Rent extraction with securities plus cash*

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Abstract

In our target-initiated theory of takeovers, a target approaches potential acquirers who privately know their stand-alone values and merger synergies, where higher synergy acquirers also tend to have larger stand-alone values. Despite their information disadvantage, targets can obtain high revenues by offering payment choices that are combinations of cash and equity, exploiting a reluctance of high-valuation acquirers to cede equity claims that induces them to bid more cash. Properly-designed cash-equity mixes can let targets costlessly screen potential acquirers, overcoming adverse selection. We also show when combining cash with securities that are more information sensitive than equities can help further.

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1 Introduction

Most empirical and theoretical research on takeovers starts with the premise that they are initiated by bidders. However, recent work by Eckbo, Norli and Thorburn (ENT, 2019) establishes that "about one half of takeover bids for public targets are initiated by the target, organizing an auction-like process to solicit potential bids." Our paper shows how the target should design such auctions, including the specifications of the cash-equity composition of bids, and it shows how the optimal design reconciles a set of empirical regularities that are elusive to explain within a single unified framework.

We consider a target that recognizes the possibility of synergies that would obtain from a merger. The target's problem is that each potential acquirer (bidder) i is privately informed about its stand-alone value V_i and the synergy $s_i(V_i)$ associated with a possible merger, where potential acquirer types with higher stand-alone values also tend to generate higher synergies. When acquirers pay with shares of the merged firm's equity, this creates an adverse selection problem, a problem that is especially severe when synergies rise slowly with stand-alone values so that the extent of information asymmetry concerning stand-alone values is high relative to that for synergy gains. Concretely, because the monetary value of an equity offer is proportional to the value of the joint firm, which is the sum of the target's value under the bidder's control plus the bidder's stand-alone value, a bidder type with lower synergies (and lower stand-alone values) may be willing to offer a higher equity share, outbidding bidder types that would provide more synergies, and hence higher NPVs, for the joint firm. This drives down the potential revenues that a target can extract.

Our paper shows how a target/seller can solve this adverse selection problem by having bidders submit bids that involve combinations of securities with differing levels of information sensitivity. The sensitivity of a security's value to the underlying cash flows is captured by the concept of steepness introduced in DeMarzo, Kremer and Skrzpacz 2005 (DKS): call options are steeper than equities, which are steeper than cash. We show that by combining securities of differing steepness, a seller can obtain high revenues despite the adverse selection, and the greater the difference in steepness between the securities, the better it is for the seller. Using combinations of different securities allows a target to overcome its information disadvantage by exploiting a reluctance of high-valuation acquirers to pay with security claims whose payments are tied more tightly to their valuations. This facilitates separation by the target of different acquirer types.

Indeed, we establish that optimally-designed combinations of cash and equity can sometimes allow the target to achieve the first-best outcome, extracting full rents—the highest-NPV bidder wins and receives *zero* rent. That is, a seller can obtain the same revenues as if it perfectly knew the private information of all bidders and just made a single take-it-or-leave-it offer to the highest-type bidder. The seller exploits the fact that low and high bidder types value cash and equity differently—high bidder types care more about ceding equity claims to their high stand-alone values. This difference means that the seller can screen lower types with high equity offers and screen higher types with high cash offers. The seller tailors its menu of possible payments so that (1) higher types choose a less steep mix that requires them to pay more cash but give up a smaller equity claim, while (2) lower types choose steeper mixes. When $s_i(V_i)$ is weakly concave, a seller can costlessly screen bidders by choosing cash-equity compositions that reduce the differential rents between high and low types to zero, extracting all synergy gains, while ensuring global incentive compatibility.

The mechanism is dominant strategy incentive compatible: if a bidder misreports and wins, its profit is negative, regardless of how other bidders bid. The mechanism also has the first-price auction property that a bidder's winning payment depends only on its own report. The combination of these properties is a novel feature of our mechanism[1] that has important implications. First, the mechanism extracts full rents even from a single bidder, absent the competition that Crémer and McLean (1985, 1988) exploit when bidders have correlated signals. Second, bidders submit bids that reveal their true types and the seller extracts full rents even when the auction is decentralized so that a seller approaches each bidder privately and bidders do not know how many rivals they face. Later, we elaborate on how these features reconcile a broad set of empirical regularities regarding takeovers.

When $s_i(V_i)$ is concave, the full extraction mechanism can be implemented by having bidders bid cash with an equity component that declines with the cash bid. The mechanism is robust in the sense that a target can extract almost all rents when it only knows the approx-

[1] By contrast, in a standard second-price auction, bidders have dominant strategies but the winner's payment depends on a losing bidder's bid; and in a standard first-price auction, the winner's payment depends on its own bid but bidders do not have dominant strategies.

imate form of $s_i(V_i)$, or when there is slight uncertainty about synergies given an acquirer's stand-alone value. When $s_i(V)$ is convex, full extraction via cash plus equity is not possible. Nonetheless, we identify the optimal cash-plus-equity mechanism in a single-bidder setting, establishing that it takes a very simple form: the seller offers a contract that consists of a fixed royalty rate plus cash.[2] Moreover, we show that the target can combine cash with securities that are steeper than equity to do better. This reflects the intuition that greater separation in steepness makes the combination of cash and a security a more effective screening device.

Our work provides a framework for understanding several empirical regularities associated with takeovers. First, it reconciles the distribution of returns between a target and acquiring firm. Researchers over different time periods and samples consistently find the target's abnormal return to be high and significant, whereas the average acquirer's is very low. The review article by Eckbo, Malenko and Thorburn (2019, EMT) reports that mean CARs to the target amount to 29% including pre-announcement run-ups due to information leakage, while Dessaint, Eckbo and Golubov (2019, DEG, p3) "find that average acquirer returns over the last four decades have remained close to zero and largely flat—both unconditionally and after controlling for the usual observable firm- and deal-specific effects." DEG (p1), citing literature survey articles by Betton, Eckbo, and Thorburn (2008a, BETa), Eckbo (2014), and Mulherin, Netter, and Poulsen (2017), summarize the state of knowledge as follows: "an expansive literature benefitting from access to large-scale electronic databases confirms that... target shareholders capture the lion's share of combined takeover gains."

Second, our model can explain the relationship between the asset composition of a bid and target/acquirer returns found in the data. As EMT's survey highlights, combinations of cash and equity payments are widely used.[3] Our paper reconciles why *both* target and acquirer returns rise with the cash share of an offer, as does their combined return (Andrade, Mitchell and Stafford (2001), Table 4). Quoting EMT's summary of the evidence for bidders (p36), "Bidder abnormal announcement returns are on average highest in all-cash offers, lowest in all-stock offers, and with mixed cash-stock offers in between"; and BETa, p355,

[2] Payments in the form of a fixed royalty rate plus cash are used in many economic transactions (e.g., off-shore oil leases and timber lease auctions often have this design (see Gorbenko and Malenko (2011) and Skrzypacz (2013)). Our model can be re-formulated to provide foundations for the optimality of such a design.

[3] See e.g., BETa, p328, "mixed cash-stock offers are pervasive across the entire (1980-2005) sample period"or Eckbo, Makaew and Thorburn (2018).

summarizes evidence that CARs to targets are greater for cash offers than equity offers. Our model delivers these results because greater cash components reveal (1) higher synergies from which only the target benefits, and (2) more positive information about the acquirer's stand-alone value, causing the acquirer to experience greater abnormal returns. Via this channel our model also reconciles the finding in Betton, Eckbo, Thompson and Thorburn (BETT, 2014), Table VII, of a strong positive correlation between target and acquirer CARs. In addition, Betton, Eckbo and Thorburn (BETb, 2008b, Table 1) find that cash use is more likely if there are multiple bids and Boone and Mulherin (BM 2007, Table IV) find that acquisitions involving equity are twice as likely as pure cash acquisitions to have one bidder. These findings obtain in our setting because more bidders raise the probability that the winning bidder has a high valuation, and bidders with higher valuations use more cash.

Third, our model provides a framework for understanding the takeover process that BM identify, where a target often privately approaches multiple potential acquirers but few bidders make public bids. EMT summarize the broad finding that public bids from multiple bidders occur in less than 10% of takeovers. Indeed, only 3.4% of all control contests for publicly-traded US targets between 1980 and 2005 involve multiple public bidders (BETa). In the optimal design that we identify, (1) each potential acquirer has a dominant strategy, and (2) the winning bidder's payment depends only on his own bid. The fact that rival public bidders are rare is reconciled by a decentralized takeover process in which a target privately approaches each potential bidder, sets the terms of payment contingent on that bidder winning, and then selects the bidder that would add the most to the target's value, with the target extracting all synergy gains. As such, our work contributes to a broader picture of understanding of auctions versus negotiation. Bulow and Klemperer (1996) highlight tradeoffs between auctions and negotiation, and our work shows ways in which the two approaches need not conflict.

Existing theories of takeovers that are based on asymmetric information (e.g., signaling) formulations in which an acquirer rather than the target has the bargaining power can only explain a subset of these features. We next discuss the security bid and takeover literature, and differentiate our empirical predictions from those of existing models.

1.1 Related Literature

DKS provide a comprehensive analysis of security-bid auctions in which payments are in securities whose values are tied to the cash flows the bidder generates. They highlight how the steepness of the security used affects a seller's revenue. They establish that if bidders have private information about the asset's value but their opportunity costs of winning (e.g., stand-alone values in takeover auctions) are equal and common knowledge, then auctions using steeper securities yield a seller greater expected revenues.[4]

Che and Kim (2010) consider the possibility that opportunity costs rise with synergies. They show that if this rise is fast enough then an extreme form of adverse selection obtains when bids are in securities: bidders with higher synergies, and thus higher NPVs, bid less because they care more about retaining claims to their stand-alone values. Steeper securities exacerbate this adverse selection problem, causing steeper securities to yield lower revenues.

The insight of our paper is that sellers can design auctions to *exploit* this adverse selection by combining securities of *differing* steepness. The gains from combining arise because the differential rents of high over low bidder types may change sign between sets of securities with differing steepness. Positive mixtures of different sets of securities can balance those differential rents to zero and achieve full surplus extraction. Consider cash-plus-equity: When bidding strategies are decreasing in pure equity auctions, low-NPV bidders may extract more rents with equity payments than high-NPV bidders. By contrast, all bidders value cash in the same way, so that, with cash, a bidder's rent always rises with his NPV. Starting from pure equity, one can reduce the equity payment and increase the cash payment so that the differential rent of a low type over a high type falls, crossing zero at some point.

Combining cash (the least steep security) with securities that are even steeper than equity can further help the seller to separate types and ensure the global incentive compatibility of the mechanism. This is because higher valuation types (relative to low valuation types) are even less willing to cede steeper securities whose value is tied more tightly to their valuations. The wider spread in steepness reduces the differential rents of higher types over lower

[4]Work on rent extraction and security-bid auctions includes Hansen (1987), Dasgupta and Tsui (2004), Board (2007), Povel and Singh (2010), Kogan and Morgan (2010), Gorbenko and Malenko (2011), Liu (2012), Skrzypacz (2013), and Sogo, Bernhardt and Liu (2016). See Burkart and Lee (2015) for security design in takeovers.

types and helps ensure global incentive compatibility. Collectively, our findings reveal how the insights of DKS on the advantages of steeper securities extend to settings with adverse selection when the seller can combine auction design with security design.

Crémer (1987) shows that when V_i is constant and common knowledge, a seller can extract full rents by reimbursing a bidder for his opportunity cost, and demanding all equity. Substantively, our mechanism relies on very different forces: the ability to screen lower types with high equity offers and higher types with high cash offers. By contrast, Crémer's mechanism has no screening—the seller only offers one payment choice, leaving each bidder type indifferent between truthful reporting and deviating. Conceptually, a further difference exists. Crémer's construction reflects the insight that with constant opportunity costs, equity bids generate more revenues than cash, so a seller should "leverage" by going short on the "inferior" cash component and demanding 100% of the "superior" equity component. Applying this logic to a setting with severe adverse selection where equity bids generate less revenues than cash bids, one might posit that a seller should "leverage" by going short on the "inferior" equity component. In fact, our mechanism uses strictly positive combinations of both equity and cash, reflecting the intuition that positive mixtures of cash and equity can balance bidders' differential rents to zero when severe adverse selection obtains.[5]

Ekmekci, Kos, and Vohra (2016) consider the sale of a firm to a single buyer who is privately informed about cash flows and the benefits of control. The seller can offer a menu of cash-equity mixtures, and the buyer must obtain a minimum equity stake to gain control. They provide sufficient conditions under which it is optimal to have the buyer acquire either the minimum stake or all shares. In contrast, we show how combinations of equity and cash—where the resulting steepness varies with the bidder type—improves seller revenues, and how auction designs that use steeper-than-equity securities can help further.

Our work also provides a counterpoint to Deb and Mishra (2014). They show that if *utility* is contractible and the type space is finite, then any dominant strategy implementable social choice rule can be implemented via a combination of cash transfers and equity transfers of *utility*. This result does not hold in our setting in which opportunity costs (stand-alone

[5]We also extend this intuition on screening and differential rents to settings where full extraction via cash plus equity is impossible. We derive the optimal mixture and provide insights into the advantages of steeper-than-equity securities.

values) are not contractible. That is, focusing on cash plus equity is not without loss of generality: we identify settings with finite types where full extraction is impossible with equity plus cash, but the seller can still extract full rents using steeper securities plus cash, and the mechanism is dominant strategy incentive compatible.

Many researchers have proposed theories of takeovers in which the acquirer has the bargaining power to explain pieces of the empirical patterns that we highlight. However, they cannot reconcile some key features of the data. First, theories in which rational bidders have substantial bargaining power (e.g., they make take-it-or-leave-it offers) have difficulty explaining why, unconditionally, acquirer abnormal returns are so low. Second, it is difficult to reconcile the paucity of public rival bidders (BETa) with standard auction or sequential auction structures, even with preemptive bidding in which a single-bidder contest results when an initial bidder has a high valuation. Contests with a single public bidder occur over 90% of the time; and this would mean that costly preemption occurs even when the initial bidder has a low valuation (and the distribution of preempting types is similar to the unconditional distribution). In contrast, a single public bidder and the unbalanced division of surplus between target and acquirer both arise naturally in a decentralized target-designed takeover process described by BM and ENT that our model can capture. In this formulation, the target privately negotiates with each potential bidder, selects the bidder that would add the most to the target's value, and extracts all synergy gains even though only one public bid is observed.

Fishman (1989) develops a bidder-initiated theory of takeovers in which an initial bidder who is privately informed about his valuations can offer either cash or debt. He shows that a high-valuation bidder offers cash to discourage rivals from acquiring information about their valuations, and then competing for the target. As in our model, his model predicts that use of more cash is associated with higher acquirer returns. However, it does not predict higher cash components are associated with higher target returns (because cash use discourages valuable competition for the target even though it provides an initial premium). In addition, his model implies that cash is *less* likely to be used when there are multiple bids, contrary to the predictions of our model and the findings in Betton, Eckbo and Thorburn (BETb, 2008b) and BMb.

Eckbo, Giammarino and Heinkel (EGH 1990) study a signaling model of takeovers in which a single bidder and the seller are both privately informed of their valuations.[6] They

[6]See also Gorbenko and Malenko (2018), where cash-equity compositions arise due to signaling and

identify a separating equilibrium in which the cash-equity composition of the bidder's offer reveals his type. As in our paper, a higher bidder type offers more cash, and is associated with higher CARs for the acquirer. However, in their model a seller accepts an offer as long as it breaks even and the acquirer extracts a large share of the surplus, in contrast to our model and the data. In addition, our model predicts that the target CAR increases in cash, but their model does not. Finally, their single-bidder model is silent on the impact of multiple bidders, whereas ours predicts that multi-bidder settings are more likely to have cash offers.

Researchers have also posited behavioral explanations for both the observed lopsided sharing of takeover synergies, and the higher CARs for cash offers than equity offers, including bidder agency costs, managerial hubris and theories based on market misvaluation (leading to bidder opportunism, with bidders paying a mis-informed target with overpriced bidder stocks). However, EMT summarize evidence against these theories, (p2-3) "Recent empirical evidence indicates that targets do not receive overpriced bidder shares," and (p39) "there is little systematic evidence of poor post-acquisition long-run performance of stock-financed takeovers."EMT also find that cash-rich and cash-poor bidders are equally likely to use equity, suggesting that cash limitations are not a fundamental driver of these patterns.

More fundamentally, our analysis underscores that what drives empirical patterns in target and acquirer returns will vary according to whether takeovers are target or acquirer initiated; both forms are common (BM, ENT), but they reflect different economic forces.[7] This means that empirical work that does not control for the initiator mixes driving forces, confounding efforts to extract the true primitive relationships that would provide directions for theory. Our analysis further points to the need to account for the asset composition of offers.

2 The model

A group of $n \geq 1$ risk-neutral bidders (acquiring firms) competes to acquire an indivisible asset—a target firm. Each bidder i ($i = 1, ..., n$) privately observes its stand-alone valuation V_i, which is independently distributed according to a continuous and strictly positive probability density f_i with support $\left[\underline{V}_i, \bar{V}_i\right]$, where $0 \leq \underline{V}_i < \bar{V}_i < \infty$. If bidder i acquires the

financial constraint considerations.

[7] Gorbenko and Malenko (2019) develop the first model that seeks to endogenize the initiator of takeovers.

target, this creates a synergy in the joint firm. The synergy is stochastic, with a distribution that can depend on the bidder's identity and the bidder's stand-alone valuation type V_i. The expected value of the synergy, s_i, is an increasing, continuous, and twice-differentiable function of V_i.[8] Hence, conditional on the winner's type V_i, the expected value of the joint firm is $V_i + V_T + s_i(V_i)$, where V_T is the target's stand-alone value. The stand-alone value V_T and the functional forms of f_i and $s_i(\cdot)$ are assumed to be common knowledge.

For each bidder i, define

$$S_i = \{V_i | s_i(V_i) > s_j(\underline{V}_j) \text{ for all } j \neq i \text{ and } s_i(V_i) > 0\}.$$

S_i is the set of V_i for which there is a positive measure of realizations of bidders' types such that selling the asset to bidder i maximizes expected social surplus relative to selling to any other bidder or having the seller retain the asset. For each bidder i, we assume that S_i is non-empty to rule out uninteresting cases in which selling to i is never socially optimal.

Our base model assumes that the target is sold via mechanisms in which the winner pays with combinations of equity and cash. Without loss of generality, we consider direct-revelation mechanisms. Let $\mathbf{z} \equiv (z_1, ...z_n)$ be the vector of reported bidder types; let $W_i(\mathbf{z})$ be the probability that bidder i wins, where $\Sigma_i W_i(\mathbf{z}) \leq 1$; and when bidder i wins, let $Q_i(\mathbf{z}) \in [0, 1]$ be the equity share that bidder i retains and let $M_i(\mathbf{z}) \in (-\infty, \infty)$ be its cash payment. The target designs $W_i(\cdot)$, $Q_i(\cdot)$, and $M_i(\cdot)$ to maximize expected profit subject to the requirement that each bidder's response be incentive compatible and individually rational.

Discussion. Our model nests as a special case the setting with a single potential acquirer, where it is natural for the target and acquirer to negotiate privately about the terms of a deal including price and means of payment. In this negotiation, the target sketches out the trade-off an acquirer would face in terms of prices and composition of payment. With a single bidder, our model can also be used to analyze the opposite setting in which the target knows its stand-alone value and synergy, while the acquirer is uninformed but has the bargaining power.

Our general setting allows us to address equilibrium outcomes when there are multiple potential acquirers. This setting allows for heterogeneity across potential acquirers. For example, one acquirer might be far more likely to generate higher synergies than another—making

[8] We show that results extend when, conditional on V_i, slight uncertainty about synergies remains.

that acquirer more likely than others, with a single bidder resulting if it matches far better.

The key tension in our model is that one side (the bidders) has private information, and the other side (the seller) is uninformed, but has the bargaining power to commit to a mechanism design. Although we pose the model in a mergers and acquisition context, the analysis applies more generally, e.g., to oil and gas lease or timber lease auctions that feature cash payments plus equity payments in the form of royalties. One can reformulate our model as an auction in which rather than there being merger synergies, each bidder faces an opportunity cost of winning, V_i, that increases with the bidder's expected total valuation of the asset, v_i. The positive relationship between private valuations and opportunity costs captures the observation that a bidder with resources that would add more value to one project may alternatively productively employ them elsewhere. In such a formulation, because opportunity costs enter bidder payoffs negatively, the curvature restrictions on $V_i(v_i)$ required for full extraction are the opposite of those needed on $s_i(V_i)$, but the analysis is otherwise identical.

2.1 Analysis

We identify necessary and sufficient conditions under which a seller can extract full rent using combinations of equity and cash—the highest-NPV bidder wins if its NPV is positive and the seller retains the asset otherwise, and all bidder types earn zero rent. We first use an example to illustrate our mechanism.

Example 1. $n \geq 1$ ex-ante identical bidders have stand-alone valuations V distributed on $[1, 2]$ and expected synergy $s(V) = 1 + V - 0.1V^2$. The target's stand-alone value is zero.

We consider the mechanism in which: (1) each bidder reports a type $z \in [1, 2]$, the highest reported type wins, (2) ties are broken randomly, and (3) the winner pays a fraction $\frac{1-0.2z}{2-0.2z}$ of equity plus a cash payment of $1+0.1z^2$. Thus, the winner retains equity claim $1-\frac{1-0.2z}{2-0.2z}$ to the firm's residual joint value $V+s(V)-(1+0.1z^2)$ after making its cash payment and it forgoes its value V as a stand-alone entity. To work out a type V bidder's reporting strategy, let

$h(V;z)$ denote its expected profit (in excess of its stand-alone value) conditional on winning:

$$\begin{aligned} h(V;z) &= \left(1-\frac{1-0.2z}{2-0.2z}\right)\left(V+s(V)-\left(1+0.1z^2\right)\right)-V \\ &= \frac{1}{2-0.2z}\left(1+2V-0.1V^2-\left(1+0.1z^2\right)\right)-V \\ &= \frac{1}{2-0.2z}\left(-0.1V^2-0.1z^2+0.2zV\right) \\ &= -\frac{0.1}{2-0.2z}\left(V-z\right)^2. \end{aligned}$$

By inspection, the maximized value of $h(V;z)$ is zero for all V, and the maximum obtains when a bidder truthfully reports $z=V$; and $h(V;z)<0$ for all other values of z. Because a bidder's profit is zero when it loses, truthful bidding is optimal regardless of how other bidders bid, i.e., truthful bidding is a weakly dominant strategy. In equilibrium, all bidders receive zero rent, the highest NPV (synergy) bidder wins and the seller extracts full rents.

Our formal analysis now (1) identifies the conditions under which full extraction via equity plus cash is possible, and (2) explores what a seller can do to maximize revenue when these conditions do not hold. We first allow cash payments to be negative. We then identify the conditions needed for full extraction when cash payments are required to be positive.

Define $\mathbf{V}_{-i} \equiv (V_1,...,V_{i-1},V_{i+1},...,V_n)$; let $W_i(z_i,\mathbf{V}_{-i})$ be the probability bidder i wins when it reports being type z_i and other bidders report $\mathbf{v}_{-i}$; let $Q_i(z_i,\mathbf{v}_{-i})$ be the equity share that bidder i retains contingent on winning; and let $M_i(z_i,\mathbf{v}_{-i})\in(-\infty,\infty)$ be its cash payment. Next, define $G_i(z_i)$ to be the probability that bidder i wins when it reports having the stand-alone value z_i and all other bidders report truthfully:

$$G_i(z_i) \equiv \int W_i(z_i,\mathbf{V}_{-i}) f_{-i}(\mathbf{V}_{-i}) d\mathbf{v}_{-i}. \tag{1}$$

Similarly, define $q_i(z_i)$ to be the expected equity share that bidder i retains conditional on winning by reporting that it has stand-alone value z_i when all others report truthfully,

$$q_i(z_i)G_i(z_i) \equiv \int Q_i(z_i,\mathbf{V}_{-i}) W_i(z_i,\mathbf{V}_{-i}) f_{-i}(\mathbf{V}_{-i}) d\mathbf{V}_{-i}, \tag{2}$$

and define ω_i to be bidder i's unconditional expected cash payment,

$$\omega_i(z_i) \equiv \int M_i(z_i, \mathbf{V}_{-i}) W_i(z_i, \mathbf{V}_{-i}) f_{-i}(\mathbf{V}_{-i}) d\mathbf{V}_{-i}. \tag{3}$$

Let $h_i(V_i, z_i)$ be bidder i's expected profit when it has stand-alone value V_i but reports z_i, and all other bidders report truthfully:

$$h_i(V_i, z_i) \equiv \left[\left(V_i + s_i(V_i) + V_T\right) q_i(z_i) - V_i\right] G_i(z_i) - \omega_i(z_i) q_i(z_i). \tag{4}$$

Bidder i's equilibrium expected profit is $h_i\left(V_i, V_i\right)$. Incentive compatibility requires

$$h_i(V_i, V_i) = \max_{z_i} h_i(V_i, z_i). \tag{5}$$

By the envelope theorem,

$$h_i(V_i, V_i) = h_i(\underline{V}_i, \underline{V}_i) + \int_{\underline{v}_i}^{v_i} (q_i(t) + \frac{ds_i(t)}{dt} q_i(t) - 1) G_i(t) dt$$

and

$$\frac{dh_i(V_i, V_i)}{dV_i} = (q_i(V_i) + \frac{ds_i(V_i)}{dV_i} q_i(V_i) - 1) G_i(V_i). \tag{6}$$

We next identify necessary and sufficient conditions for a mechanism using combinations of equity and cash to exist that extracts full rents, where we first allow cash payments to be negative. If such a mechanism exists, then

$$h_i(V_i, V_i) = 0 \quad \text{for all } v_i, \tag{7}$$

and $G_i(V_i) > 0$ if and only if $V_i \in S_i$. For $z_i \in S_i$, divide both sides of (4) by $G_i(z_i)$ to define

$$\hat{h}_i(V_i, z_i) \equiv \frac{h_i(V_i, z_i)}{G_i(z_i)} = (V_i + s_i(V_i) + V_T) q_i(z_i) - V_i - \frac{\omega_i(z_i) q_i(z_i)}{G_i(z_i)}. \tag{8}$$

Here, $\hat{h}_i(V_i, z_i)$ is bidder i's expected profit conditional on winning when it has stand-alone value V_i but reports z_i, and all other bidders report truthfully. Equations (7) and (8) yield

$$\hat{h}_i\left(V_i, V_i\right) = 0, \quad \text{for all } v_i \in S_i. \tag{9}$$

Equation (7) and the incentive compatibility condition (5) yield

$$h_i(V_i, z_i) \leq 0, \quad \text{for all } z_i \neq v_i, \tag{10}$$

and hence $\hat{h}_i(V_i, z_i) \leq 0,$ for all $z_i \in S_i$. Thus, for all $V_i \in S_i$, we have

$$\hat{h}_i\left(V_i, V_i\right) = \max_{z_i \in S_i} \hat{h}_i\left(V_i, z_i\right) = \max_{z_i \in S_i} \left(\left(V_i + s_i\left(V_i\right) + V_T\right) q_i(z_i) - V_i - \frac{\omega_i(z_i) q_i(z_i)}{G_i(z_i)} \right). \tag{11}$$

Let v_i be the pre-merger value of the merged firm under bidder i's control, as a function of V_i:

$$v_i\left(V_i\right) \equiv V_i + s_i(V_i) + V_T. \tag{12}$$

Because v_i strictly increases in V_i, it is invertible. As a result, we can express V_i as a function of v_i. We use this to re-express (11) as

$$\hat{h}_i(V_i(v_i), V_i(v_i)) = \max_{z_i \in S_i} \hat{h}_i(V_i(v_i), z_i) = \max_{z_i \in S_i} \left(v_i q_i(z_i) - V_i(v_i) - \frac{\omega_i\left(z_i\right) q_i(z_i)}{G_i(z_i)} \right).$$

Adding $V_i(v_i)$ to both sides yields that, for all $V_i(v_i) \in S_i$,

$$\hat{h}_i(V_i(v_i), V_i(v_i)) + V_i(v_i) = \max_{z_i \in S_i} \left(v_i q_i(z_i) - \frac{\omega_i\left(z_i\right) q_i(z_i)}{G_i(z_i)} \right). \tag{13}$$

By (13), $\hat{h}_i(V_i(v_i), V_i(v_i)) + V_i(v_i)$ is the maximum of a family of affine functions of v_i, so it is weakly convex for $V_i(v_i) \in S_i$. Because $\hat{h}_i\left(V_i(v_i), V_i(v_i)\right) = 0$, $V_i(v_i)$ must be weakly convex in v_i, or equivalently v_i must be weakly concave in V_i, for $V_i \in S_i$. In turn, inspection of the definition of v_i in (12), reveals that $s_i(V_i)$ must be weakly concave in V_i.[9] The necessity of the concavity of $s_i(V_i)$ for full extraction reflects that if synergies rise faster than linearly then a higher V_i type can extract strictly positive rents from a cash-equity combination designed to extract all rents from lower valuation types. Thus, we have

Lemma 1 *If $s_i(V_i)$ is not weakly concave over $V_i \in S_i$ for all i, then combinations of equity plus cash cannot extract all surplus, even if the cash component is allowed to be negative.*

[9] If the set S_i is not connected, then the requirement translates to requiring that a weakly concave function of V_i go through $\big(s_i, s_i(V_i)\big)$ for all $V_i \in S_i$. The assumptions that $s_i(V_i)$ is twice differentiable and S_i is non-empty imply that S_i must be connected if (i) of Theorem 1 hold.

We now provide all necessary conditions for there to exist a mechanism employing combinations of equity and cash that extracts full rents. These conditions are also sufficient:

Theorem 1 *A mechanism using combinations of equity and (possibly negative) cash exists that extracts full rents if and only if for all bidders i: (i) if $V_i \in S_i$, then $s_i(V_i)$ is weakly concave in V_i; and (ii) if $V_i \notin S_i$, then $s_i(V_i) \leq s_i(z_i) + s_i'(z_i)(V_i - z_i)$ for $z_i \in S_i$.*

If bidders are ex-ante identical and synergies are always positive then $V_i \in S_i$. This special case renders condition (*ii*) moot. As a result, Theorem 1 takes a simpler form:

Corollary 1 *If bidders are ex-ante identical with $s\left(\underline{V}\right) > 0$, then a mechanism using combinations of equity and (possibly negative) cash exists that extracts full rents if and only if $s(V)$ is weakly concave in V for $V \in [\underline{V}, \bar{V}]$.*

Returning to the more general case of (possibly) heterogeneous bidders, we establish key properties of the optimal mechanism.

Proposition 1 *When conditions* (*i*) *and* (*ii*) *in Theorem 1 hold, the following dominant strategy incentive compatible mechanism extracts full rents (i.e., the highest synergy bidder wins and receives zero expected profit):*

- The winning rule is $W_i(V_i, \mathbf{V}_{-i}) = \begin{cases} 1 & \text{if} \quad V_i \in S_i \text{ and } s_i(V_i) > \max_{j \neq i} s_j(V_j) \\ 0 & \text{otherwise} \end{cases}$,

  and ties are broken arbitrarily.

- The winning bidder retains equity share

$$Q_i(V_i, \mathbf{V}_{-i}) = \frac{1}{1 + \frac{ds_i(V_i)}{dV_i}}, \tag{14}$$

  and makes cash payment

$$M_i(V_i, \mathbf{V}_{-i}) = s_i(V_i) + V_T - V_i \frac{ds_i(V_i)}{dV_i}. \tag{15}$$

Proof of Theorem 1 and Proposition 1: We first prove that when the conditions in Theorem 1 hold, the mechanism in Proposition 1 extracts full rents. If bidding is truthful, then substituting for $M_i(v_i, \mathbf{v}_{-i})$ and $q_i(v_i)$ in equation (4) reveals that a bidder's equilibrium payoff is zero, regardless of his valuation, i.e.,

$$[(V_i + s_i(V_i) + V_T - M_i(V_i, \mathbf{V}_{-i}))\, Q_i(V_i, \mathbf{V}_{-i}) - V_i]\, G_i(V_i) = 0.$$

To see that truthful bidding is an equilibrium, suppose type V_i bids as if it is type z_i. If $z_i \notin S_i$, the bidder would lose, so the deviation is not optimal. If $z_i \in S_i$, then by (14) and (15),

$$q_i(z_i) = \frac{1}{1 + \frac{ds_i(z_i)}{dz_i}} \tag{16}$$

and

$$\omega_i(z_i) = \left[s_i(z_i) + V_T - z_i \frac{ds_i(z_i)}{dz_i} \right] G_i(z_i). \tag{17}$$

Then by (4),

$$\begin{aligned} h_i(V_i, z_i) &= \left[\frac{V_i + s_i\,(V_i) + V_T}{1 + \frac{ds_i(z_i)}{dz_i}} - V_i \right] G_i(z_i) - \left[\frac{s_i(z_i) + V_T - z_i \frac{ds_i(z_i)}{dz_i}}{1 + \frac{ds_i(z_i)}{dz_i}} \right] G_i(z_i) \\ &= \left[s_i(V_i) - s_i(z_i) - \frac{ds_i(z_i)}{dz_i}\,(V_i - z_i) \right] \frac{G_i(z_i)}{1 + \frac{ds_i(z_i)}{dz_i}} \le 0. \end{aligned} \tag{18}$$

Inequality (18) holds for all $V_i \in \left[\underline{V}_i, \bar{V}_i\right]$: if $V_i \in S_i$, the inequality follows because the weak concavity of $s_i(\cdot)$ in condition (i) of Theorem 1 implies that $s_i(V_i) - s_i(z_i) - \frac{ds_i(z_i)}{dz_i}\,(V_i - z_i) \le 0$ regardless of whether $V_i \ge z_i$ or $V_i < z_i$; and if $V_i \neq S_i$, the inequality follows from part (ii) of Theorem 1. Thus, deviation is not profitable for all bidder types. Note also that (14) satisfies $Q_i(\mathbf{z}) \in [0,1]$. Hence, Proposition 1 and the "if" part of Theorem 1 are established.

To prove the "only if" part of Theorem 1, assume that a full-extraction mechanism exists. The necessity of condition (i) was proved in the text. To see that (ii) must hold, note that full extraction implies that for all types $V_i \notin S_i$, bidding as if the bidder's type is $z \in S_i$ must render a non-positive profit; by (18), $G_i(z_i) > 0$ and $\frac{ds_i(z_i)}{dz_i} \ge 0$, we have (ii). ∎

Condition (i) of Theorem 1 is, in essence, a decreasing-returns-to-scale condition on synergies as a function of a bidder's stand-alone valuation. It ensures that no type in S_i—the set of types for which the social surplus from selling the asset to them is positive—wants to mimic another type in S_i, and condition (ii) ensures that types not in S_i to whom the asset should not be sold do not want to mimic a type in S_i to whom the seller might want to sell the asset. Condition (ii) is implied if condition (i) also holds for $V_i \notin S_i$: Condition (ii) is weaker than requiring (i) to hold for $V_i \notin S_i$ (given that (i) is required for $V_i \in S_i$). The difference in the restrictiveness of the two conditions reflects that for any type *in* S_i, one must ensure that no other type wants to mimic it. In contrast, for types not in S_i, such a requirement is unnecessary because mimicking a type that is not in S_i always loses.

Corollary 2 *In the full extraction mechanism (Proposition 1), higher types pay with flatter securities. That is, the equity share paid falls with type and the cash payment rises with type:*

$$\frac{d}{dV_i}(1 - Q_i(V_i, \mathbf{V}_{-i})) = \frac{1}{(1 + \frac{ds_i(V_i)}{dV_i})^2}\frac{ds_i^2(V_i)}{dV_i} \leq 0 \ \ \textit{and} \ \ \frac{dM_i(V_i, \mathbf{V}_{-i})}{dV_i} = -V_i\frac{ds_i^2(V_i)}{dV_i} \geq 0.$$

This corollary reveals the general principle underlying the gains from mixing cash and equity: a seller can tailor its menu of possible cash-equity payments so that (1) higher types that expect to generate higher synergies (and have higher stand-alone values) choose a less steep mix that requires them to give up a smaller equity claim to those stand-alone values in return for a higher cash payment;[10] while (2) lower types choose steeper mixes because they care less about ceding greater equity claims to their lower stand-alone values, and more about the cash payment. The net effect is to reduce the differential rents between high and low types.[11]

The mechanism in Proposition 1 is dominant strategy incentive compatible. A bidder's winning payment depends only on its own report, and not those of other bidders; and if a bidder misreports and wins, its profit is non-positive. Thus, the mechanism extracts full rents even from a single bidder absent any competition, and bidders would not deviate even in a complete information setting with multiple bidders. Moreover, if $s_i(V_i)$ is strictly concave over $\left[\underline{V}_i, \bar{V}_i\right]$, the equilibrium enforces itself in a strict sense: bidder i would receive a strictly negative expected profit if it deviated and won.

The cash payments specified in Proposition 1 that are needed to extract all rents could

[10] In contrast, in equity auctions where stand-alone values (or costs in a royalty auction) are constant, a higher type always gives up a *larger* claim to outbid lower types.

[11] Corollary 2 and Proposition 1 generate several empirical implications that distinguish our model from the signaling model of EGH: (1) a smaller equity share, together with a larger cash payment, is associated with a higher bidder type; (2) a higher bidder type leads to higher returns for both acquirers and targets; and (3) our results apply to more than single bidder contests, predicting that cash bid are more likely when multiple bids are observed.

be negative. In practice, as DKS highlight, mechanisms with negative cash payments could have moral hazard issues. We now provide the additional necessary and sufficient condition for the optimal mechanism to specify that the winning bidder make a strictly positive cash payment, i.e., to have $\omega_i(V_i) > 0$ for all $V_i \in S_i$:

Corollary 3 *Suppose that conditions (i) and (ii) in Theorem 1 hold. Then, if*

$$\frac{d}{dV_i}\frac{s_i(V_i)+V_T}{V_i} < 0 \quad \textit{for all } i, \tag{19}$$

the mechanism specified in Proposition 1 uses strictly positive combinations of equity and cash and extracts all rents, generating strictly higher expected revenues than either pure equity or pure cash. If these conditions do not hold then no mechanism can extract full rents using combinations of equity and strictly positive cash.

Equation (19) is equivalent to the elasticity condition $\frac{d\ln(s_i(V_i)+V_T)}{d\ln V_i} < 1$; i.e., the bidder's valuation of the asset, $s_i(V_i) + V_T$, increases in the bidder's stand-alone value V_i less than unit elastically. It is the necessary and sufficient condition for bidding strategies in second-price equity auctions to be decreasing (Che and Kim 2010).[12] This condition implies that adverse selection is severe if bidders pay with equities alone, but Corollary 3 shows that this condition in fact allows the seller to extract full rents with equity plus strictly positive cash—that is, the seller can exploit the adverse selection to its advantage. Intuitively, when adverse selection is severe, the opportunity cost of a high valuation bidder—its stand-alone valuation—rises fast enough relative to the synergy creation that strictly positive mixtures of cash and equity balance bidders' differential rents to zero.

Implementation. The full extraction mechanism can be implemented via cash plus a *decreasing* royalty-rate. By Corollary 2, the equity share paid falls with V_i and the cash payment rises with V_i. Therefore, the equity share paid can be expressed as a strictly decreasing function of the cash payment. This means that the full extraction mechanism can be implemented by having each bidder i bid in cash, and setting a reserve price of

$$r_i \equiv s_i(V_i^*) + V_T - V_i^* \frac{ds_i(V_i^*)}{dV_i}, \tag{20}$$

[12] This follows because the bidding strategy in second-price equity auctions takes the form $1 - \frac{V_i}{V_i+s_i(V_i)+V_T}$. Indifference to winning means that bidder i bids to retain the share q_i that solves $q_i(V_i + s_i(V_i) + V_T) = V_i$.

which is the cash payment in (15) for type V_i^*, where V_i^* is the zero-synergy type ($V_i^* \equiv \underline{V}_i$ if all types have positive synergies). The bidder with the highest bid exceeding the reserve wins (given ex-ante identical bidders; the approach generalizes) and pays his own bid. However, for each cash bid made, a bidder must pay an additional equity share that is uniquely determined by his cash bid via the function determined above. As with the mechanism in Proposition 1, this implementation is dominant strategy incentive compatible.

To illustrate this implementation, we revisit Example 1. Consider a first-price auction in which a bidder offers cash c plus an associated equity share

$$e(c) = \begin{cases} 1 - \frac{1}{2-0.2\sqrt{10c-10}} & \text{if} \quad c \leq 3.5 \\ 0 & \text{if} \quad c > 3.5 \end{cases}. \tag{21}$$

The cash bid must weakly exceed a reserve price of 1.1, the highest cash bid wins and pays that bid plus the associated equity.

Let $\pi(c;V)$ be type V's expected profit conditional on winning. Then

$$\begin{aligned} \pi(c;V) &= (1-e(c))(V+s(V)-c)-V \\ &= (1-e(c))\left(1+2V-0.1V^2-c\right)-V. \end{aligned}$$

Direct calculation yields that for all $V \in [1,2]$, the maximum value of $\pi(c;V)$ is zero, which obtains at

$$c = 1+0.1V^2, \tag{22}$$

and that $\pi(c;V) < 0$ for all other values of c.[13]

Because a bidder that loses earns zero profit, bidding according to (22) is optimal regardless of how other bidders bid. In equilibrium, all bidders receive zero rent. Because (22) strictly increases in V, the highest valuation bidder wins and the NPV is strictly positive: the seller extracts full rents. This example highlights how, in our first-price auction, optimal combinations of cash with an equity share that declines in the cash bid yield an equilibrium

[13]When $c \in [1.1, 3.5]$, $\pi(c;V) = \frac{1}{2-0.2\sqrt{10c-10}}\left(1+2V-0.1V^2-c\right)-V$. Equation (22) is the unique solution to the first-order condition $\frac{\partial\pi(c;V)}{\partial c} = 0$ for $c \in [1.1, 3.5]$, which yields $\pi(c;V) = 0$ for all V. Furthermore, $\frac{\partial\pi(c;V)}{\partial c} > 0$ for $c \in [1.1, 1+0.1V^2)$, and $\frac{\partial\pi(c;V)}{\partial c} < 0$ for $c \in (1+0.1V^2, 3.5]$. When $c > 3.5$, $\pi(c;V) = 1+V-0.1V^2-c < 0$ for all $V \in [1,2]$.

in which bidders employ dominant strategies.[14]

In fact, to extract full rent, it suffices for a seller to be able to commit to reserve prices: the equity share paid falls with the cash payment, so the cash payment by bidder i can be expressed as a decreasing function $\kappa_i(\cdot)$ of the equity share. The following first-price auction with a decreasing reserve extracts full rents: each bidder i offers cash/equity combination, there is an e_i-dependent reserve price for the minimum cash bid: $\max\{\kappa_i(e_i), r_i\}$, where r_i is given in (20). This reserve price falls with the share e_i offered. The bidder with the highest cash bid exceeding the reserve wins (with ex-ante identical bidders; the approach generalizes). In Example 1, this equity-dependent reserve price is $\max\left\{0.1\left(10-\frac{5}{1-e}\right)^2+1, 1.1\right\}$. The first term inside the max operator is $\kappa(e)$, which is the inverse function of (21).

Robustness. Samuelson (1987) observes that existing mechanisms that extract all (or nearly all) surplus via contingent payments have robustness concerns. First, in such mechanisms bidders earn zero (or close to zero) rents regardless of whether they report truthfully, rendering them almost indifferent to reporting any values. But then any added factor (even pure white noise) can result in an inefficient bidder type being selected as the winner. Second, such mechanisms require the winning bidder to retain zero equity share, leaving the winner prone to moral hazard, with no incentive to take actions ex post that maximize firm value.

Our full extraction mechanism addresses both of these robustness concerns. First, if $s_i(V_i)$ is strictly concave, a bidder would receive a strictly negative expected profit from deviating and winning. This makes selection of the 'right' bidder robust to small noise. Second, our mechanism almost always does not require the winning bidder to give up all equity, mitigating moral hazard concerns. We now establish a third robustness property, deriving a lower bound on seller revenues when full extraction is impossible via combinations of cash and equity. We show that if $s_i(V_i)$ is not quite weakly concave, or if there is slight uncertainty about the value of s_i conditional on V_i, then combining equity and cash can still extract almost all surplus.

We establish a simple lower-bound on a seller's revenue for the case of a single bidder that would generate positive synergies, i.e., $s(V) > 0$ for all V, so that it is socially optimal

[14] In contrast, if bidders just pay with pure cash, pure equity, or cash plus a *constant* equity share, a first-price auction does not have a dominant strategy equilibrium.

to sell to any bidder type. That is, the maximum social welfare gain is

$$\Pi \equiv \int_{\underline{V}}^{\bar{V}} s(V) f(V) dV.$$

We now define what it means for the synergy function to be close to a (weakly concave) function, and then bound the revenue loss vis à vis basing the mechanism on that function.

Definition. Consider a strictly increasing function $s(\cdot)$ defined over $V \in [\underline{V}, \bar{V}]$, and a strictly increasing function $y(\cdot)$ defined over $V \in [\underline{V} - \frac{\sqrt{2}}{2}\epsilon, \bar{V} + \frac{\sqrt{2}}{2}\epsilon]$, where $\epsilon > 0$. Then $s(\cdot)$ is in the ϵ-neighborhood of $y(\cdot)$ if, for $V_1 \in [\underline{V} - \epsilon, \bar{V} + \epsilon]$ and $V_2 \in [\underline{V}, \bar{V}]$, $V_1 + y(V_1) = V_2 + s(V_2)$ implies that $(V_1 - V_2)^2 + (y(V_1) - s(V_2))^2 \leq \epsilon^2$.

Proposition 2 *If the expected synergy function $s(V)$ is in an ϵ-neighborhood of a weakly concave function $y(V)$, then an equity-plus-cash mechanism exists that generates expected profit of at least*

$$\Pi - \sqrt{2}\epsilon.$$

The bidder reports a type $V \in [\underline{V} - \frac{\sqrt{2}}{2}\epsilon, \bar{V} + \frac{\sqrt{2}}{2}\epsilon]$, always wins the asset, retains equity share

$$Q(V) = \frac{1}{1 + \frac{dy(V)}{dV}}, \tag{23}$$

and makes cash payment

$$M(V) = y(V) + V_T - V\frac{dy(V)}{dV} - \frac{\sqrt{2}}{2}\epsilon\left(1 + \frac{dy(V)}{dV}\right). \tag{24}$$

Proof: See appendix. ∎

Intuitively, for any bidder type with stand-alone value V and expected synergy $y(V)$, in the prescribed mechanism in Proposition 2 (with $\epsilon = 0$ in (24)), the bidder's expected profit is weakly higher reporting V than reporting anything else. The proof shows that if another type has the same total valuation $V + s$ (i.e., $V' + s' = V + s$ but $V' \neq V$ and $s' \neq s$), then that other type is also better off reporting V than reporting anything else. This leaves only to guarantee that individual rationality holds. Reducing the cash payment $M(V)$ by a constant only affects individual rationality constraints and not the incentive compatibility constraints. Thus, by properly choosing this constant (setting it to be $\frac{\sqrt{2}}{2}\epsilon$), both the incentive compatibility and individual rationality constraints can be satisfied.

Proposition 2 implies that when $s(V)$ is "almost" weakly concave, a seller can extract almost all surplus. Moreover, this result holds even if s does not evolve deterministically with V. Thus, even if the seller only knows approximately how s is related to V, our mechanism still delivers close to full extraction. With multiple bidders, an additional complication may arise because, with a mis-specified model ($y_i(\cdot)$ differs from $s_i(\cdot)$), the highest synergy bidder may not always be selected. However, as noted earlier, concavity helps our mechanism offset this selection effect, and the mechanism still extracts close to full rents when ϵ is small.

2.1.1 Optimal equity-plus-cash mechanism when s(V) is not concave

We next characterize how the optimal mechanism is affected when synergies are not a concave function of stand-alone values. To illustrate, we first consider a single bidder with three possible types (V_i, s_i), i=1,2,3, where $0 < V_1 < V_2 < V_3$, and $0 < s_1 < s_2 < s_3$. Let $f_i > 0$ be the probability of a type i bidder, where $f_1 + f_2 + f_3 = 1$. When

$$\tau \equiv \frac{s_3 - s_2}{V_3 - V_2} - \frac{s_2 - s_1}{V_2 - V_1} > 0, \tag{25}$$

concavity is violated, rendering full extraction via equity plus cash impossible. Here $\tau > 0$ measures the convexity of the synergy-value relationship, approaching zero as $\frac{s_3-s_2}{V_3-V_2} \to \frac{s_2-s_1}{V_2-V_1}$, making the relationship almost linear.

We now solve for the equity-plus-cash mechanism that maximizes the seller's expected profit. Without loss of generality, let the seller offer 3 contract choices $\{c_i, e_i; p_i\}$. A bidder that selects contract i wins with probability $p_i \in [0, 1]$, in which case it pays cash c_i and equity share $e_i \in [0, 1]$. Denote the profit of a type i bidder when it chooses contract j and wins by

$$\pi_{i,j} = (1 - e_j)\,(V_i + s_i + V_T - c_j) - V_i. \tag{26}$$

To obtain type i's unconditional expected profit if it picks contract j, multiply by p_j to obtain:

$$\Pi_{i,j} = p_j \pi_{i,j} \text{ for all } i \text{ and } j. \tag{27}$$

Incentive compatibility requires that

$$\Pi_{i,i} \geq \Pi_{i,j} \text{ for all } i \text{ and } j \neq i. \tag{28}$$

Individual rationality requires that

$$\Pi_{i,i} \geq 0 \text{ for all } i. \tag{29}$$

The seller's expected profit is

$$\Pi_s = \sum_{i=1}^{3} f_i p_i \left(e_i \left(V_i + s_i + V_T - c_j\right) + c_j - V_T\right). \tag{30}$$

The seller's problem is to identify the three contracts $\{c_i, e_i; p_i\}$ for $i = 1, 2, 3$ that maximize its expected profit (30) subject to (28) and (29).

Lemma 2 *If the synergy-valuation relationship is convex so that $\tau > 0$, then it is optimal to offer a single contract consisting of a fixed royalty rate $e \in (0, 1)$ and fixed cash, with $p = 1$.*

(i) If $\frac{f_2 s_2}{\tau}$ is small so that there are sufficiently few intermediate type 2s, then it is optimal to excludes type 2s and extract all surplus from types 1 and 3.

(ii) If $\frac{f_2 s_2}{\tau}$ is bigger, then it is optimal to extract full rents from type 2 and either type 1 or type 3, which necessitates giving rents to the other type.

Corollary 4 *When $s(V)$ is convex, it is optimal to make a take-it-or-leave-it offer with a fixed equity share plus a fixed cash payment.*

An online appendix provides the details on the construction and properties of the optimal contract. When $\frac{f_2 s_2}{\tau}$ is small, the expected rent gains from type 2 are small relative to the degree of convexity, making it optimal to extract full rents from types 1 and 3, at the cost of excluding type 2 to earn profits of $\Pi_s = f_1 s_1 + f_3 s_3$. The convex synergy-valuation relationship means that any contract that satisfies type 2's individual rationality condition would leave either type 1 or type 3 with strictly positive rents that would exceed what the seller could extract from type 2, making it optimal to exclude type 2. Once $\frac{f_2 s_2}{\tau}$ is large enough, it ceases to be optimal to exclude type 2s. The optimal design is then either to extract full rents from types 1 and 2, but obtain reduced rents from type 3; or, instead, to extract full rents from types 2 and 3, but obtain reduced rents from type 1. When f_3/f_1 is relatively small, it is optimal to leave rents for type 3; and when f_3/f_1 is larger, it is optimal to leave rents for type 1. In all three cases, the shortfall in seller profits from full extraction approaches zero as $\tau \to 0$.

With three types, the seller can always implement the optimal mechanism with a single contract.[15] We now show this property extends to a continuum of types as in our main model. With 3 types, we identified the optimal mechanism in the space of all feasible contracts. In particular, we allow for any p_i between zero and 1, and show that in the optimal mechanism, $p_i \in \{0; 1\}$. That is, with 3 types, a bidder type is either always excluded or always included in the optimal mechanism. With a continuum types, to ease analysis we restrict attention to the space of contracts with this feature, i.e., $p_V \in \{0; 1\}$ for all V. We show that it remains optimal to offer a single contract when $s(V)$ is convex.

Proposition 3 *Suppose $s(V)$ is strictly convex over $\left[\underline{V}, \bar{V}\right]$ and the contract space is $p_V \in \{0; 1\}$ for all V. The optimal mechanism has the following features:*

(i) There exist two bidder types $V_1 \leq V_2$ with $V_1, V_2 \in \left[\underline{V}, \bar{V}\right]$ such that all types in $[\underline{V} \leq V_1] \cup \left[V_2, \bar{V}\right]$ participate, but intermediate types in (V_1, V_2) are excluded. If $V_1 = V_2$, then all bidder types participate. Full rents are extracted from bidder types V_1 and V_2. All other participating types earn strictly positive rents.

(ii) The optimal mechanism is implemented by a single contract with:

$$e = \frac{k}{1+k} \quad \text{and} \quad c = V_T + s\left(V_1\right) - kV_1, \tag{31}$$

where

$$k \equiv \frac{s\left(V_2\right) - s\left(V_1\right)}{V_2 - V_1} \text{ if } V_1 < V_2; \text{ and } k \equiv \frac{ds(V)}{dV}|_{V=V_1} \text{ if } V_1 = V_2. \tag{32}$$

Proposition 3 characterizes the optimal mechanism when the synergy valuation relationship is convex. It establishes that the key features of optimal mechanism in the 3 type case are general in nature, reinforcing that the optimal mechanism has a simple implementation in which the seller makes a take-it-or-leave-it offer with a fixed royalty rate plus cash.

[15] In an online appendix we show that the optimal mechanism can also be implemented by multiple contracts. Such contracts have the feature that higher types pay (weakly) higher cash and less equity.

In sum, if $s(V)$ is concave, then a seller can extract all surplus using the optimal cash-equity menu; and if $s(V)$ is convex, then full extraction using cash and equity is not possible,[16] but the optimal mechanism takes a simple form. We now show that if $s(V)$ is approximately affine in V, i.e., if it is "close" to being both concave and convex, then a seller can extract almost all surplus using a combination of a fixed royalty rate plus cash.

Result 1: Consider $n > 1$ ex-ante identical bidders with stand-alone valuations distributed according to $F(V)$ on $[\underline{V}, \bar{V}]$ with associated synergies $s(V) > 0$. The maximum social welfare gain is $\Pi = \int_{\underline{V}}^{\bar{V}} s(V) d\left(F^n(V)\right)$. Suppose $s(\cdot)$ is in an ϵ-neighborhood of an affine function $y(V) = c + bV$, with $b < \frac{ds(V)}{dV}$ at all V. Then in a second-price auction in which bidders bid with fixed royalty rate α plus cash, expected seller profit increases in $\alpha \in \left[0, \frac{b}{1+b}\right]$ and is no less than $\Pi - \sqrt{2}\epsilon$ at $\alpha = \frac{b}{1+b}$.

Proof: See appendix. ■

Even if a seller can only commit to rejecting offers that do not include a minimal equity share $\alpha \in \left(0, \frac{b}{1+b}\right)$, then the mechanism still generates more revenues than pure cash. This reflects the fact that the differential rent of a high type over a low type (by (6)) of $\left(\frac{ds(V)}{dV}(1-\alpha) - 1\right) G(v)$ is positive and decreasing in $\alpha \in \left(0, \frac{b}{1+b}\right)$. Thus, being able to commit to even a small amount of equity improves the seller's revenue.

Concavity versus convexity. One can speculate that the nature of the synergy-stand-alone valuation relationship may depend in systematic ways on target characteristics. For example, the synergy gains from acquisitions of younger and smaller target firms might hinge sensitively on the acquirer's quality, giving rise to a convex synergy relation. Concretely, a risky start-up or growth firm might succeed hugely with a high quality merger acquirer (associated with higher privately-known stand-alone values), but risk failure with a low quality acquirer. Conversely, with a mature target, the value creation might depend less sensitively on its acquirer's quality, giving rise to a concave synergy-valuation relation.

This observation can possibly reconcile the listing effect. A wide literature has documented that in acquisitions of public targets, the acquirer's return is consistent with zero. For example, BETa's analysis of **every** control contest for publicly-traded US targets between

[16] We later show that even if $s(V)$ is convex, a seller may still be able to extract full rents by combining cash with steeper-than-equity securities: the convexity of steeper-than-equity security *payments* can offset convexity in the synergy-stand-alone valuation relationship.

1980 and 2005 suggests that, on average, public targets extract all synergy gains,[17] whereas in acquisitions of private targets, the acquirer's return are positive.[18] Similarly, Faccio, McConnell and Stolin (2006) examine returns to acquisitions of listed and unlisted targets in 17 Western European countries, finding that acquirers of listed targets earn an insignificant average abnormal return of -0.38%, while acquirers of un-listed targets earn a significant average abnormal return of 1.48%. Their abstract observes that "The fundamental reason for this listing effect ... remains elusive". Our analysis suggests an explanation: with public targets the synergy relation is likely to be concave, which facilitates full extraction; whereas for private targets, the relationship is likely to be convex, which leaves rents for the acquirer.

One can contemplate other reasons for this listing effect. Quite plausibly, there is substantial information asymmetry about private targets, but not public targets. We will show that this can also contribute to the listing effect. Alternatively, as we next consider, private targets may have less bargaining power than public targets.

2.1.2 Commitment and informal auctions

Our model presumes that a seller has full commitment power, able to reject any offer outside its designed menu. Consistent with this, BM document that public targets often privately contact potential bidders, soliciting bids, and that many public transactions are seller-initiated. The data also suggest that public targets have substantial commitment power, as indicated by the minimal rents of acquiring firms highlighted earlier.

When a seller has complete commitment power, we have shown that the full extraction mechanism can be implemented by having the bidders bid with cash and pay an associated equity share that declines with the cash bid. Moreover, as we established earlier and illustrated in an alternative solution to Example 1, a seller only needs to be able to commit to reserve prices to extract full rents: the seller can employ a first-price auction in which a bidder bids cash plus a royalty rate, where the reserve price on the cash bid decreases

[17] BETa calculate combined bidder and target abnormal returns by weighting bidder and target abnormal returns by their associated market capitalizations on day -42. For a publicly-traded target and acquirer, the estimated combined CAR (-41,2) is 1.78%, but the estimated acquirer CAR (-41,2) is an insignificant -0.22%.

[18] So, too, Fuller, Netter, and Stegemoller (2002) find acquirer returns in windows around bids are lower for public targets (insignificantly negative) than for private targets (positive and significant). See also Sundaram (2006), Moeller, Schlingemann, and Stulz (2007), or Bargeron, Schlingemann, Stulz, and Zutter (2007)).

with the royalty rate, and the highest cash bid wins. In practice, sellers in auctions do set reserve prices, committing not to sell unless a bid is sufficiently high. Indeed, widespread adoption of strong takeover defenses such as poison pills in the late 1980s have endowed targets with considerable commitment power, "halt[ing] the use of hostile bids" (BETa). As Povel and Singh (2006) observe, "Deal protection devices can be used to enhance a target's commitment to the [optimal selling] procedure."[19]

Still, a private target or a subsidiary of an acquirer may have less commitment power. This leads us to consider the extreme scenario often assumed by bidder-initiated theories in which the target lacks all commitment power, unable even to set a reserve price—i.e., the acquirer has complete freedom to select the cash/equity mix to offer, and the target cannot reject any offer that leaves it with non-negative expected profit. Outcomes are sharply altered. A target's complete inability to commit results in an informal auction, as in DKS, where bidders are free to offer any cash/equity combination, and the seller chooses the most attractive bid combination ex post. DKS show in their setting that bidders will use the flattest security possible. In an online appendix, we show that the logic of their argument extends to our setting, leading bidders to offer cash only with no equity component.

In the single bidder setting often seen in mergers, a complete inability to commit generates the counterfactual implication that the acquiring firm extracts all surplus. The single acquirer now has both an information advantage (it is privately informed) and all bargaining power, so it can ensure winning and extracting full rents by offering a cash bid of V_T, leaving the target with zero profit. With the added freedom to choose any cash/equity combination, it will do no worse; but it can do no better either because in equilibrium the target correctly infers an offer's monetary value, and hence will not earn a negative profit.[20] Indeed, if the target cannot set a reserve price, then it cannot do better even if it can commit to a particular equity valuation. That is, suppose the target evaluates an offer $\{c(V), e(V)\}$ by $c + f(c, e)\, e$, where $c(V)$ is the cash amount and $e(V)$ is the share of equity to be paid, and

[19] Ahern (2012) suggests that a target also has extensive bargaining power in a customer-supply situation when the acquirer relies more on the target as a supplier or customer than the converse.

[20] This setting contrasts with our model in which one side (the bidders) has private information, and the other side (the seller) has full bargaining/commitment power, where we show that the commitment power alone can completely eliminate the other side's information advantage. Translating this result to informal auctions where a seller is privately informed about its own stand-alone value and synergy, and the acquirer is uninformed but has full bargaining power, reveals that the acquirer can still extract full rents via our mechanism.

$f(c,e) \geq 0$ is a function that can potentially depend on c and e, accepting an offer as long as its valuation of the offer $c + f(c,e)\,e \geq V_T$. Once again, because offering a cash bid of V_T ensures full rent extraction, a bidder will do no worse than full extraction.[21]

Nonetheless, as we earlier derived, a target may need only a little commitment to do well. A target that can commit to a fixed royalty rate plus cash can extract close to full rents when $s(V)$ is approximately affine. Further, when synergies are a convex function of stand-alone values, a seller cannot extract all surplus, but it can do as well as it possibly can if it can commit to a fixed royalty rate plus cash, a mechanism used in many economic transactions.

2.1.3 Two-sided private information

In an online appendix, we analyze a simple setting in which both the seller and buyer have private information about their stand-alone values. We consider a scenario with two possible bidder valuations and two possible seller valuations, where an increased spread in the possible valuations represents an increase in asymmetric information.

With two-sided private information, the menu of contracts that a seller offers may signal information to a buyer about a seller's stand-alone value. The seller cannot extract all surplus in a separating equilibrium, i.e., when different seller types offer distinct menus, revealing their types. Were it possible, then each seller type would earn the associated expected synergy as rent. But then it would be profitable for a low stand-alone valuation type to mimic a high stand-alone valuation type to earn more rents, a contradiction.

Full extraction is, however, possible in a pooling equilibrium in which both seller types offer the same menu of contracts, *as long as* information asymmetry about a seller's stand-alone value is not large: the seller uses the mechanism in Theorem 1 and Proposition 1, with the seller's expected stand-alone value replacing V_T. The seller's expected profit is the expected synergy. However, rents differ conditional on seller type: a low type earns more than full extraction, while a high type earns less. Once information asymmetry about a seller's stand-alone value is high enough, full extraction ceases to be possible: a high type seller's rents would be too low, giving it incentives to deviate, leading to strictly positive bidder rents.

[21] Indeed, if $f(c,e)$ is improperly set so that the target's evaluation of an offer's value is incorrect, a bidder might do better, leaving the target with strictly negative profit.

Thus, increased information asymmetry about a target, for example, when the target is private rather than public, can lead to increased acquirer rents, consistent with the data.[22] However, in contrast to this prediction of our target-initiated model, when the acquirer has the bargaining power, increased information asymmetry about the target *raises* the target's information advantage, counterfactually *reducing* the acquirer's return.

2.2 Combining cash with general securities

We next allow a seller to combine cash with general securities, focusing on a single bidder.[23] We show that combining cash with steeper-than-equity securities can be even more effective at rent extraction than combining cash with equity.

We first illustrate how a dominant strategy incentive compatible mechanism employing combinations of call options and cash can extract all surplus when equity plus cash cannot, underscoring how steeper securities can help in the rent extraction.

Example 2. A single bidder has possible stand-alone values, $V \in \{2, 2.6, 3\}$, each with strictly positive probability, with associated expected synergies of $s(2) = 4$, $s(2.6) = 4.4$, and $s(3) = 5$, making expected synergies a convex function of stand-alone values. The realized synergy of a type with expected synergy s is uniformly distributed on $[s-1, s+1]$. The target's stand-alone value is zero.

By Theorem 1, the convex synergy relationship means that full extraction is impossible using equity plus cash. We now show that the seller can extract full rents by combining even steeper securities with cash, using call options plus cash. The payment by a bidder who

- reports $V = 2$ consists of a call option with strike 2.25 plus 3.75 in cash;
- reports $V = 2.6$ consists of a call option with strike 2.85 plus 4.15 in cash;
- reports $V = 3$ consists of a call option with strike 3.25 plus 4.75 in cash.

When cash flow is uniformly distributed over $[v-1, v+1]$, where $v > 1$, the expected payment for a call option with strike k is: $v - k$ if $k \leq v - 1$; 0.25 if $v = k$; and 0 if $k \geq v + 1$.

[22] This fact can also be reconciled by public targets having more bargaining power than private targets.

[23] The optimal mechanism generalizes to allow the seller to extract full rents from multiple bidders, while maintaining dominant strategy incentive compatibility.

Substitution into a bidder's expected profits reveals that truth telling results in zero payoffs and deviating results in strictly negative payoffs.[24]

Thus, the mechanism is incentive compatible and all rents are extracted. In the example, were a bidder cash constrained and only able to pay with securities, the cash payment could be replaced by a debt payment.[25] The result is a hybrid security of debt plus call option: when cash flow is low, it is debt; and when cash flow is high, it is a call option. Concretely, paying "a call option with strike 2.25 plus 3.75 in cash" can be replaced by a combination of debt with face value 3.75 and call options with strike 6: $S(y) = g(y) + h(y)$, where $g(y) = min\{y, 3.75\}$ and $h(y) = max\{0, y - 6\}$; paying "a call option with strike 2.85 plus 4.15 in cash" can be replaced by a combination of debt with face value 4.15 and a call option with strike 7; paying "a call option with strike 3.25 plus 4.75 in cash" can be replaced by a combination of debt with face value 4.75 and a call option with strike 8.[26]

We now turn to our general analysis. With equity payments, only expected cash flows matter. With general securities, the payment's value depends on the details of the cash flow distribution, so more structure is needed. We assume that a type V bidder has expected synergy $s(V)$, but the realized synergy $\widetilde{s}$ is stochastic. Thus, the cash flows generated by the merged firm $y \equiv \widetilde{s} + V + V_T$, are stochastic with an expected value of $s(V) + V + V_T$.

It eases analysis to use a transformation. The expected cash flow of the target under the bidder's control, $v(V) = E[y|v] = V + s(V) + V_T$, is strictly increasing in V and hence invertible. Thus, V can be expressed as a function of v, which we denote as $V(v)$, and we use v instead of V to denote bidder type. Note that $\frac{ds(V)}{dV} > 0$ implies that $\frac{dv}{dV} > 1$, so

[24] For instance, consider a type $V = 2.6$ bidder. If it truthfully reports $V = 2.6$, then after paying 4.15 in cash, the total cash flow of the joint firm is uniformly distributed over $[2.85 - 1, 2.85 + 1]$. The value of the call option is 0.25, so the payoff of 2.85-0.25=2.6 just covers the bidder's stand-alone valuation. If it reports $V = 2$, then after paying 3.75 in cash, total cash flow is uniformly distributed over $[3.25 - 1, 3.25 + 1]$. The call option has a value of 1, so the bidder's payoff of 3.25-1<2.6 fails to cover its stand-alone valuation. If it reports $V = 3$, then after paying 4.75 in cash, total cash flow is uniformly distributed over $[2.25 - 1, 2.25 + 1]$. The call option has a value of 0, so the payoff of 2.25<2.6 fails to cover the bidder's stand-alone valuation.

[25] In this example the face value of the debt equals the corresponding cash payment, because the dispersion in the distribution of cash flows is not too great. More generally, with more dispersed cash flows, the face value of the debt or the strike value of the call may need to be adjusted.

[26] Such securities are piece-wise linear. Payments in the form of piece-wise linear securities are often used in takeovers of private targets: Cain, Denis and Denis (2011) report that "The typical earnout payment is a linear or a stepwise function of the target's performance (subject to a maximum) over the subsequent one to three years. The earnout payments are potentially quite large; on average, if the maximum earnout is paid, it would amount to 33% of the total transaction value."

$\frac{dV(v)}{dv} \in (0,1)$ for all v, and the concavity of $s(V)$ is equivalent to the *convexity* of $V(v)$. Define $\underline{v} \equiv \underline{V}+s\left(\underline{V}\right)+V_T$ and $\bar{v} \equiv \bar{V}+s\left(\bar{V}\right)+V_T$. We assume that expected synergies are positive for all bidder types, and that $\underline{V} \geq s\left(\bar{V}\right)$, so the acquirer has enough internal funds to pay cash.

We assume that cash flows are distributed according to a variant of DKS's lead example:

Assumption 1 *The cash flow is $y = \theta\left(v - \hat{v}\right) + \hat{v}$. θ is distributed over $(0, \infty)$ with a mean of 1 and independent of v, log(θ) has a log-concave density, and $\hat{v} \in [V_T + s\left(\bar{V}\right), \underline{v})$ is a constant independent of v.*

This structure has the property that $\mathrm{E}(y) = v$, and it guarantees sMLRP: the family of probability density functions for the distribution of cash flow y, conditional on bidder type, has the *strict monotone likelihood ratio property (sMLRP)*: a higher type means good news. In DKS, $\hat{v} = \dot{0}$. Here, $\hat{v} \in [V_T + s\left(\bar{V}\right), \underline{v})$ ensures that the lower bound of cash flow is high enough that after cash payment, the residual cash flows remain nonnegative.

We consider an *ordered set of securities*, $\left\{S(t,\cdot); t \in \left(\underline{t},\bar{t}\right)\right\}$, as in DKS. Each security in such a set is indexed by t: for each t, $S(t,\cdot)$ specifies the payoff of the security as a function of the stochastic cash flows generated by the asset. For example, for the set of equity securities, the index t is the equity fraction and the payoff function is $S(t,y) = ty$.

Assumption 2 *(i) for all t and $y \in [0,\infty)$, both $S(t,y)$ and $y - S(t,y)$ weakly increase in y with $0 \leq S(t,y) \leq y$.*

(ii) If $t_1 > t_2$, then $S\left(t_1,y\right) \geq S\left(t_2,y\right)$ for all $y \in (0,\infty)$, and $S\left(t_1,y\right) - S\left(t_2,y\right)$ weakly increases in $y \in (0,\infty)$.

Part (ii) says that payments increase in the security index, and that the difference in payments (weakly) widen as the cash flow rises.

The bidder pays with combinations of cash and the security. The security payment is nonnegative, so we can assume that cash payments do not exceed $s\left(\bar{V}\right) + V_T$: a bidder that paid more than this would earn negative profits. When a type v bidder pays cash $M \leq s\left(\bar{V}\right) + V_T$ plus a security $S(t,\cdot)$ let $ES(t,v;M) \equiv E\left[S(t,y-M)|v\right]$ denote its expected security payment, where $y(v)$ is the (before cash payment) stochastic cash flow from the joint firm. We assume that for all M, $ES(t,v;M)$ is strictly increasing in the security index t, differentiable in t, and twice differentiable in v. Denote the partial derivatives with respect to v by $ES_v(t, v; M)$, and the second derivative with respect to v by $ES_{vv}(t, v; M)$. We assume that the lower bound $\underline{t}$ corresponds to zero payment and the upper bound $\bar{t}$ corresponds to full payment. That is, for any v, any M, and any $\epsilon > 0$, there exist t^* and t^{**} such that $ES\left(t^*, v; M\right) < \epsilon$, $ES_v\left(t^*, v; M\right) < \epsilon$, $ES\left(t^{**}, v; M\right) > v - \epsilon$, and $ES_v\left(t^*, v; M\right) > 1 - \epsilon$. This assumption ensures a sufficient range in the security index. For instance, for call options, $\underline{t}$ corresponds to an infinite strike price and $\bar{t}$ corresponds to a zero strike. Standard securities (e.g., debt, equity, or call options) satisfy all of these conditions.

We consider direct-revelation mechanisms featuring a security plus cash. We first leave the sign of cash unrestricted. Let $W(z)$ be the probability that the bidder wins if it reports being type z; and, if it wins, let $t(z)$ be the index of the security paid and let $M(z) \in \left(-\infty, s\left(\bar{V}\right) + V_T\right]$ be the cash payment.

The bidder's expected profit when it has valuation V, but reports z is

$$h\left(v, z\right) = \left(v - V(v) - ES\left(t(z), v; M(z)\right) - M(z)\right) W(z). \tag{33}$$

If a full-extraction mechanism exists, then $h(v, v) = 0$ and $W(v) = 1$ for all v (recall that all bidder types have positive NPV). Then

$$h\left(v, z\right) = v - V(v) - ES\left(t(z), v; M(z)\right) - M(z). \tag{34}$$

Given our full extraction focus, we use (34) in lieu of (33) for h. Incentive compatibility yields

$$v \in \arg\max_z h(V, z), \quad \text{for all } v. \tag{35}$$

Thus, $h(v, z) \leq 0$ for all $z \neq v$, which, combined with $h(z, z) = 0$ for all z, yields

$$z \in \arg\max_v h(V, z), \quad \text{for all } z. \tag{36}$$

Equation (36) holds when a mechanism extracts full rents (or more generally when all bidder types receive the same rent). It is instructive to examine its relation to (35). Necessary and sufficient conditions for a mechanism to extract full rents are $W(v) = 1$, $h(v, v) = 0$ for all v, and the incentive compatibility condition (35). A similar relation holds for (36):

Lemma 3 *A mechanism extracts full rents if and only if $W(v) = 1$, $h(v, v) = 0$ for all v, and (36) holds.*

This follows directly: When $h(v, v) = 0$ for all v, (35) and (36) imply each other. Lemma 3 simplifies identification of the conditions for full extraction vis à vis working directly with (35) because, although the first-order condition of (36) is the same as the standard envelope condition, its second-order condition is more tractable than that of (35). Equation (34) and the first-order condition for (36), $\frac{\partial h(v,z)}{\partial v}|_{v=z} = 0$ yield (upon substituting v for z):

$$1 - V'(v) - ES_v\left(t(v), v; M(v)\right) = 0, \quad \text{for all } v. \tag{37}$$

The second-order condition of (36) gives

$$\frac{\partial^2 h\left(v, z\right)}{\partial v^2}|_{v=z} = -\frac{d^2 V(z)}{dz^2} - ES_{vv}\left(t(z), z; M(z)\right) \leq 0,$$

which, upon replacing z with v, yields a necessary condition for full extraction:

$$\frac{d^2 V(v)}{dv^2} \geq -ES_{vv}\left(t(v), v; M(v)\right) \tag{38}$$

for all v. When the security is equity, the right-hand side of (38) is zero: (38) just says that $V(v)$ must be weakly convex (i.e., $s(V)$ is weakly concave). By Theorem 1, this is also sufficient for full extraction via equity plus cash. For a general set of ordered securities, matters are more complicated: unlike with equities, where each security can be expressed as a "fraction" of a base-security, for a general set such as the family of call options indexed by different strikes, securities cannot be expressed as fractions of each other, complicating the analysis. This leads us to identify necessary and sufficient conditions separately and derive their implications.

The necessary condition (38) yields the following:

Corollary 5 *$V(v)$ must be strictly convex ($\frac{dV}{dv} > 0$ for all v), for combinations of (possibly negative) cash and security to extract full rent if the payoff function of each security $S(t, \cdot)$ is strictly concave.*

Proof: See appendix. ■

From Lemma 5 in DKS, if every security in an ordered set has a concave payoff function, then the set is less steep than equity. Corollary 5 shows that such securities tighten the curvature requirement on $V(v)$ for full extraction vis à vis equity.

We now identify sufficient conditions for full extraction. Specifically, we show that even when $V(v)$ is concave, as long as it is not 'too' concave, combinations of steeper-than-equity securities plus cash can extract full rents, even though cash plus equity cannot.

We focus attention to a set $\mathcal{A}$ of steeper-than-equity ordered securities. A set $\left\{S(t,\cdot);t\in\left(\underline{t},\bar{t}\right)\right\}$ of ordered securities is in $\mathcal{A}$ if and only if it is either the set of call options, or the payoff function of each security $S(t,\cdot)$ is strictly convex over $(0,\infty)$.

The full extraction condition $h(v,v)=0$ yields:

$$v-V(v)-ES\left(t(v),v;M(v)\right)-M(v)=0. \tag{39}$$

Lemma 4 *There exists at least one mechanism that satisfies local IC, and would extract full rents if it also satisfies global IC. That is, for all $v\in[\underline{v},\bar{v}]$, the system of equations (37) and (39) has at least one solution for $(t;M)$.*

Proof: See appendix. ■

To ensure global incentive compatibility of the mechanism, we define a measure of the minimum average second derivative of expected revenues generated by the security indexed by $t(v)$, where the average is taken over points between v and any other point $z\in[\underline{v},\bar{v}]$:

$$k(v)\equiv\min_{z\neq v,z\in[\underline{v},\bar{v}]}\frac{\int_v^z ES_{vv}\left(t(z),x;M(z)\right)dx}{z-v}, \tag{40}$$

where $(t(z);M\left(z\right))$ solves (37) and (39) for type z. Because the minimum of averages is no less than the minimum of individual elements,

$$k(v)\geq\min_{v,z\in[\underline{v},\bar{v}]}ES_{vv}\left(t(z),v;M(z)\right). \tag{41}$$

Define the lower bound on $k(v)$ by

$$K\equiv\min_{v\in[\underline{v},\bar{v}]}k(v). \tag{42}$$

We next provide sufficient conditions for K to be positive.

Lemma 5 *Consider any ordered set of securities in $\mathcal{A}$. If a mechanism that combines cash and these securities satisfies (37) and (39), then $K > 0$.*

Proof: See appendix. ■

From DKS, if every security in an ordered set has a convex payoff function, then the set is steeper than equity. The steeper is the security, the larger is $k(v)$. We now show that this facilitates full extraction.

Theorem 2 *Consider any ordered set of securities in $\mathcal{A}$. If a mechanism that combines cash and these securities satisfies (37) and (39), and $V''(v) \geq -K$, then that mechanism extracts full rents. That is, the mechanism is globally incentive compatible.*

Corollary 6 *Suppose bidding strategies would be strictly decreasing in a pure second-price security-bid auction (without cash payment) for the ordered set of securities in Theorem 2. Then the mechanism in Theorem 2 features strictly positive cash payments and securities, and generates strictly higher revenues than pure cash or pure securities.*

Proofs: See appendix. ■

The condition in Theorem 2 that $V''(v) \geq -K$ for all v is sufficient for full extraction via security plus cash. Equity has a linear payoff function, so $K = 0$. Thus, with equity, the condition requires $V''(v) \geq 0$, which is precisely condition (i) in our equity analysis in Theorem 1. When the set of securities is steeper than equities, K is positive by Lemma 5, relaxing the convexity requirement on $V(\cdot)$ needed for full extraction. Intuitively, greater curvature in a security's expected payoff compensates for a lack of convexity in $V(\cdot)$ (i.e., concavity in $s(V)$).

The greater curvature in the securities also relaxes the requirement in Corollary 6 for full extraction with positive cash, which requires that bidding strategies be decreasing. Steeper securities facilitate this: CK show that if equilibrium bidding strategies are decreasing for one class of ordered securities, then they are decreasing for all steeper securities.

Collectively, our findings indicate that the seller does best by combining cash with the steepest security. Tailored to different types, the mix creates wider variation in steepness, which helps reduce the differential rents between bidder types, ensuring both the global incentive compatibility of the mechanism and the positivity of the cash payment.

3 Conclusion

Returns to target and acquiring firms in takeovers offer puzzles. Multiple public bidders are rare, but targets receive almost all rents, and mean acquirer returns are very low. The cash-equity mix also matters: acquirer *and* target returns rise with the cash share of an offer. Our theory of target-initiated takeover processes reconciles these and other empirical regularities.

We consider a setting in which potential merger partners are privately informed of their stand-alone values and merger synergies, and those with higher stand-alone values also tend to generate higher expected synergies. We show that despite its information disadvantage, a target can design a payment scheme with combinations of cash and equity that extracts all rents if synergies are concavely-related to stand-alone values, exploiting a reluctance of high-valuation acquirers to cede equity claims that leads them to bid more cash. Our model delivers the result that both acquirer and target returns rise with the cash share of an offer because greater cash components reveal (1) higher synergies from which the target exclusively benefits, and (2) more positive information about the acquirer's stand-alone value, causing the acquirer to experience greater abnormal returns.

Our mechanism applies to decentralized processes in which a target privately approaches each potential acquirer, sets the terms of payment contingent on that bidder winning, and then selects the bidder that would add the most to the target's value. As a result, multiple public bidders can be rare. The robust selling mechanism is dominant strategy incentive compatible—bidders need not know anything about rivals—and a target extracts almost all rents if the synergy-valuation relationship is only approximately concave. When the synergy-valuation relationship is convex, the optimal design for cash/equity combination is simple: bidders bid cash plus a fixed equity share. Moreover, in such instances, we show that combining cash with even steeper securities like call options can help the seller further.

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5 Appendix

Proof of Corollary 3: Observe that when the conditions in Theorem 1 and (19) hold, the mechanism specified in Proposition 1 extracts full rents. Further, by (19), the cash component in (15), $s_i(V_i) + V_T - V_i \frac{ds_i(V_i)}{dV_i} = V_i^2 \frac{d}{dV_i} \frac{s_i(V_i)+V_T}{V_i}$, is strictly positive. Moreover, a seller must use strictly positive amounts of both cash and equity to extract full rents.

To prove the converse part of Corollary 3, suppose a full-extraction mechanism exists that employs strictly positive cash. For all $V_i \in S_i$, full extraction implies $G_i(v_i) > 0$ and $\frac{dh_i(V_i,V_i)}{dV_i} = 0$, which yields $q_i(V_i) = \frac{1}{1+\frac{ds_i(V_i)}{dV_i}}$ by (6). By $h_i(V_i, V_i) = 0$ and (4),

$$\omega_i(V_i) = \left(1 + \frac{ds_i(V_i)}{dV_i}\right) \left[(V_i + s_i(V_i) + V_T) \frac{1}{1 + \frac{ds_i(V_i)}{dV_i}} - V_i\right] G_i(V_i).$$

Since the cash component is positive, $\omega_i(V_i) > 0$, and hence

$$(V_i + s_i(V_i) + V_T) \frac{1}{1 + \frac{ds_i(V_i)}{dV_i}} - V_i > 0,$$

which yields (19). ■

Proof of Proposition 2: By (23) and (24), the expected profit of a type V bidder who reports as type z is

$$\begin{aligned} h(V, z) &= \frac{1}{1 + \frac{dy(z)}{dz}} \left(V + s(V) - \left(y(z) - z\frac{dy(z)}{dz} - \frac{\sqrt{2}}{2}\epsilon \left(1 + \frac{dy(z)}{dz}\right)\right)\right) - V && (43) \\ &= h^*(V^*, z) + \delta, && (44) \end{aligned}$$

where

$$h^*(V^*, z) \equiv \frac{(V^* + y(V^*)) - (y(z) + z)}{1 + \frac{dy(z)}{dz}} + z - V^*,$$

V^* is the solution to $V^* + y(V^*) = V + s(V)$, and $\delta \equiv V^* - V + \frac{\sqrt{2}}{2}\epsilon$ is independent of z. The same steps as in the proof of inequality (18) in the proof of Theorem 1 and Proposition 1 yield that for any given V^* we have (1) $z = V^*$ maximizes $h^*(V^*, z)$, and (2) $h^*(V^*, z = V^*) = 0$. Hence, reporting $z = V^*$ maximizes $h(V, z)$. Further, (43) yields $h(V, z) = \delta$, and $\delta \in \left[0, \sqrt{2}\epsilon\right]$ by the premise that $s(V)$ is in an ϵ-neighborhood of $y(V)$.

Hence, reporting $z = V^*$ satisfies individual rationality ($h(V, z = V^*) \geq 0$) and leaves each bidder type with rents no greater than $\sqrt{2}\epsilon$. ■

Proof of Proposition 3: Consider an optimal mechanism, $\{e(V), c(V); p(V)\}$. Let κ be the set of V for which $p_V = 1$. Let $h(V, z)$ be the expected profit of a bidder with stand-alone value V that reports $z \in \kappa$:

$$\begin{aligned} h(V, z) &= (1 - e(z))\,(V + s\,(V) + V_T - c(z)) - V \\ &= s(V) + V_T - (e(z)\,(V + s(V) + V_T) + (1 - e(z))\,c(z))\,. \end{aligned} \tag{45}$$

Define

$$h^*(V) \;\equiv\; \max_{z\in\kappa} h\,(V, z) \;=\; s(V) + V_T - \min_{z\in\kappa}\,(e(z)\,(V + s(V) + V_T) + (1 - e(z))\,c(z))\,.$$

A type V bidder type accepts a contract if and only if $h^*(V) \geq 0$.

To simplify solution we exploit the fact that the bidder's total gross of cash payment valuation of the target, $v\,(V) \equiv V + s(V) + V_T$, is strictly increasing in V and hence invertible. Thus, we can express V as a function of v, where the strict convexity of $s(V)$ is equivalent to the strict concavity of $V(v)$. Define $\underline{v} \equiv \underline{V} + s\,(\underline{V}) + V_T$ and $\bar{v} \equiv \bar{V} + s\left(\bar{V}\right) + V_T$.

Using $s(V) = v - V(v) - V_T$, we express $h^*(V)$ as a function of v:

$$h^{**}(v) \equiv h^*\,(V(v)) = \eta_a(v) - \eta_b(v),$$

where

$$\eta_a(v) \equiv v - V(v) \quad \text{and} \quad \eta_b(v) \equiv \min_{z\in\kappa}\,(e(z)v + (1 - e(z))\,c(z))\,. \tag{46}$$

Bidder type v participates if and only if $h^{**}(v) \geq 0$, and its expected profit when it partipates is $h^{**}(v)$. Denote the set of bidder types that participate by κ^*, which is a subset of κ (the two sets may or may not coincide). Then the seller's expected profit is

$$\pi_s = \int_\kappa s(V) f(V) dv - \int_\kappa h^{**}\,(v(V))\, f(V) dv, \tag{47}$$

where the first term is the social welfare gain, and the second term is the bidder's rents.

Next we show that a single contract can deliver the seller expected profit of at least (47).

$V(v)$ is strictly concave, so $\eta_a(v)$ is strictly convex in v. Further, $\eta_b(v)$ is weakly concave because it is the minimum of a family of affine functions. Thus, $h^{**}(v)$ is strictly convex, so there are at most two solutions to $h^{**}(v) = 0$ over $v \in [\underline{v}, \bar{v}]$. There are three possible cases:

Case 1: $h^{**}(v) = 0$ has two solutions v_1 and v_2 in $[\underline{v}, \bar{v}]$, where $v_1 < v_2$. That is, $\eta_a(v_i) = \eta_b(v_i)$ for $i = 1, 2$. Then strict convexity of $h^{**}(v)$ yields

$$h^{**}(v) < 0 \text{ for } v \in (v_1, v_2)\,; \text{ and } h^{**}(v) > 0 \text{ for } v < v_1 \text{ and } v > v_2. \tag{48}$$

Consider the single contract that takes the form specified in (31), with $V_i = V(v_i)$ $(i = 1, 2)$ in (32). Denote the expected profit of a type v bidder from this contract by $\hat{h}(v)$. Then

$$\hat{h}(v) = \eta_a(v) - \hat{\eta}_b(v), \tag{49}$$

where $\hat{\eta}_b(v)$ is given by the right-hand side of (46) with the contracts in the posited optimal mechanism replaced by the single contract:

$$\hat{\eta}_b(v) = \frac{s(V_2) - s(V_1)}{V_2 - V_1 + s(V_2) - s(V_1)} \left(v - V_1 - V_T - s(V_1)\right) + V_T + s(V_1)\,.$$

Recall that $v_i = V_i + s_i(V_i) + V_T$. Substituting yields

$$\hat{\eta}_b(v_i) = \eta_a(v_i) \text{ for } i = 1, 2. \tag{50}$$

Thus, $\hat{h}(v_i) = 0$ and

$$\hat{\eta}_b(v_i) = \eta_b(v_i) \text{ for } i = 1, 2. \tag{51}$$

We now use the property that $\hat{\eta}_b(v)$ is affine, and hence both weakly concave and weakly convex. The strict convexity of $\eta_a(v)$ and (50) yield

$$\hat{\eta}_b(v) > \eta_a(v) \text{ for } v \in (v_1, v_2)\,; \text{ and } \hat{\eta}_b(v) < \eta_a(v) \text{ for } v < v_1 \text{ or } v > v_2. \tag{52}$$

Comparing (52) and (48) reveals that the participation decision of any bidder type under the single contract is the same as that with the posited optimal contracts. Hence, analogous to (47), the seller's expected profit under the single contract is

$$\hat{\pi}_s = \int_\kappa s(V)f(V)dv - \int_\kappa (\eta_a(v) - \hat{\eta}_b(v)) f(V)dv, \tag{53}$$

which, by (47), yields

$$\hat{\pi}_s - \pi_s = \int_\kappa (\hat{\eta}_b(v) - \eta_b(v)) f(V)dv. \tag{54}$$

The weak concavity of $\eta_b(v)$, the weak convexity of $\hat{\eta}_b(v)$, and (51) yield

$$\hat{\eta}_b(v) - \eta_b(v) \geq 0 \text{ for } v < v_1 \text{ or } v > v_2,$$

which, upon plugging into (54) and noticing that κ is the union of $v < v_1$ and $v > v_2$, yields $\hat{\pi}_s \geq \pi_s$. This proves the proposition for Case 1.

Case 2: $h^{**}(v) = 0$ has one solution over $[\underline{v}, \bar{v}]$. Denote the solution by v_1, i.e., $\eta_a(v_1) = \eta_b(v_1)$. Consider three scenarios.

Scenario 1: $\frac{dh^{**}(v)}{dv}|_{v=v_1} = 0$. Then $\frac{d\eta_a(v)}{dv}|_{v=v_1} = \frac{d\eta_b(v)}{dv}|_{v=v_1}$. The convexity of $h^{**}(v)$ yields $h^{**}(v) \geq 0$, or, equivalently, $\eta_a(v_1) \geq \eta_b(v_1)$, for all v. Thus, all bidder types participate. Consider the single contract of the form given in (31) with $e = \frac{d\eta_b(v_1)}{dv}$. Denote the expected profit of a type v bidder from this contract by $\hat{h}(v)$. Then (49) holds with $\hat{\eta}_b(v)$ given by the right-hand side of (46) in which the original contract(s) are replaced by the single contract:

$$\hat{\eta}_b(v) = \frac{d\eta_b(v_1)}{dv}(v - V_1 - V_T - s(V_1)) + V_T + s(V_1), \tag{55}$$

which, by (12), yields $\hat{\eta}_b(v_1) = \eta_a(v_1)$. Hence, $\hat{\eta}_b(v_1) = \eta_b(v_1)$. Further, note that $\frac{d\hat{\eta}_b(v)}{dv} = \frac{d\eta_b(v_1)}{dv}$, hence $\frac{d\hat{\eta}_b(v)}{dv} = \frac{d\eta_a(v_1)}{dv}$. Using the facts that $\hat{\eta}_b(v)$ is affine, $\eta_a(v)$ is convex, $\hat{\eta}_b(v_1) = \eta_a(v_1)$ and $\frac{d\hat{\eta}_b(v_1)}{dv} = \frac{d\eta_a(v_1)}{dv}$, we have $\eta_a(v) \geq \hat{\eta}_b(v)$ for all v. Thus, all bidder types participate given this single contract, just as with the original contract(s). Further, because $\eta_b(v)$ is concave, $\hat{\eta}_b(v_1) = \eta_b(v_1)$ and $\frac{d\hat{\eta}_b(v_1)}{dv} = \frac{d\eta_b(v_1)}{dv}$, we have $\eta_b(v) \leq \hat{\eta}_b(v)$ for all v. Thus, the seller's expected profit is weakly higher under the single contract. Next, $e = \frac{d\eta_b(v_1)}{dv} = \frac{d\eta_a(v_1)}{dv}$ yields, by (12), $e = 1 - \frac{dV}{dv} = 1 - \frac{1}{\frac{dv}{dV}} = \frac{\frac{ds}{dV}|_{V=V_1}}{1+\frac{ds}{dV}|_{V=V_1}}$, where $V_1 = V(v_1)$. By $e = \frac{k}{1+k}$ ((31)), we have $k = \frac{ds(V_1)}{dV}$, consistent with (32). This proves the proposition for Scenario 1.

Scenario 2: $\frac{dh^{**}(v)}{dv}|_{v=v_1} < 0$. Then $\frac{d\eta_a(v_1)}{dv} < \frac{d\eta_b(v_1)}{dv}$. Because $h^{**}(v) = 0$ has one solution over $[\underline{v}, \bar{v}]$, $h^{**}(v) < 0$ for all $v > v_1$ and $h^{**}(v) > 0$ for all $v < v_1$. Thus, only bidder types

$v \leq v_1$ participate. Then $v_1 > \underline{v}$, else no bidder type participates, a contradiction to the premise that the mechanism is optimal.

Sub-case 1: $v_1 < \bar{v}$. Consider the single contract that takes the form given in (31) with $V_1 = V(v_1)$ and $V_2 = V(v_2)$, where $v_2 \equiv \bar{v}$, in (32). Let $\hat{h}(v)$ be the expected profit of a type v bidder from this contract. Then (49) holds with $\hat{\eta}_b(v)$ given by (55). Similar arguments as in Case 1 yield that $\eta_a(v_i) = \hat{\eta}_b(v_i)$ for $i = 1, 2$, and that $\hat{h}(v) < 0$ if and only if $v \in (v_1, \bar{v})$. Thus, bidder types $v \leq v_1$ participate given this contract, just as they did with the original contracts. The only difference is that, with the single contract, bidder type $\bar{v}$ (of measure zero) participates with rents fully extracted, which weakly increases the seller's expected profit. Now consider types $v \leq v_1$. Because $\eta_a(\bar{v}) < \eta_b(\bar{v})$, we have $\hat{\eta}_b(\bar{v}) < \eta_b(\bar{v})$. Further, (i) $\hat{\eta}_b(v_1) = \eta_b(v_1)$, (ii) $\eta_b(v)$ weakly concave and (iii) $\hat{\eta}_b(v)$ affine, together yield $\frac{d\hat{\eta}_b(v)}{dv} < \frac{d\eta_b(v_1)}{dv}$ and $\hat{\eta}_b(v) > \eta_b(v)$ for all $v < v_1$. Thus, the seller's expected revenue is strictly higher than with the posited optimal contract(s).

Sub-case 2: $v_1 = \bar{v}$. Consider the single contract that takes the form given in (31) with $e = \frac{d\eta_a(v_1)}{dv}$. A similar argument shows the seller's expected revenue is strictly higher with this contract. Thus, Scenario 2 contradicts the premise that the original contract(s) were optimal.

Scenario 3: $\frac{dh^{**}(v)}{dv}|_{v=v_1} > 0$. To show that scenario 3 contradicts the premise of optimality of the mechanism follow the arguments for scenario 2, using the single contract taking the form specified in (31), where if $v_1 = \underline{v}$ then $e = \frac{d\eta_a(v_1)}{dv}$, and if $v_1 > \underline{v}$ then $k = \frac{s(V(v_1))-s(V(\underline{v}))}{V(v_1)-V(\underline{v})}$ in (32). These results prove the proposition for Case 2.

Case 3: $h^{**}(v) = 0$ has one solution over $[\underline{v}, \bar{v}]$. Then either (1) $h^{**}(v) < 0$ for all $v \in [\underline{v}, \bar{v}]$, or (2) $h^{**}(v) > 0$ for all $v \in [\underline{v}, \bar{v}]$. Neither scenario can be optimal: in (1) no bidder type participates; and in (2) one can add a small positive cash payment to all contracts so that all bidder types still participate but earn strictly less rents. This completes the proof. ■

Proof of Result 1: Let $\beta(V)$ denote the cash bid in the second-price auction by type V,

$$\beta(V) = s(V) + V_T - \frac{\alpha}{1-\alpha}V,$$

which strictly increases in V, so the highest bidder type wins. The lowest type $\underline{V}$ earns no rents, and the differential rent of a high type over a low type (by (6)) is positive and strictly decreasing in $\alpha \in \left(0, \frac{b}{1+b}\right)$. Denote the highest and second-highest bidder types by V_1 and

V_2. Consider $\alpha = \frac{b}{1+b}$. The winner's profit is

$$\pi = \frac{1}{1+b}\left(s\left(V_1\right) - s\left(V_2\right)\right) - \frac{b}{1+b}\left(V_1 - V_2\right).$$

For $i = 1, 2$ define V_i^* such that $s\left(V_i\right) + V_i = y\left(V_1^*\right) + V_1^*$. Because $\frac{1}{1+b}\left(y\left(V_1^*\right) - y\left(V_2^*\right)\right) - \frac{b}{1+b}\left(V_1^* - V_2^*\right) = 0$, we can rewrite π as

$$\pi = \left(\frac{1}{1+b}\left(s\left(V_1\right) - y\left(V_1^*\right)\right) - \frac{b}{1+b}\left(V_1 - V_1^*\right)\right) + \left(\frac{b}{1+b}\left(V_2 - V_2^*\right)\right) - \frac{1}{1+b}\left(s\left(V_2\right) - y\left(V_2^*\right)\right).$$

For the first term on the right-hand side,

$$\frac{1}{1+b}\left(s\left(V_1\right) - y\left(V_1^*\right)\right) - \frac{b}{1+b}\left(V_1 - V_1^*\right) = \left(s\left(V_1\right) - y\left(V_1^*\right)\right) \leq \frac{\sqrt{2}}{2}\epsilon.$$

Similarly, the second term is also no greater than $\frac{\sqrt{2}}{2}\epsilon$. Hence $\pi \leq \sqrt{2}\epsilon$, proving the result. ■

Proof of Corollary 5: For a security with index $t(z)$, denote its payoff function $S\left(t(z), \cdot\right)$ by $\hat{S}\left(\cdot\right)$ for notational ease. Denote the distribution of θ by $f_\theta(\cdot)$. Then

$$ES(t(z), v; M(z)) = \int_0^\infty \hat{S}(\theta\left(v - \hat{v}\right) + \hat{v} - M(z)) f_\theta(\theta) d\theta.$$

Hence,

$$ES_{vv}(t(z), v; M(z)) = \int_0^\infty \theta^2 \hat{S}''(\theta\left(v - \hat{v}\right) + \hat{v} - M(z)) f_\theta(\theta) d\theta.$$

If $\hat{S}(\cdot)$ is strictly concave, then $ES_{vv}(t(z), v; M(z)) < 0$ for all v, z. Then the necessary condition (38) establishes the corollary. ■

Proof of Lemma 4:

Claim 1: At any v, for any $M < v - V(v)$, there is a $t^*\left(M; v\right)$ that solves the re-written form of (39)

$$v - V(v) - ES(t, v; M) - M = 0. \tag{56}$$

Furthermore, $t^*(M; v)$ decreases in M.

Proof of Claim 1: From the assumption of a sufficient range in the security index, the left-hand side of (56) approaches $v - V(v) - M > 0$ when t is low enough. Similarly, the left-hand side approaches $v - V(v) - (v - M) - M = -V\left(v\right) < 0$ when t is high enough. Thus, there exists a smallest t that satisfies (56), which we denote by $t^*(M;v)$ ($ES(t,v;M)$ could be flat at t^*). Because $\frac{\partial ES(t(v),v;M)}{\partial M} \in [-1,0]$, $\frac{\partial(v-V(v)-ES(t,v;M)-M)}{\partial M} \leq 0$. Because $ES(t,v;M)$ increases in t, $t^*(M;v)$ decreases in M. ■

For any $y \geq 0$, by Claim 1 and Assumption 2, $S(t^*(M;v),y)$ decreases in M. Because $S(t^*(M;v),y)$ is bounded between 0 and y, $S(t^*(M;v),y)$ reaches a limit as $M \to v - V(v)$. Define $S^*(y) \equiv \lim_{M \to v-V(v)} S(t^*(M;v),y)$.

Claim 2: $S^*(y) = 0$ at all $y \geq 0$.

Proof of Claim 2: Suppose that $S^*(y^*) = \delta > 0$ at some y^*. Then $S^*(y) \geq \delta$ for all $y \geq y^*$. Then $S(t^*(M;v),y) \geq \delta$ for all M and $y \geq y^*$. Then

$$\begin{aligned} ES\left(t^*(M),v;M\right) &\geq \delta E\left[\Pr\text{obability}\left(\widetilde{y(v)} \geq y^* + M\right)\right] \\ &\geq \delta E\left[\Pr\text{obability}\left(\widetilde{y(v)} \geq y^* + v - V(v)\right)\right] > 0, \end{aligned}$$

where $\widetilde{y(v)}$ is the stochastic (before cash payment) cash flow of bidder type v. Then as M approaches $v - V(v)$, the left-hand side of (56) becomes strictly negative, a contradiction. ■

Claim 3: $\lim_{M \to v-V(v)} 1 - V'(v) - ES_v\left(t^*(M),v;M\right) > 0$.

Proof: Define $\kappa \equiv 1 - V'(v) > 0$. By the assumption of sufficient range in index, for any $M \in [v - V(v) - 1, v - V(v)]$, there exists a $\hat{t}$ such that $ES_v(\hat{t},v;M) < \kappa$. By Claim 2, as M approaches $v - V(v)$, $t^*(M;v)$ must become less than $\hat{t}$. ■

For any $y \geq 0$, by Claim 1 and Assumption 2, the right-derivative of $S\left(t^*(M;v),y\right)$ with respect to y decreases in M. Because this quantity is bounded between 0 and 1, it reaches a limit as M approaches $-\infty$. Denote this limiting value by function $d(y)$. Because security payoffs are convex, $d(y)$ is nondecreasing and cannot exceed 1. Hence when y goes to infinity, $d(y)$ approaches a limit, which we denote by d^*.

Claim 4: d^* =1.

Proof: Suppose, instead, that $d^* < 1$. Then by Assumption 2, Claim 1 and convexity in security payoffs, for all M and y, $S(t^*(M;v),y)$ is no greater than $d^* y$. Then the left-hand side of (56) is strictly positive for M sufficiently negative, a contradiction. ■

By Claim 4, there exists a $y^* > 0$ such that $d(y^*) > 1 - \frac{V'(v)}{3}$. Hence there exists M^* such that the right-derivative of $S\left(t^*(M;v),\cdot\right)$, evaluated at y^*, exceeds $1 - \frac{V'(v)}{2}$ for all

$M < M^*$. From the convexity of the payoff function, the right-derivative of $S\left(t^*(M;v),\cdot\right)$ exceeds $1-\frac{V'(v)}{2}$ for all $M < M^*$ and all $y > y^*$.

For $M < \min\{M^*, -y^*\}$, denote the payoff function $S\left(t^*(M),\cdot\right)$ by $\hat{S}(\cdot)$ for notational ease. Then

$$ES\left(t^*(M),v;M\right) = E\left[\hat{S}\left(\widetilde{y\left(v\right)}-M\right)\right]$$

and

$$ES\left(t^*(M),v+\Delta v;M\right) = E\left[\hat{S}\left(y\,(\widetilde{v+\Delta}v)-M\right)\right],$$

where $\Delta v > 0$. Because the distribution of $y\,(\widetilde{v+\Delta}v)$ first-order stochastically dominates that of $\widetilde{y\left(v\right)}$, $y\,(\widetilde{v+\Delta}v)$ can be expressed as the sum of random variable $\widetilde{y\left(v\right)}$ plus another random variable $\widetilde{\epsilon}$, where $\widetilde{\epsilon}$ is nonnegative and $\mathrm{E}[\widetilde{\epsilon}] = \Delta v$. Thus

$$\begin{aligned} ES\left(t^*(M),v+\Delta v;M\right) &= E\left[\hat{S}\left(\widetilde{y\left(v\right)}+\widetilde{\epsilon}-M\right)\right] \\ &\geq E\left[\hat{S}\left(\widetilde{y\left(v\right)}-M\right)+\left(1-\frac{V'(v)}{2}\right)\widetilde{\epsilon}\right] \\ &= E\left[\hat{S}\left(\widetilde{y\left(v\right)}-M\right)\right]+\left(1-\frac{V'(v)}{2}\right)\Delta v. \end{aligned}$$

Hence, $ES_v\left(t^*(M),v;M\right) \geq \left(1-\frac{V'(v)}{2}\right)$. Then $1-V'(v)-ES_v\left(t^*(M),v;M\right) < 0$. By Claim 3, there exists M such that $1-V'(v)-ES_v\left(t^*(M),v;M\right) = 0$. This proves Lemma 4. ■

Proof of Lemma 5: By a similar argument as in the proof of Corollary 5, if $\hat{S}(\cdot)$ is strictly convex, then $ES_{vv}(t(z),v;M(z)) > 0$ for all v, z. Thus, (41) and (42) yield $K > 0$.

Now consider call options. Let $t(z) > 0$ denote the strike price (abusing notation slightly as a higher strike price corresponds to a lower payoff). If $t(z)-\hat{v}+M(z) > 0$, then

$$ES(t(z),v;M(z)) = \int_{\frac{t(z)+M(z)-\hat{v}}{v-\hat{v}}}^{\infty}\left(\theta\left(v-\hat{v}\right)+\hat{v}-M(z)-t(z)\right)f_\theta(\theta)d\theta.$$

Therefore,

$$ES_v(t(z),v;M(z)) = \int_{\frac{t(z)+M(z)-\hat{v}}{v-\hat{v}}}^{\infty}\theta f_\theta(\theta)d\theta$$

and

$$ES_{vv}(t(z),v;M(z)) = \frac{\left(t(z)+M(z)-\hat{v}\right)^2}{\left(v-\hat{v}\right)^3}f_\theta\left(\frac{t(z)+M(z)-\hat{v}}{v-\hat{v}}\right) > 0.$$

If $t(z) - \hat{v} + M(z) \leq 0$, we have $ES_v(t(z), v; M(z)) = 1$ and $ES_{vv}(t(z), v; M(z)) = 0$. When $z = v$, we must have $t(z) - \hat{v} + M(z) > 0$, or else the left-hand side of (37) would be $-V'(v) < 0$, a contradiction. Thus, $ES_{vv}(t(z), z; M(z)) > 0$ for all z. $ES_{vv}(t(z), v; M(z))$ is continuous in v, so $ES_{vv}(t(z), v; M(z)) > 0$ if $|z - v|$ is small. Because $ES_{vv}(t(z), v; M(z))$ is nonnegative for all z and v, (40) yields $k(v) > 0$ for all v. Thus, (42) yields $K > 0$. ■

Proof of Theorem 2: By the premise that the mechanism satisfies (37) and (39) (Lemma 4 guarantees the existence of such mechanism), we only need to establish global incentive compatibility. By Lemma 3, it suffices to show that $\frac{\partial h(v,z)}{\partial v} \geq 0$ for $v < z$ and that $\frac{\partial h(v,z)}{\partial v} \leq 0$ for $v > z$. We have $\frac{\partial h(v,z)}{\partial v} = l(v, z)$, where

$$l(v, z) \equiv 1 - V'(v) - ES_v(t(z), v; M(z)),$$

and (37) yields $l(z, z) = 0$. Thus, for $z \neq v$,

$$\begin{aligned} l(v,z) &= l(v,z) - l(z,z) \\ &= -V'(v) - ES_v(t(z), v; M(z)) + V'(z) + ES_v(t(z), z; M(z)) \\ &= \int_v^z V''(x)dx + \frac{ES_v(t(z), z; M(z)) - ES_v(t(z), v; M(z))}{z - v}(z - v) \\ &= \int_v^z V''(x)dx + \frac{\int_v^z ES_{vv}(t(z), x; M(z))dx}{z - v}(z - v) \end{aligned}$$

If $z > v$, then by (40)

$$l(v,z) \geq \int_v^z V''(x)dx + k(v)(z - v) \geq \int_v^z V''(x)dx + K(z - v) = \int_v^z (V''(x) + K)\,dx \geq 0,$$

and if $z < v$, then

$$l(v,z) \leq \int_v^z V''(v)dv + k(v)(z - v) \leq \int_v^z V''(v)dv + K(z - v) = \int_v^z (V''(x) + K)\,dx \leq 0.$$

Thus, $\frac{\partial h(v,z)}{\partial v} \geq 0$ for $v < z$, and $\frac{\partial h(v,z)}{\partial v} \leq 0$ for $v > z$. Thus, $h(v, z) \leq 0$, $\forall z \neq v$. ■

Proof of Corollary 6:

Claim 1: $ES_v(t, v; M)$ is increasing in t for all v.

Proof: For $t_1 > t_2$, we have

$$ES_v(t_1, v; M) - ES_v(t_2, v; M) = \int [S(t_1, y - M) - S(t_2, y - M)]g_v(y|v))dy,$$

where $g(\cdot|v)$ denotes the pdf of cash flows by bidder type v, and g_v denotes its derivative. Because g_v integrates to zero, there exists a y^* such that $g_v(y^*|v) = 0$. $\frac{g_v(y|v)}{g(y|v)}$ is monotone increasing in y by the sMLRP. Thus, for $y < y^*$, $\frac{g_v(y|v)}{g(y|v)} < 0$ and hence $g_v(y|v) < 0$; and for $y > y^*$, $\frac{g_v(y|v)}{g(y|v)} > 0$ and hence $g_v(y|v) > 0$. Then since $[S(t_1, y - M) - S(t_2, y - M)]$ is weakly increasing by Assumption 2,

$$\int_{y<y^*} [S(t_1, y - M) - S(t_2, y - M)]g_v(y|v)dy \geq \int_{y<y^*} [S(t_1, y^* - M) - S(t_2, y^* - M)]g_v(y|v)dy$$

and

$$\int_{y>y^*} [S(t_1, y - M) - S(t_2, y - M)]g_v(y|v)dy \geq \int_{y>y^*} [S(t_1, y^* - M) - S(t_2, y^* - M)]g_v(y|v)dy.$$

Adding yields Claim 1: $ES_v(t_1, v; M) - ES_v(t_2, v; M)$

$$\geq \int_{y<y^*} [S(t_1, y^* - M) - S(t_2, y^* - M)]g_v(y|v)dy + \int_{y>y^*} [S(t_1, y^* - M) - S(t_2, y^* - M)]g_v(y|v)dy = 0.$$

Next, consider any v^*. To show that $M(v^*)$ is strictly positive, suppose that $M(v^*) \leq 0$. Then by (56), $ES(t(v^*), v^*; M(v^*)) \geq v^* - V(v^*)$. Then by Assumption 2 and $v^* - V(v^*) > 0$, there exists a $\hat{t} \leq t(v^*)$ such that $ES(\hat{t}, v^*; M(v^*)) = v^* - V(v^*)$. Observe that if bidding strategies in a second-price security-bid auction are strictly decreasing, they are still strictly decreasing when the stochastic cash flows for all types v are shifted up by the same amount $|M(v^*)|$. Thus, by the premise of the corollary on the decreasing bidding strategy, $ES(\hat{t}, v^*; M(v^*)) = v^* - V(v^*)$ implies $ES_v(\hat{t}, v^*; M(v^*)) > 1 - V'(v^*)$ (Lemma 1, Che and Kim 2010). By $\hat{t} \leq t(v^*)$ and Claim 1, $ES_v(t(v^*), v^*; M(v^*)) > 1 - V'(v^*)$, contradicting (37). Thus, $M(v^*) > 0$. Moreover, pure security or pure cash cannot extract full rents because when a seller can combine them, it must use strictly positive amounts of both to extract full rents. ■

6 Online Appendix: Optimal equity-plus-cash mechanism when s(V) is not concave (1 bidder, 3 type case)

Proposition 4 *When $\tau > 0$, the optimal equity-plus-cash mechanism has these features:*

(i) If there are sufficiently few intermediate type 2s so that either

$$f_1 \geq f_3 \frac{V_3 + s_3 - (V_2 + s_2)}{V_2 + s_2 - (V_1 + s_1)} \text{ and } \tau f_3 \frac{(V_3 - V_2)(V_2 - V_1)}{V_3 + s_3 - (V_1 + s_1)} \geq f_2 s_2 \tag{57}$$

or

$$f_1 < f_3 \frac{V_3 + s_3 - (V_2 + s_2)}{V_2 + s_2 - (V_1 + s_1)} \text{ and } \tau f_1 \frac{(V_3 - V_2)(V_2 - V_1)}{V_3 + s_3 - (V_1 + s_1)} \geq f_2 s_2, \tag{58}$$

the seller excludes type 2s and extracts all surplus from types 1 and 3, earning expected profit

$$\Pi_s = f_1 s_1 + f_3 s_3. \tag{59}$$

(ii) With more type 2s so that neither (57) *nor* (58) *hold, and*

$$f_1 \geq f_3 \frac{V_3 + s_3 - (V_2 + s_2)}{V_2 + s_2 - (V_1 + s_1)} \tag{60}$$

so that type 1s are relatively more abundant than type 3s, the seller extracts all surplus from types 1 and 2, leaving rents to type 3, earning expected profit

$$\Pi_s = f_1 s_1 + f_2 s_2 + f_3 s_3 - \tau f_3 \frac{(V_3 - V_2)(V_2 - V_1)}{V_3 + s_3 - (V_1 + s_1)}. \tag{61}$$

If, instead, inequality (60) is reversed, then the seller extracts all surplus from types 2 and 3, leaving rents to type 1, earning expected profit

$$\Pi_s = f_1 s_1 + f_2 s_2 + f_3 s_3 - \tau f_1 \frac{(V_3 - V_2)(V_2 - V_1)}{V_3 + s_3 - (V_1 + s_1)}. \tag{62}$$

(iii) In all cases, the optimal mechanism can be implemented by a single contract with $p = 1$ *and* $e \in (0, 1)$*. More generally, the optimal mechanism can be implemented by multiple contracts so that the higher type pays (weakly) higher cash and less equity share.*

Proof of Proposition 4: First note that for any i, if $p_i > 0$, (29) yields that $\pi_{ii} \geq 0$; and if

$p_i = 0$, then the details of contract i do not affect (28), (29), or (30). Hence, it is without loss to assume that $\pi_{ii} \geq 0$ for all i. Set $j = 2$ in (28). Then by $p_1 \leq 1$, $p_2 \leq 1$ and $\pi_{ii} \geq 0$, we have

$$p_1 \pi_{11} \geq \max\{p_2 \pi_{12}, 0\}, \tag{63}$$

and

$$p_3 \pi_{33} \geq \max\{p_2 \pi_{32}, 0\}. \tag{64}$$

By (26),

$$\begin{aligned}
& \frac{V_3 + s_3 - (V_2 + s_2)}{V_3 + s_3 - (V_1 + s_1)} \pi_{12} + \frac{V_2 + s_2 - (V_1 + s_1)}{V_3 + s_3 - (V_1 + s_1)} \pi_{32} - \pi_{22} \\
= & \frac{V_3 + s_3 - (V_2 + s_2)}{V_3 + s_3 - (V_1 + s_1)} (\pi_{12} - \pi_{22}) + \frac{V_2 + s_2 - (V_1 + s_1)}{V_3 + s_3 - (V_1 + s_1)} (\pi_{32} - \pi_{22}) \\
= & \frac{V_3 + s_3 - (V_2 + s_2)}{V_3 + s_3 - (V_1 + s_1)} [(1 - e_2)(v_1 - v_2) + V_2 - V_1] + \frac{V_2 + s_2 - (V_1 + s_1)}{V_3 + s_3 - (V_1 + s_1)} [(1 - e_2)(v_3 - v_2) + V_2 - V_3] \\
= & \frac{V_3 + s_3 - (V_2 + s_2)}{V_3 + s_3 - (V_1 + s_1)} (V_2 - V_1) - \frac{V_2 + s_2 - (V_1 + s_1)}{V_3 + s_3 - (V_1 + s_1)} (V_3 - V_2) \\
= & \frac{1}{V_3 + s_3 - (V_1 + s_1)} ((V_3 - V_2 + s_3 - s_2)(V_2 - V_1) - (V_2 - V_1 + s_2 - s_1)(V_3 - V_2)) \\
= & \tau,
\end{aligned} \tag{65}$$

where τ is defined in (25). By (65) and $\pi_{22} \geq 0$, we have

$$\frac{V_3 + s_3 - (V_2 + s_2)}{V_3 + s_3 - (V_1 + s_1)} \pi_{12} + \frac{V_2 + s_2 - (V_1 + s_1)}{V_3 + s_3 - (V_1 + s_1)} \pi_{32} \geq \tau. \tag{66}$$

Rewrite the seller's expected profit (30) as

$$\begin{aligned}
\Pi_s &= f_1 p_1 (s_1 - \pi_{11}) + f_2 p_2 (s_2 - \pi_{22}) + f_3 p_3 (s_3 - \pi_{33}) && (67) \\
&\leq f_1 p_1 s_1 + f_2 p_2 s_2 + f_3 p_3 s_3 - p_2 (f_1 \max\{\pi_{12}, 0\} + f_2 \pi_{22} + f_3 \max\{\pi_{32}, 0\}) && (68) \\
&\leq f_1 s_1 + f_2 p_2 s_2 + f_3 s_3 - p_2 (f_1 \max\{\pi_{12}, 0\} + f_3 \max\{\pi_{32}, 0\}) && (69) \\
&= f_1 s_1 + f_3 s_3 + p_2 (f_2 s_2 - \pi^*) && (70)
\end{aligned}$$

where the first inequality follows from (63) and (64), the second inequality follows from

$p_1 \leq 1$, $p_3 \leq 1$ and $\pi_{22} \geq 0$, and

$$\pi^* \equiv f_1 \max\{\pi_{12}, 0\} + f_3 \max\{\pi_{32}, 0\}. \tag{71}$$

Next we bound π^* from below.

Claim:

$$\pi^* \geq \begin{cases} \tau f_3 \frac{V_3+s_3-(V_1+s_1)}{V_2+s_2-(V_1+s_1)} & \text{if} \quad f_1 \geq f_3 \frac{V_3+s_3-(V_2+s_2)}{V_2+s_2-(V_1+s_1)} \\ \tau f_1 \frac{V_3+s_3-(V_1+s_1)}{V_3+s_3-(V_2+s_2)} & \text{if} \quad f_1 < f_3 \frac{V_3+s_3-(V_2+s_2)}{V_2+s_2-(V_1+s_1)} \end{cases} \tag{72}$$

To prove the claim, we consider 3 cases.

Case 1: Suppose that

$$0 \leq \pi_{12} \leq \tau \frac{V_3 + s_3 - (V_1 + s_1)}{V_3 + s_3 - (V_2 + s_2)}. \tag{73}$$

Then (66) yields $\pi_{32} \geq 0$, and (71) yields

$$\begin{aligned} \pi^* &= f_1 \pi_{12} + f_3 \pi_{32} \\ &\geq f_1 \pi_{12} + f_3 \left(\frac{V_3 + s_3 - (V_1 + s_1)}{V_2 + s_2 - (V_1 + s_1)} \tau - \frac{V_3 + s_3 - (V_2 + s_2)}{V_2 + s_2 - (V_1 + s_1)} \pi_{12} \right) \\ &= \frac{V_3 + s_3 - (V_1 + s_1)}{V_2 + s_2 - (V_1 + s_1)} f_3 \tau + \left(f_1 - f_3 \frac{V_3 + s_3 - (V_2 + s_2)}{V_2 + s_2 - (V_1 + s_1)} \right) \pi_{12}, \end{aligned}$$

where the inequality follows from (66). Under (73), if $f_1 \geq f_3 \frac{V_3+s_3-(V_2+s_2)}{V_2+s_2-(V_1+s_1)}$ then

$$\begin{aligned} \pi^* &\geq \frac{V_3 + s_3 - (V_1 + s_1)}{V_2 + s_2 - (V_1 + s_1)} f_3 \tau \\ &= \tau f_3 \frac{V_3 + s_3 - (V_1 + s_1)}{V_2 + s_2 - (V_1 + s_1)} \end{aligned}$$

and if $f_1 < f_3 \frac{V_3+s_3-(V_2+s_2)}{V_2+s_2-(V_1+s_1)}$, then

$$\begin{aligned} \pi^* &\geq \frac{V_3 + s_3 - (V_1 + s_1)}{V_2 + s_2 - (V_1 + s_1)} f_3 \tau + \left(f_1 - f_3 \frac{V_3 + s_3 - (V_2 + s_2)}{V_2 + s_2 - (V_1 + s_1)} \right) \left(\tau \frac{V_3 + s_3 - (V_1 + s_1)}{V_3 + s_3 - (V_2 + s_2)} \right) \\ &= \tau f_1 \frac{V_3 + s_3 - (V_1 + s_1)}{V_3 + s_3 - (V_2 + s_2)}. \end{aligned}$$

This proves the claim for Case 1.

Case 2: $\pi_{12} < 0$. Then (66) yields $\pi_{32} > \tau \frac{V_3+s_3-(V_1+s_1)}{V_2+s_2-(V_1+s_1)}$, which we substitute into (71) to obtain

$$\pi^* \geq f_3 \tau \frac{V_3 + s_3 - (V_1 + s_1)}{V_2 + s_2 - (V_1 + s_1)}, \tag{74}$$

so the first line of (72) is satisfied. Next, suppose that $f_1 < f_3 \frac{V_3+s_3-(V_2+s_2)}{V_2+s_2-(V_1+s_1)}$. Then plugging $f_3 > f_1 \frac{V_2+s_2-(V_1+s_1)}{V_3+s_3-(V_2+s_2)}$ into (74) yields the second line of (72). This proves the claim for Case 2.

Case 3: $\pi_{12} > \tau \frac{V_3+s_3-(V_1+s_1)}{V_3+s_3-(V_2+s_2)}$. Plugging this condition into (71) yields

$$\pi^* \geq f_1 \tau \frac{V_3 + s_3 - (V_1 + s_1)}{V_3 + s_3 - (V_2 + s_2)}, \tag{75}$$

so the second line of (72) is trivially satisfied. Next, suppose that $f_1 \geq f_3 \frac{V_3+s_3-(V_2+s_2)}{V_2+s_2-(V_1+s_1)}$. Plugging this condition into (74) yields the second line of (72). This proves the claim for Case 3.

Then, from (72) and (70),

$$\Pi_s \leq \begin{cases} f_1 s_1 + f_2 s_2 + f_3 s_3 - \tau f_3 \frac{V_3+s_3-(V_1+s_1)}{V_2+s_2-(V_1+s_1)} & \text{if} \quad f_1 \geq f_3 \frac{V_3+s_3-(V_2+s_2)}{V_2+s_2-(V_1+s_1)} \text{ and } \tau f_3 \frac{V_3+s_3-(V_1+s_1)}{V_2+s_2-(V_1+s_1)} < f_2 s_2 \\ f_1 s_1 + f_3 s_3 & \text{if} \quad f_1 \geq f_3 \frac{V_3+s_3-(V_2+s_2)}{V_2+s_2-(V_1+s_1)} \text{ and } \tau f_3 \frac{V_3+s_3-(V_1+s_1)}{V_2+s_2-(V_1+s_1)} \geq f_2 s_2 \\ f_1 s_1 + f_3 s_3 & \text{if} \quad f_1 < f_3 \frac{V_3+s_3-(V_2+s_2)}{V_2+s_2-(V_1+s_1)} \text{ and } \tau f_1 \frac{V_3+s_3-(V_1+s_1)}{V_3+s_3-(V_2+s_2)} \geq f_2 s_2 \\ f_1 s_1 + f_2 s_2 + f_3 s_3 - \tau f_1 \frac{V_3+s_3-(V_1+s_1)}{V_3+s_3-(V_2+s_2)} & \text{if} \quad f_1 < f_3 \frac{V_3+s_3-(V_2+s_2)}{V_2+s_2-(V_1+s_1)} \text{ and } \tau f_1 \frac{V_3+s_3-(V_1+s_1)}{V_3+s_3-(V_2+s_2)} < f_2 s_2 \end{cases}. \tag{76}$$

Now, consider the following single contract with $p = 1$ and (a) if either (57) or (58) holds then

$$e = \frac{s_3 - s_1}{V_3 + s_3 - V_1 - s_1}, c = V_T + s_1 - \frac{eV_1}{1-e},$$

and (b) if neither (57) nor (58) holds, and (60) holds then

$$e = \frac{s_2 - s_1}{V_2 + s_2 - V_1 - s_1}, c = V_T + s_1 - \frac{eV_1}{1-e},$$

and (c) if neither (57) nor (58) holds, and (60) is reversed then

$$e = \frac{s_3 - s_2}{V_3 + s_3 - V_2 - s_2}, c = V_T + s_3 - \frac{eV_3}{1-e}.$$

It is simple to show that this contract satisfies the properties stated in the proposition that in case (i), bidder types 1 and 3 receive zero rents, and type 2 would receive strictly negative profit and hence not participate; in case (ii) all bidder types receive nonnegative expected profit, and type 3 earns strictly positive rent when (60) holds, while type 1 earns strictly positive rent when (60) is reversed. The seller's expected profit achieves the upper bound on Π_s specified in (76), so the mechanism is optimal, and (59), (61), and (62) hold.

Next, consider the following menu of two contracts with $p = 1$ for both contracts, and

$$e_1 \geq \frac{s_3 - s_1}{V_3 + s_3 - V_1 - s_1}, c_1 = V_T + s_1 - \frac{e_1 V_1}{1 - e_1}, \tag{77}$$

$$e_3 \leq \frac{s_3 - s_1}{V_3 + s_3 - V_1 - s_1}, c_3 = V_T + s_3 - \frac{e_3 V_3}{1 - e_3} \tag{78}$$

if either (57) or (58) holds; and

$$e_{23} = \frac{s_2 - s_1}{V_2 + s_2 - V_1 - s_1}, c_{23} = V_T + s_2 - \frac{e_{23} V_2}{1 - e_{23}}$$

$$e_1 \geq e_{23},\ c_1 = V_T + s_1 - \frac{e_1 V_1}{1 - e_1}$$

if neither (57) nor (58) holds, and (60) holds; and

$$e_{12} = \frac{s_3 - s_2}{V_3 + s_3 - V_2 - s_2}, c_{12} = V_T + s_2 - \frac{e_{12} V_2}{1 - e_{12}}$$

$$e_3 \leq e_{12},\ c_3 = V_T + s_3 - \frac{e_3 V_3}{1 - e_3}$$

if neither (57) nor (58) holds, and (60) is reversed. The index "23" means that types 2 and 3 receive the same contract, and the index "12" has an analogous interpretation. It is easy to show that this menu of contracts induces the same acceptance/rejection decision from the bidder and acheives the same revenue, as the earlier single contract. Thus, this menu also implements the optimal mechanism. Lastly we show that a higher type pays (weakly) more cash and a smaller equity share. In the case where either (57) or (58) holds, (77) and (78) yield $e_3 \le e_1$, and

$$\begin{aligned}
c_3 - c_1 &= s_3 - \frac{e_3 V_3}{1-e_3} - \left(s_1 - \frac{e_1 V_1}{1-e_1} \right) \\
&= s_3 + V_3 - \frac{V_3}{1-e_3} - \left(s_1 + V_1 - \frac{V_1}{1-e_1} \right) \\
&\ge s_3 + V_3 - \frac{V_3}{1-e_3} - \left(s_1 + V_1 - \frac{V_1}{1-e_3} \right) \\
&\ge s_3 + V_3 - s_1 - V_1 - \frac{V_3 - V_1}{1 - \left(\frac{s_3 - s_1}{V_3 + s_3 - V_1 - s_1} \right)} \\
&= 0.
\end{aligned}$$

One can similarly show for the other two cases that the higher type pays (weakly) higher cash and less equity. ■

7 Online Appendix: Informal Auctions

Lemma 6 *There is no pooling unless the bid is pure cash: for any V_1 and V_2, if $c(V_1) = c(V_2)$ and $e(V_1) = e(V_2)$, then either $V_1 = V_2$ or $e(V_1) = e(V_2) = 0$.*

Proof: Suppose instead that $e(V_1) = e(V_2) > 0$, and that multiple types bid $\{c(V_1), e(V_2)\}$. Denote the set of all such types by τ. Then the monetary value that the target assigns to the bid is $c(V_1)+e(V_2)\mathrm{E}[s(V)|V \in \tau]$. Now the highest type in τ can strictly benefit by deviating to a pure cash bid of dollar amount $c(V_1)+e(V_2)\mathrm{E}[s(V)|V \in \tau]$, because the target will assign the same monetary value for this cash bid as for this bidder's equilibrium bid. Therefore, the probability of winning is the same, but the bidder pays strictly less (because if it used equity, its equity payment would be $e(V_2)\, s(V) > e(V_2)\, E[s(V)|V \in \tau]$), a contradiction.

In light of the lemma, for any V, the monetary value that the target assigns to $\{c(V), e(V)\}$ is $c(V) + e(V)s(V)$. Next, when a bidder of type V decides on a bid, it has the option to mimic a type $V' = V - dV$ just below it. Such a deviation has two effects. First, the deviation reduces the probability of winning to that of type V'. Second, the deviation changes the expected payment if it wins from $c(V)+e(V)s(V)$ to $c(V')+e(V')\, s(V)$. On the margin, these two effects must balance out (or else there is a profitable deviation). Type V can also deviate to a cash bid of amount $c(V') + e(V')\, s(V')$. Since the seller values this cash bid the same as the bid $\{c(V'), e(V')\}$ by type V', the marginal effect on the probability of winning is the same as if the bidder deviates to $\{c(V'), e(V')\}$. However, unless $e(V') = 0$, the monetary value would be strictly less because $s(V) > s(V')$. Thus, type V would profit by deviating to a pure cash bid unless $e(V') = 0$. Consequently, an equilibrium only involves cash bids. ■

8 Online Appendix: Two-sided Private Information

We analyze a setting with two bidder and two seller types. Let the bidder's possible stand-alone values and synergies be (V_{Ai}, s_i), for $i = 1, 2$, with $0 < V_{A1} < V_{A2}$, and $0 < s_1 < s_2$. Let $f_{Ai} > 0$ be the probability of a type i bidder, where $f_{A1} + f_{A2} = 1$. Let the seller's possible stand-alone values be V_{Ti}, $i = 1, 2$, where $0 < V_{T1} < V_{T2}$. Let $f_{Tj} > 0$ be the probability of type j seller, where $f_{T1} + f_{T2} = 1$. Each seller type offers a menu of contracts, $\{c_i, e_i; p_i\}_{i=1,2}$, one for each bidder type. When a bidder selects contract i, it wins with probability $p_i \in [0, 1]$; and when the bidder wins, it pays cash c_i and equity share $e_i \in [0, 1]$.

Denote the menu of contracts offered by a type j seller by $\{c_{ji}, e_{ji}; p_{ji}\}_{i=1,2}$. The equilibrium is pooling if seller types 1 and 2 offer the same menu; that is, if $c_{1i} = c_{2i}$, $e_{1i} = e_{2i}$, and $p_{1i} = p_{2i}$ for $i = 1, 2$. The equilibrium is separating otherwise.

Given the menu of contracts $\{c_k, e_k; p_k\}_{k=1,2}$ offered by the seller, each type i bidder forms beliefs about the seller's expected stand-alone value, V_T. Denote these beliefs by

$$\theta_i(\{c_k, e_k; p_k\}_{k=1,2}) \in [V_{T1}, V_{T2}]\,; i = 1, 2. \tag{79}$$

The expected (net) profit of a type i bidder that chooses contract $\{c_k, e_k; p_k\}$ is

$$\Pi = p_k\left((1 - e_k)\left(V_{Ai} + s_i + \theta_i - c_k\right) - V_{Ai}\right). \tag{80}$$

On an equilibrium path, $\theta_i = E\left[V_T | \{c_k, e_k; p_k\}_{k=1,2}\right]$. Thus, in a pooling equilibrium,

$$\theta_i(\{c_{jk}, e_{jk}; p_{jk}\}_{k=1,2}) = E\left[V_T\right] \text{ for } j = 1, 2 \text{ and } i = 1, 2; \tag{81}$$

and in a separating equilibrium in which a type j seller offers menu $\{c_{jk}, e_{jk}; p_{jk}\}_{k=1,2}$,

$$\theta_i(\{c_{jk}, e_{jk}; p_{jk}\}_{k=1,2}) = V_{Tj} \text{ for } j = 1, 2 \text{ and } i = 1, 2. \tag{82}$$

Let $\Pi_{i,k,j}$ be the expected profit of type i bidder when it chooses contract k in the menu:

$$\Pi_{i,k,j} = p_{jk}\left((1 - e_{jk})\left(V_{Ai} + s_i + \theta_i - c_{jk}\right) - V_{Ai}\right), \tag{83}$$

where θ_i satisfies (81) or (82). Incentive compatibility for a type i bidder requires

$$\Pi_{i,i,j} \geq \Pi_{i,k,j} \text{ for all } i,\ j \text{ and } k \neq i, \tag{84}$$

Note that this trivially holds in a pooling equilibrium. Individual rationality requires

$$\Pi_{i,i,j} \geq 0 \text{ for all } i, j. \tag{85}$$

The equilibrium expected profit of a type i bidder, integrated over the two seller types, is$\Pi_{b,i} = \sum_{j=1}^{2} f_{Tj}\Pi_{i,i,j}$. To obtain the unconditional equilibrium expected profit of bidders, integrate over their types to obtain $\Pi_b = \sum_{i=1}^{2} f_{Ai}\Pi_{b,i}$. To obtain the expected profit of a type j seller from offering the menu offered by a type k seller, integrate over the two buyer types:

$$\pi_{s,j,k} = \sum_{i=1}^{2} f_{Ai}p_{ki}\left(e_{ki}\left(V_{Ai} + s_i + \theta_i - c_{ki}\right) + c_{ki} - V_{Tj}\right), \tag{86}$$

where $\theta_i = V_{Tk}$ if the equilibrium is separating, and $\theta_i = E\left[V_T\right]$ if the equilibrium is pooling.

In equilibrium, a type j seller's expected profit is $\pi_{s,j,j}$. Incentive compatibility requires that it not be profitable for a type j seller to offer the menu offered by a type $k \neq j$ seller:

$$\pi_{s,j,j} \geq \pi_{s,j,k} \text{ for } k \neq j \text{ and both } j. \tag{87}$$

Unlike the bidder, which can only choose between the two contracts offered, a seller can deviate by offering any arbitrary menu of contracts. The optimality of the mechanism for a seller requires that the expected profit of each seller type weakly exceed what she can get from offering any other menu. Complications arise in imposing this requirement because, for any menu, the set of equilibria and equilibrium payoffs depend on the possible off-equilibrium-path beliefs. We only impose a minimal (and necessary) requirement for seller optimality: for any off-equilibrium-path offer $\left\{c'_{ji}, e'_{ji}; p'_{ji}\right\}_{i=1,2}$ made by a type j seller, if her expected profit is at least $\Pi'_{s,\min}$ for every bidder belief that satisfies (79), then $\Pi_{s,j} \geq \Pi'_{s,\min}$.

Proposition 5 *Suppose that*

$$\frac{s_2 - s_1}{V_{A2} - V_{A1}} > \frac{f_{A1}}{(1 - f_{A1})}, \tag{88}$$

and that there is sufficient information asymmetry on V_T that

$$V_{T2} - V_{T1} > \phi\left(V_{A1}, V_{A2}, s_1, s_2, f_{A1}, f_{T1}\right), \tag{89}$$

where the function ϕ is defined in equation (101) of the proof. Then the bidder's expected profit is strictly positive in any equilibrium: $\Pi_b > 0$.

Proposition 5 reflects the intuition that when information asymmetry about a seller's stand-alone value is high enough, full extraction by the seller is impossible even in a pooling equilibrium because the high type seller's rents would be too low, providing it incentives to deviate. Failure of full extraction need not imply that a bidder will earn positive rents, because a seller may sell only to one bidder type and exclude the other type. In such a case, the bidder earns no rents even though the seller does not extract full rents. Condition (88) rules out such a case by ensuring that it is not optimal for the seller to exclude a type 2 bidder and only sell to a type 1 bidder. Condition (88) holds as long as the probability of a type 1 bidder, f_{A1}, is not too high. This requirement is not that restrictive: with $n > 2$ bidder types, the condition ensuring that the bidder earns strictly positive rents is still that the probability of a low type 1 bidder is not too high, which is naturally satisfied when n is large.

Proof of Proposition 5. From (83),

$$\Pi_{1,2,j} - \Pi_{2,2,j} = p_{j2}\left(\left(V_{A2} - V_{A1}\right) - \left(1 - e_{j2}\right)\left(\left(V_{A2} - V_{A1}\right) + s_2 - s_1\right)\right).$$

From bidder incentive compatibility, $\Pi_{1,1,j} \geq \Pi_{1,2,j}$, so the difference in the equilibrium expected profits of bidder types 1 and 2, conditional on seller type j, satisfies:

$$\Pi_{1,1,j} - \Pi_{2,2,j} \geq p_{j2}\left(\left(V_{A2} - V_{A1}\right) - \left(1 - e_{j2}\right)\left(\left(V_{A2} - V_{A1}\right) + s_2 - s_1\right)\right).$$

Summing over seller types, yields:

$$\Pi_{b,1} - \Pi_{b,2} \geq \sum_{j=1}^{2} f_{Tj} p_{j2}\left(\left(V_{A2} - V_{A1}\right) - \left(1 - e_{j2}\right)\left(\left(V_{A2} - V_{A1}\right) + s_2 - s_1\right)\right).$$

Defining

$$\Delta \equiv p_{22}\left(V_{A2} - V_{A1}\right) - p_{22}\left(1 - e_{22}\right)\left(\left(V_{A2} - V_{A1}\right) + s_2 - s_1\right), \tag{90}$$

$\Pi_{b,2} \geq 0$ yields

$$\Pi_{b,1} \geq \sum_{j=1}^{2} f_{Tj} p_{j2} \left((V_{A2} - V_{A1}) - (1 - e_{j2}) \left((V_{A2} - V_{A1}) + s_2 - s_1 \right) \right) \geq f_{T2} \Delta. \tag{91}$$

So, too, incentive compatibility for a type $j = 1$ seller yields $\pi_{s,1,1} \geq \pi_{s,1,2}$. By (86),

$$\pi_{s,1,2} - \pi_{s,2,2} = \sum_{i=1}^{2} f_{Ai} p_{2i} (1 - e_{2i}) (V_{T2} - V_{T1}).$$

Thus, the expected profit of type 1 seller exceeds that of type 2 seller by at least:

$$\begin{aligned} \pi_{s,1,1} - \pi_{s,2,2} &\geq (V_{T2} - V_{T1}) \sum_{i=1}^{2} f_{Ai} p_{2i} (1 - e_{2i}) \\ &\geq (V_{T2} - V_{T1}) f_{A2} p_{22} (1 - e_{22}). \end{aligned}$$

Re-arranging yields $p_{22} (1 - e_{22}) \leq \frac{\pi_{s,1,1} - \pi_{s,2,2}}{(V_{T2} - V_{T1}) f_{A2}}$. Substituting this inequality into the last term in (90), yields

$$\begin{aligned} \Delta &\geq p_{22} (V_{A2} - V_{A1}) - \frac{\pi_{s,1,1} - \pi_{s,2,2}}{(V_{T2} - V_{T1}) f_{A2}} \left((V_{A2} - V_{A1}) + s_2 - s_1 \right) & (92) \\ &\geq p_{22} (V_{A2} - V_{A1}) - \frac{\pi_{s,1,1}}{(V_{T2} - V_{T1}) f_{A2}} \left((V_{A2} - V_{A1}) + s_2 - s_1 \right). & (93) \end{aligned}$$

The seller's unconditional equilibrium expected profit, summed over its two types, is

$$\begin{aligned} \pi_s &= f_{T1} \pi_{s,1,1} + f_{T2} \pi_{s,2,2} & (94) \\ &\geq f_{T1} \pi_{s,1,1}. & (95) \end{aligned}$$

Because a seller's unconditional expected profit cannot exceed the full extraction amount,

$$\pi_{s,1,1} \leq f_{A1} s_1 + f_{A2} s_2,$$

which, by (95), yields $\pi_{s,1,1} \leq \frac{f_{A1} s_1 + f_{A2} s_2}{f_{T1}}$. Substituting this inequality into (93) yields

$$\Delta \geq p_{22} (V_{A2} - V_{A1}) - \frac{f_{A1} s_1 + f_{A2} s_2}{(V_{T2} - V_{T1}) f_{A2} f_{T1}} \left((V_{A2} - V_{A1}) + s_2 - s_1 \right). \tag{96}$$

Next, we bound p_{22}. If a seller type offers a menu consisting of the single contract with $\left(c, e = 1 - \frac{V_{A2}}{V_{A2}+s_2+V_{T1}-c}; p = 1\right)$, where c is sufficiently negative then by (80), this contract will yield both bidder types with nonnegative expected profit. Hence, both bidder types will accept the offer. When c is sufficiently negative, the seller's expected profit approaches $f_{A2}s_2 + f_{A1}(V_{A1} + s_1 - V_{A2})$. Thus, optimality of the mechanism requires

$$\pi_{s,j,j} \geq f_{A2}s_2 + f_{A1}(V_{A1} + s_1 - V_{A2}), \; j = 1, 2. \tag{97}$$

Suppose the equilibrium is separating. Then conditional on a type 2 seller, the maximum welfare surplus is $s_1 + p_{22}s_2$. The bidder's individual rationality condition yields $\pi_{s,2,2} \leq s_1 + p_{22}s_2$, which, by (97), yields

$$s_1 + p_{22}s_2 \geq f_{A2}s_2 + f_{A1}(V_{A1} + s_1 - V_{A2}). \tag{98}$$

Substitute for $f_{A2} = 1 - f_{A1}$, define $\delta \equiv \frac{s_2 - s_1}{V_{A2} - V_{A1}} - \frac{f_{A1}}{(1-f_{A1})}$ and solve the inequality for

$$p_{22} \geq \frac{\delta (V_{A2} - V_{A1})(1 - f_{A1})}{s_2}. \tag{99}$$

Note that $\delta > 0$ from the premise of the proposition in (88).

Now consider a pooling equilibrium. Then both seller types offer the same menu, so $p_{21} = p_{22}$. Unconditional on seller type, the maximum welfare surplus is again $s_1 + p_{22}s_2$. Individual rationality of bidders yields that a seller's unconditional expected equilibrium profit satisfies $\pi_s \leq s_1 + p_{22}s_2$, where $\pi_s = f_{T1}\pi_{s,1,1} + f_{T2}\pi_{s,2,2}$. Then (97) and (94) yield (98) and (99).

Thus, (99) holds in both separating and pooling equilibria. Plugging (99) into (96) yields

$$\Delta \geq \frac{\delta (V_{A2} - V_{A1})^2 (1 - f_{A1})}{s_2} - \frac{f_{A1}s_1 + f_{A2}s_2}{(V_{T2} - V_{T1}) f_{A2} f_{T1}} ((V_{A2} - V_{A1}) + s_2 - s_1). \tag{100}$$

This yields $\Delta > 0$ when (89) holds, i.e., when $V_{T2} - V_{T1} > \phi$, where

$$\phi \equiv \frac{(f_{A1}s_1 + f_{A2}s_2)\, s_2 ((V_{A2} - V_{A1}) + s_2 - s_1)}{f_{A2} f_{T1} (V_{A2} - V_{A1})^2 (1 - f_{A1})\, \delta}. \tag{101}$$

By (91) and $\Pi_{b,2} \geq 0$, the proposition follows. ■