Strategic commitment by an informed speculator[1]

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Abstract

We analyze a model of speculation by an informed trader who can commit to his trading strategy in a Kyle-style dealership market. The informed trader commits to a trading strategy and market makers price competitively given knowledge of the functional form of the trading strategy, but not the trader's private information. We provide conditions under which a unique equilibrium obtains. We characterize the (non-negative) value of this strategic commitment, showing constructively that it can be strictly positive. We then derive necessary and sufficient closed-form conditions for the informed trader not to be able to profit from commitment. This imposes conditions on model primitives—the distributions of the fundamental value and noise trades—that are satisfied by linear equilibria, e.g., when both distributions are Normal.

1 Introduction

Consider a speculator who has designed a sophisticated trading algorithm. The algorithm may rely on information gleaned elsewhere, but otherwise has minimal real time human interaction—in essence, the trading algorithm is fixed over long durations although its information inputs arrive at a far higher frequency. The speculator trades in a market with high frequency market makers who seek to reverse engineer the speculator's algorithm to uncover her trading strategy. One's first instinct might be that this reverse engineering should harm the speculator. But reflection suggests that this might not be so, as the market makers are learning the form of the speculator's strategy, rather her information, itself. In effect, by unraveling the speculator's strategy, this raises the possibility that market makers may be providing the speculator a first-mover advantage, possibly yielding her higher profits.[1,2]

We analyze speculation by an informed trader who can commit to her trading strategy in a Kyle-style (Kyle, 1985) competitive dealership market. Specifically, the informed trader can commit to the functional form of her trading strategy; market makers then price competitively given knowledge of the functional form of the trading strategy (e.g., the linear parameter of a linear trading strategy), but not the trader's private information, setting the informationally-efficient pricing rule as a function of the net order flow from the speculator and noise traders. In the equilibrium of this Stackelberg setting (Kyle, A., 2007, private communication), the speculator's trading strategy maximizes expected profits subject to market makers setting consistent informationally-efficient pricing rules. We establish existence of a unique equilibrium and characterize the value of strategic commitment by an informed trader.

To do this, we build on Boulatov and Livdan (2022), who establish that a unique Nash equilibrium exists in the single-period trading model of Kyle (1984, 1985). The original Kyle (1985) static trading model examines a Nash equilibrium in which a monopolistic informed trader chooses a possibly non-linear trading strategy to maximize profits and competitive market makers simultaneously choose a possibly non-linear pricing rule that generates zero expected market-maker profits conditional on any net order flow. Kyle (1985) shows that there is only one equilibrium in which the trading strategy and pricing rule are both linear functions. Using mild regularity conditions, Boulatov and Livdan prove the existence and

[1] An illustration of an explicit open construction of strategies is RavenPack, which posts the algorithm used to read news and score sentiment on its website.

[2] Indeed, market makers have strong incentives to unravel a speculator's trading strategy, else they will misprice relative to other market makers who do unravel.

uniqueness of a Nash equilibrium without imposing linearity assumptions and for a broad class of pdfs for the fundamental value v and random aggregate noise trader demand u. We extend this analysis to our Stackelberg setting, establishing that a unique equilibrium exists.

We then characterize the value of strategic commitment. We ask: when can a speculator earn higher expected profits if she can commit to a particular trading strategy? In this setting, market makers break even in expectation, so the question becomes: when can committing to a trading strategy allow a speculator to extract greater profits from the noise traders? It is immediate that the speculator can do at least as well as in the standard Nash setting—the speculator can always commit to the Nash trading strategy to earn the same expected profit—but under what circumstances can the speculator do no better with commitment?

We establish constructively that the speculator can earn strictly higher expected profits in the Stackelberg equilibrium than in the Nash equilibrium. Specifically, we solve explicitly for equilibrium outcomes in the Cho and Karoui (2000) model, which maintains all assumptions of Kyle (1985) save that v has a Bernoulli distribution rather than a normal distribution, showing that the ability to commit to a trading strategy has value.

We then show, surprisingly, the speculator cannot always profit from an ability to commit to her trading strategy. We derive closed-form analytic conditions on the pdfs of v and u for a speculator not to be able to profit from commitment, i.e., for the Nash and Stackelberg equilibria to coincide. Of note, these conditions are satisfied for linear equilibria, in particular when as in Kyle (1985) settings, the pdf f_v is equal to some linear rescaling of f_u. In our proof, we formulate the equilibrium problem as a fixed point problem of a particular functional, defined in an appropriate function space. We explicitly construct the pricing functional for an arbitrary informed trading strategy, and derive its properties. We then show the knife-edge nature of this result: we prove that in the vicinity of linear equilibria, for almost all distributions f_v and f_u, commitment yields higher speculator profits.

The closest related research is Biais and Germain (2002), who analyze a setting with discrete (bad, zero, good) private information and equally likely liquidity trades of $-L$, 0, or L. The competitive market makers see a pair of orders, but do not know which one is from the informed trader. Biais and Germain characterize when commitment by the speculator to a mixed trading strategy (only trading when she has private information with a probability $\alpha < 1$) has value. In particular, commitment raises informed profits whenever the speculator is sufficiently likely to have information, as price then moves less when she trades L or $-L$:

the less aggressive strategy raises expected ex-ante profits by reducing market reaction.

2 Model

As a benchmark, recall the classic static Kyle (1985) model. In the model, a single risk-neutral informed trader privately observes an asset's liquidation value v drawn from a distribution with mean zero and pdf $f_v(\cdot)$. Liquidity traders cumulatively trade a quantity u, drawn independently from a distribution with zero mean and pdf $f_u(\cdot)$. After observing v, the speculator chooses a quantity x to trade. The quantity x is a "market order," in the sense that it depends on v, but not the equilibrium price. The speculator does not see the level of noise trade u before trading. Market makers know the joint distribution of v and u but do not observe either realization. Instead, they only observe the net order flow $y = x+u$, and then set a competitive price that yields zero expected profits conditional on the net order flow observed.

The Bayesian Nash equilibrium (BNE) in Kyle (1985) is formally defined by two functions, a trading strategy $X^*(\cdot)$ and a pricing rule $P^*(\cdot)$, that satisfy, respectively, a profit-maximization condition and a market-efficiency condition. The profit-maximization condition states that the speculator's order $x = X^*(v)$ maximizes her expected profits given the pricing rule $P^*(\cdot)$ i.e.,

$$X^*(v) = \arg\max_x E_u[(v - P^*(x+u))x|v]. \tag{1}$$

The market efficiency condition states that market makers expect zero profits given the observed net order flow $y = x+u$ and taking the informed trader's trading strategy as given, i.e.,

$$P^*(y) = E[v|X^*(v) + u = y]. \tag{2}$$

With Normal pdfs $f_v(\cdot)$ and $f_u(\cdot)$, the proof that a unique BNE exists in which $X^*(\cdot)$ and $P^*(\cdot)$ are linear functions is simple (see, e.g, Kyle (1985)). The equilibrium trading strategy and pricing rule take the forms

$$X^*(v) = \frac{\sigma_u}{\sigma_v} v \quad \text{and} \quad P^*(y) = \frac{1}{2}\frac{\sigma_v}{\sigma_u} y. \tag{3}$$

Note that a linear trading strategy implies a linear pricing rule and vice versa. Thus, in any equilibrium, *either both* the trading strategy and pricing rule are linear *or neither* are.

To examine general non-linear trading strategies and pricing rules, it is useful to introduce notation that describes the reaction function of market makers to a possibly non-linear trading strategy of the speculator. We re-write pricing rule (2) to emphasize the functional dependence on the conjectured speculator's strategy $X_c(\cdot)$:

$$P(y, X_c) = E[v|X_c(v) + u = y]. \tag{4}$$

The notation $P(y, X_c)$ indicates that the price depends on both a scalar argument given by the aggregate order flow y and a function argument given by the demand function X_c that the market makers believe the informed trader is using. Our analysis makes use of functionals, i.e. functions mapping both scalars and other functions into scalars. To keep notation clear, we place scalar arguments in front of functional arguments, as in (4). For clarity, we generally use lower-case letters to denote scalars and upper case letters to denote functions or functionals, except for pdfs, where we use lower case letters to avoid confusion with cdfs.

Let $\overline{P}(x, X_c)$ denote the expected price obtained by the speculator when she trades x and market makers believe she is using trading strategy X_c. The functional $\overline{P}(x, X_c)$ is defined by

$$\overline{P}(x, X_c) = E_u\left[P\left(x + u, X_c\right)\right]. \tag{5}$$

When the trading strategy $X(\cdot)$ is linear, the functionals $P(y, X)$ and $\overline{P}(x, X)$ are identical linear functions because the zero-mean noise term u has no effect when P is linear in y. When, instead, the trading strategy $X(\cdot)$ is non-linear, the functions $P(y, X)$ and $\overline{P}(x, X)$ generally differ from each other and do not have simple closed-form expressions. When P is a non-linear function of y, the noise term u has the effect of making $\overline{P}$ a smoothed version of P.

We analyze two strategic settings, the Nash equilibrium of Kyle (1985) and the Stackelberg equilibrium described by Kyle (1983).[3] To facilitate analysis, we redefine the equilibrium concepts as fixed point problems involving functionals.

Suppose the speculator observes the realization v and trades the quantity x, while market makers conjecture that the speculator follows strategy X_c and set the informationally-efficient pricing rule, $P(x + u, X_c))$. The speculator's expected payoff $\pi_I(v, x, P)$ is given by

$$\pi_I(v, x, P) = E_u[x(v - P(x + u, X_c))]. \tag{6}$$

When the speculator determines the *functional form* of her best response $X(\cdot)$ *before* seeing the specific realization of v,[4] we formulate her ex-ante expected payoff, (6), integrating over

[3] We discuss later structural similarities and distinctions between this model and that of Rochet and Vila (1994) who consider a informed trader who also sees the level of noise trade u and can condition trade on u.

[4] After the realization of v is observed, substituting it into X yields the optimal traded quantity $x = X(v)$.

all realizations of v, as a functional of her actual strategy X and the market makers' conjecture X_c in (4). The speculator's *ex-ante expected* payoff, denoted $\Pi_I(X, X_c)$, takes the form

$$\begin{aligned} \overline{\pi}_I(X, X_c) &= E_v[\pi_I(v, X(v), X_c)] \\ &= E_{v,u}[x(v - P(x+u, X_c))]. \end{aligned} \tag{7}$$

Definitions of BNE and BSE. The Nash equilibrium trading strategy, denoted X_N, is defined by the fixed-point condition

$$X_N = \arg\max_X \; \overline{\pi}_I \left(X, X_N\right). \tag{8}$$

That is, in a Nash equilibrium, when the speculator takes as given the pricing by market makers based on their beliefs about the trading strategy she is going to use, the speculator indeed chooses that same trading strategy. Although the reaction-function notation emphasizes the choice of the function $X(\cdot)$, condition (8) leads to a definition of Nash equilibrium that is logically equivalent to that in Kyle (1985), defined above in equations (1) and (2). The two definitions are equivalent because the speculator's optimization problem decomposes into separate state-by-state optimization problems for each realization of v.

Suppose now that the speculator can commit to her trading strategy. Then the speculator's Stackelberg equilibrium trading strategy, denoted X_S, is given by the fixed-point

$$X_S = \arg\max_X \; \overline{\pi}_I(X, X). \tag{9}$$

Market makers still price according to the market-efficiency condition (4), but the speculator now accounts for the functional dependence of their pricing rule on her trading strategy.

3 First-Order Conditions

In the Stackelberg setting, we must account for the functional dependence of price when we derive the analogue of first-order conditions (FOC) for the payoff (7). The speculator's payoff depends on the pricing rule, which, in turn, depends on the functional form of her trading strategy. The speculator may be able to increase her payoff by adjusting the functional form of her trading strategy $X(\cdot)$.

Define the partial derivative of the price functional

$$\overline{P}'(x, X) = \frac{\partial}{\partial x}\overline{P}(x, X). \tag{10}$$

To describe the price sensitivity to variation in the speculator's trading strategy, we use the notion of *functional differentiation*. The functional differential of the price at a strategy $X(\cdot)$ is defined by

$$\delta\overline{P}\left(x, X; \delta X\right) = \lim_{\varepsilon \to 0} \left\{ \frac{\overline{P}\left(x, X + \varepsilon\delta X\right) - \overline{P}(x, X)}{\varepsilon} \right\}, \tag{11}$$

provided that the limit (11) exists for every $\delta X(\cdot)$ (from the same functional space), and defines a functional, linear and bounded in $\delta X(\cdot)$. This definition corresponds to the *weak*, or *Gateaux differential* (see, e.g., Kolmogorov and Fomin (1999), Saati (1981)). Equation (11) can be viewed as an extension of the *directional derivative* of functions depending on several variables to the case where some of the arguments are functions. The differential (11) measures the price sensitivity to the *functional form* of the speculator's trading strategy $X(\cdot)$. In contrast, (10) describes the *quantity* variation component, which does not depend on the functional form of the strategy. The definition below of the full variation of the price functional can be viewed as an extension of the full differential of a function depending on several variables, when the full differential is obtained as a sum of partial differentials with respect to all variables. In our setting, one of the "variables" is the function representing the speculator's trading strategy. For this reason, the full variation of the price functional is given by

$$D\overline{P}\left(x, X, \delta x; \delta X\right) = \overline{P}'(x, X)\delta x + \delta\overline{P}\left(x, X; \delta X\right). \tag{12}$$

For a given x and X, $D\overline{P}\left(x, X, \delta x; \delta X\right)$ defines a linear function of δx and a linear functional of δX.

When the speculator can commit to her trading strategy, her expected payoff (7) has two functional arguments, one corresponding to her actual strategy, and one corresponding to the strategy conjectured by the market makers. Using the definition in (11), $\delta_1\overline{\pi}_I\left(X, Y; \delta X\right)$ and $\delta_2\overline{\pi}_I\left(X, Y; \delta Y\right)$ are the functional differentials of the speculator's expected payoff with respect to the first and second functional arguments, respectively. Correspondingly, the full variation of the speculator's expected payoff is given by

$$\delta\overline{\pi}_I\left(X, Y; \delta X, \delta Y\right) = \delta_1\overline{\pi}_I\left(X, Y; \delta X\right) + \delta_2\overline{\pi}_I\left(X, Y; \delta Y\right). \tag{13}$$

Using these definitions, we obtain the following first-order conditions for the speculator's profit maximization problem in the Nash and Stackelberg settings. A necessary condition for a Nash equilibrium is that for all variations δX belonging to the basic functional set, we have:

$$\begin{aligned} 0 &= \delta_1\overline{\pi}_I(X_N, X_N; \delta X) \\ &= E_v\left[\left\{v - \overline{P}(X_N(v), X_N) - X_N(v)\overline{P}'(X_N(v), X_N)\right\}\delta X(v)\right]. \end{aligned} \tag{14}$$

A necessary condition for a Stackelberg equilibrium is that for all variations δX, we have:

$$\begin{aligned} 0 &= \delta_1 \overline{\pi}_I \left(X_S, X_S; \delta X\right) + \delta_2 \overline{\pi}_I \left(X_S, X_S; \delta X\right) \\ &= E_v \left[\left\{ v - \overline{P}\left(X_S(v), X_S\right) - X_S(v) \overline{P}'\left(X_S\left(v\right), X_S\right) \right\} \delta X(v) \right] - \\ &\quad E_v \left[X_S(v) \delta_2 \overline{P}\left(X_S\left(v\right), X_S\right) \right]. \end{aligned} \tag{15}$$

The first-order conditions (14) and (15) are analogous to, but distinct from, the first-order condition obtained in Rochet and Vila (1994), where an insider also observes uninformed (liquidity demand) that she can condition her strategy on. In our Kyle (1985) setting, insiders do not observe liquidity demand, rendering the analysis fundamentally different.

In the Stackelberg setting, the first-order condition (15) contains the additional structural derivative of the price functional $\delta_2 \overline{P}\left(X_S(v), X_S\right)$, reflecting that when the speculator changes her trading strategy, she internalizes the effect on market maker pricing in her profit maximization problem. Define $Q(y, X) = E\left[X(v)|X(v) + u = y\right]$ to be the expected value of $X(v)$ from the perspective of market makers who observe the net order flow y; and define

$$J(v, x, X) = \frac{\partial}{\partial x} E_u \left[Q(x+u, X) \left\{ v - P(x+u, X) \right\} \right] \tag{16}$$

to be the derivative of the speculator's profits with respect to x, accounting for how x affects market maker inferences about $X(\cdot)$. Using (14) and (15) we have

Proposition 1: *For any v, the first-order conditions for the speculator's strategy are given by*

$$0 = v - \overline{P}\left(X_N(v), X_N\right) - X_N(v) \overline{P}'(X_N(v), X_N), \tag{17}$$

for the Nash setting, and

$$0 = v - \overline{P}(X_S(v), X_S) - X_S(v) \overline{P}'(X_S(v), X_S) - J\left(v, X_S(v), X_S\right), \tag{18}$$

for the Stackelberg setting.

Proof: See the Appendix. □

The J term on the right-hand side of (18) corresponds to the marginal expected informed trader's profits in the information set of market makers. Technically, the J-term comes from the structural variation component $\delta_2 \overline{P}\left(X(v), X(\cdot)\right)$ in the Stackelberg setting, which is absent in the informed trader's optimization problem in the Nash setting. That is, the J-term

(16) describes how the speculator accounts for the market makers' reaction to her actions in her optimal strategy. The J-term represents the difference between full functional variation in the Stackelberg setting and the ordinary differential in the Nash setting.

To develop intuition, we show that the optimality condition in the Stackelberg setting can equivalently be formulated as maximizing the speculator's "effective payoff":

Result 1: *A speculator's optimization problem in the Stackelberg setting can be expressed as:*

$$max_x \pi_{eff}(x), \tag{19}$$

where the speculator's effective payoff is given by

$$\pi_{eff}(X(v)) = E_u[(X(v) - E[X(v)|X(v) + u = y])\{v - P(X(v) + u, X)\}]. \tag{20}$$

Proof: Substituting (16) for J into the speculator's first-order conddition, (18), yields

$$\left.\frac{\partial}{\partial x}\right|_{x=X(v)} \{E_u[x(v - P(x+u, X))] - E_u[Q(x+u, X)\{v - P(x+u, X)\}]\} = 0.$$

Factoring terms yields

$$\left.\frac{\partial}{\partial x}\right|_{x=X(v)} E_u[(x - Q(x+u, X))(v - P(x+u, X))] = 0, \tag{21}$$

which can be interpreted as an optimality condition for a speculator with "effective" payoff

$$\pi_{eff}(x) = E_u[(x - Q(x+u, X))(v - P(x+u, X))]. \qquad \square \tag{22}$$

The speculator's effective payoff (22) has a simple economic interpretation. In the Nash setting, the speculator's optimization problem takes the form $\max_x E_u[x(v - P(x+u, X))]$. Because $Q(x+u, X) = E[X(v)|X(v) + u = y]$ is the market makers' best estimate of the speculator's market order $X(v)$, the term $x - Q(x+u, X)$ in (22) represents the *unexpected* component of the speculator's order from the informational perspective of market makers. The speculator's effective payoff is the expected value of the error in the market makers' forecast of the speculator's trade.[5] In the Stackelberg setting, the FOC (18) has the following interpretation. The market makers are trying to minimize the speculator's profits, and the speculator knows this. The expression (22) for the speculator's effective payoffs reveals

[5]For related forecast error results see Bernhardt and Miao (2004) or Bernhardt, Seiler and Taub (2010).

that she anticipates the market makers' actions, and hence maximizes the difference between her actual profits and the market makers' forecast.

Comparing (22) with the speculator's Nash objective, we observe that with commitment, the speculator's optimization problem explicitly takes into account that market makers can partially "undo" the effect of speculation by anticipating the speculator's orders. This effect may lead the speculator to trade less aggressively. When this is so, one can expect that the equilibrium with commitment is characterized by smaller price impacts and hence reduced information efficiency compared to the Nash case, as we will show can occur.

We next illustrate the distinction between the two components of the full variation of the price functional (12) in the classical Normal setting.

Example 1: Suppose $f_v \sim N(0, \sigma_v^2)$ and $f_u \sim N(0, \sigma_u^2)$ and strategies are linear $X(v) = \beta v$.

Kyle (1985) shows that the conjectured linear speculator strategy $X_c(v) = \beta_c v$ leads to the linear informationally-efficient pricing rule

$$P(y, X_c) \;=\; \lambda\left(X_c\right) y, \qquad \text{where } \lambda(X_c) = \frac{\beta_c}{\beta_c^2 + \beta_0^2} \text{ and } \beta_0 = \sigma_u/\sigma_v. \tag{23}$$

The expected price functional is also linear

$$\overline{P}\left(x, X_c\right) = \lambda\left(X_c\right) x. \tag{24}$$

Combining (12), (23), and (24), we obtain for the linear strategy $X(v) = \beta v$

$$\Delta\overline{P}\left(x, dx, X; \delta X\right) = \lambda dx + x\delta\lambda\left(X, \delta X\right), \tag{25}$$

where the last term on the right-hand side, $\delta\lambda\left(X; \delta X\right)$, only depends on the functional variations of the linear strategies $\delta X(v) = v\delta\beta$. In other words, $\delta\lambda\left(X; \delta X\right)$ depends on the variation of β, but not v. Then (12) and (25) yield that for linear variations $\delta X(v) = v\delta\beta$, the price derivative and the structural variation components are given by

$$\overline{P}'(x, X) \;=\; \lambda(\beta) = \frac{\beta}{\beta^2 + \beta_0^2} \quad \text{and} \quad \delta\overline{P}\left(x, X; \delta X\right) \;=\; x\frac{\partial\lambda}{\partial\beta}\delta\beta. \tag{26}$$

Foreshadowing future results, we exploit the explicit linear solutions for the speculator's trading strategy and market maker pricing to show that commitment has no added value when both liquidation values and noise trade are normally distributed. Around the optimal linear strategy, $\beta = \beta_0$, we have

$$\left[\frac{\partial\lambda}{\partial\beta}\right]_{\beta=\beta_0} = \frac{\partial}{\partial\beta}\left[\frac{\beta}{\beta^2 + \beta_0^2}\right]_{\beta=\beta_0} = 0. \tag{27}$$

Substituting this into (26) yields that the variation of the price functional with respect to linear variations of the informed trader's strategy, $\delta X(v) = \delta\beta v$, vanishes at $X(v) = \beta_0 v$:

$$\delta\overline{P}(x, X; \delta X) = 0. \tag{28}$$

Therefore, $E_v\left[x\delta\overline{P}(x, X; \delta X)\right] = E_v\left[v^2\right]\beta^2\frac{\partial\lambda}{\partial\beta}\delta\beta = 0$, which implies that the Stackelberg term J vanishes in equilibrium. Directly evaluating, $Q(y, X) = E_{v|y}[X(v)] = \lambda\beta y$, and we confirm that $J = E_u\left[\lambda\beta v - 2\lambda^2\beta y\right] = \lambda\beta v(1 - 2\lambda\beta) = 0$. $\square$

3.1 Equilibrium Existence

We next pose the informed trader's optimization in both Nash and Stackelberg settings as fixed-point problems. We impose regularity conditions on the distributions to ensure the existence of equilibria. Specifically, we impose assumptions on the densities $f_v(\cdot)$ and $f_u(\cdot)$ that rely on regularly-varying functions (RVF) and slowly-varying functions (SVF) in the Karamata sense (see, e.g., Seneta (2019), Takesi and Maric (2006), or Karamata (1962)).

DEFINITION 1A (Seneta, 2019): *A function $f(x)$ defined, positive, and measurable on $x \geq A > 0$, is said to be regularly varying of index ρ if $\lim\limits_{x\to\infty}\frac{f(\lambda x)}{f(x)} = \lambda^\rho$ for some real $\rho \in R$ and any $\lambda > 0$.*

To establish the existence of Nash and Stackelberg equilibria we adopt the regularity conditions on the distributions of priors and noise trade imposed by Boulatov and Livdan (2022):

1. Probability density functions $f_v(\cdot)$ and $f_u(\cdot)$ are smooth and have infinite support.

2. $f_v(x) = C_v \exp\left(-\psi_v(|x|)\right)$ and $f_u(x) = C_u \exp\left(-\psi_u(|x|)\right)$ where (i) ψ_v is convex (the pdf f_v is thus log-concave), (ii) both ψ_v and ψ_u are measurable with respect to f_v, and (iii) ψ_u is measurable with respect to f_u. Also, both ψ_v and ψ_u are RVF with indices $a_v > 1$ and a_u, respectively.

3. $f_u(u)$ is an analytic function for $u \in C$ (except for, possibly, the point $u = 0$) satisfying the condition $|f_u(x + iy)| \leq g_u(x)M(y)$, where the function $g_u(\cdot)$ is a pdf, ψ_u is measurable with respect to g_u, and $M(y)$ is finite, $|M(y)| < \infty$, for any finite $|y| < \infty$.

For example, Normal pdfs satisfy these regularity conditions, corresponding to $f_v(x) = C_v \exp\left(-\psi_v(|x|)\right)$ and $f_u(x) = C_u \exp\left(-\psi_u(|x|)\right)$ where $\psi_v(|x|) = \psi_u(|x|) = \frac{x^2}{2}$, implying that ψ_v and ψ_u are RVF with indices $a_v = a_u = 2$. The conditions rule out fat-tailed distributions with infinite support and asymmetric distributions. Condition 3 says that the analytic extension of $f_u(\cdot)$ in a complex plane remains bounded and without fat tails along the real axis. In essence, equilibrium existence requires that expected trading profits be bounded—when the tails are too fat, speculators observe high values of v sufficiently frequently and with fat-tailed noise trade can submit aggressive orders without being easily detected by market makers, resulting in unbounded profits and hence an unraveling of equilibrium.

Theorem 1 *The Bayesian Nash and Stackelberg equilibria* (X_N, P_N) *and* (X_S, P_S) *exist given the regularity conditions 1–3 on the distributions. The informed trader's optimal Stackelberg strategy* X^* *satisfies the first-order condition:* $\nabla_{X^*(v)}\overline{\pi}_I(X^*, X^*) = 0$.

Proof: See the Appendix. □

Boulatov and Livdan (2022) prove that the technical conditions 1–3 above ensure that a unique BNE exists. To extend this result to establish existence of the Stackelberg equilibrium, we first observe that the functional $\overline{\pi}_I(X, X)$ is *weakly continuous*. Therefore, its values on each weakly converging sequence $\{X_n\}$ converge (Krasnoselskii, 1964). Because $\overline{\pi}_I$ is weakly continuous, it assumes its upper and lower bounds on any finite ball $\|X\| \leq r$ (Krasnoselskii, 1964).

Second, exploiting the explicit form of the pricing rule, we observe that when the norm of the speculator's strategy $\|X\|$ increases unboundedly, i.e., when $\|X\| \to \infty$, then $\overline{\pi}_I(X, X) \to 0$. Economically, this means that strategies with infinite norm are not profitable, implying that the optimal strategy has a finite norm. This also means that the speculator's expected profit is bounded from above, $\overline{\pi}_I(X, X) \leq a < +\infty$. Because $\overline{\pi}_I$ is weakly continuous, we have $\max_{\|X\|\leq h} \overline{\pi}_I(X, X) = \overline{\pi}_I^*$, where $\overline{\pi}_I^* \leq a < +\infty$ is the maximal value of $\overline{\pi}_I(X, X)$ in the entire strategy space. Then, because the optimum $\overline{\pi}_I^*$ of $\overline{\pi}_I(X, X)$ exists and is finite, the corresponding optimal strategy X^* has a finite norm and $\overline{\pi}_I(X, X)$ takes its maximal value on this optimal strategy, $\overline{\pi}_I^* = \overline{\pi}_I(X^*, X^*)$. Finally, because $\overline{\pi}_I(X, X)$ is uniformly differentiable, its gradient at the optimal strategy X^* vanishes, $\nabla_{X^*(v)}\overline{\pi}_I(X^*, X^*) = 0$ (Krasnoselskii, 1964).

3.2 Market maker isoprofit curves

To provide intuition about the speculator's optimization problem in the Nash and Stackelberg settings, we now allow for arbitrary pricing functions $P(y)$ that need not equal the informationally-efficient pricing function, $P(y, X_c) = E[v|y = X_c(v) + u]$. We consider a speculator's profit functionals in this extended strategy space. The informationally-efficient pricing rules and corresponding conjectured strategies form an *isoprofit curve* in the functional space characterized by the expected profits of market makers being identically zero along the curve. Expected market maker profits equal

$$\begin{aligned} \overline{\pi}_M(X, X_c) &= E_{v,u}[y(P(y, X_c) - v)] \\ &= E_y E_{v|y}[y(P(y, X_c) - v)] \\ &= E_y[y(P(y, X_c) - P(y, X))], \end{aligned} \tag{29}$$

where $P(y, X) = E_{v|y}[v] = E[v|y = X(v) + u]$ is the expected value of v conditional on the net order flow y. Informational efficiency requires that the marginal profit be zero, i.e., the price is the conditional expectation of v given the correct conjecture, $X_c = X$. We prove in Proposition 2 that this condition holds locally, even for arbitrary variations of the pricing rule.

Consider the Stackelberg setting with commitment, $X_c = X$, and define the *isoprofit curve* $\Gamma_C(X, P)$ in the functional space of pricing rules $P(y, X)$ as $\overline{\pi}_M(X, P) \equiv \mathit{const} = C$, when $(X, P) \in \Gamma_C(X, P)$. This is a direct extension of the notion of a curve in finite dimensional Euclidean space to the case where the "point" (X, P) is characterized by two functions defined in a Banach space. In particular, the zero-profit isoprofit curve for market makers, $\overline{\pi}_M(X, P) \equiv 0$, contains all pairs of speculator trading strategies and corresponding informationally-efficient pricing rules, i.e., $(X, P) \in \Gamma_0(X, P)$.

Suppose that a pair (X, P) belongs to Γ_C, i.e. $(X, P) \in \Gamma_C(X, P)$. Consider any small variation of the speculator's strategy δX. Then the resulting variation in the pricing rule δP should lead the point $(X + \delta X, P + \delta P)$ to still belong to the same isoprofit curve Γ_C. Evaluating the variation of (29) along Γ_C, and noting that $\delta y = \delta(X(v) + u) = \delta X(v)$, we obtain

$$\delta\overline{\pi}_M(X, P) = E_{v,u}[\delta X(v)(P(y, X) - v + yP'(y, X)) + y\delta P(y, X)] = 0. \tag{30}$$

In particular, along Γ_0, we use the explicit form of the informationally-efficient pricing rule $P_e(y, X) = E[v|y]$ to obtain:

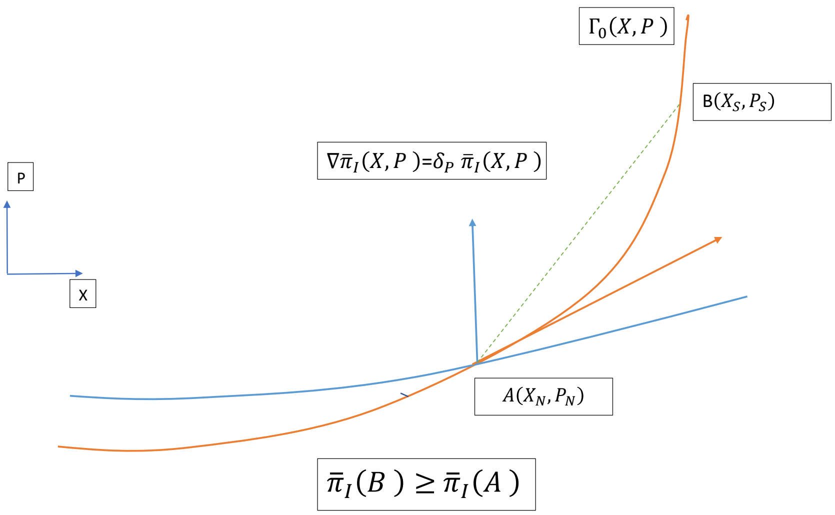
Proposition 2: *The informationally-efficient pricing rule $P_e(y, X)$ satisfies the isoprofit condition*

$$\delta\overline{\pi}_M(X, P_e) = E_{v,u}[\delta X(v)(P_e(y, X) - v + yP_e'(y, X)) + y\delta P_e(y, X)] = 0, \tag{31}$$

for expected market maker profits.

Proof: See the Appendix. □

Figure 1: Optimization by speculator in Nash and Stackelberg settings



The Nash and Stackelberg equilibrium points are (X_N, P_N) and (X_S, P_S), respectively. The Nash equilibrium is given by intersection of the blue curve representing the speculator's best response to an arbitrary pricing rule P and the red curve representing informationally-efficient pricing given an arbitrary informed trading strategy X. The Stackelberg equilibrium $(X_S, P_S) = \underset{(X,P)\in\Gamma_0}{\arg\max}\bar{\pi}_I(X, P)$, is obtained by maximizing the speculator's expected profit *along the zero profit curve* Γ_0 *for market makers.*

Figure 1 illustrates the economics. Recall that the speculator's FOC in the Nash setting takes the form

$$\delta_1\bar{\pi}_I(X, X_c) = \delta_X E_{u,v}[X(v)(v - P(X(v) + u, X_c)] = 0. \tag{32}$$

This must hold also in equilibrium at the "fixed" point of (X, X_c) where $X = X_c$. In the Nash setting, the functional form of pricing rule is pinned down by the market makers' conjecture X_c; it is not a choice variable for the speculator who takes it as given in her optimization problem. Then the equilibrium point (X_N, P_N) is obtained as an intersection of two curves in the functional space (X, P), as depicted in Figure 1. The blue curve in Figure 1 is a "response function" of the speculator's trading strategy X to some arbitrary pricing rule P set by market makers. That is, the blue curve illustrates a set of pricing rules and corresponding speculator trading strategies in the functional space associated with the FOC (32).

The red curve in Figure 1 is $\Gamma_0(X,P)$, which represents a set (X,P_e) of arbitrary speculator trading strategies X and associated informationally-efficient pricing rules $P_e\,(y,X)$. It can be viewed as the response function P_e of market makers to an arbitrary speculator trading strategy X. The Nash equilibrium point (X_N,P_N) is given by the intersection of the two curves, i.e., by the intersection of the two reaction functions, and is characterized by both FOCs holding, i.e., by the speculator's optimality condition and the informational efficiency condition coming from the informational-efficient pricing rule set by market makers.

Now consider a Stackelberg equilibrium point (X_S,P_S) as illustrated in Figure 1. The speculator is, in effect, constrained to choosing a point on the isoprofit curve Γ_0 for which market makers expect zero profit, i.e., for which the pricing rule is informationally efficient. That is, a speculator's optimization problem can be represented as maximizing her expected profit *along the zero profit curve* Γ_0 *for market makers.* Its solution, $(X_S,P_S) = \arg\max_{(X,P)\in\Gamma_0} \overline{\pi}_I(X,P)$, or, equivalently, $X_S = \arg\max_{X} \overline{\pi}_I(X,X)$ gives the Stackelberg equilibrium. By construction, moving away from the Stackelberg equilibrium (X_S,P_S) along Γ_0 cannot lead to higher speculator profits.

3.3 Speculator profit and commitment

The ability to commit to a trading strategy can never harm a speculator—she can always commit to her Nash strategy, in which case market maker pricing is unchanged, implying that the speculator's profits are unchanged state by state. We next constructively establish that a speculator can sometimes earn strictly higher expected profits in the Stackelberg equilibrium than in the Nash equilibrium. Further, since market makers expect zero profits in both types of equilibria, this implies that noise trader losses are strictly higher in the Stackelberg setting.

To do this, we compare speculator profits in the Nash and Stackelberg equilibria of the Cho and Karoui (2000) model. Cho and Karoui replace the assumption in Kyle (1985) that v has a normal distribution with the assumption that v has a symmetric Bernoulli distribution,

$$v = \begin{cases} a, & prob. = \frac{1}{2} \\ -a, & prob. = \frac{1}{2}, \end{cases} \tag{33}$$

maintaining all other assumptions of Kyle (1985). Boulatov and Livdan (2022) prove that this model has a unique Nash equilibrium, and that equilibrium trading strategies and pricing are non-linear.

The pricing rule in the Cho and Karoui (2000) model remains both a smooth analytic function of total order flow and a smooth analytic functional in the speculator's strategy (Boulatov and Livdan (2022)), even though the Bernoulli distribution does not satisfy our distributional assumptions. As a result, the model is (almost) analytically tractable.

It is straightforward to verify that with a symmetric Bernoulli prior (33), the speculator's strategy takes the form

$$X(v) = \begin{cases} \beta, & v = a \\ -\beta, & v = -a \end{cases}. \tag{34}$$

That is, the speculator's optimal strategy is an *odd* function of v. Following Boulatov and Livdan (2022), we obtain, as in Cho and Karoui (2000), the pricing rule

$$\begin{aligned} P(y) &= a\frac{\frac{1}{2}\exp\left(\frac{(y-\beta)^2}{2\sigma_u^2}\right) - \frac{1}{2}\exp\left(\frac{(y+\beta)^2}{2\sigma_u^2}\right)}{\frac{1}{2}\exp\left(\frac{(y-\beta)^2}{2\sigma_u^2}\right) + \frac{1}{2}\exp\left(\frac{(y+\beta)^2}{2\sigma_u^2}\right)} \\ &= a\frac{\exp\left(\frac{y\beta}{\sigma_u^2}\right) - \exp\left(-\frac{y\beta}{\sigma_u^2}\right)}{\exp\left(\frac{y\beta}{\sigma_u^2}\right) + \exp\left(\frac{y\beta}{\sigma_u^2}\right)} = a\tanh\left(\frac{\beta y}{\sigma_u^2}\right). \end{aligned} \tag{35}$$

To analyze the Nash equilibrium, we use the speculator's optimality condition

$$\overline{P}\left(X(v)\right) + X(v)\overline{P}'\left(X(v)\right) = v,$$

which involves the expected price functional $\overline{P}\left(X(v)\right) = \mathbb{E}_u\left[P\left(X(v)+u\right)\right]$. Using (35), we solve for the expected price functional

$$\overline{P}(x) = a\mathbb{E}_u\left[\tanh\left(\frac{\beta}{\sigma_u^2}\left(x+u\right)\right)\right]. \tag{36}$$

Because $\tanh(\cdot)$ is an odd function and the normal $N(0,\sigma_u^2)$ distribution of the noise trade is symmetric, it follows from (36) that $\overline{P}(x)$ is an odd function, i.e., $\overline{P}\left(-x\right) = -\overline{P}(x)$. Rescaling the informed trader's strategy and the noise demand by σ_u, we obtain

$$\overline{P}(x) = a\mathbb{E}_u\left[\tanh\left(\beta\left(x+u\right)\right)\right]. \tag{37}$$

As in Cho and Karoui (2000), we obtain the speculator's FOCs in the form

$$\begin{aligned} 1 &= \mathbb{E}_u\left[\tanh\left(\frac{\beta}{\sigma_u^2}\left(\beta+u\right)\right)\right] + \left(\frac{\partial}{\partial x}\right)_{x=\beta}\mathbb{E}_u\left[\tanh\left(\frac{\beta}{\sigma_u^2}\left(x+u\right)\right)\right], \\ -1 &= \mathbb{E}_u\left[\tanh\left(\frac{\beta}{\sigma_u^2}\left(-\beta+u\right)\right)\right] + \left(\frac{\partial}{\partial x}\right)_{x=-\beta}\mathbb{E}_u\left[\tanh\left(\frac{\beta}{\sigma_u^2}\left(x+u\right)\right)\right], \end{aligned} \tag{38}$$

where the parameter a cancels out. Because the expected price (36) is an odd function of β, it follows that the two FOCs in (38) are equivalent (reflecting the symmetric prior and signal structure). After simplifying, evaluating the derivative, and rescaling one more time, tedious algebra yields the following fixed point condition

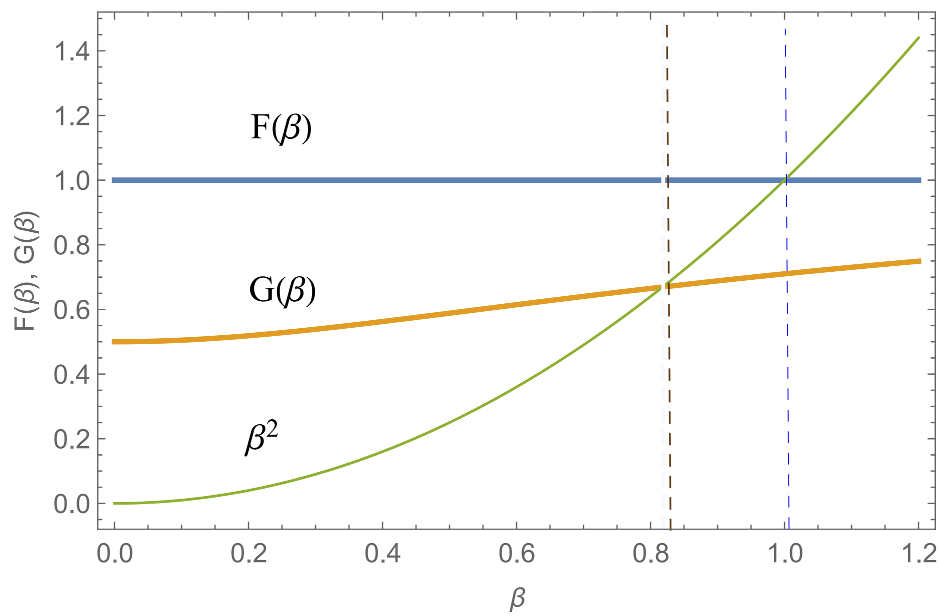
$$\beta^2 = \frac{1 - \mathbb{E}_u \left[\tanh\left(\beta^2 + \beta u\right)\right]}{1 - \mathbb{E}_u \left[\tanh^2\left(\beta^2 + \beta u\right)\right]}, \ u \sim N\left(0, 1\right). \tag{39}$$

Integration yields that $\mathbb{E}_u \left[\tanh\left(\beta^2 + \beta u\right)\right] = \mathbb{E}_u \left[\tanh^2\left(\beta^2 + \beta u\right)\right]$ for all β and hence the right-hand side of (39) is given by

$$F(\beta) = \frac{1 - \mathbb{E}_u \left[\tanh\left(\beta^2 + \beta u\right)\right]}{1 - \mathbb{E}_u \left[\tanh^2\left(\beta^2 + \beta u\right)\right]} \equiv 1. \tag{40}$$

In Figure 1, the functions β^2 and $F(\beta)$ are given by the thin green and blue lines, respectively. Hence, the solution of (39) is given by $\beta = 1$. This completely characterizes the Bayesian Nash equilibrium (BNE).

Figure 2: FOC for equilibrium with commitment and Nash



The functions β^2 and $F(\beta)$ are given by the thin green and blue lines, respectively. The function $G(\beta)$ is presented by an orange line. The solution of (39) is $\beta = 1$. The solution $\beta_S \approx 0.8$ of (42) is smaller than $\beta_N = 1$ solving (39), i.e., $\beta_S < \beta_N = 1$.

Now consider the Stackelberg setting. In this case, the speculator maximizes:

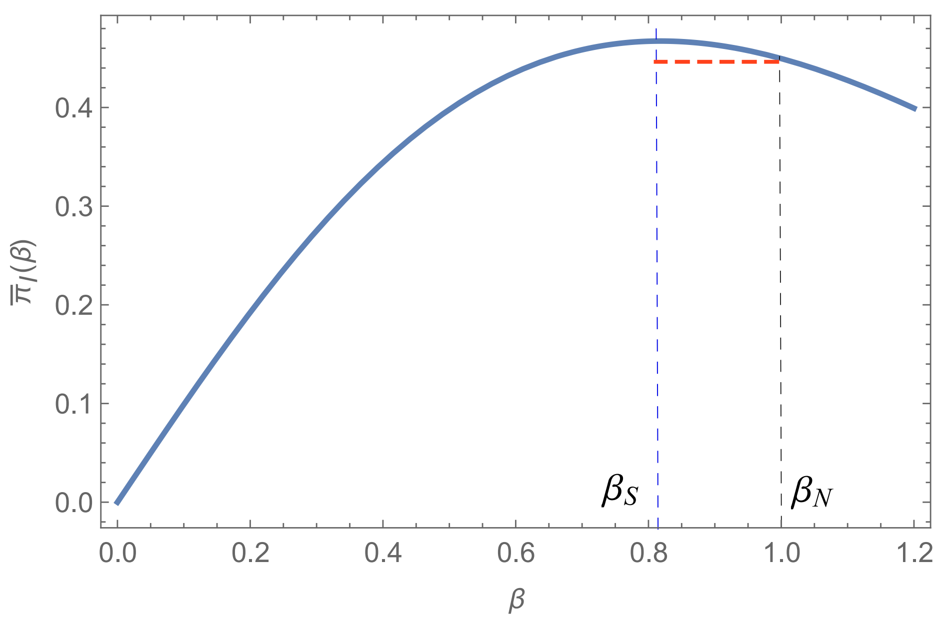
$$\overline{\pi}_I(\beta) = a\beta\left(1 - \mathbb{E}_u\left[\tanh\left(\beta\left(x + u\right)\right)\right]\right), \ u \sim N\left(0, 1\right). \tag{41}$$

The associated FOCs are given by

$$\begin{aligned} \beta^2 &= G(\beta), \\ G(\beta) &= \frac{1 - \mathbb{E}_u \left[\tanh\left(\beta^2 + \beta u\right)\right]}{2\mathbb{E}_u \left[\left(1 - \tanh^2\left(\beta^2 + \beta u\right)\right)\left(1 - \tanh\left(\beta^2 + \beta u\right)\right)\right]}. \end{aligned} \tag{42}$$

The function $G(\beta)$ is given by the orange line in Figure 2. The Stackleberg solution β_S of (42) is smaller than the Nash solution β_N solving (39), i.e., $\beta_S < \beta_N = 1$. Indeed, plugging $\beta = 1$ into the right-hand side of (42), yields $G(1) = 0.65 < 1$. Thus, as Figure 2 illustrates, the solution for β in (42) is less than 1, implying that the speculator's expected Stackelberg profits in equation (41) strictly exceed her Nash profits. Figure 3 plots those profits, revealing that they are maximized by $\beta_S \approx 0.8 < \beta_N = 1$, i.e., the speculator can increase profits by committing to a less aggressive trading strategy, loosely consistent with intuition from Biais and Germain (2002). Equation (37) also indicates that since $\beta_S < \beta_N$, price impacts are smaller in the Stackelberg equilibrium than in the Nash equilibrium, implying that the Stackelberg equilibrium features greater residual market uncertainty (lower information efficiency).

Figure 3: Speculator profits in equilibrium with commitment



This figure shows the speculator's profits (41) as function of her trading intensity β. The speculator's expected profit has a maximum at $\beta_S \approx 0.8 < \beta_N = 1$.

The speculator's equilibrium expected profits are *higher* when she can commit to her strategy than in the BNE, whenever the two equilibria differ. With commitment the speculator cannot increase her equilibrium expected profits by deviating from the Nash equilibrium allocation (X_N, P_N) to another allocation with zero expected market maker profit if and only if the two equilibria correspond, $(X_N, P_N) = (X_S, P_S)$. Conversely, for the Nash equilibrium (X_N, P_N) to correspond to the Stackelberg equilibrium, it means that

$$\delta_k \overline{\pi}_I (X_N, X_N; \delta X_k) = 0, \quad \text{for both } k = 1, 2. \tag{43}$$

From (31), the market price efficiency condition takes the form

$$\delta_1 \overline{\pi}_M (X_N, X_N; \delta X_1) + \delta_2 \overline{\pi}_M (X_N, X_N; \delta X_2) = 0. \tag{44}$$

We can now provide explicit conditions under which strategic commitment by the informed speculator does not have value:

Proposition 3: *The Stackelberg equilibrium outcome* (X_S, P_S) *is the same as the Nash equilibrium outcome* (X_N, P_N) *if and only if*

$$\begin{aligned} \left(\frac{\partial}{\partial x}\right)_{x=X_N(v)} E_u \left[x \left\{v - P(x+u, X_N)\right\}\right] &= 0, \\ \left(\frac{\partial}{\partial x}\right)_{x=X_N(v)} E_u \left[Q(x+u, X_N) \left\{v - P(x+u, X_N)\right\}\right] &= 0. \end{aligned} \tag{45}$$

Proof: The equilibria correspond if and only if

$$\delta_k \overline{\pi}_I (X_N, X_N; \delta X_k) = 0, \quad k = 1, 2. \tag{46}$$

The first condition of (45) follows from $\delta_1 \overline{\pi}_I (X_N, X_N; \delta X_1) = 0$ and is equivalent to the result of Proposition 1 for the FOC in the Nash setting, i.e.,

$$v - \overline{P}(X_N(v), X_N) - X_N(v) \overline{P}'(X_N(v), X_N) = 0. \tag{47}$$

The second condition of (45) reduces to $J(v, x, X) = 0$, which is required for the FOCs for the Nash and Stackelberg to be the same, and hence for the resulting equilibrium outcomes to match, which also follows from Proposition 1. □

The first condition of (45) is the FOC for the Nash setting, and the second condition reduces to $J(v, x, X) = 0$, implying that the Nash and Stackelberg FOCs are identical and hence so are their equilibrium outcomes. Surprisingly, this correspondence of equilibrium outcomes holds in the classical finance setting where the Nash equilibrium takes a linear form:

Proposition 4: *The Nash equilibrium is linear when the pdf $f_v(\cdot)$ is equal to a linear rescaling of $f_u(\cdot)$, i.e., when $\gamma f_u(\gamma u) = f_v(u)$ for some positive real $\gamma > 0$ and any real u.*[6]

The J term vanishes if the Nash equilibrium is linear, implying that Nash and Stackelberg equilibria yield the same equilibrium outcome.

Proof: We conjecture and verify that the equilibrium trading strategy is linear, $X(v) = \beta v$ with trading intensity parameter $\beta = \gamma$. The informationally-efficient pricing rule is then:

$$P_e(y, X) = \frac{\int v f_v(v) f_u(y - \beta v)\, dv}{\int f_v(v) f_u(y - \beta v)\, dv}. \tag{48}$$

Substituting $\beta = \gamma$, equation (48) yields

$$\begin{aligned} P_e(y, X) &= \frac{\int v f_v(v) f_u\left(\gamma\left(\frac{y}{\gamma} - v\right)\right) dv}{\int f_v(v) f_u\left(\gamma\left(\frac{y}{\gamma} - v\right)\right) dv} \\ &= \frac{\int v f_v(v) f_v\left(\frac{y}{\gamma} - v\right) dv}{\int f_v(v) f_v\left(\frac{y}{\gamma} - v\right) dv}. \end{aligned} \tag{49}$$

Using the new integration variable $v' = \frac{y}{\gamma} - v$, we have $v = \frac{y}{\gamma} - v'$, and (49) yields:

$$\begin{aligned} P_e(y, X) &= \frac{\int v f_v(v) f_v\left(\frac{y}{\gamma} - v\right) dv}{\int f_v(v) f_v\left(\frac{y}{\gamma} - v\right) dv} = \frac{\int \left(\frac{y}{\gamma} - v'\right) f_v\left(\frac{y}{\gamma} - v'\right) f_v(v')\, dv'}{\int f_v\left(\frac{y}{\gamma} - v'\right) f_v(v')\, dv'} \\ &= \frac{y}{\gamma} - P_e(y, X). \end{aligned} \tag{50}$$

Solving for $P_e(y, x)$ yields $P_e(y, X) = \lambda y$, with $\lambda = \frac{1}{2\gamma}$. Note that $\lambda = \frac{1}{2\gamma}$ is consistent with the speculator's FOC, $2\lambda\gamma = 1$. Thus, the linear equilibrium exists (and uniqueness follows from Boulatov and Livdan (2022)).

We now prove that the J term vanishes if the Nash equilibrium is linear. From (45),

$$\begin{aligned} \left(\frac{\partial}{\partial x}\right)_{x = X_N(v)} E_u\left[x\left\{v - P(x + u, X_N)\right\}\right] &= 0, \\ \left(\frac{\partial}{\partial x}\right)_{x = X_N(v)} E_u\left[Q(x + u, X_N)\left\{v - P(x + u, X_N)\right\}\right] &= 0. \end{aligned}$$

[6] See Carre, Collin-Dufresne and Gabriel (2022).

The speculator's trading strategy is $X_N(v) = \gamma v$ and $P(y, X_N) = E_{v|y}[v] = \lambda y$. Therefore,

$$Q(y, X_N) = E_{v|y}[X_N(v)] = \lambda\gamma y. \tag{51}$$

Using the short-hand notation $Q(y) = Q(x+u, X_N)$ and $P(y) = P(x+u, X_N)$, we have

$$\begin{aligned} & \left(\frac{\partial}{\partial x}\right)_{x=X_N(v)} E_u\left[Q(x+u, X_N)\left\{v - P(x+u, X_N)\right\}\right] \\ = \; & E_u\left[Q'(y)\left\{v - P(y)\right\} - Q(y)P'(y)\right]. \end{aligned} \tag{52}$$

where $y = X_N(v) + u$. Finally, we obtain

$$\begin{aligned} J &= E_u\left[Q'(y)\left\{v - P(y)\right\} - Q(y)P'(y)\right] \\ &= E_u\left[Q'(y)v - \frac{\partial}{\partial y}\left\{P(y)Q(y)\right\}\right] \\ &= E_u\left[\lambda\gamma v - 2\lambda^2\gamma y\right] = \lambda\gamma v\left(1 - 2\lambda\gamma\right) = 0. \end{aligned} \tag{53}$$

The last equality in (53) follows from the speculator's FOC, $2\lambda\gamma = 1$. □

Thus, we see that in a common class of financial models—those with normally distributed uncertainty where the speculator's equilibrium trading strategy takes a linear form—the speculator is unable to profit from an ability to commit to the form of her trading strategy.

To conclude we show the knife-edge nature of this result. Specifically, we show that if the Nash equilibrium strategy deviates just slightly from linearity then Nash and Stackelberg equilibria typically cease to correspond, in which case the ability to commit to a trading strategy has strictly positive value. The speculator's Nash equilibrium strategy can slightly deviate from linearity, $X(v) = \beta v + \delta X(v)$, when the pdfs slightly deviate from the rescaling condition that the pdfs f_v and f_u satisfy $\gamma f_u(\gamma u) = f_v(u)$ for some $\gamma > 0$ and any $u \in R$.

To show this, we consider a small variation of the noise distribution pdf,[7] $f_u^{(1)}(u) = f_u(u) + \delta f_u(u)$, such that the scaling condition $\gamma f_u(\gamma u) = f_v(u)$ with the same scaling parameter γ ceases to hold. From Proposition 3, Nash and Stackelberg equilibria correspond if and only if the J term vanishes at equilibrium, i.e. $J(v, x, X) = 0$. We have:

Proposition 5 *In the vicinity of linear equilibria, the set of variations δf_u that lead to a zero J term has measure zero; in contrast, the set of variations δf_u that lead to a non-zero J term has positive measure. Thus, in the vicinity of linear equilibria, speculator profit with commitment almost always exceeds profits in Nash equilibria.*

[7] Equivalently, we could assume a small variation of f_v.

Proof: See the Appendix. □

4 Conclusion

We analyze strategic commitment by an informed speculator in a Kyle-style competitive dealership market. The speculator commits to a trading strategy and market makers price competitively given knowledge of the functional form of the trading strategy, but not the speculator's private information. We provide conditions under which a unique equilibrium obtains. We characterize the (non-negative) value of this strategic commitment, showing constructively that it is strictly positive in a setting where, rather than receiving a normal signal, the informed trader receives a binary signal about the asset's value and noise trade is normally distributed, so that the resulting informationally-efficient pricing function is non-linear. We then derive necessary and sufficient closed-form conditions for the informed trader not to be able to profit from commitment. This imposes conditions on model primitives—the distributions of the fundamental value and noise trade—that are satisfied by linear equilibria, e.g., when both distributions are Normal as in the classical Kyle model, but not when equilibria are only "almost linear".

5 Appendix A: Proofs

Proof of Proposition 1: The FOC in the Nash setting (17) follows directly from (14). We now derive the FOC (18) in the Stackelberg setting, which takes into account the price functional variation component. The FOC expressed in terms of the first functional variation of (15) yields

$$\begin{aligned} 0 &= E_v\left[\left\{v - \overline{P}\left(X_S(v), X_S\right) - X_S(v)\overline{P}' X_S\left(v\right), X_S\right\}\delta X(v)\right] \\ &\quad - E_{v,u}\left[X_S(v)\delta P\left(y, X_S, \delta X\right)\right], \end{aligned} \tag{54}$$

where $y = X_S(v) + u$ and we use $\delta P\left(y, X\right)$ as short-hand notation for the functional variation

$$\delta P\left(y, X\right) = \frac{\int \left(v - P\left(y, X\right)\right) f_v(v)\left(-\frac{\partial}{\partial y}\right) f_u\left(y - X(v)\right)\delta X(v) dv}{\int f_v\left(v\right) f_u\left(y - X(v)\right) dv}, \tag{55}$$

and we omit the irrelevant argument δX. Defining $Q\left(y, X\right) = E_{v|y}\left[X(v)\right]$, we have

$$\begin{aligned} E_{v,u}[X(v)\delta P\left(y, X\right)] &= \int\int f_v(v) f_u\left(y - X(v)\right) X(v)\delta P\left(y, X\right) dy dv \\ &= E_v\left[\int Q\left(y, X\right)\left(v - P\left(y, X\right)\right)\left(-\frac{\partial}{\partial y}\right) f_u\left(y - X(v)\right)\delta X(v) dy\right] \\ &= E_v\left[\delta X(v)\int\left(\frac{\partial}{\partial y} Q\left(y, X\right)\left(v - P\left(y, X\right)\right)\right) f_u\left(y - X(v)\right) dy\right] \\ &= E_v\left[\delta X(v) E_u\left[\frac{\partial}{\partial y} Q\left(y, X\right)\left(v - P\left(y, X\right)\right)\right]\right] \\ &= E_{v,u}\left[\delta X(v)\frac{\partial}{\partial y} Q\left(y, X\right)\left(v - P\left(y, X\right)\right)\right]. \end{aligned}$$

Thus,

$$E_{v,u}[X(v)\delta P\left(y, X\right)] = E_{v,u}\left[\delta X(v)\frac{\partial}{\partial y} Q\left(y, X\right)\left(v - P\left(y, X\right)\right)\right], \tag{56}$$

or substituting

$$J(v, x, X) = E_u\left[\frac{\partial}{\partial y} Q\left(y, X\right)\left(v - P\left(y, X\right)\right)\right] = \frac{\partial}{\partial x} E_u\left[Q\left(x + u, X\right)\left(v - P\left(x + u, X\right)\right)\right],$$

we have

$$E_{u,v}\left[X(v)\delta P\left(y, X\right)\right] = E_v\left[\delta X(v) J(v, x, X)\right], \tag{57}$$

for arbitrary variations $\delta X\left(v\right)$. Application of the main principle of variation calculus (Kolmogorov and Fomin, 1999) to (57) yields the result of Proposition 1. □

Proof of Proposition 2: The informationally-efficient pricing rule and its functional derivative with respect to the informed trader's strategy are given by (Boulatov and Livdan, 2022):

$$P_e(y,X) = \frac{\int v f_v(v) f_u(y - X(v))\,dv}{\int f_v(v) f_u(y - X(v))\,dv}, \tag{58}$$

and

$$\delta P_e(y,X) = \frac{\int (v - P_e(y,X)) f_v(v) \left(-\frac{\partial}{\partial y}\right) f_u(y - X(v))\,\delta X(v) dv}{\int f_v(v) f_u(y - X(v))\,dv}. \tag{59}$$

Therefore,

$$\begin{aligned}
E_{v,u}[y\delta P_e(y,X)] &= \int\int f_v(v) f_u(y - X(v))\, y\delta P_e(y,X)\,dydv \qquad (60)\\
&= \int\int y(v - P_e(y,X)) f_v(v)\left(-\frac{\partial}{\partial y}\right) f_u(y - X(v))\,\delta X(v) dvdy\\
&= \int f_v(v)\delta X(v) \int \left(\frac{\partial}{\partial y} y(v - P_e(y,X))\right) f_u(y - X(v))\,dydv\\
&= E_v\left[\delta X(v) E_u\left[\frac{\partial}{\partial y} y(v - P_e(y,X))\right]\right]\\
&= E_{v,u}\left[\delta X(v)\frac{\partial}{\partial y} y(v - P_e(y,X))\right]\\
&= E_{v,u}\left[\delta X(v)\left(v - P_e(y,X) - yP_e'(y,X)\right)\right].
\end{aligned}$$

Thus,

$$E_{v,u}[y\delta P_e(y,X)] = -E_{v,u}\left[\delta X(v)\left(P_e(y,X) - v + yP_e'(y,X)\right)\right], \tag{61}$$

or

$$E_{v,u}[y\delta P_e(y,X) + \delta X(v)\left(P_e(y,X) - v + yP_e'(y,X)\right)] = 0. \qquad \square \tag{62}$$

Proof of Theorem 1. Boulatov and Livdan (2022) establish existence of the BNE given the regularity conditions 1–3.

For the Stackelberg equilibrium, we first observe that the functional $\overline{\pi}_I(X,X)$ is *weakly continuous*. Therefore, its values on each weakly converging sequence $\{X_n\}$ converge (Krasnoselskii, 1964). In fact, we know that $\overline{\pi}_I(X,X)$ has a uniformly continuous gradient and hence is uniformly differentiable.

Claim: Because $\overline{\pi}_I$ is weakly continuous, it assumes its upper and lower bounds on any finite ball $\|X\| \leq r$ (Krasnoselskii, 1964).

Proof: The *gradient* of the functional $\overline{\pi}_I(X,X)$ is a completely continuous operator (Krasnoselskii, 1964). That is, by (15) and (18), the speculator's expected profit functional has gradient

$$\begin{aligned} \nabla_{X(v)}\overline{\pi}_I(X,X) &= v - A_S\left(v, X(v), X\right), \\ \text{where } A_S\left(v, X(v), X\right) &= \overline{P}\left(X\left(v\right), X\right) + X(v)\overline{P}'\left(X\left(v\right), X\right) + J\left(v, X(v), X\right), \end{aligned} \tag{63}$$

which is equivalent to saying that the full variation of $\overline{\pi}_I(X,X)$ takes the form

$$\delta\overline{\pi}_I\left(X, X; \delta X(v)\right) = E_v\left[\delta X(v)\left(v - \overline{P}\left(X(v), X\right) - X(v)\overline{P}'\left(X(v), X\right) - J\left(v, X(v), X\right)\right)\right],$$

obtained in Proposition 1. The operator $A_S(v,x,X) = \frac{\partial}{\partial x}\left(x\overline{P}(X,X)\right) + J\left(v, X(v), X\right)$ is completely continuous. Boulatov and Livdan (2022) show the first term $\frac{\partial}{\partial x}\left(x\overline{P}(X,X)\right)$ is completely continuous given regularity conditions 1–3.

The J-term given by (16) is also completely continuous. That is, (16) says that

$$\begin{aligned} J(v,x,X) &= E_u\left[\frac{\partial}{\partial y}Q\left(y,X\right)\left(v - P\left(y,X\right)\right)\right], \\ &= \int_{-\infty}^{+\infty} dy f_u\left(y-x\right)\frac{\partial}{\partial y}Q\left(y,X\right)\left(v - P\left(y,X\right)\right). \end{aligned} \tag{64}$$

Therefore, J is given by a superposition, $J = GR$, with $R = \frac{\partial}{\partial y}Q\left(y,X\right)\left(v - P\left(y,X\right)\right)$ and linear operator G acting on a vector $F \in L_1$ and defined by

$$G\left(x;F\right) = \int_{-\infty}^{+\infty} dy f_u\left(y-x\right)F(y) = \mathbb{E}_u\left[F\left(x+u\right)\right].$$

The linear operator G is characterized by a kernel $g\left(x,y\right) = f_u\left(y-x\right)$. Since $g\left(x,y\right)$ is continuous, G is completely continuous (Krasnosel'skii, 1964, p. 19). The pricing rule $R\left(y,X\right)$ is continuous with respect to the functional argument X. It is even Frechet differentiable for any nontrivial strategy $X \neq 0$, and hence is smooth in the functional sense. Therefore, the operator J is completely continuous as a superposition of a continuous and a completely continuous operators (Krasnoselskii, 1964, p. 46). □

Second, we use the explicit form of the pricing rule, to establish:

Claim: When the norm of the speculator's trading strategy $\|X\|$ increases unboundedly, $\|X\| \to \infty$, then $\overline{\pi}_I(X,X) \to 0$.

Proof: We have

$$\overline{\pi}_I(X,X) = E_{u,v}[X(v)(v - P_e(X(v)+u,X))],$$

with

$$P_e(X(s)+u,X) = \frac{\int v f_v(v) f_u(X(s) - X(v) + u)\,dv}{\int f_v(v) f_u(X(s) - X(v) + u)\,dv}. \tag{65}$$

Suppose X belongs to a finite ball $\|X\| \leq r$, and consider a transformation $X \to tX$. In the limit $t \to \infty$, we have asymptotically $f_u(tz) \to \frac{1}{t}\delta(z)$. Taking into account that, for any differentiable function g, $\delta(g(s)) = \frac{\delta(s-s_0)}{g'(s_0)}$, with $g(s_0) = 0$ and $g'(s_0) \neq 0$, we obtain

$$\begin{aligned} P_e(tX(s)+u,tX) &= \frac{\int v f_v(v) f_u(tX(s) - tX(v) + u)\,dv}{\int f_v(v) f_u(tX(s) - tX(v) + u)\,dv} \\ &= \frac{\int v f_v(v) f_u\left(t\left(X(s) - X(v) + \frac{u}{t}\right)\right)dv}{\int f_v(v) f_u\left(t\left(X(s) - X(v) + \frac{u}{t}\right)\right)dv} \to s, \end{aligned} \tag{66}$$

and hence

$$\overline{\pi}_I(X,X) \to E_{u,v}[X(v)(v-v)] = 0.$$

Because any vector with infinitely large norm, $\|X\| \to \infty$, can be obtained from the vector $\|X_0\| \leq r$ via a transformation $X = tX_0$, this result means that $\overline{\pi}_I(X,X) \to 0$ for any X with norm $\|X\| \to \infty$. □

It follows that strategies with infinite norms are not profitable, and thus the optimal strategy must have a finite norm. Because the expected profit functional $\overline{\pi}_I$ is continuous and finite in any finite region $\|X\| < +\infty$, this also means that the speculator's expected profit is bounded from above, $\overline{\pi}_I(X,X) \leq a < +\infty$. If r is sufficiently large, then $\max\limits_{\|X\| \leq r} \overline{\pi}_I(X,X) = \max\limits_{\|X\| \leq h} \overline{\pi}_I(X,X)$, for some $h < r$ (Krasnoselskii, 1964). Because $\overline{\pi}_I$ is weakly continuous, $\max\limits_{\|X\| \leq h} \overline{\pi}_I(X,X) = \overline{\pi}_I^*$, where $\overline{\pi}_I^* \leq a < +\infty$ is a maximal value of $\overline{\pi}_I(X,X)$ in the entire strategy space.

To summarize, the optimum $\overline{\pi}_I^*$ of $\overline{\pi}_I(X,X)$ exists and is finite, and the associated optimal strategy X^* has a finite norm, with $\overline{\pi}_I^* = \overline{\pi}_I(X^*,X^*)$. Finally, because $\overline{\pi}_I(X,X)$ is uniformly differentiable, its gradient at the optimal strategy X^* vanishes, i.e., $\nabla_{X^*(v)}\overline{\pi}_I(X^*,X^*) = 0$ (Krasnoselskii, 1964). □

Proof of Proposition 5: We proceed with a series of lemmas. We first show that as a result of this variation, $\delta f_u(u) \neq 0$, the initial linear Nash equilibrium (X,P) defined by the speculator's strategy $X = \beta v$ and pricing rule $P = \lambda y$ transforms into the (generally nonlinear) equilibrium $\left(X^{(1)}, P^{(1)}\right)$ with $X^{(1)}(v) = \beta v + \delta X(v)$ and $P^{(1)}(y) = \lambda y + \delta P(y)$, where the variations δX and δP are uniquely defined and continuously and smoothly depend on the variation $\delta f_u(u)$. More precisely, the variations δX and δP are expressed through $\delta f_u(u)$ by means of continuous linear operators in a Banach space:

Lemma 1 *With a small variation of the noise distribution pdf, $f_u^{(1)}(u) = f_u(u) + \delta f_u(u)$, the initial linear Nash equilibrium (X, P) transforms into $\left(X^{(1)}, P^{(1)}\right)$ with $X^{(1)}(v) = \beta v + \delta X(v)$ and $P^{(1)}(y) = \lambda y + \delta P(y)$, as $\delta P = R_P \delta f_u$ and $\delta X = R_X \delta f_u$, where R_P and R_X are linear operators in Banach space defined by*

$$\delta P(y) = \int_{-\infty}^{+\infty} dy' R_P(y, y') \delta f_u(y'), \text{ and } \delta X(v) = \int_{-\infty}^{+\infty} dv' R_X(v, v') \delta f_u(v'),$$

with the linear operators R_P and R_X defined in the proof below.

Proof: The pricing rule variation is given by

$$\begin{aligned} \delta P(y, X) &= \frac{\int (v - P(y, X)) f_v(v) \left(\delta f_u(y - X(v)) - \frac{\partial}{\partial y} f_u(y - X(v)) \delta X(v)\right) dv}{\int f_v(v) f_u(y - X(v)) dv} \\ &= \frac{\int (v - \lambda y) f_v(v) \left(\delta f_u(y - \beta v) - f_u'(y - \beta v) \delta X(v)\right) dv}{\int f_v(v) f_u(y - \beta v) dv}. \end{aligned} \tag{67}$$

Because FOC (17) always holds in the Nash equilibrium, we have

$$v = \left(\frac{\partial}{\partial x}\right)_{x = X(v)} x \overline{P}(x, X). \tag{68}$$

Using the inverse function $V(\cdot)$ property, $V(X(v)) = v$, we have

$$\begin{aligned} \delta X(v, X) &= -\frac{1}{V'(X(v))} \left(\frac{\partial}{\partial x}\right)_{x = X(v)} x \delta \overline{P}(x, X) \\ &= -\frac{1}{2\lambda} \frac{\partial}{\partial v} \delta \overline{P}(\beta v) = -\frac{1}{2\lambda} E_u\left[\frac{\partial}{\partial v} \delta P(\beta v + u)\right], \end{aligned} \tag{69}$$

where we take into account that the variation is around the linear strategy $X(v) = \beta v$.

Combining (69) and (67) yields a closed-form linear nonuniform integral equation with respect to the pricing rule variation δP as

$$(1 - K) \delta P = L \delta f_u, \tag{70}$$

with the linear operators K and L:

$$\begin{aligned} K\delta P(y) &= \frac{\int (v-\lambda y) f_v(v) f_u'(y-\beta v) \frac{1}{2\lambda} E_u\left[\frac{\partial}{\partial v}\delta P(\beta v+u)\right] dv}{\int f_v(v) f_u(y-\beta v)\, dv} \\ &= \frac{\beta}{2\lambda} \frac{\int\int (v-\lambda y) f_v(v) f_u'(y-\beta v) f_u(y'-\beta v) \frac{\partial}{\partial y'}\delta P(y')\, dv dy'}{\int f_v(v) f_u(y-\beta v)\, dv} \\ &= -\frac{1}{4\lambda^2} \frac{\int\int (v-\lambda y) f_v(v) f_u'(y-\beta v) f_u'(y'-\beta v)\, \delta P(y')\, dv dy'}{\int f_v(v) f_u(y-\beta v)\, dv}, \end{aligned} \tag{71}$$

and

$$\begin{aligned} L\delta f_u(y) &= \frac{\int (v-\lambda y) f_v(v) \delta f_u(y-\beta v)\, dv}{\int f_v(v) f_u(y-\beta v)\, dv} \\ &= \frac{1}{\beta} \frac{\int \left(\frac{y}{2}-\xi\right) f_u(y-\xi)\, \delta f_u(\xi)\, d\xi}{\int f_u(\xi) f_u(y-\xi)\, d\xi}, \end{aligned} \tag{72}$$

where in the last line, we introduce a dummy integration variable $\xi = y - \beta v$ and use the scaling property $f_v\left(\frac{1}{\beta}u\right) = \beta f_u(u)$.

Now consider (70), which is a nonuniform Fredholm type integral equation (Kolmogorov and Fomin, 1999, p.116-117). Then the following result[8] holds (Kolmogorov and Fomin, 1999, p.120): **either** (1) the equation (70) has a unique solution δP for any right-hand side given by $L\delta f_u$, which means that the operator $I - K$ is invertible, **or** (2) the uniform integral equation $(1-K)\,\delta P = 0$ has a nontrivial solution.

We can exclude the second possibility, as it would imply the existence of other Nash equilibrium infinitesimally close to the initial linear one, even though the distribution of noise trade did not change, contradicting the uniqueness of Nash equilibrium under conditions 1–3 established by Boulatov and Livdan (2022).

Thus, we have $\delta P = (1-K)^{-1} L\delta f_u = R_P \delta f_u$ with the linear operator $R_P = (1-K)^{-1} L$. Making use of (69), we also obtain $\delta X = M\delta P = M R_P \delta f_u = R_X \delta f_u$, with the linear operator $R_X = M R_P$ and $M\delta P = \frac{\beta}{2\lambda}\int dy f_u'(y-\beta v)\,\delta P(y)$. Based on the Riesz representation theorem (Balakrishnan, 1971), the linear operators R_P and R_X are represented as $R_P \delta f_u = \int_{-\infty}^{+\infty} dy' R_P(y,y')\,\delta f_u(y')$ and $R_X \delta f_u = \int_{-\infty}^{+\infty} dv' R_X(v,v')\,\delta f_u(v')$, respectively. □

We next observe that there are some "directions" in the functional space of variations δf_u along which the equilibrium remains linear, the J term remains equal zero, $J = 0$, and therefore commitment does not lead to higher expected speculator profits. We then show

[8] Also referred to as a Fredholm Alternative.

that these "directions" in the functional space are "rare" comprising a measure zero set, while variations δf_u for which the J term is nonzero comprise a positive measure set.

Recall that in our initial linear Nash equilibrium $X(v) = \beta v$, $P(y) = \frac{1}{2\beta} y$, the equilibrium trading intensity β equals the scaling parameter, $\beta = \gamma$, with the scaling condition $\gamma f_u(\gamma u) = f_v(u)$. Then we have the following result:

Lemma 2 *If the variation δf_u of the noise distribution satisfies the condition*

$$\delta f_u(u) = -\delta k \frac{\partial}{\partial u} u f_u(u), \tag{73}$$

with a real parameter $\delta k = \frac{\delta\gamma}{\gamma}$, then the initial linear equilibrium $X(v) = \beta v$, $P(y) = \frac{1}{2\beta} y$ transfroms into a linear one with $\beta^{(1)} = \beta + \delta\gamma$, and hence the J term remains zero.

Proof: (73) implies that with the small deviation $\delta\gamma \to 0$, the scaling property of the distribution still holds, but with a "shifted" scaling parameter

$$(\gamma + \delta\gamma)\left(f_u\left((\gamma + \delta\gamma) u\right) + \delta f_u(\gamma u)\right) = f_v(u) + o(\delta\gamma), \tag{74}$$

where we use the notation $o(\delta\gamma)$ for the second-order terms associated with the small change, $\delta\gamma$. Thus, the scaling property of the distributions still holds, but with a "shifted" scaling parameter $\gamma^{(1)} = \gamma + \delta\gamma$. Indeed, (74) says that

$$(\gamma + \delta\gamma)\left(f_u(\gamma u) + f_u'(\gamma u) u\delta\gamma + \delta f_u(\gamma u)\right) = f_v(u) + o(\delta\gamma), \tag{75}$$

or

$$(\gamma + \delta\gamma)\left(f_u(\gamma u) + f_u'(\gamma u) u\delta\gamma\right) + \gamma\delta f_u(\gamma u) = f_v(u) + o(\delta\gamma). \tag{76}$$

Using $\gamma f_u(\gamma u) = f_v(u)$, this simplifies to

$$\delta\gamma\left(f_u(\gamma u) + \gamma u f_u'(\gamma u)\right) + \gamma\delta f_u(\gamma u) = 0, \tag{77}$$

equivalent to (73).

Now, we directly verify that under the transformation (73), the initial linear equilibrium transforms into a linear one with $\beta^{(1)} = \beta + \delta\gamma$. The variation of the pricing rule is

$$\begin{aligned} \delta P(y, X) &= \frac{\int (v - \lambda y) f_v(v)\left(\delta f_u(y - \beta v) - f_u'(y - \beta v)\,\delta X(v)\right) dv}{\int f_v(v) f_u(y - \beta v)\, dv} \\ &= -\delta k \frac{\int (v - \lambda y) f_v(v) f_u(y - \beta v)\, dv}{\int f_v(v) f_u(y - \beta v)\, dv} \\ &\quad + \frac{\int (v - \lambda y) f_v(v) f_u'(y - \beta v)\left(-\delta k y + \delta k \beta v - \delta X(v)\right) dv}{\int f_v(v) f_u(y - \beta v)\, dv}, \end{aligned} \tag{78}$$

which, with re-organization becomes

$$\begin{aligned}\delta P(y,X) &= -\delta k\frac{\int(v-\lambda y)\,f_v(v)f_u(y-\beta v)\,dv}{\int f_v(v)\,f_u(y-\beta v)\,dv}\\ &\quad -\delta ky\frac{\int(v-\lambda y)\,f_v(v)f_u'(y-\beta v)\,dv}{\int f_v(v)f_u(y-\beta v)\,dv}\\ &\quad +\frac{\int(v-\lambda y)\,f_v(v)f_u'(y-\beta v)\,(\delta k\beta v-\delta X(v))\,dv}{\int f_v(v)f_u(y-\beta v)\,dv}.\end{aligned} \tag{79}$$

Consider the right-hand side of (79). The first term vanishes since it equals $-\delta k\,(E[v|y]-P(y))=0$. The second term equals $-\delta kyP'(y)=-\frac{\delta\gamma}{\gamma}P(y)=\frac{\delta\lambda}{\lambda}P(y)$, with $\frac{\delta\lambda}{\lambda}=-\frac{\delta\gamma}{\gamma}=-\frac{\delta\beta}{\beta}$, consistent with the linear equilibrium condition $\lambda=\frac{1}{2\beta}$. The third term also vanishes if the strategy variation satisfies $\delta k\beta v=\delta X(v)$, or $\frac{\delta X(v)}{X(v)}=\delta k=\frac{\delta\beta}{\beta}$. This holds if and only if the speculator's strategy remains linear, $X^{(1)}(v)=\beta v+\delta\beta v=(\beta+\delta\beta)\,v$. □

Finally we prove that the special "shifts" in the functional space like (73) along which the J term remains zero, are "rare":

Lemma 3 *For almost all variations δf_u, the J term is nonzero, implying that the equilibrium becomes nonlinear, and hence commitment almost always pays off.*

Proof: Define $I(v,x,X)=\frac{\partial}{\partial x}J(v,x,X)$. In equilibrium, $J(v,x,X)=J(X^*(v),X^*)$ and $I(v,x,X)=I(X^*(v),X^*)$, where $X^*(v)$ is an equilibrium trading strategy. To simplify notation, we drop the star and write $X^*(v)$ as $X(v)$. With the notation $\overline{Q}(x,X)=E_u[Q(x+u,X)]$, the identical transformations yield

$$\begin{aligned}I(v,x,X) &= E_u\left[Q(x+u,X)\{v-P(x+u,X)\}\right]\\ &= E_u\left[\left(Q(x+u,X)-\overline{Q}(x,X)+\overline{Q}(x,X)\right)\{v-P(x+u,X)\}\right]\\ &= -E_u\left[\left(Q(x+u,X)-\overline{Q}(x,X)\right)\left(P(x+u,X)-\overline{P}(x,X)\right)\right]\\ &\quad +\overline{Q}(x,X)\left\{v-\overline{P}(x,X)\right\}\\ &= -Cov_u\left[Q(x+u,X),P(x+u,X)\right]+\overline{Q}(x,X)\left\{v-\overline{P}(x,X)\right\}.\end{aligned} \tag{80}$$

When the pdf of noise trade distribution f_u shifts to $f_u+\delta f_u$, the initial linear Nash equilibrium shifts to a new one, with $Q(y,X)=\lambda\beta y+\delta Q(y,X)$, $P(y,X)=\lambda y+\delta P(y,X)$. Then the last equality of (80) yields, in the first order limit with respect to the variations

δQ and δP:

$$
\begin{aligned}
I(v,x,X) &= -Cov_u\left[\left(\lambda\beta\left(x+u\right)+\delta Q\left(x+u,X\right)\right),\left(\lambda\left(x+u\right)+\delta P\left(x+u,X\right)\right)\right] \\
&\quad +\overline{Q}\left(x,X\right)\left\{v-\overline{P}\left(x,X\right)\right\} \\
&= -\lambda E_u\left[u,\left(\delta Q\left(x+u,X\right)+\beta\delta P\left(x+u,X\right)\right)\right] \\
&\quad +\overline{Q}\left(x,X\right)\left\{v-\overline{P}\left(x,X\right)\right\}+o\left(\delta Q,\delta P\right).
\end{aligned}
\tag{81}
$$

We must evaluate the variation $\delta I = I - I_0$, where I_0 corresponds to the initial noise distribution f_u. One can show that $\delta Q\left(y,X\right) = \beta\delta P\left(y,X\right)+\delta\varphi\left(y,X\right)$, where $\delta\varphi\left(y,X\right) = E_{v|y}\left[\delta X(v)\right]$. That is,

$$
\begin{aligned}
\delta P\left(y,X\right) &= \frac{\int\left(v-\lambda y\right)f_v(v)\left(\delta f_u\left(y-\beta v\right)-f_u'\left(y-\beta v\right)\delta X(v)\right)dv}{\int f_v(v)f_u\left(y-\beta v\right)dv}, \\
\delta Q\left(y,X\right) &= \frac{\int\beta\left(v-\lambda y\right)f_v(v)\left(\delta f_u\left(y-\beta v\right)-f_u'\left(y-\beta v\right)\delta X(v)\right)dv}{\int f_v(v)f_u\left(y-\beta v\right)dv} \\
&\quad +\frac{\int f_v(v)f_u\left(y-\beta v\right)\delta X(v)dv}{\int f_v(v)f_u\left(y-\beta v\right)dv},
\end{aligned}
\tag{82}
$$

and hence $\delta Q\left(y,X\right) = \beta\delta P\left(y,X\right)+\delta\varphi\left(y,X\right)$. In what follows, we omit the functional variable X and use the short-hand notation $\delta Q\left(y\right)$ and $\delta P\left(y\right)$ for variations of the expected trading strategy and pricing rule, respectively. With this notation, transforming the last term and taking into account that in equilibrium, $x = X\left(v\right)$, the FOC is satisfied as $v-\overline{P}\left(x\right) = x\overline{P}'\left(x\right)$ and $2\lambda\beta = 1$, we obtain

$$
\begin{aligned}
\delta I &= -E_u\left[u\delta P\left(x+u,X\right)\right]-\lambda E_u\left[u\delta\varphi\left(x+u,X\right)\right] \\
&\quad +\frac{\delta\overline{Q}\left(x\right)}{x}x\left(v-\overline{P}\left(x,X\right)\right)+o\left(\delta Q,\delta P\right),
\end{aligned}
\tag{83}
$$

where $\delta\overline{\varphi}\left(x,X\right) = E_u\left[\delta\varphi\left(x+u,X\right)\right]$. Differentiating with respect to x yields

$$
\begin{aligned}
J &= -E_u\left[u\delta P'\left(x+u\right)\right]-\lambda E_u\left[u\delta\varphi'\left(x+u\right)\right] \\
&\quad +\lambda x\left(v-\lambda x\right)\frac{\partial}{\partial x}\left(\frac{\delta\overline{Q}\left(x\right)}{x}\right)+o\left(\delta Q,\delta P\right) \\
&= -E_u\left[u\delta P'\left(x+u\right)\right]-\lambda E_u\left[u\delta\varphi'\left(x+u\right)\right] \\
&\quad -\lambda E_u\left[\beta\delta P\left(x+u\right)+\delta\varphi\left(x+u\right)-x\left(\beta\delta P'\left(x+u\right)+\delta\varphi'\left(x+u\right)\right)\right]+o\left(\delta Q,\delta P\right).
\end{aligned}
\tag{84}
$$

Now we have auxilliary results

$$\begin{aligned} E_u\left[u\delta P'(x+u)\right] &= \int du f_u(u)\, u\frac{\partial}{\partial x}\delta P(x+u) \\ &= \int du f_u(u)\, u\frac{\partial}{\partial u}\delta P(x+u) \\ &= -\int dy\delta P(y)\frac{\partial}{\partial y}\left((y-x) f_u(y-x)\right), \end{aligned} \tag{85}$$

and $\delta\varphi(y) = -\frac{1}{2\lambda}E_{v|y}\left[\frac{\partial}{\partial v}\left(v\delta\overline{P}(\beta v)\right)\right]$, which yields

$$\begin{aligned} \delta\varphi(y) &= -\frac{1}{2\lambda}\frac{\int dv f_v(v) f_u(y-\beta v)\frac{\partial}{\partial v}\left(v\delta\overline{P}(\beta v)\right)}{\int dv f_v(v) f_u(y-\beta v)} \\ &= \frac{1}{2\lambda}\frac{\int dv v\delta\overline{P}(\beta v)\frac{\partial}{\partial v}\left(f_v(v) f_u(y-\beta v)\right)}{\int dv f_v(v) f_u(y-\beta v)}, \end{aligned} \tag{86}$$

where we integrate by parts in the last line. Introducing a new dummy integration variable $\xi = \beta v$ and using the scaling property of the noise distribution, we finally obtain

$$\begin{aligned} \delta\varphi(y) &= -\beta\int dy' K_\varphi(y,y')\,\delta P(y'), \\ K_\varphi(y,y') &= \frac{\int d\xi f_u(\xi) f_u(y-\xi)\frac{\partial}{\partial\xi}\left(\xi f_u(y'-\xi)\right)}{\int d\xi f_u(\xi) f_u(y-\xi)}. \end{aligned} \tag{87}$$

Substituting the first-order variations δQ and δP into (84) yields:

$$\begin{aligned} J &= -\frac{1}{2}E_u\left[2u\delta P'(y) - u\frac{\partial}{\partial y}\int dy' K_\varphi(y,y')\,\delta P(y')\right] \\ &\quad -\frac{1}{2}E_u\left[\delta P(y) - \int dy' K_\varphi(y,y')\,\delta P(y')\right] \\ &\quad -\frac{1}{2}E_u\left[-x\delta P'(y) + x\frac{\partial}{\partial y}\int dy' K_\varphi(y,y')\,\delta P(y')\right], \end{aligned} \tag{88}$$

or

$$\begin{aligned} J &= -\int dy f_u(y-x)(y-x)\frac{\partial}{\partial y}\delta P(y) \\ &\quad +\frac{1}{2}\int dy f_u(y-x)(y-x)\frac{\partial}{\partial y}\int dy' K_\varphi(y,y')\,\delta P(y') \\ &\quad -\frac{1}{2}\int dy f_u(y-x)\left(\delta P(y) - \int dy' K_\varphi(y,y')\,\delta P(y')\right) \\ &\quad +\frac{1}{2}x\int dy f_u(y-x)\left(\delta P'(y) - \frac{\partial}{\partial y}\int dy' K_\varphi(y,y')\,\delta P(y')\right) \end{aligned} \tag{89}$$

Integrating the first and second terms by parts yields

$$
\begin{aligned}
J &= \int dy \frac{\partial}{\partial y}\left(\left(y-x\right)f_u\left(y-x\right)\right)\delta P\left(y\right) \\
&-\frac{1}{2}\int dy \int dy' \frac{\partial}{\partial y}\left(\left(y-x\right)f_u\left(y-x\right)\right)K_\varphi\left(y,y'\right)\delta P\left(y'\right) \\
&-\frac{1}{2}\int dy f_u\left(y-x\right)\left(\delta P\left(y\right)-\int dy' K_\varphi\left(y,y'\right)\delta P\left(y'\right)\right) \\
&+\frac{1}{2}x\int dy f_u\left(y-x\right)\left(\delta P'\left(y\right)-\frac{\partial}{\partial y}\int dy' K_\varphi\left(y,y'\right)\delta P\left(y'\right)\right),
\end{aligned} \tag{90}
$$

which can be put in the form

$$
J=\frac{1}{2}\int \Gamma\left(y,x\right)\delta P\left(y\right)dy, \tag{91}
$$

with

$$
\begin{aligned}
\Gamma\left(y,x\right) &= 2\frac{\partial}{\partial y}\left(\left(y-x\right)f_u\left(y-x\right)\right) \\
&+\int dy'\left(y'-x\right)f_u\left(y'-x\right)\frac{\partial}{\partial y'}K_\varphi\left(y',y\right) \\
&-f_u\left(y-x\right)+\int dy' f_u\left(y'-x\right)K_\varphi\left(y',y\right) \\
&-x\frac{\partial}{\partial y}f_u\left(y-x\right)-x\int dy' f_u\left(y'-x\right)\frac{\partial}{\partial y'}K_\varphi\left(y',y\right).
\end{aligned} \tag{92}
$$

Next, note that an arbitrary variation δf_u leads to a new equilibrium and causes a variation of the pricing rule $\delta P = R_P \delta f_u$, where R_P is a linear operator according to (67). Since a variation δf_u translates into a variation δP, we now show that for almost all variations δf_u the variation of δP is nonzero, and hence the J term is almost always nonzero.

This stability property follows from the Euler-Lagrange lemma (Young, 1969), which says that the integral (91) could be zero for arbitrary variation $\delta P\left(y\right)$ if and only if the function $\Gamma\left(y,x\right)$ is identically zero for any real y and x, i.e., $\Gamma\left(y,x\right)\equiv 0$, for $\forall y, x \in R$. As follows from (92), this is equivalent to the following integro-differential equation with respect to the pdf f_u:

$$
\begin{aligned}
f_u\left(y-x\right) &= 2\frac{\partial}{\partial y}\left(\left(y-x\right)f_u\left(y-x\right)\right) \\
&+\int dy'\left(y'-x\right)f_u\left(y'-x\right)\frac{\partial}{\partial y'}K_\varphi\left(y',y\right) \\
&+\int dy' f_u\left(y'-x\right)K_\varphi\left(y',y\right) \\
&-x\frac{\partial}{\partial y}f_u\left(y-x\right)-x\int dy' f_u\left(y'-x\right)\frac{\partial}{\partial y'}K_\varphi\left(y',y\right).
\end{aligned} \tag{93}
$$

Integrating both parts with respect to $x \in (-\infty; +\infty)$, we obtain

$$\begin{aligned}\int f_u(y-x)\,dx &= -1 = 2\frac{\partial}{\partial y}\int (y-x)\,f_u(y-x)\,dx \\ &+\int dy'\int (y'-x)\,f_u(y'-x)\,dx\frac{\partial}{\partial y'}\int K_\varphi(y',y)\,dy \\ &-\int dy' K_\varphi(y',y) \\ &-\int dy'\int dx x f_u(y'-x)\,\frac{\partial}{\partial y'}K_\varphi(y',y)\,.\end{aligned} \tag{94}$$

The first and second terms on the right-hand side of (94) vanish because, by assumption, the noise distribution has zero first moment, $\int f_u(u)\,udu = 0$. Collecting the last two terms on the right-hand side yields

$$\begin{aligned}&\int dy' K_\varphi(y',y) + \int dy'\int dx x f_u(y'-x)\,\frac{\partial}{\partial y'}K_\varphi(y',y) \\ = &\int dy' K_\varphi(y',y) + \int dy'\int dx\,(y'-(y'-x))\,f_u(y'-x)\,\frac{\partial}{\partial y'}K_\varphi(y',y) \\ = &\int dy' K_\varphi(y',y) + \int dy' y'\frac{\partial}{\partial y'}K_\varphi(y',y) = \int dy'\frac{\partial}{\partial y'}\left(y' K_\varphi(y',y)\right) = 0,\end{aligned} \tag{95}$$

where the last equality holds because $|y|\,K_\varphi(y',y) \to 0$, when $|y| \to \infty$.

Hence we conclude that $\int \Gamma(y,x)\,dx = -1$ and the equation (93) cannot hold for arbitrary variations δP. This means that (91) is not zero for the price variations δP of nonzero measure, and it can be zero for only special variations of zero measure.

Further note that

$$\int K_\varphi(y',y)\,dy = \frac{\int d\xi f_u(\xi)\,f_u(y'-\xi)\,\frac{\partial}{\partial \xi}\left(\xi \int f_u(y-\xi)\,dy\right)}{\int d\xi f_u(\xi)\,f_u(y'-\xi)} = 1. \tag{96}$$

Using this, we show, as above, that $\int \Gamma(y,x)\,dy = 0$. In particular, integrating (93) with respect to y note that all terms except for the third one on the right-hand side vanish, yielding

$$\int f_u(y-x)\,dy = 1 = \int dy' f_u(y'-x)\int K_\varphi(y',y)\,dy = 1,$$

and hence $\int \Gamma(y,x)\,dy = 0$. One can also show that $\int \Gamma(y,x)\,ydy = 0$ reflecting that the J term vanishes in linear equilibria. Using (91), we investigate the condition $J = 0$ further. The condition

$$g(x) = \int \Gamma(y,x)\,\delta P(y)\,dy = 0, \tag{97}$$

can be viewed as a linear integral equation with respect to the variation δP. As we know, it has a nontrivial linear solution corresponding to the linear equilibrium, $\delta P(y) = \delta\lambda y$.

However, the set of solutions of (97) forms a zero measure set in the space of all price variations δP. In particular, we now show that it can only have linear solutions of the form $\delta P(y) = \mu y + \nu$ with real parameters $\mu, \nu \in R$. Going back to (89), we observe that

$$2J = -\int dy f_u (y-x)(y-x)\frac{\partial}{\partial y}\delta P(y) - \int dy f_u (y-x)\,\Phi(y,x,\delta P), \tag{98}$$

with

$$\Phi(y,x,\delta P) = \left(1 + (y-2x)\frac{\partial}{\partial y}\right)\left(\delta P(y) - \int dy' K_{\varphi}(y,y')\,\delta P(y')\right). \tag{99}$$

Therefore, the condition $J = 0$ yields

$$\int dy f_u (y-x)(y-x)\frac{\partial}{\partial y}\delta P(y) + \int dy f_u (y-x)\,\Phi(y,x,\delta P) = 0. \tag{100}$$

One can show that the integral equation $\Phi(y,x,\delta P) = 0$ has only linear solutions, which also satisfy $\int dy f_u (y-x)(y-x)\frac{\partial}{\partial y}\delta P(y) = 0$. To summarize, the linear solutions satisfy (98), and it has no other solutions.

We conclude that (93) cannot hold. Therefore, (91) is not zero for price variations δP of nonzero measure, and is zero for only special variations of zero measure. $\square$

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