash flows upon winning. Both scenarios incentivize bidders to submit more aggressive *cash* bids. For perspective, with ex-ante identical bidders, nondiscriminatory fixed-parameter auctions exhibit the same properties if all bidders change together—revenues rise if all bidders switch to the same steeper security, or if their standalone values are reduced by the same amount—but if only a single bidder's standalone value is reduced, revenues eventually fall below those in a cash auction.

It follows directly that a seller in a floating-parameter auction who can select the set of ordered securities used by a given bidder would choose the steepest feasible set of securities for that bidder—see DKS for why sets of particularly steep securities may be infeasible (moral hazard, etc.), possibly for bidder-specific reasons.[3] More subtly,

[3]Also, when bidders finance their cash bids by issuing securities to the market after the auction—a scenario that our floating-parameter design incorporates—bidders may have exogenous reasons for using different types of security: for example, institutional rigidities (e.g., tax, bankruptcy risk,

we prove that a *bidder's* payoff is always higher when it uses flatter securities that are in the bidder's feasible set—this result is not immediate because a bidder who uses flatter securities retains a higher share of payoffs, but is also less likely to win.

The key distinction between the two designs is that bidders submit security bids in fixed-parameter auctions, but they submit cash bids in floating-parameter auctions. If bidders have different characteristics, some will bid more aggressively than others. More aggressive bidding always benefits a seller in floating-parameter auctions—no matter how bidders differ, more cash is always good—but not in fixed-parameter auctions where the security bid with the highest face value may not have the highest monetary value, as the bid could be from an incentivized bidder with a low valuation. Put differently, bids in fixed-parameter auctions are made in different and non-transparent units (e.g., the monetary values of equity claims depend on a firm's specific features and private information); to properly compare bids, a seller must solve *each* bidder's optimization problem. By contrast, floating-parameter designs convert all bids into a common unit—cash. This makes it **easy to evaluate bids** even when bidders have arbitrarily different characteristics and use different types of securities.

The ability of our nondiscriminatory mechanism to accommodate heterogeneous bidders differentiates it from existing work in fundamental ways. Its virtues include:

1. It is simple, detail-free and invariant to the environment: a seller uses the same mechanism no matter how bidders differ ex ante, or how their securities differ.

2. It removes legal and moral hazard concerns about discriminatory auctions. In takeovers, it lets a seller satisfy its fiduciary duties to accept the "highest" bid even if bidders have very different characteristics.

3. It alleviates information demands. Collectively, the market only needs to learn about the winner, and it can acquire the information post-auction, after a winner's selection. By contrast, in a fixed-parameter auction, to discriminate properly, a seller must know all information about all bidders pre-auction.

4. It always generates higher revenues than cash, and is robust to bidder heterogeneity, exhibiting the property that revenues rise when any single bidder uses a steeper security, or has a lower standalone value.

or leverage concerns) can lead to the use of debt versus equities; see e.g., Faccio and Masulis (2005).

The advantages of floating-parameter designs reflect that the *market can do things that a seller cannot*: (i) a seller cannot discriminate ex ante due to legal/moral hazard concerns about discriminatory auctions, while the market effectively discriminates via its ex-post assessment of the winner; (ii) the market responds to details ex post, so the selling mechanism can be detail-free ex ante and a seller doesn't *ever* have to learn.

We specialize to equity auctions to establish ways in which revenues in our simple floating-parameter design are close to those in the complicated *optimal* auction design with heterogeneous agents (Liu, 2016); optimal designs for securities other than equity are currently unknown. In the optimal equity design, when bidders have the same synergy distribution but differ in standalone values, those with lower standalone values have higher winning probabilities.[4] This reflects that a bidder's informational advantage in an equity auction *grows* with its standalone value, making it optimal to reward smaller bidders that the seller can better exploit. Floating-parameter auctions share this feature—in floating-parameter auctions, a smaller bidder has a greater incentive to signal high synergies because it pays a higher equity share when it wins, and is thus more concerned with the exchange-rate set by the market. This leads smaller bidders to submit more aggressive bids, making them more likely to win but *with no need for a seller to explicitly discriminate against larger bidders*.

We establish that a floating-parameter auction with a *common* reserve price can generate almost as much revenue as the optimal mechanism (which requires discriminatory reserves as well as discriminatory selection for bids above reserves) when bidders have the same synergy distribution but differ in standalone values. We show that with uniformly-distributed synergies, as the standalone values of the target and bidders grow large vis-à-vis the dispersion in synergies,[5] the difference in expected revenues between the optimal and floating-parameter designs goes to zero. Floating-parameter auctions implement the optimal mechanism precisely: (1) both designs select the same winner; and (2) the heterogeneous reserves for different bidders in the optimal design correspond to the *same* cash bid with floating-parameter offers. It follows that floating-parameter auctions can select the right bidder and achieve the optimum.

Thus, we prove that revenues in the floating-parameter design converge to those

[4]By way of contrast, standalone values are irrelevant in cash auctions.

[5]This scenario captures takeover settings—Betton et al.'s (2008) analysis of all control contests for publicly-traded US targets between 1980 and 2005 finds that, on average, the combined standalone values of the target and acquiring firm are over 50 times the size of the synergies.

in the optimal mechanism both when standalone values grow very large or very small. We also numerically analyze cases in which synergies are neither large nor small relative to standalone values, and find that revenue losses remain small.

Relation to Literature. Floating-parameter auctions were first analyzed in Liu (2012). Liu studies an ascending-price format with equity payments, showing how signaling incentives arise and interact with the multiple-bid nature of ascending-price auctions. By contrast, we characterize the revenue properties of second-price, floating-parameter auctions for general securities, allowing for arbitrary heterogeneity in bidder characteristics and the types of securities they use.[6]

Researchers have stressed the desirability of robust mechanisms that do not depend on agents' common knowledge so as to work well in a range of settings (Hurwicz 1972, Wilson 1985, Dasgupta and Maskin 2000, Bergemann and Morris 2005). With heterogeneous bidders, a seller in a fixed-parameter auction must know *all* details of bidders ex-ante and discriminate accordingly, else the auction can generate lower revenues than cash auctions. In contrast, floating parameter auctions are detail-free—the high bid wins and pays the second-highest bid. Moreover, floating-parameter auctions shift informational burdens from the seller to the capital market, also reducing total informational demands as the market only needs to learn after the auction about the winner—there is no need to learn about losing bidders. In addition, bidders have weakly dominant strategies. Thus, not only does a seller need not know anything about bidders, but bidders do not need to know anything about rival bidders.

Despite its modest informational demands and simple structure, our floating-parameter design always generates higher revenues than cash auctions. Thus, our paper also relates to research on robust auctions that obtain maximum revenue guarantees in worst case scenarios about unknown distributions of bidder valuations or bidder beliefs and equilibrium selection (Bergemann, Brooks, and Morris (2016, 2017, 2019), Brooks and Du (2021), Du (2018)). Indeed, with our mechanism, revenues when bidders have different standalone values and use different types of securities

[6] More generally, our paper also relates to a vast literature on security auction design that includes Hansen (1985), Crémer (1987), Samuelson (1987), Dasgupta and Tsui (2004), Povel and Singh (2010), Gorbenko and Malenko (2011), Abhishek, Hajek, and William (2015), Sogo, Bernhardt and Liu (2016), Sogo (2017) and Cong (2020), among others. Dasgupta and Tsui (2003), Gorbenko and Malenko (2014) and Boulatov and Severinov (2021) analyze auctions with asymmetric bidders, and Biais et al. (2007), Ekmekci, Kos and Vohra (2016), Burkart and Lee (2015), Lee and Rajan (2018), and Eckbo, Malenko and Thorburn (2020) analyze auction/security design in financing and takeover settings.

always exceed those in a homogeneous setting where bidders have the same (highest) standalone value and use the same (flattest) set of securities.

The nondiscriminatory nature of floating-parameter designs also makes them robust to moral hazard concerns, as they treat the cash bids of all bidders identically, *precluding the possibility of rigging outcomes*. In contrast, with discriminatory auctions, a biased designer (e.g., a firm's board) can favor its desired winner in ways that are undetectable absent knowledge of its information.

Our auction design with market feedback closely relates to the two-stage mechanism structure of Mezzetti (2004) in which outcomes (allocations) are determined first, and then payments are determined, depending on the information revealed in the first stage. In Mezzetti, as in our paper, agents may have heterogeneous characteristics and transfers are contingent on the realized decision-outcome payoffs. In our auction design, it is the market's assessment of the value of merged entity in the second stage that determines the security payment of the auction winner at the first stage.

Our paper relates to Dasgupta and Maskin (2000) in that they simplify auction designs by making bids contingent on other bidders' valuations, whereas we do so by *making bids contingent on post-auction market prices*.[7] Thus, we contribute to work on the role of financial markets, in particular to how market feedback affects real outcomes (Bond et al. 2012; Bond and Goldstein 2015; Goldstein and Yang 2019) and how takeover bids affect stock prices (Bagnoli and Lipman 1996). To our knowledge, we are the first to show how capital markets can be used to simplify auction designs.

2 Illustrative Example

We use an example with equity payments to illustrate outcomes for the different auction designs. Suppose a seller's standalone value is zero and bidders' expected synergies are independently distributed according to a uniform distribution on $[0, 1]$. Consider a bidder i with standalone value $X_i > 0$ and expected synergy θ_i. In a second-price cash auction, i has a weakly dominant strategy to bid its true valuation,

[7] This relates our paper to work on the performance of "simple contracts" relative to the optimal complex menu. McAfee, McMillan, and Reny (1989) derive conditions under which the optimal selling procedure in a multiple-bidder setting can be implemented via a simple mechanism where a seller makes offers only to one bidder. Rogerson (2003) and Chu and Sappington (2007) identify conditions in principal-agent settings under which limited menus extract over 70% of the surplus. See also Chu, Leslie and Sorensen (2011), Bose, Pal and Sappington (2011), Chassang (2013), and Carroll (2015).

$\beta_i^{cash}(\theta_i) = \theta_i$, just breaking even if it wins and pays its own bid. Similarly, in a second-price fixed-parameter equity auction i bids the share $\beta_i^{fix}(\theta_i) = \frac{\theta_i}{X_i+\theta_i}$ that would break even in the event that i paid its bid.

In a floating-parameter equity auction, the equilibrium cash bid also breaks even when it is the bid paid, i.e., when the two highest bids are equal; but when this happens, the market does not know that the winner is about to lose, inferring only that its synergies are at least as high as those associated with the bid (and likely higher). This logic pins down i's bidding strategy to be $\beta_i^{flt}(\theta_i) = \theta_i(1 + \frac{1-\theta_i}{2(X_i+\theta_i)})$, which strictly increases in θ_i and exceeds $\beta_i^{cash}(\theta_i) = \theta_i$. To understand this bidding strategy, observe that if i has expected synergy θ_i^* and wins at a price that equals its bidding strategy $\beta_i^{flt}(\theta^*)$, then the market infers that $\beta_i^{flt}(\theta_i) \geq \beta_i^{flt}(\theta^*)$. This translates to $\theta_i \geq \theta^*$, i.e., the market believes that θ_i is uniformly distributed over $\left[\theta^*, 1\right]$. This corresponds to an expected synergy of $\frac{1+\theta_i^*}{2}$, which exceeds i's true type θ_i^*. The market assesses the value of the joint firm at $X_i + \frac{1+\theta_i^*}{2}$, so i pays equity share $\beta_i^{flt}(\theta_i^*)/\left(X_i + \frac{1+\theta_i^*}{2}\right) = \frac{\theta_i^*}{X_i+\theta_i^*}$, which has dollar value $\frac{\theta_i^*}{X_i+\theta_i^*}(X_i + \theta_i^*) = \theta_i^*$, i.e., i breaks even.

To illustrate our revenue equivalence result for ex-ante identical bidders, suppose there are two bidders with the same standalone value, $X_1 = X_2 \equiv X$, that we let go to zero to simplify algebra. In the fixed-parameter auction, both bidders bid close to 100%, regardless of their types. But since X is not quite zero, the higher-type bidder will bid slightly more, hence winning the auction. Thus, expected seller revenue is $\mathrm{E}[\max\{\theta_1, \theta_2\} \times 100\%] = \frac{2}{3}$, where $\max\{\theta_1, \theta_2\}$ is value of the combined firm under the winner's control, and 100% is the losing bid. In the analogous floating-parameter auction, as X goes to zero, $\beta_i^{flt}(\theta_i)$ reduces to $\frac{1}{2}(1+\theta_i)$ for all $\theta_i \in (0, 1]$, so the higher-type bidder still wins. The winner pays the lower-type bidder's bid, so expected seller revenue is $\frac{1}{2}(1 + E[\min\{\theta_1, \theta_2\}]) = \frac{1+1/3}{2} = \frac{2}{3}$, just as in the fixed-parameter auction.

To see why and how revenue equivalence breaks down with heterogenous bidders, suppose that while both X_1 and X_2 go to zero, they do so at different rates. To ease calculations, suppose the ratio X_1/X_2 also goes to zero (e.g., X_1 declines quadratically, while X_2 declines linearly). Then, in the fixed-parameter equity auction, while both bidders' equity offers pool toward 100%, bidder 1's offer is slightly larger (when $\theta_1 > 0$), even if $\theta_1 < \theta_2$. Thus, bidder 1 wins with probability one, and pays 100% of the combined firm (bidder 2's bid). Hence, expected seller revenue is only $\mathrm{E}[\theta_1 \times 100\%] = \frac{1}{2}$. By contrast, bidding strategies in the floating-parameter auction remain separating (rather than pooling as in the fixed-parameter auction); i.e., $\beta_i^{flt}(\theta_i) = \frac{1+\theta_i}{2}$ strictly increases in θ_i. Thus the higher-type bidder still wins, regardless of differences in the rates with which standalone values approach zero, so expected seller revenue remains $\frac{2}{3}$. Further, with more than two bidders, the advantages of floating-parameter auctions grow even larger. If the standalone value of one bidder goes to zero much faster than those of other bidders, expected revenue in a fixed-parameter auction remains $\frac{1}{2}$ but revenue in the floating-parameter auction rises past $\frac{2}{3}$.

3 The Model

The asset being auctioned has a value of V_T if the seller retains it. There are n bidders who are risk neutral and possibly heterogeneous. Bidder i has a standalone value of $X_i > 0$. If bidder i acquires the asset, then it yields a contractible stochastic payoff of Z_i that equals the sum of the combined standalone values of the seller and bidder plus the synergies or value added by bidder i's control. Bidder i receives an independently-distributed signal Θ_i of the value added if i wins, where $\Theta_i \sim F_i(\cdot)$ with full support over $\left[\underline{\theta}_i, \overline{\theta}_i\right]$ and $\overline{\theta}_i > \underline{\theta}_i \geq 0$. Conditional on $\Theta_i = \theta_i$, the expected value of Z_i is

$$E(Z_i|\Theta_i = \theta_i) = X_i + V_T + \theta_i.$$

Thus, the expected value added if bidder i wins is $E(Z_i|\Theta_i = \theta_i) - X_i - V_T = \theta_i$. Bidder i is privately informed about θ_i, which we refer to as bidder i's private type or expected synergy. Conditional on θ_i, Z_i is distributed according to a density $h_i(\cdot|\theta_i)$ with full support on $[0, \infty)$, where the family $\{h_i(\cdot|\theta_i)\}$ has the *strict monotone likelihood ratio property (sMLRP)*: $h_i(z|\theta_i)/h_i(z|\theta_i')$ is increasing in z for $\theta_i > \theta_i'$, i.e., higher signals represent better news.

As in DKS, to simplify analysis, we make additional technical assumptions: (i) for all i, the conditional density function $h_i(z|\theta)$ is twice differentiable in z and θ; and (ii), the functions $zh_i(z|\theta)$, $|z\frac{\partial h_i(z|\theta)}{\partial z}|$, and $|z\frac{\partial^2 h_i(z|\theta)}{\partial^2 z}|$ are integrable on $z \in (0, \infty)$. These assumptions are weak and allow us to take derivatives "through" expectation operators.

Ordered sets of securities. The winner pays the seller with a security from an *ordered set* whose elements are indexed by a parameter s. Let $S(s, z)$ denote the value of security s when the cash flow is z. An ordered set of securities is a set $\mathcal{S} = \{S(s, \cdot) : s \in [\underline{s}, \overline{s}]\}$ such that (i) for all s, $S(s, z)$ and $z - S(s, z)$ are nonnegative and weakly increasing in z; (ii) for any given bidder i, $ES^i(s,\theta) \equiv E(S(s,Z_i)|\Theta_i = \theta)$, i.e., the expected value of security $S(s,\cdot)$ derived from cash flows generated by bidder i conditional on $\Theta_i = \theta$, is differentiable and strictly increasing in both arguments; and (iii) there is sufficient range in the index that $ES^i(\underline{s},\underline{\theta}_i) \leq \underline{\theta}_i + V_T$ and $ES^i(\overline{s},\overline{\theta}_i) > V_T + \overline{\theta}_i$ for all i.

For example, if the base security is equity, then $S(s,z) = sz$, and if it is debt, then $S(s,z) = min\{s,z\}$. Note that, with heterogeneous bidders, $ES^i(s,\theta)$ depends on the identity i of a bidder.

We use the notion of steepness introduced in DKS: an ordered set of securities $\mathcal{S}_A$ is *steeper* than $\mathcal{S}_B$ if for all $s_A \in [\underline{s}_A, \overline{s}_A]$, $s_B \in [\underline{s}_B, \overline{s}_B]$ and bidders i, $ES^i_A(s_A,\theta^*) = ES^i_B(s_B,\theta^*)$ implies that $\partial ES^i_A(s_A,\theta^*)/\partial\theta > \partial ES^i_B(s_B,\theta^*)/\partial\theta$. Thus, if a bidder with type θ^* expects to pay the same amount with the two securities, then when that bidder has a higher type $\theta > \theta^*$, it expects to pay strictly more with the steeper security than with the flatter security. That is, the payment of the steeper security is tied more tightly to the bidder's valuation.

We depart from existing theories on security-bid auctions to consider bidders who pay with securities from (possibly) different sets. We use $\mathcal{S}^i$ to denote the ordered set of securities used by bidder i and use $\underline{s}_i$ and $\overline{s}_i$ to denote the corresponding bounds on the security parameter. $\mathcal{S}^i$ could be determined exogenously prior to the auction. Alternatively, $\mathcal{S}^i$ could be specified prior to the auction by the seller, who—in light of the result that revenues are higher when any bidder uses a steeper set of ordered securities—would select for a given bidder the steepest set of feasible ordered securities (the feasible sets of securities may differ across bidders due to heterogeneities in the nature of bidders' moral hazard concerns or institutional rigidities; see footnote 3).

Bidders maximize long-run (i.e., after all information is revealed) expected profits. If the asset is not sold, all agents receive zero profit. If the asset is sold, then a losing bidder's profit is zero. When winner i pays the seller with security s, the seller's expected revenue is $ES^i(s,\theta_i)$, i.e., seller's expected profit is $ES^i(s,\theta_i) - V_T$, and the winner's expected profit is

$$\pi_i = V_T + \theta_i - ES^i(s,\theta_i), \tag{1}$$

where $V_T + \theta_i$ is the expected value of the asset under the bidder's control.

Our floating parameter auction differs from a standard fixed-parameter auction in how the winner i and the parameter s of the security paid are determined.

Fixed-parameter auctions. In a fixed-parameter auction, each bidder i submits a bid $s_i \in [\underline{s}_i, \bar{s}_i]$, which is a parameter from its ordered set $\mathcal{S}^i$. The seller commits to an evaluation rule for selecting the winner and for determining the parameter of the security that the winner pays. The evaluation rule is a function of all bids $\{s_i\}_{i=1}^n$, the sets of securities that bidders use $\{\mathcal{S}^i\}_{i=1}^n$, and it may also depend on any characteristics of bidders, $(X_i,\ F_i(\cdot),\ h_i(\cdot|\cdot))$, that are in the seller's information set.

When bidders have the same characteristics and pay with the same type of security, standard first- or second-price formats are sensible ways to conduct a fixed-parameter auction. For example, in a second-price auction, the highest s bid wins and the winner pays the second-highest s bid. In contrast, when bidders have different characteristics or use different securities, the literature (e.g., Hansen 1985; DKS) has long recognized that the evaluation rule must explicitly account for the heterogeneities, else security-bid auctions can generate even lower revenues than cash auctions. A seller must know the details of $X_i,\ F_i(\cdot),\ h_i(\cdot|\cdot)$ for all i, and its evaluation rule must discriminate—treating bidders differently—according to both these details and the security types used. The optimal design is sufficiently complicated that it is only known for equities, and no one has studied settings where bidders pay with different types of securities.

Floating-parameter auctions. In a *floating-parameter second-price auction* bidders submit *monetary* bids, the highest bid wins and the cash price p equals the second-highest bid, which is publicly announced. However, the winner pays this price not with cash, but rather with a security whose parameter is determined after the auction by a competitive capital market that incorporates all relevant public information.

We allow for cash reserve prices, $\{r_j\}_{j=1}^n$, set by the seller. Let β_i be the highest bid made. If $\beta_i \geq r_i$, then bidder i wins with a cash price of $p = \max\{\max_{j\neq i}\{\beta_j\}, r_i\}$. The asset is not sold if $\beta_i < r_i$. Ties are broken uniformly. After the auction, the cash price p becomes public information. The market forms beliefs about winner i's type θ_i based on p, i's characteristics (X_i, F_i, and $h_i(\cdot|\cdot)$), and the type of security used. Winner i pays with a security from $\mathcal{S}^i$. The parameter s of the security paid (e.g., the face value of a debt; the share of equity; the strike price of a call option) is such that the security's value, as determined by the capital market, equals the winning price:

$$E\left[ES^i(s, \theta_i)|i \text{ wins at price } p\right] = p, \tag{2}$$

where the left-hand side is the expected value of security s given the market's beliefs about the winner's type based on the winning price and the winner's identity.[8]

Discussion. In both standard and floating-parameter security-bid auctions, bidders pay with securities from a pre-specified set. The key difference is that bidders in standard security-bid auctions offer securities whose *parameters* are fixed, whereas bidders in our mechanism offer securities whose *cash values* are fixed (guaranteed). That is, bidders offer securities whose parameters float so as to deliver that cash value, as determined by the market. Thus, in contrast to fixed-parameter offers where the bid is an index s from an ordered set of securities, with floating-parameter offers, the bid is a cash amount, but the ultimate payment is a security whose index (e.g., equity share) is such that the post-auction market value of the security equals the winning cash price. Alternatively and equivalently, the bid could be a cash payment that the winner must fully finance by issuing securities from the specified set to the market.

The market's post-auction pricing of the winning firm's securities, which reflects the market's post-auction beliefs about the winner's type, can be used to determine the security index for the winner's payment. For example, suppose the ordered set of securities for bidder i (i.e., $\mathcal{S}^i$) is the set of equities, and the parameter s is the fraction of the combined firm. If θ^* is the market's post-auction belief about the expected synergy generated by bidder i when i wins at price p, then bidder i pays the seller share

$$s = \frac{p}{V_T + V_i + \theta^*} \tag{3}$$

of the joint firm. In a takeover with equity payments,[9] the process works as follows: Suppose prior to the takeover, bidder i has N shares outstanding; and that after i wins but before it pays the seller by issuing new shares, the capital market prices i's existing shares at

$$p_{post} = \frac{V_T + V_i + \theta^* - p}{N}. \tag{4}$$

Then bidder i issues new shares to target shareholders priced at p_{post} and the total value of new shares equals p. That is, i issues p/p_{post} new shares to pay target shareholders. Thus, after the payment, target shareholders hold a share $\frac{p/p_{post}}{N+p/p_{post}}$ of the

[8] For completeness of description, we assume that in the event that the left-hand side of (2) is less than p even when $s = \bar{s}_i$, then the solution to (2) is $s = \bar{s}_i$ (i.e., the winner pays the seller with $\bar{s}_i$). We prove that this will not happen; Proposition 1 shows that, in equilibrium, $s \in [\underline{s}_i, \bar{s}^i)$.

[9] The use of equity is common in takeover settings: Andrade, Mitchell, and Stafford (2001) report that 58% of mergers and acquisitions are paid entirely in equity, and 70% involve at least some equity.

joint firm. This share, by (4), equals $\frac{p}{V_T+V_i+\theta^*}$, which is identical to (3).

This floating-parameter offer design corresponds to "collars" in equity payments that are often used in takeovers. Officer (2004) finds that two-thirds of collars guarantee a dollar value if a bidder's stock price stays within specified bounds around the effective merger date, and the other third details a constant exchange ratio over a range of bidder stock prices, with adjustment outside the bounds. Floating-rate offers correspond to an infinite range for the first type of collar and a zero range for the second. In this regard, our analysis of floating parameter auctions helps provide theoretical foundations for takeovers with collars.

Floating-rate offers also correspond to settings where bidders bid with cash but finance the cash bids by issuing securities after the auction. In such settings, bidders bid more aggressively than in cash auctions in order to induce better post-auction financing terms—the financing terms at the time of bidding "float". For instance, with debt financing,[10] the winner pays a seller with cash, but finances the cash payment (the winning price) by issuing debt after the auction. In such a case, bidding strategies and expected seller revenue are exactly the same as in a floating-parameter auction in which the winner pays with debt whose value, as determined by the market post auction, equals the winning price. In practice, cash acquisitions often involve external financing with debt or equity issuance.[11] Our model also applies to *hybrid* settings, for example where some bidders pay with collared equities and others make cash bids that are subsequently financed with a debt issue, or even settings where some bidders pay with internal cash and others pay with securities. Our model's general nature reflects that floating-parameter designs convert security bids to cash bids, allowing security bids of any type, as well as cash bids, to be evaluated *on the same footing*.

4 Analysis of bidding with general securities

Denote bidder i's bidding strategy in the floating-parameter security auction by $\beta_i(\theta_i)$. We now impose mild conditions under which $\beta_i(\theta_i)$ is uniquely determined.

[10] An example is a leveraged buyout in which the acquirer pays the cash price by issuing bonds (see Shleifer and Vishny (1990) for the widespread use of leveraged buyouts in takeover waves). See footnote 3 for other scenarios where institutional rigidities lead to debt financing.

[11] Martynova and Renneboog (2009) report that one-third of such cash acquisitions are subsequently financed with securities, of which about 30% are financed with equity and 70% with debt.

Assumption 1: In equilibrium bidders do not play weakly dominated strategies.

As long as bidders do not play weakly dominated strategies, any equilibrium bidding strategy must be weakly increasing (for bids above the reserve):

Lemma 1 *Under Assumption 1, the bidding strategy in any equilibrium is weakly increasing: if* $\theta_i^a > \theta_i^b$ *and* $\beta_i(\theta_i^b) \geq r_i$*, then* $\beta_i(\theta_i^a) \geq \beta_i(\theta_i^b)$*.*

The market forms beliefs about the winner's type based on the winning price and winner identity. Without loss of generality, we denote the market's belief about winner i's expected synergy θ_i by a cumulative density function, $M(\theta_i; p, i)$, which has support over $\left[\underline{\theta}_i, \bar{\theta}_i\right]$. We impose weak restrictions on the market's beliefs. In our setting, all $p \in \left[\min_{\theta_i \in \left[\underline{\theta}, \bar{\theta}\right]} \beta_i(\theta_i), \max_{\theta_i \in \left[\underline{\theta}, \bar{\theta}\right]} \beta_i(\theta_i)\right]$ are on the equilibrium path, whereas any p outside that interval is off the equilibrium path. For any p on the equilibrium path, the market's belief must be consistent with the bidder's strategy. That is, Bayes' rule must hold: the market's belief about winner i's type θ_i is such that θ_i is in the set $\{\theta_i : \beta_i(\theta_i) \geq p\}$. By Lemma 1, the market's belief reduces to $\theta_i \geq \hat{\theta}_i(p)$, where $\hat{\theta}_i(p)$ is the minimum value of θ_i that satisfies $\beta_i(\theta_i) \geq p$. Thus, $M(\cdot; p, i)$ corresponds to the original distribution $F_i(\cdot)$ truncated at $\hat{\theta}_i(p)$.

By Lemma 1, for any $p_1 > p_2$ on the equilibrium path, $M(\cdot; p_1, i)$ first-order stochastically dominates $M(\cdot; p_2, i)$. We also require this property to hold for p off the equilibrium path.

Assumption 2: The market's belief about winner i's type weakly increases in the winning price: for any prices $p_1 > p_2$, $M(\cdot; p_1, i)$ weakly first-order stochastically dominates $M(\cdot; p_2, i)$.

Under Assumptions 1 and 2, bidding strategies are strictly increasing and the equilibrium is unique:[12]

Proposition 1 *Under Assumptions 1 and 2, bidding strategies are uniquely determined. Bidder i's bidding strategy is strictly increasing, given by*

$$\beta_i(\theta_i) = E_{\theta;i}[ES^i(s, \theta) | \theta \geq \theta_i], \tag{5}$$

[12]More precisely, bidding strategies are uniquely determined for bids above the reserve. Define θ_i^* to be $\beta_i(\theta_i^*) \equiv r_i$, where r_i is the cash reserve price for i. Then (5) applies to all $\theta_i \geq \theta_i^*$. A bid for $\theta_i < \theta_i^*$ will be below r_i so its exact value is inconsequential. Thus, we can regard (5) as applying to all θ_i.

where $E_{\theta;i}$ denotes the expectation over θ given $\theta \sim F_i$, and s solves

$$ES^i(s, \theta_i) = \theta_i + V_T, \tag{6}$$

which has a unique solution for $s \in [\underline{s}_i, \bar{s}^i)$. The market's belief about winner i's type, conditional on the winning price p, is that

$$\begin{cases} \theta_i = \bar{\theta}_i & \text{if} \quad p > \beta_i(\bar{\theta}_i) \\ \theta_i \geq \beta_i^{(-1)}(p) & \text{if} \quad p \in \left[\beta_i(\underline{\theta}_i), \beta_i\left(\bar{\theta}_i\right)\right] \\ \theta_i \geq \underline{\theta}_i & \text{if} \quad p < \beta_i(\underline{\theta}_i) \end{cases} . \tag{7}$$

Proof: See the appendix. □

To convey the intuition for Proposition 1, we provide a heuristic derivation of the optimal bidding strategy in the absence of reserve prices (a complete proof is in the appendix). Suppose bidder i wins at p. Then the index s of the security paid, as subsequently determined by the market, solves

$$E_{\theta;i}[ES^i(s, \theta) | i \text{ wins at } p] = p,$$

where $E_{\theta;i}$ is the expectation over bidder i's expected synergy given $F_i(\cdot)$. This is equivalent to

$$E_{\theta;i}[ES^i(s, \theta) | \beta_i(\theta) \geq p] = p. \tag{8}$$

Winner i's expected profit is given by (1), in which s solves (8). Standard reasoning yields that, as in any second-price auction, the optimal bid leaves a bidder indifferent between winning and paying that bid and losing. Consider the scenario in which bidder i with synergy θ_i wins at a price that equals his strategy $\beta_i(\theta_i)$ (i.e., the highest losing bid ties with i's winning bid of $\beta_i(\theta_i)$). Then the condition $\beta_i(\theta) \geq p$ in (8) becomes $\beta_i(\theta) \geq \beta_i(\theta_i)$, which reduces to $\theta \geq \theta_i$ (the strictly increasing strategy is invertible). Replacing p with $\beta_i(\theta_i)$ on the right-hand side of (8) yields (5), and the winner's indifference condition (setting (1) to zero) yields (6), which pins down the index s in (5).

Even though the winner pays with a security in the floating-parameter design, the seller's expected revenue is exactly the same as if the bidder paid the winning cash price. This reflects the law of iterated expectations and the fact that the market's expected value of the security payment equals the cash price. Thus, the expected revenue in the floating-parameter design is the same as in a *hypothetical* cash auction in which bidders bid according to Proposition 1. However, bidding in floating-parameter designs is more aggressive than in standard second-price pure-cash auctions—bidders bid more than their true values, as $E_{\theta;i}[ES^i(s,\theta)|\theta \geq \theta_i] \geq ES^i(s,\theta_i)$. Therefore, $\beta_i(\theta_i) \geq \theta_i + V_T$, and this inequality is strict for $\theta_i < \bar{\theta}_i$.

Intuitively, bidder i's strategy in a second-price auction is determined by outcomes at the point where i barely wins (i.e., where its bid equals the highest losing bid). At this point the market does not know that i is about to lose, inferring only that i is some type $\theta \geq \theta_i$. This means that if bidder i barely wins, it would profit from being grouped with higher types (i.e., the market overestimates i's type and hence assigns a lower security for it to pay), incentivizing i to bid above its true valuation. Note also that bidders employ weakly-dominant strategies in Proposition 1. This implies that a bidder does not need to know anything about other bidders or the securities they use.

To establish a benchmark we compare revenues from floating- and fixed-parameter auctions in a setting with ex-ante identical bidders who use the same set of securities.

Definition 1 *(heterogeneity) Bidders are ex-ante identical if and only if they have the same standalone value, distribution of expected synergy, distribution of future cash flows conditional on the expected synergy, and use the same set of securities:* $X_i = X_j$, $F_i = F_j$, $h_i(\cdot|\theta) = h_j(\cdot|\theta)$, *and* $\mathcal{S}^i = \mathcal{S}^j$ *for all* i, j, *and* θ. *Bidders are heterogeneous if they are not ex-ante identical.*

Proposition 2 *With ex-ante identical bidders and a common ordered set of securities, floating- and fixed-parameter second-price auctions yield the same expected revenue.*

Proof: Abusing notation, let θ_1 and θ_2 denote the highest and second highest signals, respectively. Because bidders are ex-ante identical, we suppress the superscript i in ES and subscript i in $E_{\theta;i}$. In standard fixed-parameter auctions, the bid "s^{fix}" by bidder 2 satisfies (6) with $i = 2$. The expected value of the winner's payment is $ES(s^{fix}, \theta_1)$. Hence, conditional on the second-highest signal being θ_2, the expected revenue is $E_\theta[ES(s^{flt}, \theta)|\theta \geq \theta_2]$. In the floating-parameter auction, conditional on the second-highest signal being θ_2, the expected revenue is the cash bid of bidder 2. By Proposition 1, this cash bid of bidder 2 is $E_\theta[ES(s^{flt}, \theta)|\theta \geq \theta_2]$, where s^{flt} satisfies the same (6) with $i = 2$. By the law of iterated expectation, the result follows. $\square$

The proposition establishes a revenue equivalence between second-price fixed- and floating-parameter auctions when bidders are ex-ante identical. One can also prove an analogous result for first-price auctions: With ex-ante identical bidders and any common ordered set of securities, floating- and fixed-parameter first-price auctions generate the same expected revenue.[13]

The proof of Proposition 2 reveals how and where this revenue equivalence breaks down when bidders are heterogeneous: the proof's logic uses the fact that with ex-ante identical bidders, (i) the winner is the same (highest-signal bidder) in both fixed- and floating-parameter formats, and (ii) the expectations of the synergies $E_{\theta;i}$ and security values ES^i are independent of the bidder's identity i. With heterogeneous bidders, these features no longer hold, so revenue equivalence breaks down.

This breakdown turns out to be desirable. The literature has long recognized that fixed-parameter auction designs must incorporate discriminatory adjustments when bidders are ex-ante heterogeneous, else they could generate even lower revenues than cash auctions (see e.g., Hansen 1985; DKS). This contrasts with our nondiscriminatory floating-parameter design, which, as we will show, always generates higher revenues than cash auctions, regardless of how bidders differ ex ante.

To benchmark comparisons with our nondiscriminatory floating-parameter design, we first provide sufficient conditions under which nondiscriminatory second-price fixed-parameter auctions (henceforth "nondiscriminatory fixed-parameter auctions") generate low revenues. For general securities, a security's value depends on the distribution of cash flows conditional on a bidder type, $h_i(\cdot|\theta_i)$. To ease presentation, we assume that the cash flow distribution scales with the total valuation:

Assumption 3: For each bidder i, the final cash flow is given by $\tau(\theta_i + X_i + V_T)$, where τ is a random variable over $(0, \infty)$ with a mean of 1.

Proposition 3 *Consider two bidders 1 and 2 with identical synergy distributions on support $[\underline{\theta}, \overline{\theta}]$. Suppose the difference in bidder standalone values is large enough that*

$$X_2 - X_1 > \max\left\{\frac{\bar{\theta} - \underline{\theta}}{\underline{\theta} + V_T} X_1, \frac{E[\theta] - \underline{\theta}}{\underline{\theta} + V_T}(X_1 + E[\theta] + V_T)\right\}.$$

Then, under Assumption 3, for nondiscriminatory fixed-parameter auctions,

[13] Proof available upon request. For the first-price format, as with the second-price format, we assume that the winning price—the highest bid with a first-price format—is publicly announced.

(i) equity generates less revenues than cash.

(ii) steeper-than-equity securities generate less revenue than equities, and the revenue strictly decreases as securities grow even steeper.

Proof: See the appendix. $\square$

In contrast to Proposition 3, we now establish that floating-parameter auctions are robust to bidder heterogeneities even when bidders differ arbitrarily in their characteristics and use different types of securities. We first show that when any bidder i uses a steeper set of securities, its cash bids are more aggressive, reflecting that steeper securities extract relatively more revenues from types with higher valuations than bidder i.

Proposition 4 *Suppose the ordered set of securities $\mathcal{S}_A^i$ is steeper than $\mathcal{S}_B^i$ for bidder i. Then i bids more aggressively in floating-parameter auctions with $\mathcal{S}_A^i$ than with $\mathcal{S}_B^i$:*

$$\beta_{(A)i}(\theta_i) \geq \beta_{(B)i}(\theta_i),$$

for all θ_i, with strict inequality for all $\theta_i < \bar{\theta}_i$.

Proof. Proposition 1 yields that (i)

$$\beta_{(A)i}(\theta_i) = E_{\theta;i}[ES^i(s_A, \theta)|\theta \geq \theta_i], \tag{9}$$

where s_A solves

$$ES^i(s_A, \theta_i) = \theta_i + V_T, \tag{10}$$

and (ii)

$$\beta_{(B)i}(\theta_i) = E_{\theta;i}[ES^i(s_B, \theta)|\theta \geq \theta_i], \tag{11}$$

where s_B solves

$$ES^i(s_B, \theta_i) = \theta_i + V_T. \tag{12}$$

Here s_A and s_B denote the corresponding security in sets $\mathcal{S}_A^i$ and $\mathcal{S}_B^i$. By (10) and (12),

$$ES_A(s_A, \theta_i) = ES_B(s_B, \theta_i). \tag{13}$$

By the property of steeper securities, a bidder who expects to pay the same amount with a steeper security as with a flatter security for a given private valuation expects to pay strictly more with the steeper security than the flatter security if its private valuation is higher. Thus, (13) yields $ES_A(s_A, \theta) > ES_B(s_B, \theta)$ for all $\theta > \theta_i$, and hence

$$E_{\theta;i}\left[ES_A(s_A, \theta)|\theta \geq \theta_i\right] \geq E_{\theta;i}\left[ES_B(s_B, \theta)|\theta \geq \theta_i\right],$$

where the inequality is strict for $\theta_i < \bar{\theta}_i$. Noting that reserve prices do not affect bidding strategies, it follows that the proposition holds for any given reserve prices. $\square$

Definition 2 *(profile of security sets) Let $\{\mathcal{S}^i\}_{i=1}^n$ denote the profile of ordered sets of securities used by the n bidders. We say profile A is steeper than B if $\mathcal{S}_A^j$ is weakly steeper than $\mathcal{S}_B^j$ for all $j \in \{1, ..., n\}$, with strict inequality for at least one j.*

Thus, the profile of the security sets is steeper if at least one bidder switches to using a steeper set of securities, while all other bidders use the same set of securities as before. We now derive our central characterization result, showing how the steepness of the (possibly heterogeneous) sets of securities used by bidders affects seller revenues.

Theorem 1 *Suppose the profile of ordered sets of securities A is steeper than B. Then, regardless of how bidders differ in their characteristics or the types of securities used, expected revenue is higher with A than with B given any fixed set of reserve prices $\{r_j\}_{j=1}^n$ (or no reserves).*

Proof. The proof of Proposition 4 shows that for any set of reserve prices (or no reserves), when the profile of security sets is steeper, some bidders place more aggressive cash bids and the cash bids of all other bidders are weakly higher. Because the expected value of the security payment equals the winning cash price, the theorem follows by the law of iterated expectations. $\square$

Corollary 1 *Under the conditions of Proposition 3, nondiscriminatory floating-parameter auctions deliver higher revenues than nondiscriminatory fixed-parameter auctions.*

The corollary is immediate since seller revenues in floating-parameter auctions always exceed those in cash auctions.

Since Theorem 1 holds when the reserve prices are optimal for security profile B, the revenue dominance result is only reinforced if a seller uses the optimal reserves for A. Importantly, while an auctioneer needs to know bidder attributes to set optimal reserves, Theorem 1 does not require the auctioneer to know how to set optimal reserves: the revenue ranking holds for *any* set of reserve prices. Thus, an auctioneer can ensure that steeper securities generate higher expected revenues just by setting the same reserves as with less-steep securities or cash auctions.

The performance of security auctions is more sensitive to bidder heterogeneity than is the performance of cash auctions, and the literature has made little progress in identifying optimal designs, much less their revenue properties, for general sets of securities, especially when bidders pay with different types of securities. In this regard, our floating-parameter design suggests a *simple yet sensible* way to conduct security-bid auctions with heterogeneous bidders. As Theorem 1 shows, the insights in DKS that steeper securities generate greater revenues extend from their homogeneous bidder setting to our floating-parameter auction design with arbitrarily heterogeneous bidders.

Our floating-parameter mechanism is nondiscriminatory and detail-free. The intuition for the revenue superiority and simple design reflects that floating-parameter offers have (1) cash-like properties that allow for simple auction rules, but (2) unlike cash payments, these cash-equivalent payments induce signaling incentives. Steeper securities increase signaling incentives, leading to more aggressive cash bids. More aggressive bidding *by any bidder* always benefits a seller in floating-parameter auctions—no matter how bidders differ, more cash is always good—but not in fixed-parameter auctions where winner selection depends only on the face values of bids, and the winner could be an incentivized (small) bidder with a low valuation—possibly leading to significantly sub-optimal allocations that sharply reduce revenue.

The auction design shifts information acquisition burdens from the auctioneer to the market, exploiting the market's inferences about bidders' types based on the observed bidding process. Indeed, because information acquisition occurs post-auction, the market only needs to gather information about the winning bidder, as the attributes of other bidders are not relevant for assessing the winner's type.

We next numerically illustrate Theorem 1 and its implications.

Example 1: The target's market value is 3. There are two bidders, 1 and 2, with market values X_1 and X_2, that can create synergies from a merger. Each bidder i's expected synergy, θ_i, is drawn from a uniform distribution on $[1, 2]$. The cash flow distribution conditional on θ_i is the same as in the lead example from DKS: the final cash flow is distributed according to $\tau_i (\theta_i + X_i + V_T)$, where τ_i is a lognormal distribution with a mean of 1 and standard deviation of 0.5. The second-price auction has no reserve price, and bidders use the same type of securities. We examine four types of securities (cash, debt, equity, and call) for different combinations of X_1 and X_2.

Table 1: Floating Parameter Auction Revenues When Bidders Use the Same Security

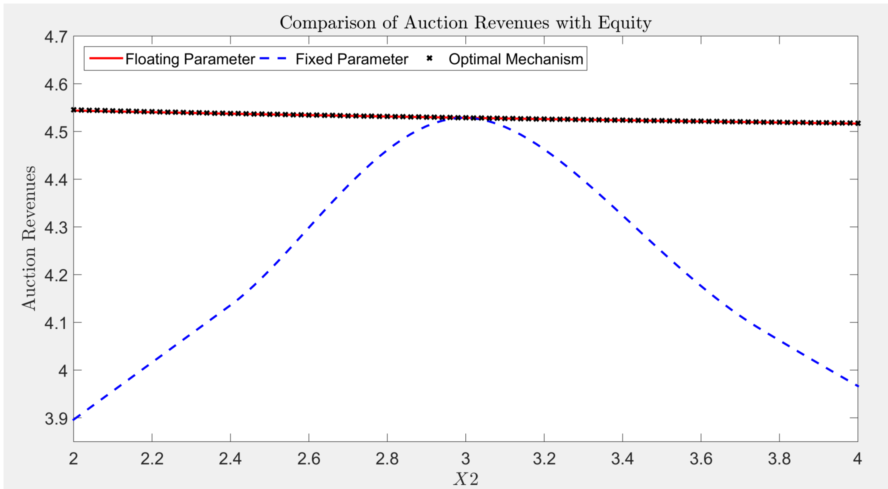
Market Capitalization	Cash	Debt	Equity	Call
$X_1 = 3, X_2 = 3$	4.33	4.36	4.53	4.66
$X_1 = 3, X_2 = 4$	4.33	4.36	4.52	4.66
$X_1 = 3, X_2 = 6$	4.33	4.35	4.50	4.65
$X_1 = 6, X_2 = 6$	4.33	4.34	4.47	4.63

Table 1 illustrates that, consistent with Theorem 1, with a floating-parameter design, steeper securities always generate more revenues, even with heterogeneous bidders. Thus, call options generate the highest revenues, and cash generates the lowest.

Table 1 also illustrates that given any common security used by bidders, the floating-parameter design exhibits the property that revenues decrease monotonically and slowly in X_i for any single bidder i. This monotonicity property *with respect to a single bidder* implies that revenues with $X_1 \neq X_2$ *always exceed* those in a homogeneous setting when X_1 and X_2 are replaced by $\max(X_1, X_2)$. We will prove that this property holds in more general setting where bidders use different securities.

Using a fine grid, Figure 1 illustrates the revenue advantages of floating-parameter designs when bidders use equities and have different standalone values. In the setting of Example 1, we fix bidder 1's market capitalization at 3 and vary bidder 2's market capitalization from 2 to 4. Even with no reserve, revenues from floating-parameter equity auctions decline very slowly as bidder 2's market capitalization rises. In contrast, revenues from the analogous nondiscriminatory fixed-parameter design are sensitive to differences in bidders' market capitalizations, and are notably lower than those from the floating-parameter design once market capitalizations differ by more than tiny amounts. Figure 1 also plots payoffs from the optimal equity auction, showing that revenue losses from using the floating-parameter auction rather than the optimal design are negligible, almost visually indistinguishable from zero.

Figure 1: Revenue Comparisons of Floating, Fixed and Optimal Equity Auctions



Notes: This figure plots expected revenues in floating-parameter, nondiscriminatory fixed-parameter, and optimal equity auctions as a function of bidder 2's market value. The target's market value is 3, as is bidder 1's market value. Each bidder's expected synergy is drawn from a uniform distribution on $[1, 2]$.

In both floating- and fixed-parameter designs when bidders have the same standalone value, seller revenues always rise if we reduce this value uniformly. However, sharp differences emerge if only one bidder's standalone value is reduced: Proposition 3 shows that with a nondiscriminatory fixed-parameter design, fixing one bidder's standalone value and reducing the other bidder's standalone value sufficiently, can cause revenues to fall even below those in cash auctions. In contrast, for standard securities in nondiscriminatory floating-parameter auctions, decreasing *any* single bidder's standalone value *always increases* seller revenues:

Proposition 5 *Suppose a bidder i uses equity, call options or debt securities. Then, under Assumption 3, reductions in i's standalone value increase i's bidding strategy, which strictly increases expected seller revenue in a floating-parameter auction.*

Proof: See the appendix.

A bidder with a smaller standalone value pays a larger fraction of its cash flows upon winning. In turn, this raises the value of being perceived by the market as a higher type, increasing the bidder's signaling incentives, and hence its cash bid.

We next illustrate how the steepness of securities affects revenues in floating-parameter auctions when different bidders may use different types of securities.

Example 2: The target's market value is 3. The bidders, 1 and 2, have market values $X_1 = 3$ and $X_2 = 6$. The expected synergy θ_i that a bidder i can create is drawn from a uniform distribution on $[1, 2]$. Conditional on θ_i, the final cash flow is distributed according to $\tau_i(\theta_i + X_i + V_T)$, where τ_i is a lognormal distribution with a mean of 1 and standard deviation of 0.5. We consider seller revenues when bidders use different combinations of cash, debt, equity and call securities.

Table 2: Floating-Parameter Auction Revenues when Bidders Use Different Securities

Securities Used	Expected Revenues	Securities Used	Expected Revenues
($\mathcal{S}_1$=Cash, $\mathcal{S}_2$=Cash)	4.333		
($\mathcal{S}_1$=Cash, $\mathcal{S}_2$=Debt)	4.336	($\mathcal{S}_1$=Debt, $\mathcal{S}_2$=Cash)	4.348
($\mathcal{S}_1$=Debt, $\mathcal{S}_2$=Debt)	4.351		
($\mathcal{S}_1$=Equity, $\mathcal{S}_2$=Debt)	4.42	($\mathcal{S}_1$=Debt, $\mathcal{S}_2$=Equity)	4.41
($\mathcal{S}_1$=Equity, $\mathcal{S}_2$=Equity)	4.50		
($\mathcal{S}_1$=Equity, $\mathcal{S}_2$=Call)	4.58	($\mathcal{S}_1$=Call, $\mathcal{S}_2$=Equity)	4.55
($\mathcal{S}_1$=Call, $\mathcal{S}_2$=Call)	4.65		

Notes: Market capitalizations of the two bidders are $(X_1 = 3, X_2 = 6)$.

Table 2 illustrates the result in Theorem 1 that even when bidders differ ex ante, seller revenues are always higher when (at least) one bidder switches to using a steeper set of securities, regardless of which bidder switches. More subtly, while revenue always increases when a bidder switches to steeper securities, which bidder switches matters for the size of the increase: whether revenues are higher when the smaller bidder uses the steeper security hinges on the steepness of the securities. When switching from cash to debt, revenues are higher if the smaller bidder switches due to the greater reduction in its informational advantage (i.e., a bidder's standalone value affects its informational advantage when it uses debt, but not when it uses cash); in contrast when switching from equity to call options, revenues are higher if the larger bidder

2 switches, because bidder 1's informational advantage with equity is already small (due to its smaller standalone value and the feature that equities are not too flat).

Collectively, the results in Theorem 1, Proposition 5 and the examples highlight the robustness of floating-parameter designs to bidder heterogeneity: revenues exhibit a monotonicity property *with respect to any single bidder both in terms of the bidder's standalone value and the type of security used*. Thus, revenues when bidders have different standalone values and use different types of securities *always exceed* those in a homogeneous setting in which bidders have the same largest standalone value and use the same flattest set of securities.

Theorem 1 establishes that when a seller can select the type of security that each bidder pays, seller revenue is maximized when a bidder uses the steepest set that is feasible for that bidder. But what about the converse? Suppose that prior to receiving information each bidder can commit to the type of securities that it uses: will a bidder always want to use the flattest set that is feasible for itself? Existing studies of fixed-parameter securities auctions with ex-ante identical bidders can give a partial answer by imposing the constraint that all bidders use the same set of securities. Specifically, when bidders collectively decide on the same set of securities, they are best off with the flattest set that is feasible. This follows directly: flatter securities reduce seller profit, and a seller's loss translates to the collective gain of bidders, which, with ex-ante identical bidders who use the same set of securities, is equally divided among bidders.

Now consider floating-parameter designs with heterogeneous bidders who make choices individually and may use different types of securities. The question becomes: given the security sets of other bidders, would a bidder always gain from unilaterally switching to flatter securities? The answer is less direct, as there are opposing forces: with a flatter set, a bidder's expected profit if it wins is higher, but the bidder is less likely to win. Nonetheless, we find that on balance, a bidder benefits from a flatter set:

Proposition 6 *Consider any bidder i, and fix the (possibly different) ordered sets of securities of the other bidders. Suppose that $\mathcal{S}_A^i$ is steeper than $\mathcal{S}_B^i$ and that both sets are feasible for bidder i. Then for any given set of reserve prices $\{r_j\}_{j=1}^n$, bidder i's expected profit in the floating-parameter auction with $\mathcal{S}_B^i$ exceeds that with $\mathcal{S}_A^i$.*

Proof: See the appendix. $\square$

We next specialize to equity securities. We use of the fact that the optimal (revenue-maximizing) equity mechanism that adjusts for bidder heterogeneities is known (Liu 2016) to identify conditions under which expected revenues in second-price floating-parameter equity auctions approach the theoretical optimum.

5 Equity Auctions

We compare revenues from optimal and floating-parameter equity auctions when bidders share the same distribution of synergies but differ in standalone values.

We first derive the bidding strategy for the floating-parameter format with equity securities. With equity payments, only expected cash flows matter. Let s_i denote the equity share of the joint firm that bidder i pays. Then

$$ES^i(s_i, \theta_i) = s_i(V_T + X_i + \theta_i).$$

Substituting this into (6) yields

$$s_i = \frac{V_T + \theta_i}{V_T + X_i + \theta_i}.$$

Then Proposition 1 yields

$$\beta_i(\theta_i) = (V_T + \theta_i)\,\frac{V_T + X_i + E_{\theta;i}[\theta|\theta \geq \theta_i]}{V_T + X_i + \theta_i}. \tag{14}$$

Now consider the optimal equity mechanism. Its construction depends on the virtual valuation of each bidder:

$$\phi_i(\theta_i) \equiv \theta_i - \frac{X_i\,(1 - F_i(\theta_i))}{(X_i + \theta_i + V_T)\,f_i(\theta_i)} - \frac{X_i \int_{\underline{\theta}_i}^{\bar{\theta}_i} (1 - F_i(t))\,dt}{(X_i + \theta_i + V_T)^2\,f_i(\theta_i)}, \tag{15}$$

where $f_i(\cdot)$ is the pdf of $F_i(\cdot)$. The virtual valuation (15) represents the rent in terms of seller profit—revenue minus the seller's standalone value V_T—that the seller can extract from a bidder who pays with equities.[14]

Under the standard regularity condition that $\phi_i(\cdot)$ strictly increases over $\left[\underline{\theta}_i, \bar{\theta}_i\right]$ for all i, it is optimal to select the bidder with the highest virtual valuation as

[14] Equation (15) is equation (19) of Liu (2016) minus V_T. This subtraction eases exposition by effectively changing the seller's reservation value from V_T to 0.

the winner, as long as this valuation is nonnegative: the winner i is such that $\phi_i(\theta_i) \geq \max_{j \neq i} \{\phi_j(\theta_j)\}$ and $\phi_i(\theta_i) \geq 0$. The equity share s_i that the winner i pays is

$$s_i = 1 - \frac{X_i}{X_i + \phi_i^{-1}(\max\{\max_{j \neq i}\{\phi_j(\theta_j)\}, 0\})}. \tag{16}$$

$\phi_i^{-1}(\max\{\max_{j \neq i}\{\phi_j(\theta_j)\}, 0\})$ is the smallest value of θ_i that corresponds to a non-negative virtual valuation that lets i win against bidders with signals $(\theta_1 ... \theta_{i-1,} \theta_{i+1} ... \theta_n)$: $\phi_i^{-1}(\cdot)$ is the truncated inverse of $\phi_i(\cdot)$ with $\phi_i^{-1}(x) \equiv \underline{\theta}_i$ if $x < \phi_i(\underline{\theta}_i)$ and $\phi_i^{-1}(x) \equiv \bar{\theta}_i$ if $x > \phi_i(\bar{\theta}_i)$.

The construction of the optimal equity mechanism underscores the undesirable features highlighted earlier about general security auctions that generate high revenues in the presence of bidder heterogeneities: the optimal rules are complicated, bidder-specific, and informationally demanding for an auctioneer to determine. There are three bidder-specific aspects of the optimal equity mechanism. First, the optimal reserves are heterogeneous when $\phi_i^{-1}(0)$ differs across i, and ϕ_i is especially sensitive to X_i. Second, when multiple bidder types are above their respective reserves, the optimal winner selection rule is discriminatory in nature. In particular, when the distribution of synergies is the same for all bidders, the optimal mechanism favors bidders with lower standalone values, i.e., ϕ_i in equation (15) typically decreases in X_i. This reflects that a bidder's informational advantage in an equity auction grows with its standalone value,[15] making it optimal for a seller to reward smaller bidders that the seller can better exploit. Third, determination of the winner's payment (equation 16) is also discriminatory in nature as it depends on the bid and identity of the highest losing bidder. These features underscore that with *any* bidder heterogeneity, the optimal fixed-parameter mechanism ceases to be a standard nondiscriminatory auction.

We now compare seller profits in optimal equity and floating-parameter equity auctions. With heterogeneous bidders, fixed-parameter designs require discriminatory reserves. To facilitate comparisons, we impose a reserve price for floating parameter

[15] Bidders derive informational advantages from their private information about their synergies. The impacts of synergies on a winner's payoffs differ in cash and equity auctions. At the margin, a bidder in a cash auction is the residual claimant: for a given bid, a \$1 increase in the synergy translates to a \$1 increase in the winning payoff. By contrast, for a given equity bid, the benefit of an increase in the synergy is shared in proportion to the fraction of equity surrendered. Thus, a bidder's informational advantage in an equity auction is reduced and is scaled by the equity stake the bidder would retain upon winning. When bidders have different standalone values, lower-standalone-value bidders have smaller informational advantages because they retain smaller equity stakes.

auctions, using a common reserve to preserve the auction's non-discriminatory nature:

Definition 3 *With a floating-parameter equity auction design, we impose the following common reserve price for all bidders:*

$$r^{flt} = \max_i \left\{ \beta_i(\max\left\{ \phi_i^{-1}(0), \underline{\theta}_i \right\}) \right\} + V_T. \tag{17}$$

With this common reserve,[16] the auction design remains simple and nondiscriminatory in its reserve, winner selection, and determination of the winner's payment. We next prove that with uniformly distributed synergies, seller profits with floating-parameter offers and common reserve price r^{flt} are identical to those in the optimal mechanism both when the standalone values of bidders and the seller are far larger than the extent of synergies (as Betton et al. (2008) find is the norm in takeovers), or far smaller. In these two limiting scenarios, virtual valuations and bidding strategies in floating-parameter auctions take simpler forms, facilitating comparisons.

We use $\pi_s^{opt;equity}$, $\pi_s^{opt;cash}$ and π_s^{flt} respectively to denote a seller's expected *profit* under (i) the optimal equity mechanism, (ii) the optimal cash mechanism, and (iii) floating-parameter offers with common reserve price r^{flt}. Note that when V_T goes to infinity, expected profit (i.e., expected revenue minus V_T) remains bounded (it cannot exceed $\bar{\theta}$, the highest possible NPV). Note also that $\pi_s^{opt,cash}$ is independent of k since the standalone values of bidders and seller do not affect seller profit in cash auctions.

Proposition 7 *Suppose that the synergies of all bidders are drawn from a uniform distribution over* $\left[\underline{\theta}, \bar{\theta}\right]$. *Let* $V_T = kV_T^*$ *and* $X_i = kX_i^*$ $(i = 1, ..., n)$, *where* k *is a scaling factor and* $V_T^* > 0$ *and* $X_i^* > 0$ $(i = 1, ..., n)$ *are constants. Then*

$$\lim_{k\to\infty} \pi_s^{flt} = \lim_{k\to\infty} \pi_s^{opt,equity} \tag{18}$$

$$> \pi_s^{opt,cash} \tag{19}$$

[16] None of our earlier results requires a seller to know anything about bidders. In this section, when we compare revenues with the optimal mechanism to determine the optimal *common* reserve for the floating parameter design, a seller only needs to know who has the highest value of $\beta_i(\max\left\{\phi_i^{-1}(0), \underline{\theta}_i\right\})$ and what that value is.

and

$$\lim_{k\to 0}\pi_s^{flt} = \lim_{k\to 0}\pi_s^{opt,equity} \tag{20}$$
$$> \pi_s^{opt,cash}. \tag{21}$$

Proof: See the appendix. $\square$

When synergies are drawn from uniform distributions, the virtual valuation in equation (15) becomes

$$\phi_i(\theta_i) = \theta_i - \frac{X_i}{(V_T + X_i + \theta_i)}\left(\bar{\theta} - \theta_i\right) - \frac{1}{2}\frac{X_i}{(V_T + X_i + \theta_i)^2}\left(\bar{\theta} - \theta_i\right)^2, \tag{22}$$

and in the floating-parameter equity auction, strategies (equation 14) become

$$\beta_i(\theta_i) = V_T + \frac{1}{2}\bar{\theta} + \frac{1}{2}\left[\theta_i - \frac{X_i}{V_T + X_i + \theta_i}\left(\bar{\theta} - \theta_i\right)\right]. \tag{23}$$

First, consider the case where k goes to infinity. Defining

$$\Delta_i(\theta_i) \equiv \theta_i - \frac{X_i}{V_T + X_i + \theta_i}\left(\bar{\theta} - \theta_i\right), \tag{24}$$

equation (23) becomes $\beta_i(\theta_i) = V_T + \frac{1}{2}\bar{\theta} + \frac{1}{2}\Delta(\theta_i)$. This implies that $\beta_i(\theta_i)$ and $\Delta(\theta_i)$ have the same ordering: $\beta_i(\theta_i) > \beta_j(\theta_j)$ if and only if $\Delta(\theta_i) > \Delta(\theta_j)$.

Comparing equations (24) and (22) shows that $\Delta_i(\theta_i)$ is precisely the first two terms in $\phi_i(\theta_i)$. Moreover, the third term in $\phi_i(\theta_i)$ is typically far smaller than the second term, so that $\Delta_i(\theta_i)$ and $\phi_i(\theta_i)$ tend to be very close. Under the condition of Proposition 7 that standalone values are far larger than synergies, the third term in the virtual valuation goes to zero, so that $\Delta_i(\theta_i)$ equals $\phi_i(\theta_i)$. Thus, floating-parameter and optimal mechanisms select the same winner, leading to identical allocations.

Now consider the case where k goes to zero. The virtual valuation (equation (22)) becomes

$$\phi_i(\theta_i) = \theta_i. \tag{25}$$

The bidding strategy in floating-parameter auction becomes $\beta_i(\theta_i) = V_T + \frac{1}{2}\bar{\theta} + \frac{1}{2}\theta_i = V_T + \frac{1}{2}\bar{\theta} + \frac{1}{2}\phi_i(\theta_i)$. Thus, $\beta_i(\theta_i)$ and $\phi_i(\theta_i)$ have the same ordering: floating-parameter and optimal mechanisms select the same winner, leading to identical allocations.

We can similarly see why a common reserve in floating-parameter auctions achieves optimality. In the optimal equity mechanism, the reserve types for bidders i and j are such that $\phi_i(\theta_i^*) = \phi_j(\theta_j^*) = 0$. Because virtual valuations depend on standalone values, when $X_i \neq X_j$, we have $\theta_i^* \neq \theta_j^*$. In particular, in Proposition 7 when k goes to infinity (standalone values are large relative to possible synergies), the virtual valuation in (22) reduces to

$$\phi_i(\theta_i) = \theta_i - \frac{X_i^*}{V_T^* + X_i^*} \left(\bar{\theta} - \theta_i\right), \tag{26}$$

which decreases in X_i^*. Thus, if $X_i^* < X_j^*$, then $\phi_i(\theta_i^*) = \phi_j(\theta_j^*) = 0$ implies $\theta_i^* < \theta_j^*$ (i.e., allocations favor smaller bidders). It follows that when stand-alone values differ across i, the optimal mechanism features discriminatory reserves. Now consider floating-parameter auctions. When k grows arbitrarily large, $\phi_i(\theta_i^*) = \phi_j(\theta_j^*) = 0$ leads to $\Delta(\theta_i^*) = \Delta(\theta_j^*)$ and hence $\beta_i(\theta_i^*) = \beta_j(\theta_j^*)$. Thus, a common reserve price of $\beta_i(\theta_i^*)$ (which equals $\beta_j(\theta_j^*)$) in floating-parameter auctions selects the same set of *heterogeneous* reserve types (where $\theta_i^* \neq \theta_j^*$) as the optimal mechanism. Similarly, when k goes to 0, $\phi_i(\theta_i^*) = \phi_j(\theta_j^*) = 0$ leads to $\beta_i(\theta_i^*) = \beta_j(\theta_j^*)$, so a common reserve price of $\beta_i(\theta_i^*)$ again implements the reserve in the optimal mechanism.

To understand (19) and (21), note that the virtual valuation for cash auctions (Myerson (1981)) is

$$\phi_i^{cash}(\theta_i) = \theta_i - (\bar{\theta} - \theta_i),$$

which is independent of k. Direct comparison with (26) and (25) shows that $\phi_i^{cash}(\theta_i)$ is strictly less than the virtual valuation for equities (for all $\theta_i < \bar{\theta}$).[17] Thus, the optimal equity mechanism generates strictly higher seller profits than the optimal cash mechanism, regardless of how bidders differ ex ante. As floating-parameter auctions generate the same revenues as the optimal equity mechanism when k is arbitrarily large or arbitrarily small, inequalities (19) and (21) follow.

To illustrate Proposition 7 for the case where k goes to infinity, we consider a two-bidder example in which synergies are uniformly distributed on $[0, 1]$. The optimal

[17] Note that if we were to have $V_T^* = 0$ (ruled out by the premise in Proposition 7), then the virtual valuation for equity auctions, (26), becomes $2\theta_i - \bar{\theta}$, which is same as the virtual valuation for cash auctions. Hence we would have $\lim_{k\to\infty} \pi_s^{opt,equity} = \pi_s^{opt,cash}$. Intuitively, a bidder's informational advantage in equity auctions rises as its standalone value increases; in the limit where a bidder's standalone value is arbitrarily larger than the seller's standalone value and the extent of synergies, a bidder's informational advantages approach those in cash auctions.

cash mechanism features a reserve of $\frac{1}{2}+V_T$, and seller profit is $\frac{5}{12}$. Now consider equity mechanisms with $V_T^*=X_1^*=2X_2^*$. With large k, (26) yields $\phi_1(\theta_1)=\frac{3}{2}\theta_1-\frac{1}{2}$ and $\phi_2(\theta_2)=\frac{4}{3}\theta_2-\frac{1}{3}$. Hence, in the optimal equity mechanism, the reserve types, which solve $\phi_1(\theta_1^*)=\phi_2(\theta_2^*)=0$, are $\theta_1^*=\frac{1}{3}$ and $\theta_2^*=\frac{1}{4}$. Here $\theta_1^*\neq\theta_2^*$ reflects the discriminatory nature of the optimal equity mechanism. Seller profit is $\mathrm{E}[\max\{\phi_1(\theta_1),\phi_2(\theta_2),0\}]=\frac{13}{24}$. Next consider the floating-parameter auction, where the bidding strategy is given by (23). Taking the limit as k goes to infinity yields $\beta_1(\theta_1^*)=\beta_2(\theta_2^*)$ (i.e., $\beta_1(1/3)=\beta_2(1/4)$), which equals $V_T+\frac{1}{2}$. Direct calculation shows that with a uniform reserve price of this value ($V_T+\frac{1}{2}$), the floating-parameter auction generates a seller profit of $\frac{13}{24}$, just as with the optimal equity mechanism, which exceeds the profit of $\frac{5}{12}$ in optimal cash auctions by thirty percent.

It follows that the ratio $\frac{\pi_s^{opt,eqity}-\pi_s^{flt}}{\pi_s^{opt,equity}-\pi_s^{opt,cash}}$, which measures how close revenues in floating-parameter auctions are to those in the optimal equity mechanism, goes to zero both when k goes to infinity and when k goes to zero. Unreported numerical calculations suggest that for intermediate values of k, differences in revenues between floating-parameter and optimal mechanisms remain negligible. Even when valuations are not large relative to synergies, the common distribution of synergies is not uniform, and bidders' standalone values differ by several multiples, the ratio $\frac{\pi_s^{opt,eqity}-\pi_s^{flt}}{\pi_s^{opt,equity}-\pi_s^{opt,cash}}$ is tiny (less than 10^{-3} for very large bidder differences).

We next establish that the results in Proposition 7 extend quite generally: when the standalone values of bidders go to zero so that their standalone values are much smaller than the sum of the seller's standalone value and expected synergies—e.g., bidders could be private equity firms that are far smaller than a target and pay the cash price by issuing debt—then floating parameter auctions yield the same revenues as the optimal auction, even if (1) the rates at which standalone values go to zero vary across bidders, (2) the common distribution for expected synergies is arbitrary (not necessarily uniform), (3) bidders offer general securities (not necessarily equity), or (4) different bidders use different types of securities.

Proposition 8 *Suppose that the expected synergies of all bidders are drawn from the same distribution $F(\cdot)$, conditional cash flow distributions follow Assumption 3, and bidders offer general securities that can differ across bidders. Then, when X_i goes to* *zero for all i, possibly at different rates, the following holds:*

$$\lim_{X_i \to 0 \text{ for all } i} \pi_s^{flt} = \lim_{X_i \to 0 \text{ for all } i} \pi_s^{opt,fixed} \tag{27}$$

$$> \pi_s^{opt,cash}, \tag{28}$$

where $\pi_s^{opt,fixed}$ is seller profit in an optimal fixed-parameter security auction.

To prove this, we show that under the proposition's premise, floating-parameter auctions deliver first-best seller profit (full rent extraction). We first show that for any security set $\mathcal{S}^i$ that bidder i uses, the solution s to (6) must approach the "full security" (a security that pays $S(z) = z$ for all z, i.e., paying out all cash flows) as X_i goes to 0. To see this, note that under Assumption 3, for any given s, the left-hand-side of (6) decreases as X_i decreases. To maintain equality, the solution s to (6) must increase as X_i decreases, and hence must reach a limit as X_i goes to 0. Next, note that if this limit did not correspond to a full security, then as X_i gets arbitrarily close to 0, the left-hand side would be strictly less than the right-hand side, leading to a contradiction.

Next, observe that when X_i goes to zero and s is a full security, the term $ES^i(s, \theta)$ on the right-hand side of (5) becomes $V_T + \theta$, for any security set $\mathcal{S}^i$. Hence (5) yields

$$\lim_{X_i \to 0} \beta_i(\theta_i) = V_T + E_\theta[\theta | \theta \geq \theta_i], \tag{29}$$

where E_θ is the expectation over $\theta \sim F(\cdot)$. The bidding strategy is strictly increasing, so the highest-type bidder wins and pays the second-highest bid. The seller's expected profit is

$$\lim_{X_i \to 0 \text{ for all } i} \pi_s^{flt} = E\left[E_\theta[\theta | \theta \geq Y_2]\right], \tag{30}$$

where Y_2 denotes the second-highest value among the n random draws $\{\theta_1, \theta_2, ..., \theta_n\}$, and E is the expectation over the realizations of these n random draws. Further,

$$E_\theta[\theta | \theta \geq Y_2] = E[Y_1 | Y_2],$$

where Y_1 is the highest of the n random draws $\{\theta_1, \theta_2, ..., \theta_n\}$, and E is the expectation over the realizations of $\{\theta_1, \theta_2, ..., \theta_n\}$ (which differs from E_θ). Here, $E[Y_1 | Y_2]$ is the expected value of Y_1 conditional on a given value of Y_2. Plugging this into (30) yields

$$\lim_{X_i \to 0 \text{ for all } i} \pi_s^{flt} = E\left[E[Y_1|Y_2]\right] = E\left[Y_1\right],$$

where the second equality follows from the law of iterated expectations. This means that the floating-parameter auction extracts the full NPVs (highest expected synergies). Equation (28) follows because cash auctions cannot extract full rents.

To understand (27), note that although the optimal fixed-parameter mechanism is unknown for general securities, it cannot do better than full extraction. However, it can do no worse than floating-parameter auctions, so (27) holds.

Nondiscriminatory fixed-parameter auctions also extract full rents if all X_i's go to zero at the same rate and bidders pay with the same type of securities. However, if the rates at which X_i's go to zero vary with i, fixed-parameter auctions must incorporate discriminatory adjustments to select the right winner. By contrast, our floating-parameter auction extracts full rents regardless of the rates at which X_i's go to zero. This difference reflects that when X_i goes to 0, the bidding strategy in fixed-parameter auctions approaches pooling: all bidder types offer the full security (i.e., bids are insensitive to bidder types so that even small differences in standalone values of bidders can lead to selection of low types), whereas the bidding strategy in the floating-parameter auction (equation (29)) remains strictly increasing (separating). Fixed-parameter auctions similarly need adjustments when bidders pay with different types of securities even when bidders have the same standalone value,[18] whereas floating-parameter auctions do not.

These features highlight the simple and robust virtues of our floating-parameter design. The floating-parameter design circumvents the need for ex-ante discrimination by the seller by *shifting* the burden of discrimination to the market via the market's post-auction inferences. A seller uses the same mechanism regardless of how bidders differ ex ante, or how their securities differ. This invariance reflects that floating-parameter designs convert bids of heterogeneous bidders that are measured

[18]For instance, consider three bidders with the same standalone value, $X_1 = X_2 = X_3 \equiv X$, but bidder 1 offers debt, bidder 2 offers equity and bidder 3 offers call options. As X goes to zero, then regardless of their types, bidder 1's offer of the face value of debt approaches infinity, bidder 2's offer of equity fraction approaches one, and the strike price in bidder 3's offer approaches zero. This means that winner selection requires delicate adjustments, which will depend sensitively on the precise value of X and the details of the conditional cash flow distributions $h(\cdot|\cdot)$.

in different units (e.g., equity claims whose monetary values depend on a firm's specific attribute) into a common unit—cash—at the outset; whereas fixed-parameter auctions do the conversion ex post—after bids are made.

We end this section by thinking beyond Proposition 8 to settings where bidders' standalone values are not close to zero and they pay with general securities. We make two observations. First, when bidders use the same type of security (e,g., they all use debt or they all use call options), smaller bidders in floating-parameter auctions are more likely to win (Proposition 5). This reflects that smaller bidders in floating-parameter auctions have greater incentives to signal because they would pay a larger fraction of cash flows upon winning, and thus are more concerned with the market's beliefs. As a result, smaller bidders bid more aggressively and hence are more likely to win. Second, if bidders use different types of securities, *bidders who use steeper securities in floating-parameter auctions are more likely to win*, reflecting that steeper securities induce more aggressive bidding (Proposition 4; a point not previously highlighted). Both features raise seller profits in floating parameter mechanisms, and we conjecture that the optimal mechanism for general securities shares these features.[19]

6 Conclusions

A serious drawback of optimal mechanisms with heterogeneous bidders is that they invariably require complex discriminatory winner selection and payment rules, and they impose implausible information demands on the auctioneer, as the optimal design is sensitive to the features of each bidder. These concerns especially manifest themselves in security-bid auctions, where the monetary values of bids depend on the fine details of the underlying cash flow distributions, and nondiscriminatory designs risk selecting the wrong winning bidder and lower expected seller revenue than cash auctions.

We identify a simple mechanism that imposes minimal information burdens on the auctioneer and is invariant to the environment, yet generates high revenues regardless of how bidders differ in their ex-ante attributes or the securities they use. In our second-price floating-parameter auction, bidders submit cash bids, the high

[19]Intuitively, bidders' informational advantages decline with reductions in standalone values and with the use of steeper securities because both increase the tying of security payments to the winner's type. This makes it optimal for a seller to favor smaller bidders and bidders who use steeper securities.

bid wins and pays with securities priced in a competitive capital market that sees the winning price. That is, bidders offer securities with guaranteed cash equivalent values, and the market determines the security parameter paid so that the security's expected value equals the winning cash price. Key virtues of the design include:

1. It is detail-free and nondiscriminatory, removing legal and moral hazard concerns about discriminatory auctions. In takeovers, our mechanism lets a seller satisfy fiduciary duties requiring it to accept the "highest" bid, even if bidders use different securities or differ in other ways. The seller uses the same mechanism no matter how bidders differ ex ante, or how their securities differ.

2. It alleviates information demands. A seller can be uninformed (but need not be). Collectively, the market only needs to learn about the winner, and it can acquire the information post-auction, after a winner's selection.

3. It is *practically relevant*. It applies to both collars and cash bids that are financed after the auction with securities, as occurs for a large share of takeovers.

4. It always generates higher revenues than cash, and is robust to bidder heterogeneity, exhibiting the properties that revenues rise when any single bidder uses a steeper security, or has a lower standalone value.

5. It delivers optimal revenues when bidders only differ in stand-alone values and have either small or large stand-alone values; and numerically we find that revenues are close-to-optimal when bidders' stand-alone values are at intermediate levels, even if they differ substantially.

More broadly, the message of our paper is that an auctioneer can greatly simplify auction design with heterogeneous bidders by linking auction outcomes to post-auction market prices, thereby obtaining high revenues. To our knowledge, our paper is the first to establish these points, providing a starting point for future research.

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7 Appendix

Proof of Lemma 1: Denoting by $s_i(p)$ the solution to (2) when bidder i wins, we rewrite (1), the expected profit of winner i with θ_i, as

$$\pi_i(\theta_i, p) = V_T + \theta_i - ES^i(s_i(p), \theta_i). \tag{31}$$

Below, to economize on language, we say a security is a "full security" if its payoff function $S(z) = z$ for all z; that is, if the security pays out 100% of the cash flows.

Claim 1: $\pi_i(\theta_i^a, p) \geq \pi_i(\theta_i^b, p)$ for all $p \geq r_i$, where the inequality is strict unless the security corresponding to $s_i(p)$ is a full security.

Proof: We use Lemma 1 of DKS. In our setting this lemma implies that the derivative of $ES^i(s_i(p), \theta_i)$ with respect to θ_i is strictly less than 1 if the security corresponding to $s_i(p)$ is not a full security. With a full security, the derivative equals 1. Given our premise that $\theta_i^a > \theta_i^b$, Claim 1 follows.

Claim 2: If the security corresponding to $s_i(p)$ is a full security, then $\pi_i(\theta_i, p) < 0$.

Proof: With a full security, (31) yields $\pi_i(\theta_i, p) = V_T + \theta_i - (V_T + \theta_i + X_i) = -X_i < 0$.

Claim 3: If $\beta_i(\theta_i^b) > r_i$, then for any $\hat{b} \in [r_i, \beta_i(\theta_i^b))$, there is a positive measure of $p \in \left[\hat{b}, \beta_i(\theta_i^b)\right]$ such that the security corresponding to $s_i(p)$ is not a full security.

Proof: Denote the expected profit of bidder i of type θ_i when it bids some $b \geq r_i$ by $\hat{\pi}_i(\theta_i, b)$, and denote the cumulative density function for the highest bid among all bidders other than i by $K(\cdot)$. Then

$$\hat{\pi}_i(\theta_i, b) = \int_{r_i}^{b} \pi_i(\theta_i, t) dK(t) + \pi_i(\theta_i, r_i) K(r_i).$$

Consider bidder i with θ_i^b. If i bids its equilibrium bid $\beta_i(\theta_i^b)$, then i's expected profit is

$$\hat{\pi}_i(\theta_i^b, \beta_i(\theta_i^b)) = \int_{r_i}^{\beta_i(\theta_i^b)} \pi_i(\theta_i, t) dK(t) + \pi_i(\theta_i, r_i) K(r_i).$$

If bidder i instead deviates and bids $\hat{b}$, the gain from the deviation is

$$\hat{\pi}_i(\theta_i^b, \hat{b}) - \hat{\pi}_i(\theta_i^b, \beta_i(\theta_i^b)) = -\int_{\hat{b}}^{\beta_i(\theta_i^b)} \pi_i(\theta_i^b, t) dK(t).$$

Now suppose that contrary to Claim 3, for all (full measure) $p \in \left[\hat{b}, \beta_i(\theta_i^b)\right]$, the security corresponding to $s_i(p)$ is a full security. Then Claim 2 would yield that the deviation gain $\hat{\pi}_i(\theta_i^b, \hat{b}) - \hat{\pi}_i(\theta_i^b, \beta_i(\theta_i^b)) \geq 0$ for all functions $K(t)$, with strict inequality for some $K(\cdot)$. Thus, bidding the posited equilibrium bid $\beta_i(\theta_i^b)$ would be weakly dominated by bidding $\hat{b}$, a violation of Assumption 1. This proves Claim 3.

Claim 4: Suppose $\beta_i(\theta_i^b) > r_i$. Then for bidder i with type $\theta_i^a > \theta_i^b$, bidding $\beta_i(\theta_i^b)$ weakly dominates bidding any $\hat{b} \in [r_i, \beta_i(\theta_i^b))$.

Proof: The difference in i's expected profit when it bids $\beta_i(\theta_i^b)$ and when it bids $\hat{b}$ is:

$$
\begin{aligned}
\hat{\pi}_i(\theta_i^a, \beta_i(\theta_i^b)) - \hat{\pi}_i(\theta_i^a, \hat{b}) &= \int_{\hat{b}}^{\beta_i(\theta_i^b)} \pi_i(\theta_i^a, t) dK(t) \\
&= \int_{\hat{b}}^{\beta_i(\theta_i^b)} (\pi_i(\theta_i^a, t) - \pi_i(\theta_i^b, t)) dK(t) + \int_{\hat{b}}^{\beta_i(\theta_i^b)} \pi_i(\theta_i^b, t) dK(t) \\
&\geq \int_{\hat{b}}^{\beta_i(\theta_i^b)} (\pi_i(\theta_i^a, t) - \pi_i(\theta_i^b, t)) dK(t) \\
&\geq 0.
\end{aligned}
$$

The second equation follows by subtracting and adding $\int_{\hat{b}}^{\beta_i(\theta_i^b)} \pi_i\left(\theta_i^b, t\right) dK(t)$; the third equation follows because the added term is nonnegative (by individual rationality of type θ_i^b); the fourth equation follows from Claim 1. By Claims 1 and 3, a strict inequality holds in the fourth equation for some function $K(\cdot)$—ones with a positive support over the region of prices for which the corresponding security is not a full security. Thus, for type θ_i^a, bidding $\beta_i(\theta_i^b)$ weakly dominates bidding $\hat{b}$. This proves Claim 4.

Using the same arguments as those for Claim 4, it is also strightforward to show that (1) if $\beta_i(\theta_i^b) > r_i$, then for type θ_i^a, bidding $\beta_i(\theta_i^b)$ weakly dominates non-participation (bidding below r_i); and (2) if $\beta_i(\theta_i^b) = r_i$, then for type θ_i^a, bidding $\beta_i(\theta_i^b)$ weakly dominates non-participation. This proves Lemma 1. $\square$

Proof of Proposition 1: We first prove that the bidding strategy is strictly increasing under Assumptions 1 and 2.

Claim 1: If bidder i of type θ_i wins at price $p = \beta_i(\theta_i)$, then (i) the bidder's expected profit $\pi_i(\theta_i, p)$ (in (31)) must be nonnegative; and (ii) the security corresponding to $s_i(p)$ is not a full security.

Proof: If $\pi_i(\theta_i, p)$ were strictly negative, then following standard arguments for second-price auctions, bidding the posited equilibrium bid $\beta_i(\theta_i)$ would be weakly dominated by bidding slightly less than $\beta_i(\theta_i)$, a violation of Assumption 1. This proves part (i). To prove part (ii), note that if it were a full security, then by Claim 2 in the proof of Lemma 1, i's expected profit would be strictly negative if it wins at $\beta_i(\theta_i)$.

Claim 2: Any equilibrium bidding strategy is strictly increasing: if $\theta_i^a > \theta_i^b$ and $\beta_i(\theta_i^b) \geq r_i$, then $\beta_i(\theta_i^a) > \beta_i(\theta_i^b)$.

Proof: Suppose instead that Claim 2 does not hold. Then by Lemma 1, the equilibrium bid of type θ_i^a must equal that of type θ_i^b; i.e., $\beta_i(\theta_i^a)$=$\beta_i(\theta_i^b)$. Next we show that a contradiction would arise, thereby establishing Claim 2.

- By Claim 1 above and Claim 1 in the proof of Lemma 1, $\pi(\theta_i^a, \beta_i(\theta_i^b)) > \pi(\theta_i^b, \beta_i(\theta_i^b)) \geq 0$; that is, type θ_i^a's expected profit from winning at $\beta_i(\theta_i^a) = \beta_i(\theta_i^b)$ is strictly positive.

- We now show that bidding slightly above $\beta_i(\theta_i^a)$ is a profitable deviation for type θ_i^a. First, suppose that when winning at a higher (than $\beta_i(\theta_i^a)$) price, the market's beliefs about i's synergies are the same as if i wins at $\beta_i(\theta_i^a)$. By the continuity of the profit function, if i wins at a price that exceeds $\beta_i(\theta_i^a)$ by an arbitrarily small amount, i's profit must still be strictly positive. Second, by Assumption 2, the market's beliefs are weakly more optimistic when a bidder wins at a higher price. Thus, relaxing the premise that beliefs are the same from winning at a slightly higher price as those from winning at $\beta_i(\theta_i^a)$ can only further increase i's expected profit. Hence, following standard arguments for second-price auctions, bidding i's posited equilibrium bid $\beta_i(\theta_i^a)$ is weakly dominated by bidding slightly above $\beta_i(\theta_i^a)$, violating Assumption 1.

Claim 3: If the bidding strategy is given by (5), then the market's belief is given by (7).

Proof: The second and third equations in (7) follow from the consistency requirements on the equilibrium path—market beliefs must satisfy Bayes rule. The first equation in (7) follows from Assumption 2 that the market's beliefs are non-decreasing in the winning price.

Claim 4: For any θ_i, (6) has a unique solution for $s \in [\underline{s}_i, \bar{s}^i)$.

Proof: We use Lemma 1 of DKS, which implies that for any s, the derivative of $ES^i(s, \theta_i)$ with respect to θ_i is no greater than 1. Then our model premise of sufficient range in the index ($ES^i(\underline{s},\underline{\theta}_i) \leq \underline{\theta}_i + V_T$ and $ES^i(\overline{s},\overline{\theta}_i) > V_T + \overline{\theta}_i$) implies that $ES^i(\underline{s},\theta_i) \leq \theta_i + V_T$ and $ES^i(\overline{s},\theta_i) > V_T + \theta_i$. Then Claim 4 follows by the intermediate value theorem and the property that $ES^i(s,\theta_i)$ is strictly increasing in s.

We next prove the remainder of the proposition.

Step 1. First consider settings with no reserve prices. We show that bidding according to (5) comprises an equilibrium. Note that $\beta_i(\theta_i)$ as defined in (5) is strictly increasing in θ_i because, for any $\theta_1 > \theta_2$, the distribution of θ conditional on $\theta \geq \theta_1$ strictly first-order stochastically dominates the distribution of θ conditional on $\theta \geq \theta_2$.

Consider the strategy of a generic bidder i with type θ_i. We first show that if i bids above $\beta_i(\theta_i)$ and wins at a price $p > \beta_i(\theta_i)$, then its expected profit is negative.

Case 1: $p \leq \beta_i(\bar{\theta}_i)$. Then, there exists some $\theta^* > \theta_i$ such that $p = \beta_i(\theta^*)$. Let s^* denote the resulting security index of its payment as determined by the market. Because the expected value of payment equals the winning price $\beta_i(\theta^*)$, s^* satisfies

$$E_{\theta;i}[ES^i(s^*,\theta)|\theta \geq \theta^*] = \beta_i(\theta^*).$$

Note that if bidder i were of type θ^* and won at price $p = \beta_i(\theta^*)$, then the index of the security paid would be s^*, and its expected profit would be zero (because bidder i of type θ^* is indifferent between winning at $\beta_i(\theta^*)$ and losing). Thus, (1) yields

$$V_T + \theta^* - ES^i(s^*,\theta^*) = 0. \tag{32}$$

Next we show that

$$ES^i(s^*,\theta^*) - ES^i(s^*,\theta_i) \leq \theta^* - \theta_i. \tag{33}$$

Let Z_i^* and Z_i denote the random cash flows associated with $\Theta_i = \theta^*$ and $\Theta_i = \theta_i$, respectively. Then $ES^i(s^*,\theta^*) - ES^i(s^*,\theta_i) = E[S(s^*,Z_i^*)] - E[S(s^*,Z_i)]$. Because the distribution of Z_i^* first-order stochastically dominates that of Z_i (due to the sMLRP), it follows that Z_i^* can be expressed as the sum of random variable Z_i plus another random variable $\widetilde{\epsilon}$, where $\widetilde{\epsilon}$ is nonnegative and $\mathrm{E}[\widetilde{\epsilon}] = \theta^* - \theta_i$. Thus,

$$E[S(s^*,Z_i^*)] = E[S(s^*,Z_i+\widetilde{\epsilon})] \leq E[S(s^*,Z_i)+\widetilde{\epsilon}] = E[S(s^*,Z_i)] + \theta^* - \theta_i.$$

This establishes (33), which, by (32), (1), and the inequality $\theta^* > \theta_i$ yields that its expected profit is negative, $V_T + \theta_i - ES^i(s^*,\theta_i) < 0$.

Case 2: $p > \beta_i(\bar{\theta}_i)$. Because the market's belief about the winner's type cannot exceed $\bar{\theta}_i$, the bidder's expected profit does not exceed what it would be if it won at $p = \beta_i(\bar{\theta}_i)$, which was shown to be negative in Case 1.

We now show bidder i's expected profit is positive if it wins at a price $p < \beta_i(\theta_i)$.

Case 1: $p \geq \beta_i(\underline{\theta}_i)$. Then, there exists $\theta^* < \theta_i$ so that $p = \beta_i(\theta^*)$. Let s^* denote the security index of his payment as determined by the market. Then (32) follows by a similar argument as in the above step. By (32) and (1), its expected profit is positive.

Case 2: $p < \beta_i(\underline{\theta}_i)$. Because the market's belief about the winner's type is the same as if the bidder wins at $p = \beta_i(\underline{\theta}_i)$ (see third equation in (7); this reflects that winning at a price below $\beta_i(\underline{\theta}_i)$ is on the equilibrium path—see Claim 3), i's expected profit is no less than what would be if it wins at $p = \beta_i(\underline{\theta}_i)$, which is positive from Case 1.

Combining the results above and the nature of second-price auction that contingent on winning, the price does not depend on winner's own bid, yields that bidding according to (5) is weakly dominant, constituting an equilibrium. Furthermore, note that by Claim 2 and the arguments given in the main text, in any equilibrium the bidding strategy must be given by (5). Hence the equilibrium is unique.

Step 2. Given reserve prices $\{r_j\}_{j=1}^n$, it is routine to show that the equilibrium holds in which bidder i bids according to (5) when $\beta_i(\theta_i) \geq r_i$; and i places a non-valid bid that is less than r_i when $\beta_i(\theta_i) < r_i$. Because the exact value of a non-valid bid is inconsequential, bidding according to (5) constitutes an equilibrium for all θ_i and equilibrium is unique (up to the indeterminancy of non-valid bids). $\square$

Proof of Proposition 3: First consider equity. Bidder 1 bids $\frac{\theta_1+V_T}{\theta_1+V_T+X_1} \geq \frac{\underline{\theta}+V_T}{\underline{\theta}+V_T+X_1}$, and bidder 2 bids $\frac{\theta_2+V_T}{\theta_2+V_T+X_2} \leq \frac{\bar{\theta}+V_T}{\bar{\theta}+V_T+X_2}$. Because $X_2 > X_1 + \frac{\bar{\theta}-\underline{\theta}}{\underline{\theta}+V_T} X_1 = \frac{\bar{\theta}+V_T}{\underline{\theta}+V_T} X_1$, we have $\frac{\underline{\theta}+V_T}{\underline{\theta}+V_T+X_1} > \frac{\bar{\theta}+V_T}{\bar{\theta}+V_T+X_2}$, so bidder 1 wins against bidder 2. Expected seller revenue is

$$
\begin{aligned}
E\left[\theta_1 + V_T + X_1\right] E\left[\frac{\theta_2 + V_T}{\theta_2 + V_T + X_2}\right] &= \left(E\left[\theta\right] + V_T + X_1\right) E\left[1 - \frac{X_2}{\theta_2 + V_T + X_2}\right] \\
&\leq \left(E\left[\theta\right] + V_T + X_1\right) \left(1 - \frac{X_2}{E\left[\theta\right] + V_T + X_2}\right) \\
&= \left(E\left[\theta\right] + V_T + X_1\right) \left(\frac{E\left[\theta\right] + V_T}{E\left[\theta\right] + V_T + X_2}\right) \\
&= \frac{E\left[\theta\right] + V_T + X_1}{E\left[\theta\right] + V_T + X_2} \left(E\left[\theta\right] + V_T\right),
\end{aligned}
$$

where the inequality follows from Jensen's inequality. The premise that $X_2 - X_1 > \frac{E[\theta]-\underline{\theta}}{V_T+\underline{\theta}}\left(X_1 + E\left[\theta\right] + V_T\right)$ yields

$$\begin{aligned}
\frac{E\left[\theta\right] + V_T + X_1}{E\left[\theta\right] + V_T + X_2}\left(E\left[\theta\right] + V_T\right) &< \frac{E\left[\theta\right] + V_T + X_1}{E\left[\theta\right] + V_T + X_1 + \frac{E[\theta]-\underline{\theta}}{V_T+\underline{\theta}}\left(X_1 + E\left[\theta\right] + V_T\right)}\left(E\left[\theta\right] + V_T\right) \\
&= \frac{E\left[\theta\right] + V_T + X_1}{\frac{V_T+E[\theta]}{V_T+\underline{\theta}}\left(X_1 + E\left[\theta\right] + V_T\right)}\left(E\left[\theta\right] + V_T\right) \\
&= V_T + \underline{\theta}.
\end{aligned}$$

Thus, expected seller revenue in an equity auction is less than $V_T + \underline{\theta}$. Because expected revenue in a cash auction exceeds $V_T + \underline{\theta}$, part (i) of the proposition follows.

Lemma 2 *Consider nondiscriminatory fixed-parameter auctions using ordered securities $\mathcal{S}_A$ and $\mathcal{S}_B$, where $\mathcal{S}_A$ is steeper than $\mathcal{S}_B$. Let $X_1 < X_2$. If bidder 1 with expected synergy θ_1 wins against bidder 2 with expected synergy θ_2 under $\mathcal{S}_B$, then (i) bidder 1 also wins against bidder 2 under $\mathcal{S}_A$, and (ii) seller revenue under $\mathcal{S}_A$ is strictly less than that under $\mathcal{S}_B$ given θ_1 and θ_2.*

Proof of Lemma 2: By Assumption 3, the cash flow distribution depends only on the expected total valuation of the bidder (bidder identity is irrelevant). Abusing notation we suppress the superscript i in ES, and use $ES(s, \theta_i + X_i + V_T)$ to denote the expected value of the security with index s. Let s_B^1 denote bidder 1's optimal bid under $\mathcal{S}_B$. Then s_B^1 solves

$$ES_B(s_B^1, \theta_1 + X_1 + V_T) = X_1. \tag{34}$$

Because bidder 1 wins against bidder 2 under $\mathcal{S}_B$, we have $ES_B(s_B^1, \theta_2 + X_2 + V_T) > X_2$. Hence there exists

$$v^* < \theta_2 + X_2 + V_T \tag{35}$$

such that

$$ES_B(s_B^1, v^*) = X_2. \tag{36}$$

Because $X_1 < X_2$, we have

$$\theta_1 + X_1 + V_T < v^*. \tag{37}$$

Let s_A^1 denote the optimal bid of bidder 1 using the steeper securities, $\mathcal{S}_A$:

$$ES_A(s_A^1, \theta_1 + X_1 + V_T) = X_1. \tag{38}$$

By (34), (38), (36), and the properties of steeper securities, $ES_A(s_A^1, v^*) > ES_B(s_B^1, v^*)$. Hence, by (37), $ES_A(s_A^1, \theta_2 + X_2 + V_T) > X_2$. Thus, bidder 1 also wins against bidder 2 under $\mathcal{S}_A$.

Analogously, bidder 2's bids under $\mathcal{S}_A$ and $\mathcal{S}_B$ solve

$$ES_A(s_A^2, \theta_2 + X_2 + V_T) = X_2 \tag{39}$$

and

$$ES_B(s_B^2, \theta_2 + X_2 + V_T) = X_2. \tag{40}$$

Seller revenues under $\mathcal{S}_A$ and $\mathcal{S}_B$ are, $ES_A(s_A^2, \theta_1+X_1+V_T)$ and $ES_B(s_B^2, \theta_1+X_1+V_T)$. Further, (35) and (37) yield $\theta_1 + X_1 + V_T < \theta_2 + X_2 + V_T$, which, by the properties of steeper securities, yields that revenue under $\mathcal{S}_A$ is strictly less than that under $\mathcal{S}_B$. This proves the lemma.

Part (i) showed that bidder 1 wins against bidder 2 in equity auctions. From the lemma and the law of iterated expectations, part (ii) of the proposition follows. □

Proof of Proposition 5: Consider any bidder i. Recall that $ES^i(s, \theta) \equiv E(S(s, Z_i)|\Theta_i = \theta)$ is the expected value of security with index s, derived from cash flows generated by that bidder conditional on $\Theta_i = \theta$. To ease notation we replace "ES^i" with "g, which is a function of s, the standalone value X, and the expected synergy θ of bidder i. We derive a sufficient condition for the bidding strategy to decrease in X:

Lemma 3 *If the ratio* $\frac{\frac{\partial g}{\partial X}}{\frac{\partial g}{\partial s}}$ *strictly decreases in the expected synergy* θ *at any given* s *and* X*, then the bidder's bidding strategy in floating-parameter design weakly decreases in* X*, where the decrease is strict for all* $\theta < \bar{\theta}_i$*.*

Proof of Lemma 3. Use the notation of this proof to rewrite (6) as:

$$g(s, X, \theta = \theta_i) = \theta_i + V_T, \tag{41}$$

where θ_i is a constant. In (41), s is an implicit function of X, which we write as $s(X)$; that is, $s = s(X)$ and X satisfies (41).

Next examine the derivative of $g(s = s(X), X, \theta)$ with respect to X:

$$\frac{dg(s(X), X, \theta)}{dX} = \frac{\partial g}{\partial X}|_{s=s(X)} + \frac{\partial g}{\partial s}|_{s=s(X)} \frac{ds(X)}{dX}, \tag{42}$$

where,

$$\frac{ds(X)}{dX} = -\frac{\frac{\partial g(s,X,\theta_i)}{\partial X}}{\frac{\partial g(s,X,\theta_i)}{\partial s}}|_{s=s(X)},$$

which follows by applying the implicit function theorem on (41). Plugging the above into (42) yields

$$\begin{aligned} \frac{dg(s(X), X, \theta)}{dX} &= \frac{\partial g(s, X, \theta)}{\partial X}|_{s=s(X)} - \frac{\partial g(s, X, \theta)}{\partial s}|_{s=s(X)} \frac{\frac{\partial g(s,X,\theta_i)}{\partial X}}{\frac{\partial g(s,X,\theta_i)}{\partial s}}|_{s=s(X)} \\ &= \frac{\partial g(s, X, \theta)}{\partial s}|_{s=s(X)} \left\{ \frac{\frac{\partial g(s,X,\theta)}{\partial X}}{\frac{\partial g(s,X,\theta)}{\partial s}} - \frac{\frac{\partial g(s,X,\theta_i)}{\partial X}}{\frac{\partial g(s,X,\theta_i)}{\partial s}} \right\} |_{s=s(X)}. \end{aligned} \tag{43}$$

Use the notation in this proof to rewrite (5), the bidding startegy when the bidder's expected synergy is θ_i, as

$$\beta_i(\theta_i) = E[g(s(X), X, \theta)|\theta \geq \theta_i]. \tag{44}$$

For any $\theta > \theta_i$, by the premise of the lemma that the ratio $\frac{\frac{\partial g}{\partial X}}{\frac{\partial g}{\partial s}}$ strictly decreases in θ, the term inside the curly brackets on the right-hand side of (43) is strictly negative. Hence, by $\frac{\partial g(s,X,\theta)}{\partial s}|_{s=s(X)} > 0$, $\frac{dg(s(X),X,\theta)}{dX} < 0$. Thus, by (44), we have $\frac{d}{dX}\beta_i(\theta_i) \leq 0$, where strict inequality holds for $\theta_i < \bar{\theta}_i$. This proves the lemma.

Next we show that for debt and call, the ratio $\frac{\frac{\partial g}{\partial X}}{\frac{\partial g}{\partial s}}$ strictly decreases in θ. First note that both $\frac{\partial g(s,X,\theta)}{\partial s}$ and $\frac{\partial g(s,X,\theta)}{\partial X}$ are strictly positive.

Consider debt. Let s denote the face value of debt. We have

$$g = \theta + X + V_T - E\left[\max\left(\left(\left(\theta + X + V_T\right)\tau - s\right), 0\right)\right],$$

which yields

$$\frac{\partial g}{\partial s} = \text{prob}\left(\tau \geq \frac{s}{\theta + X + V_T}\right)$$

and

$$\frac{\partial g}{\partial X} = 1 - E\left[\tau|\tau \geq \frac{s}{\theta + X + V_T}\right] \text{prob}\left(\tau \geq \frac{s}{\theta + X + V_T}\right),$$

where "prob" denotes the probability. $\text{prob}\left(\tau \geq \frac{s}{\theta+X+V_T}\right)$ increases in θ, as does $E\left[\tau|\tau \geq \frac{s}{\theta+X+V_T}\right]\text{prob}\left(\tau \geq \frac{s}{\theta+X+V_T}\right)$ (to see this note that the term equals $\int_{\frac{s}{\theta+X+V_T}}^{\infty} \tau dR(\tau)$, where $R(\tau)$ is the cdf of τ, and then differentiate with respect to θ). Hence, $\frac{\partial g}{\partial s}$ increases in θ and $\frac{\partial g}{\partial X}$ decreases in θ. Thus, the ratio $\frac{\frac{\partial g}{\partial X}}{\frac{\partial g}{\partial s}}$ strictly decreases in θ.

Next, consider call options. Let K denote the strike price. Given our onvention that a larger s means a larger security, but a larger K means a smaller call, we set $s = -K$ to preserve the order. We have

$$g = E\left[\max\left(\left(\left(\theta + X + V_T\right)\tau - K\right), 0\right)\right],$$

which yields

$$\frac{\partial g}{\partial s} = \text{prob}\left(\tau \geq \frac{K}{\theta + X + V_T}\right)$$

and

$$\frac{\partial g}{\partial X} = E\left[\tau|\tau \geq \frac{K}{\theta + X + V_T}\right] \text{prob}\left(\tau \geq \frac{K}{\theta + X + V_T}\right).$$

Hence

$$\frac{\frac{\partial g}{\partial X}}{\frac{\partial g}{\partial s}} = E\left[\tau|\tau \geq \frac{K}{\theta + X + V_T}\right],$$

which strictly decreases in θ.

Thus, debt and call satisfy the premise of Lemma 3. Therefore, the bidding strategy in the floating-parameter design strictly increases as X decreases for $\theta_i < \bar{\theta}_i$. Further, following readily from (14), the above statement also holds for equity. Because more agressive cash bidding by any bidder leads to a higher expected profit, the proposition follows. □

Proof of Proposition 6. We use the property that the expected monetary value of the winner's payment equals the winning cash price, conditional on the winning price and the winner's identity. Then the law of iterated expectations yields that a bidder's expected profit in the floating-parameter design is the same as that in a "hypothetical" cash auction in which bidders bid cash according to Proposition 1, and that the winner pays the second-highest bid directly with cash.

Let Y_1 denote the highest valid cash bid (i.e., the highest bid exceeding the reserves) among the $n-1$ bidders other than bidder i, and let $K(Y_1)$ denote the cdf of Y_1. We use π_A^i and π_B^i to denote bidder i's expected profit under $\mathcal{S}_A^i$ and $\mathcal{S}_B^i$, respectively. Then for $J \in \{A, B\}$,

$$\pi_J^i = \int_{\theta_J^*}^{\bar{\theta}_i} \int_{\min_{j\neq i}\{r_j\}}^{\beta_{i;J}(\theta_i)} (V_T + \theta_i - \max(Y_1, r_i))\, dK(Y_1)\, dF_i(\theta_i),$$

where $\beta_{i;J}(\theta_i)$ denotes bidder i's bidding strategy under security set $\mathcal{S}_J^i$, and $\theta_J^* \equiv \underline{\theta}_i$ if $r_i \le \beta_{i;J}(\underline{\theta}_i)$ and $\beta_{i;J}(\theta_J^*) \equiv r_i$ if $r_i > \beta_{i;J}(\underline{\theta}_i)$.

Because $\mathcal{S}_A^i$ is steeper than $\mathcal{S}_B^i$, we have $\theta_A^* \le \theta_B^*$. Rewrite π_A^i as

$$\pi_A^i \equiv \pi_{A;1}^i + \pi_{A,2}^i,$$

where

$$\pi_{A;1}^i \equiv \int_{\theta_A^*}^{\theta_B^*} \int_{\min_{j\neq i}\{r_j\}}^{\beta_{i;A}(\theta_i)} (V_T + \theta_i - \max(Y_1, r_i))\, dK(Y_1)\, dF_i(\theta_i) \tag{45}$$

and

$$\pi_{A;2}^i \equiv \int_{\theta_B^*}^{\bar{\theta}_i} \int_{\min_{j\neq i}\{r_j\}}^{\beta_{i;A}(\theta_i)} (V_T + \theta_i - \max(Y_1, r_i))\, dK(Y_1)\, dF_i(\theta_i). \tag{46}$$

We first show that $\pi_{A;1}^i \le 0$. If $\theta_A^* = \theta_B^*$, then $\pi_{A;1}^i = 0$. If $\theta_A^* < \theta_B^*$, then it must be that $\theta_B^* > \underline{\theta}_i$ (else $\theta_A^* = \theta_B^* = \underline{\theta}_i$). Thus, $r_i = \beta_{i;B}(\theta_B^*)$. Since $\beta_{i;B}(\theta_B^*) \ge V_T + \theta_B^*$,

$$V_T + \theta_B^* - r_i \le 0. \tag{47}$$

By (45), we have

$$\begin{aligned}
\pi_{A;1}^i &\le \int_{\theta_A^*}^{\theta_B^*} \int_{\min_{j\neq i}\{r_j\}}^{\beta_{i;A}(\theta_i)} (V_T + \theta_i - r_i)\, dK(Y_1)\, dF_i(\theta_i) \\
&\le \int_{\theta_A^*}^{\theta_B^*} \int_{\min_{j\neq i}\{r_j\}}^{\beta_{i;A}(\theta_i)} (V_T + \theta_B^* - r_i)\, dK(Y_1)\, dF_i(\theta_i) \\
&\le 0,
\end{aligned}$$

where the first inequality follows from $\max(Y_1, r_i) \ge r_i$, and the final inequality follows from (47).

Next, (46) yields

$$\begin{aligned}
\pi_{A;2}^{i}-\pi_{B}^{i} &= \int_{\theta_B^*}^{\bar{\theta}_i}\int_{\beta_{i;B}(\theta_i)}^{\beta_{i;A}(\theta_i)}\left(V_T+\theta_i-\max\left(Y_1,r_i\right)\right)dK\left(Y_1\right)dF_i\left(\theta_i\right)\\
&\leq \int_{\theta_B^*}^{\bar{\theta}_i}\int_{\beta_{i;B}(\theta_i)}^{\beta_{i;A}(\theta_i)}\left(V_T+\theta_i-Y_1\right)dK\left(Y_1\right)dF_i\left(\theta_i\right)\\
&\leq \int_{\theta_B^*}^{\bar{\theta}_i}\int_{\beta_{i;B}(\theta_i)}^{\beta_{i;A}(\theta_i)}\left(V_T+\theta_i-\beta_{i;B}(\theta_i)\right)dK\left(Y_1\right)dF_i\left(\theta_i\right)\\
&\leq 0,
\end{aligned}$$

where the first inequality follows from $\max\left(Y_1,r_i\right)\geq Y_1$, and the final inequality follows from $\beta_{i;B}(\theta_i)\geq V_T+\theta_i$. By $\pi_{A;2}^{i}-\pi_{B}^{i}\leq 0$ and $\pi_{A;1}^{i}\leq 0$, we have $\pi_{A}^{i}\leq\pi_{B}^{i}$. $\square$

Proof of Proposition 7: Theorem 1 in Liu (2016) shows the seller's expected profit in any incentive-compatible equity mechanism decomposes into the sum of three terms. The first two terms vanish for both the optimal mechanism and the floating-parameter mechanism (the first term vanishes when all bidders i with $\underline{s}_i$ earn zero rent, and the second term vanishes when losing bidders do not pay). Hence, for both the floating-parameter and optimal mechanisms, the expected seller profit equals the third term, which (after adjusting for the different notation in this paper) is given by:

$$\pi_s=\int_{\chi}\left[\sum_{i=1}^{n}W_i\left(\theta_1,\theta_2,...,\theta_n\right)\phi_i(\theta_i)\right]f_1\left(\theta_1\right)...f_n\left(\theta_n\right)d\theta_1...d\theta_n, \tag{48}$$

where $W_i\left(\theta_1,\theta_2,...,\theta_n\right)$ is the probability that bidder i wins when bidders' reported types are $(\theta_1,\theta_2,...,\theta_n)$, and $\Sigma_i W_i\leq 1$ for all $(\theta_1,\theta_2,...,\theta_n)$.

Now we prove equation (18). Referring to (24), we have that for all i and θ_i,

$$\begin{aligned}
\Delta_i(\theta_i)-\phi_i(\theta_i) &= \frac{1}{2}\frac{X_i}{\left(V_T+X_i+\theta_i\right)^2}\left(\bar{\theta}-\theta_i\right)^2 && (49)\\
&\leq \frac{1}{2}\frac{X_i\bar{\theta}^2}{\left(V_T+X_i+\theta_i\right)^2}\\
&\leq \frac{1}{2}\frac{\bar{\theta}^2}{V_T+X_i}=\frac{1}{2k}\frac{\bar{\theta}^2}{V_T^*+X_i^*}\\
&\leq \frac{1}{2k}\frac{\bar{\theta}^2}{V_T^*+\min_i\left\{X_i^*\right\}}, && (50)
\end{aligned}$$

where the right-hand-side of (50) is independent of i and approaches zero as k goes to infinity. Note also that by (49), $\Delta_i(\theta_i) - \phi_i(\theta_i) > 0$ for all i and θ_i. Thus, for any $\epsilon > 0$, there exist a $k^*(\epsilon)$ such that for all $k > k^*(\epsilon)$,

$$0 < \Delta_i(\theta_i) - \phi_i(\theta_i) < \epsilon \tag{51}$$

for all i and θ_i.

Claim 1: For any $\epsilon > 0$ and any $i, j \in \{1, ..., n\}$ such that $\Delta_i(\theta_i) \geq \Delta_j(\theta_j)$, $\phi_i(\theta_i) > \phi_j(\theta_j) - \epsilon$ for all $k > k^*(\epsilon)$.

To establish this claim, note that for all $k > k^*$,

$$\begin{aligned} \phi_j(\theta_j) - \phi_i(\theta_i) &= [\phi_j(\theta_j) - \Delta_j(\theta_j)] + [\Delta_j(\theta_j) - \Delta_i(\theta_i)] + [\Delta_i(\theta_i) - \phi_i(\theta_i)] \\ &\leq 0 + 0 + [\Delta_l(s_l) - \phi_l(s_l)] \\ &< \epsilon, \end{aligned} \tag{52}$$

where the last inequality follows from (51), establishing Claim 1.

Given any realization of the synergies $(\theta_1, \theta_2, ..., \theta_n)$, assume bidder m has the highest $\phi_i(\theta_i)$ and bidder l has the highest $\Delta_i(\theta_i)$ among all bidders, where m and l may or may not be the same bidder. By (23) and (24), $\beta_i(\theta_i) = V_T + \frac{1}{2}\bar{\theta} + \frac{1}{2}\Delta(\theta_i)$ and hence bidder l has the highest $\beta_i(\theta_i)$ among all bidders. To simplify presentation, we ignore ties because they occur over a space with measure zero, and hence do not contribute to the integration in (48). In the optimal mechanism, m wins if $\phi_m(\theta_m) \geq 0$, and the asset is not sold if $\phi_m(\theta_m) < 0$. In the floating-parameter mechanism, given the uniform reserve r^{flt} (equation 17), l wins if $\beta_l(\theta_l) \geq r^{flt}$, and the asset is not sold if $\beta_l(\theta_l) < r^{flt}$. Referring to the term inside the bracket in (48), we prove the following claim.

Claim 2. Given any realization of the synergies $(\theta_1, \theta_2, ..., \theta_n)$, for any $\epsilon > 0$, the following holds for all $k > k^*(\epsilon)$:

$$\left[\sum_{i=1}^{n} W_i^{opt}(\theta_1, \theta_2, ..., \theta_n)\, \phi_i(\theta_i)\right] - \left[\sum_{i=1}^{n} W_i^{flt}(\theta_1, \theta_2, ..., \theta_n)\, \phi_i(\theta_i)\right] \leq 2\epsilon \tag{53}$$

To prove the claim, we consider the following 4 cases.

Case 1. The asset is sold in both the optimal and floating-parameter mechanisms. Then $\sum_{i=1}^{n} W_i^{opt}(\theta_1, \theta_2, ..., \theta_n)\, \phi_i(\theta_i) = \phi_m(\theta_m)$ and $\sum_{i=1}^{n} W_i^{flt}(\theta_1, \theta_2, ..., \theta_n)\, \phi_i(\theta_i) = \phi_l(\theta_l)$. Because $\Delta_l(\theta_l) \geq \Delta_m(\theta_m)$, Claim 1 yields Claim 2.

Case 2. The asset is sold in neither the optimal nor floating-parameter mechanisms. Then both terms on the left-hand side of (53) vanish, so (53) holds trivially.

Case 3. The asset is sold in the optimal mechanism but not the floating-parameter mechanism. Then the left-hand side of (53) becomes $\phi_m(\theta_m)$. The fact that the asset is not sold in the floating-parameter mechanism means that $\beta_l^{flt}(\theta_l) < r^{flt}$. Consider two scenarios below.

Scenario 1. $\beta_l(\theta_l) < \beta_l(\max\left\{\phi_l^{-1}(0), \underline{\theta}_l\right\}$. This means $\phi_l(\theta_l) < 0$. Because $\Delta_l(\theta_l) \geq \Delta_m(\theta_m)$, by Claim 1, $\phi_m(\theta_m) < \epsilon$. This establishes Claim 2.

Scenario 2. $\beta_l(\theta_l) < \beta_j(\max\left\{\phi_j^{-1}(0), \underline{\theta}_j\right\}$ *for some* $j \neq l$. There are two subcases. In the first subcase, $\beta_l^{flt}(\theta_l) < \beta_j(\left\{\phi_j^{-1}(0)\right\})$ and $\phi_j^{-1}(0) \geq \underline{\theta}_j$. Because $\phi_j\left\{\phi_j^{-1}(0)\right\} = 0$, by $\beta_l(\theta_l) < \beta_j(\left\{\phi_j^{-1}(0)\right\})$ and Claim 1, $\phi_l(\theta_l) < \epsilon$. Then by Claim 1 again and $\Delta_l(\theta_l) \geq \Delta_m(\theta_m)$, $\phi_m(\theta_m) < 2\epsilon$, which establishes Claim 2. In the second subcase, $\beta_l(\theta_l) < \beta_j(\underline{\theta}_j)$. But this second subcase can't happen because $\beta_l(\theta_l) \geq \beta_j(\theta_j) \geq \beta_j(\underline{\theta}_j)$.

Case 4. The asset is sold in the floating-parameter mechanism but not in the optimal mechanism. We show that this case cannot happen. The fact that it is sold in the floating-parameter mechanism means that $\beta_l(\theta_l) \geq r^{flt} \geq \beta_l(\max\left\{\phi_l^{-1}(0)\right\}$, implying that $\theta_l \geq \max\left\{\phi_l^{-1}(0)\right\}$ and $\phi_l(\theta_l) \geq 0$. Because bidder m has the highest $\phi_i(\theta_i)$—higher than that of bidder l, we have $\phi_m(\theta_m) \geq 0$. Thus, the asset should be sold in the optimal mechanism, a contradiction. This completes the proof of Claim 2.

By Claim 2 and (48), $\pi_s^{opt} - \pi_s^{II,flt} < \epsilon$. As ϵ is arbitrary, equation (18) holds. The remainder of the proposition is proved in the main text, establishing the proposition. ∎