Auctioning control and cash-flow rights separately*

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Abstract

We consider a classical auction setting in which an asset/project is sold to buyers who privately receive independently-distributed signals about expected payoffs, and payoffs are more sensitive to the signal of the bidder who controls the asset. We show that a seller can increase revenues by sometimes allocating cash-flow rights and control to different bidders. Allocating both cash-flow rights and control to the highest bidder only maximizes seller revenue when his signal sufficiently exceeds the next highest. When signals are closer, the seller optimally splits rights between the top two bidders, with the second-highest bidder receiving control if and only if the degree of common values in asset payoffs is high enough.

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1 Introduction

The starting point for this paper is the classical setting of Myerson (1981), in which a seller seeks to sell a single asset/project to risk neutral buyers who privately receive independently-distributed signals about the asset's expected future cash flows. When a bidder wins control of the asset, the asset's payoffs hinge on both his signal and those of rival bidders. The existing literature characterizes optimal mechanisms in settings where only the bidder who controls the project receives cash flows (either receiving all cash flows or splitting them with the seller), but *none* of the other bidders receives any cash flows.

Our point of departure is to recognize that a seller can do better by sometimes allocating control rights and cash-flow rights to the project to different bidders. Specifically, we investigate whether and when a seller does better to select one bidder to run the project but select another bidder to receive some or all of the cash flows generated by the project. We establish that as long as expected cash flows are more sensitive to the signal of the bidder who controls the project than to those of rival bidders—so that project payoffs have both private value and common value components—then expected seller revenues are *strictly* higher when the seller sometimes allocate control and cash-flow rights to different bidders. To our knowledge our paper is the first to propose a "separation" mechanism of this form, to establish its revenue advantages, and to identify the sources of those advantages.

To highlight how outcomes in our separation framework differ from those in the "no-separation" frameworks of existing studies, we focus on symmetric settings with ex-ante identical bidders. Even in such simple settings, sharp differences emerge. In the classical no-separation framework, given a mild regularity condition, it is optimal for a seller to always award both the control and cash-flow rights to the highest bidder. This result reflects that (i) allocating control to the bidder with the highest signal maximizes social welfare, and (ii) allocating cash flows to the highest bidder reduces rents earned by bidders with lower signals, thereby minimizing bidders' total rents (reflecting the standard envelope theorem logic). Each of these effects increases seller revenues. Our paper derives the surprising result that a seller can do better than that: separating control and cash-flow rights among different bidders facilitates rent extraction, the benefit of which will sometimes strictly outweigh the costs of not assigning both cash flow rights and control to the (same) highest bidder.

We first identify and impose a regularity condition under which local incentive compatibility implies global incentive compatibility in our separation mechanism. We establish that a seller should award both rights to the bidder with the highest signal *only* when the highest signal exceeds the second-highest signal by a *sufficiently large* amount. When, instead, the two highest signals are closer, the seller does better to split the two rights between the bidders with the two highest signals. The optimal nature of the division—which bidder gets which right—hinges on the private vs. common value composition. Specifically, when common values comprise a sufficiently high share of the total project value, a seller does best to assign control to the second-highest bidder and cash-flow rights to the highest bidder; but when common values comprise a smaller share, a seller does best to reverse this allocation, assigning control to the highest bidder and cash-flow rights to the second-highest bidder.

The intuition for the benefit of separation is that when a bidder's valuation is more sensitive to his own signal than to those of other bidders, the payoff of the project to the bidder who receives the cash flows is less sensitive to his signal when he does not control the project. This reduced sensitivity decreases a bidder's informational advantage, increasing seller revenues. To see this most clearly, suppose the two highest signals are equal. In the classical no-separation framework, assigning both rights to either of the two bidders is optimal. By contrast, in our framework, separation is strictly better: splitting the two rights between the two bidders strictly dominates assigning both rights to a single bidder. This reflect that with a tie, the two costs of separation—the inefficiency due to allocating control to a lower signal bidder, and the increased bidder rents due to allocating cash flows to a lower signal bidder—are both zero, leaving only the benefit from the reduced sensitivity of a bidder's payoff to his signal.

When the two highest signals differ and the difference is not too large, separation continues to dominate no-separation, but the way in which rights are split matters: the optimal way of splitting depends on which cost of splitting is larger. In turn, this depends on the importance of common values relative to private values. When the common value share of project value is sufficiently large, the efficiency cost of allocating control to the second-highest bidder rather than the highest is small enough that it is optimal to assign control to the second-highest bidder and cash flows to the highest bidder. When, instead, the private value share is high enough, efficiency considerations make the opposite allocations optimal—the highest bidder is assigned control and the second-highest bidder receives cash flow rights. It only becomes optimal to assign both rights to the highest bidder when his signal exceeds the second highest by enough that the smaller of the two costs of splitting exceeds the benefits.

In sum, when the regularity condition holds so that local incentive compatibility implies global incentive compatibility, then with significant common values, the bidder with the highest signal is always assigned cash flow rights, but receives control if and only if his signal sufficiently exceeds the second-highest signal. Conversely, when common values are smaller, the bidder with the highest signal is always assigned control, but receives cash flow rights if and only if his signal sufficiently exceeds the second-highest signal.

This leads to the question—when will the regularity condition hold so that local incentive compatibility implies global incentive compatibility? We show that when common values are substantial enough so that revenue-maximization calls for the bidder with the highest signal to always receive cash flow rights, the standard decreasing inverse hazard assumption on the distribution of bidder signals is enough to ensure that regularity holds for any number of bidders. This reflects that bidders only extract rents from cash flow rights, and when cash flows are always awarded to the highest bidder, a bidder with a higher signal is more likely to receive cash flows, and this provides the necessary global monotonicity.

When, instead, the degree of private values is high enough that the bidder with the second-highest signal sometimes receives cash flow rights, global incentive compatibility may be violated. This is because assigning cash flows to the second-highest bidder, rather than the highest, makes it harder to satisfy regularity, which requires a bidder with a higher signal to expect more cash flows. Whether regularity holds depends on the signal distribution, the number of bidders, and the degree of common values. With uniformly-distributed signals, we show the regularity condition becomes easier to satisfy if there are more bidders. This feature contrasts with the no-separation framework, where, with ex-ante identical bidders and independent signals, the regularity condition depends only on the signal distribution.[1]

Importantly, even when the regularity condition does not hold, we construct (subopti-

[1]To see the dependence on the number of bidders, note that with two bidders, the second-highest bidder is the lowest bidder, so assigning cash flows to him goes strongly against regularity. In contrast, with many bidders, the second-highest bidder tends to have a relatively high value, making it easier to satisfy regularity.

mal) separation mechanisms that still yield higher seller revenues than optimally-designed no-separation mechanisms. In particular, as long as the standard decreasing inverse hazard regularity condition holds, the seller can earn strictly higher profits by always assigning the highest bidder cash flow rights, but giving control to the second-highest bidder when his signal is close enough to the highest. That is, the separation mechanism that is optimal when common values matter sufficiently also generates strictly higher revenues than the optimal no-separation mechanism when private values matter more.

One can design this separation mechanism so that only the bidder receiving cash flow rights makes a payment, and a bidder who controls the project but does not receive cash flow rights is compensated with a cash payment that just covers his costs of running the project—in essence, the seller contracts out the running of the project and selling the composite product to the bidder receiving cash flow rights. We provide a simple two-stage auction implementation for this separation mechanism. The first stage is a standard English auction. In the second stage, the seller offers the auction winner a choice of whether to have both cash flow rights and control, or just to retain cash flow rights and have the second-highest bidder run the project in return for a reduced cash payment. Our separation mechanism has the added virtue that it is an ex-post equilibrium in the sense of Bergemann and Morris (2008)—it is ex-post incentive compatible with no bidder regrets. Moreover, it always generates (weakly and sometimes strictly) higher revenue than the optimal no-separation mechanism event by event. Thus, its advantages extend directly to risk-averse sellers.

Other researchers have considered splitting rights when bidders obtain private benefits of control. Ekmekci, Kos and Vohra (2016) consider the problem of selling a firm to a *single* buyer who is privately informed about post-sale cash flows and the benefits of control. The seller can offer a menu of cash-equity mixtures, and the buyer must obtain a minimum claim to cash flow rights (with the seller retaining any residual cash flow rights) to gain control rights. They provide sufficient conditions for the optimal mechanism to take the form of a take-it-or-leave-it offer for either the minimum stake or for all shares. In contrast, we consider a classical multi-bidder auction setting in which there are no benefits from control and bidders receive signals that have both private and common value components. We allow the seller to allocate control and cash-flow rights to different *bidders*, showing that this increases seller revenues, and we characterize the optimal allocations and how they vary with primitives—the levels and compositions of the signals that bidders receive.

The literature has examined optimal designs of no-separation auctions with common valuations in many settings. McAfee, McMillan, and Reny (1989) derive conditions under which, with common values, the optimal no-separation selling procedure is implemented by a simple mechanism in which a seller solicits reports from one bidder and offers the asset to another bidder. Mezzetti (2004) studies efficient auction designs with interdependent valuations. Bergemann, Brooks, and Morris (2016) and Brooks and Du (2018) identify robust auctions in pure common value settings that yield maximum revenue guarantees. Lauermann and Speit (2022) study bidding in common-value auctions with an unknown number of bidders.

Other literature has examined the consequences of separating ownership and control in the market for corporate control in the context of agency issues, free riding problems and information aggregation (see, e.g., Bagnoli and Lipman 1988, Ekmekci and Kos 2016, Voss and Kulms 2022). Our paper contributes to this literature by identifying an advantage of the separation of ownership and control from the perspective of optimal auction design.

2 Model

$n > 1$ risk-neutral bidders bid for an asset/project in an auction designed by a risk-neutral seller. The project can be controlled (run) by only one bidder who incurs a publicly-known opportunity cost $\tau \geq 0$ from running the project that then generates future cash flows. Bidders do not discount future cash flows, whereas the seller values only current cash payments from the auction, discounting future cash flows to zero.

Each bidder i privately receives a signal t_i that is informative about the (undiscounted) future cash flows of the project. Each t_i is distributed independently and identically according to a strictly positive and continuous pdf f over $[\underline{t}, \bar{t}]$, with associated cdf F. Valuations are interdependent and linearly related to signals. Using $v_i(t_1, ..., t_n)$ to denote the value (expected future cash flows) of the project if bidder i controls the asset, we have

$$v_i(t_1, ..., t_n) = \sum_j W_{ij} t_j.$$

Here, $W_{ij} = A_n \equiv \frac{1}{1+(n-1)\rho}$ for $i = j$, $W_{ij} = \rho A_n$ for $i \neq j$, and $\rho \in (0,1)$, yielding $v_i = A_n\big(t_i + \rho \sum_{j\neq i} t_j\big)$. $\rho < 1$ implies that a bidder's valuation is more sensitive to his own signal than to a rival bidder's signal. The constant A_n does not affect analysis, but it facilitates interpretation as it ensures $\sum_j W_{ij} = 1$. We sometimes refer to t_i as bidder i's type. We use $\mathbf{t} \equiv (t_1, t_2, ...t_n)$ to denote the vector of all bidder types, $f(\mathbf{t}) \equiv \Pi_{i=1}^n f(t_i)$ to denote the joint density of $\mathbf{t}$, and $f_{-i}(\mathbf{t}_{-i}) \equiv \Pi_{j\neq i} f(t_j)$ to denote the joint density of bidder types other than i.

Our key departure from the literature is to consider settings in which a seller can allocate control and cash-flow rights to different bidders. That is, a bidder who does not control the project may nonetheless receive some or all of the future cash flows generated. Formally, we consider direct-revelation mechanisms that allow for the separation of control from cash-flow rights. Let $R_j(\mathbf{t}) \in [0,1]$ be the probability that bidder j is assigned control when bidders report $\mathbf{t}$. Let $Q_{ji}(\mathbf{t}) \in [0,1]$ be the fraction of the total cash flow that i gets when bidders report $\mathbf{t}$ and control is assigned to j.[2] Let $M_i(\mathbf{t})$ be i's expected cash payment to the seller when bidders report $\mathbf{t}$. Note that for a given $\mathbf{t}$, i's cash payment can be contingent on whether i receives the control, and $M_i(\mathbf{t})$ takes expectations over these possibilities. Cash payments are made at the current (auction) time, and the seller does not discount these payments.

We require for all $\mathbf{t}$ that

$$\sum_j R_j(\mathbf{t}) \leq 1 \tag{1}$$

and

$$\sum_i Q_{ji}(\mathbf{t}) \leq 1 \text{ for all } j. \tag{2}$$

One can interpret $\sum_j R_j(\mathbf{t}) < 1$ as the seller retaining the project with some probability, in which case the project does not generate any cash flows. Similarly, one can interpret $\sum_i Q_{ji}(\mathbf{t}) < 1$ as the seller receiving a fraction $(1 - \sum_i Q_{ji}(\mathbf{t}))$ of the future cash flows. The seller does not value this fraction because she discounts future cash flows to zero.

[2]If given report $\mathbf{t}$, j is never assigned control, then the value of $Q_{ji}(\mathbf{t})$ is irrelevant.

2.1 Equilibrium requirements, payoffs and seller's objective

Define $U_i(t_i, t_i'; \mathbf{t}_{-i})$ to be bidder i's expected profit when he is type t_i and reports t_i', and all other bidders truthfully report $\mathbf{t}_{-i}$:

$$U_i(t_i, t_i'; \mathbf{t}_{-i}) \equiv \sum_j R_j(t_i'; \mathbf{t}_{-i}) Q_{ji}(t_i'; \mathbf{t}_{-i}) v_j(\mathbf{t}) - M_i(t_i'; \mathbf{t}_{-i}) - \tau R_i(t_i'; \mathbf{t}_{-i}).$$

The first term $\sum_j R_j Q_{ji} v_j$ on the right-hand side is the expected value of the cash flows awarded to bidder i, where the summation over j reflects that bidders other than i may run the project when i receives cash flows. The second term is the expected value of payments that i makes to the seller, and the third term is the expected costs that i incurs from running the project, which is τ multiplied by the probability that i is assigned control.

Integrating over $\mathbf{t}_{-i}$, define $\bar{U}_i(t_i, t_i')$ to be bidder i's expected profit when he has type t_i but reports t_i' and all other bidders report truthfully:

$$\begin{aligned} \bar{U}_i(t_i, t_i') &\equiv \int_{\Omega_{n-1}} U_i(t_i, t_i'; \mathbf{t}_{-i}) f_{-i}(\mathbf{t}_{-i}) d\mathbf{t}_{-i} \\ &= \int_{\Omega_{n-1}} \sum_j R_j(t_i'; \mathbf{t}_{-i}) Q_{ji}(t_i'; \mathbf{t}_{-i}) v_j(t_i; \mathbf{t}_{-i}) f_{-i}(\mathbf{t}_{-i}) d\mathbf{t}_{-i} \\ &\qquad - m_i(t_i') - \tau \int_{\Omega_{n-1}} R_i(t_i'; \mathbf{t}_{-i}) f_{-i}(\mathbf{t}_{-i}) d\mathbf{t}_{-i}, \end{aligned} \tag{3}$$

where $\Omega_{n-1} \equiv [\underline{t}, \bar{t}]^{n-1}$ denotes the space of integration for the $n-1$ other bidders and

$$m_i(t_i') \equiv \int_{\Omega_{n-1}} M_i\left(t_i'; \mathbf{t}_{-i}\right) f_{-i}\left(\mathbf{t}_{-i}\right) d\mathbf{t}_{-i} \tag{4}$$

is the expected cash payment by bidder i when he reports t_i'.

The equilibrium expected profit for bidder i of type t_i is $\bar{U}_i(t_i, t_i)$. Equilibrium requires that both the (interim) incentive compatibility condition,

$$\bar{U}_i(t_i, t_i) = \max_{t_i'} \bar{U}_i\left(t_i, t_i'\right), \tag{5}$$

and the (interim) individual rationality condition,

$$\bar{U}_i(t_i, t_i) \geq 0, \tag{6}$$

hold for all i and t_i. We later show that our optimal auction design satisfies the stronger requirements of ex-post rationality and (for large enough ρ) ex-post incentive compatibility.

The seller's expected revenue is the sum of the cash payments of all bidders:

$$\pi_s = \sum_{i=1}^{n} \int_{\underline{t}}^{t} m_i(t_i) f(t_i) dt_i. \tag{7}$$

Our objective is to identify the mechanism that maximizes expected seller revenue (7) subject to the feasibility conditions (1) and (2), and the incentive compatibility (5) and individual rationality (6) conditions.

2.2 Discussion

We can equivalently write i's valuation as $v_i = A_n \left(\rho \sum_j t_j + (1-\rho)t_i\right)$, i.e., a bidder's valuation is the sum of common value and private value components. Here, ρ measures the degree of common valuations: a higher ρ implies a higher degree of common valuations. The assumption of linear valuations facilitates analysis and is common in the literature on auctions with interdependent valuations (e.g., Povel and Singh 2006, Bergemann and Morris 2007, Bergemann, Shi and Valimaki 2009, Gorbenko and Malenko 2022). Our key insight is that the separation between bidders of control and cash-flow rights helps rents extraction as long as a bidder's valuation is more sensitive to his own signal than to those of other bidders, i.e., there is some private value component to signals. In our model, $\rho = \frac{\partial v_j(\mathbf{t})}{\partial t_i} / \frac{\partial v_j(\mathbf{t})}{\partial t_j} < 1$ for $i \neq j$ measures the sensitivity of a bidder j's valuation to some other bidder i's signal relative to his own signal.[3] With a linear valuation structure, ρ is constant, but the logic underlying our findings extends to non-linear settings: separation facilitates rents extraction when the relative sensitivity $\left(\frac{\partial v_j(\mathbf{t})}{\partial t_i} / \frac{\partial v_j(\mathbf{t})}{\partial t_j}\right)$ varies with j, i and $\mathbf{t}$, as long as it is less than one for some j, i and $\mathbf{t}$.

Our model focuses on ex-ante identical bidders in order to make clear the counter-intuitive point that always assigning cash-flow rights and control to the same (highest) bidder does not maximize seller revenues. Our analysis and findings generalize to heterogeneous bidders.

From a theoretical perspective, our separation mechanism is a novel formulation that extends the existing framework of mechanism design by incorporating the assignments of rights into the design consideration. In practice, it loosely corresponds to settings in which, for example, an entrepreneur "sells" a project idea to a syndicated VC group, where the lead VC is

[3]See (12) for how $\frac{\partial v_j(\mathbf{t})}{\partial t_i}$ enters the analysis.

directly involved in the project management, while the other VCs only contribute funding in return for claims to future cash flows.[4] Our model assumes that the project is only run by the bidder, but it easily extends to settings where the seller and bidder jointly run the project.[5]

We analyze an "unconstrained" setting with no restrictions on the share of cash flows received by the bidder who runs the project. As we later touch on, our qualitative findings extend to the setting of Ekmekci, Kos and Vohra (2016) where the controller must receive a minimum stake—it is optimal for the top two bidders to sometimes share the cash flows.

Our model considers an impatient seller who does not value retention of cash flows. This impatience assumption is standard in the security design literature where a seller owns an asset that generates future cash flows, but has a higher discount rate than buyers, creating gains to trade (see, e.g., Biais and Mariotti 2005). Our analysis immediately extends to settings where the seller is as patient as bidders, and the project generates goods or services (instead of cash flows) that the seller values less than bidders, creating gains to trade as in standard auction models. Moreover, even if the project generates cash flows and the seller is as patient as bidders, our insights extend when the seller is penniless and has to raise cash to cover an upfront investment (or other liquidity need). Existing studies have examined such settings when cash-flow rights and control are split between the seller and a single bidder in the sense that the seller awards both control and a share of cash-flow rights to the (same) highest bidder in exchange for the cash needed for investment but no other bidder receives cash flows. Our insights apply here, too: a seller can do better by sometimes splitting a share of cash-flow rights between the two highest bidders—while retaining the remaining cash flows for herself, or awarding control to the second-highest bidder rather than the highest.

2.3 An example

Before starting our formal analysis, we provide a simple example that illustrates how our separation mechanism generates higher revenues than optimal no-separation mechanisms.

Example 1: Two bidders receive independently and identically uniformly distributed sig-

[4] It also loosely corresponds to the separation in a limited partner framework, in which the limited partners provide capital and other inputs, delegating the running of the business to the general partner.

[5] In such a setting, we interpret the "cash flows from the project when it is run by a bidder" as "cash flows from the project when it is run by the bidder in conjunction with the seller".

nals with support $[\tau+1,\tau+2]$, where τ is the cost of running the project. The value of the project when run by bidder i is $v_i(t_i, t_{-i}) = \frac{2}{3}t_i + \frac{1}{3}t_{-i}$, where t_{-i} is the rival bidder's signal.

First consider a no-separation English auction (with no reserve price), where the higher bidder wins both control and cash-flow rights and pays the lower (second-highest) bid. In the symmetric equilibrium, bidder i will exit at price $v_i(t_i, t_i) - \tau = t_i - \tau$. Thus, seller revenue is $\min\{t_1, t_2\} - \tau$, yielding expected seller revenue of $E[\min\{t_1, t_2\}] - \tau = \frac{4}{3}$. Standard results in the literature show that among all no-separation mechanisms, this auction maximizes expected seller revenues (see discussions after equation (36)).

Next consider the following direct-revelation separation mechanism (which we will show is the optimal separation mechanism for this example).

Separation Mechanism. Without loss of generality let t^h and t^l denote the higher and lower reported types. Then:

- If t^h is sufficiently higher than t^l so that $t^h \geq \frac{1}{2}\left(t^l + \tau + 2\right)$, then the bidder with higher report receives both cash-flow rights and control, and he pays the seller

$$\frac{5}{6}\left(t^l - \tau\right) + \frac{1}{3} \tag{8}$$

  in cash. The bidder with lower report receives nothing and pays nothing.

- If t^h exceeds t^l by less, so that $t^h \in [t^l, \frac{1}{2}\left(t^l + \tau + 2\right))$, the bidder reporting t^h receives cash-flow rights only and pays the seller t^l in cash. The bidder reporting t^l receives control and the seller pays him τ in cash to cover his costs of running the project.

We first show the revenue dominance of the separation mechanism, assuming incentive compatibility. We then prove that truth telling does, in fact, constitute an equilibrium for this separation mechanism. *In fact, a bidder's incentive compatibility condition holds not just in an interim sense (integrating over the rival's possible signals), but also in an ex post sense (given the actual realization of the rival's signal).*

To calculate the separation mechanism's revenue, assume without loss of generality that $t_1 \geq t_2$. If bidder 1's signal is not sufficiently higher than bidder 2's, i.e., if $t_1 \in [t_2, \frac{(t_2+\tau+2)}{2})$, then bidder 2 runs the project in the separation mechanism and net seller revenue is $t_2 - \tau$, which is the same as in the no-separation mechanism. However, if bidder 1's signal is even higher, $t_1 \geq \frac{(t_2+\tau+2)}{2}$, so that bidder 1 runs the project, then net seller revenue is $\frac{5}{6}(t_2-\tau)+\frac{1}{3}$ by (8), which exceeds revenue in the no-separation mechanism (of $t_2-\tau$) by

$$\frac{5}{6}(t_2-\tau)+\frac{1}{3}-(t_2-\tau)=\frac{1}{6}(\tau+2-t_2)\geq 0, \tag{9}$$

where the inequality is strict for all $t_2 < \tau+2$. Thus, expected seller revenue in the separation mechanism strictly exceeds that in the no-separation mechanism. Direct calculation yields that expected seller revenue in the separation mechanism is $1\frac{7}{18}$, which exceeds that from the no-separation mechanism by 1/18.[6]

Proof of ex-post incentive compatibility for the separation mechanism. Without loss of generality consider a generic bidder 1 (not necessarily the auction winner) who has signal t_1 but reports t_1', and assume bidder 2 reports his true type t_2. We show bidder 1 is weakly better off reporting his true type t_1 for *each* possible realization of t_2.

Claim 1: A bidder who does not receive cash flow rights receives zero profit.

Proof: If a bidder receives neither cash flows nor control, then his payment is zero and hence his profit is zero; if a bidder only receives control, he incurs the cost of running the project but receives a payment that just offsets this cost, again resulting in zero profit. □

Claim 2: If bidder 1 receives both cash flow rights and control, his profit is

$$\frac{2}{3}t_1+\frac{1}{3}t_2-\left(\frac{5}{6}(t_2-\tau)+\frac{1}{3}\right)-\tau=\frac{2}{3}t_1-\frac{1}{2}t_2-\frac{1}{6}\tau-\frac{1}{3}. \tag{10}$$

Proof: Bidder 1's profit is the value of the cash flows when he runs the project, which is $\frac{2}{3}t_1+\frac{1}{3}t_2$, less his payment (in (8)) and his cost of running the project. □

Claim 3: If bidder 1 receives cash flow rights but not control, his profit is

$$\frac{2}{3}t_2+\frac{1}{3}t_1-t_2=\frac{1}{3}(t_1-t_2). \tag{11}$$

[6] To see this revenue difference, observe that it does not vary with τ, so we can simplify calculations by letting $\tau=-1$ (in this case, interpret $|\tau|$ as the benefit of control rather than the cost of running the project), making signals uniform on $[0,1]$. Denote the higher and lower realizations by t_1 and t_2, and let $G(t_2)$ be cdf of t_2. Then $G(t_2)=1-(1-t_2)^2$. Conditional on t_2, t_1 is uniform over $[t_2,1]$, and the extra payment occurs when $t_1>\frac{1+t_2}{2}$, which occurs with probability 0.5. Thus, the expected revenue difference between the two mechanisms is

$$\int_0^1 0.5\left(\frac{1}{6}(1-t_2)\right)dG(t_2)=\int_0^1\frac{1}{12}(1-t_2)\,dG(t_2)=\int_0^1\frac{1}{6}(1-t_2)^2\,dt_2=1/18,$$

where the first expression comes from plugging $\tau=-1$ into (9).

Proof: Bidder 1's profit is the value of the cash flows when bidder 2 runs the project, which is $\frac{2}{3}t_2 + \frac{1}{3}t_1$, less his payment t_2. $\square$

These claims hold regardless of whether bidder 1 reports his type truthfully.

We group realizations of t_2 into four possible cases.

Case 1: t_2 is such that $t_1 \geq \frac{t_2+\tau+2}{2}$. Then bidder 1 receives both cash flow rights and control if he reports truthfully. By (10) in Claim 2, bidder 1's equilibrium profits are

$$\begin{aligned} \frac{2}{3}t_1 - \frac{1}{2}t_2 - \frac{1}{6}\tau - \frac{1}{3} &\geq \frac{2}{3}\left(\frac{t_2+\tau+2}{2}\right) - \frac{1}{2}t_2 - \frac{1}{6}\tau - \frac{1}{3} \\ &= -\frac{t_2}{6} + \frac{1}{6}(\tau+2) \geq 0; \end{aligned}$$

where the first inequality follows from the premise that $t_1 \geq \frac{t_2+\tau+2}{2}$, and the second inequality follows from $t_2 \leq \tau+2$. Thus, bidder 1's equilibrium profit is nonnegative. We now examine bidder 1's deviation profit when he reports t_1'. There are four subcases.

(i) If $t_1' \geq \frac{t_2+\tau+2}{2}$, then deviation leads to the same allocations and payment for bidder 1, so his profit equals his equilibrium profit.

(ii) If $t_1' \in (t_2, \frac{t_2+\tau+2}{2})$, then bidder 1 upon deviation receives cash flow rights and bidder 2 receives control. By Claim 3, bidder 1's deviation profit is given by (11). The net profit from deviation is the difference between (11) and his equilibrium profit in (10):

$$\begin{aligned} \frac{1}{3}(t_1 - t_2) - \left(\frac{2}{3}t_1 - \frac{1}{2}t_2 - \frac{1}{6}\tau - \frac{1}{3}\right) &= -\frac{t_1}{3} + \frac{1}{6}t_2 + \frac{1}{6}\tau + \frac{1}{3} \\ &\leq -\frac{1}{3}\left(\frac{t_2+\tau+2}{2}\right) + \frac{1}{6}t_2 + \frac{1}{6}\tau + \frac{1}{3} = 0, \end{aligned}$$

where the inequality follows from $t_1 \geq \frac{t_2+\tau+2}{2}$.

(iii) If $t_1' < t_2$, bidder 1 does not receive cash flow rights, so his profit is zero by Claim 1.

(iv) If $t_1' = t_2$, then there are two ways to break the tie. (a) Bidder 1 receives cash flow rights and bidder 2 receives control. Then the same argument as in subcase (ii) applies, showing that the deviation is not profitable. (b) Bidder 1 receives control and bidder 2 receives cash flow rights. Then bidder 1's profit is zero by Claim 1.

Case 2: t_2 is such that $t_1 \in (t_2, \frac{t_2+\tau+2}{2})$. Then bidder 1 receives cash flow rights but bidder 2 has control in equilibrium. By Claim 3, bidder 1's equilibrium profit is $\frac{1}{3}(t_1 - t_2) > 0$. Now consider bidder 1's deviation profit from reporting $t_1' \neq t_1$. There are three subcases.

(i) If $t_1' \geq \frac{t_2+\tau+2}{2}$, then bidder 1 receives both cash flow rights and control. His deviation profit is given by Claim 2. The net profit from deviation is the difference between (10) in Claim 2 and his equilibrium profit of $\frac{1}{3}(t_1 - t_2)$, yielding:

$$\begin{aligned} \frac{2}{3}t_1 - \frac{1}{2}t_2 - \frac{1}{6}\tau - \frac{1}{3} - \frac{1}{3}(t_1 - t_2) &= \frac{1}{3}t_1 - \frac{1}{6}t_2 - \frac{1}{6}\tau - \frac{1}{3} \\ &< \frac{1}{3}\frac{t_2+\tau+2}{2} - \frac{1}{6}t_2 - \frac{1}{6}\tau - \frac{1}{3} = 0, \end{aligned}$$

where the inequality follows from $t_1 < \frac{t_2+\tau+2}{2}$.

(ii) If $t_1' \in (t_2, \frac{t_2+\tau+2}{2})$, then deviation leads to same allocations and same payment for bidder 1. Thus, bidder 1's profit equals his equilibrium profit.

(iii) If $t_1' < t_2$, then bidder 1 receives no cash flow rights, so his profit is zero by Claim 1.

(iv) If $t_1' = t_2$, then regardless of how the tie is resolved, arguments similar to those in subcases (ii) and (iii) yield that deviation is not profitable.

Case 3: t_2 is such that $t_1 < t_2$. Then bidder 1 does not receive cash flow rights in equilibrium, so his equilibrium profit is zero by Claim 1. We now examine bidder 1's deviation profit from reporting t_1'. By Claim 1, the only possible profitable deviation must result in bidder 1 receiving cash flow rights, and hence that $t_1' \geq t_2$. There are two subcases.

(i) Bidder 1 receives both cash flow rights and control. By Claim 2, his profit is

$$\frac{2}{3}t_1 - \frac{1}{2}t_2 - \frac{1}{6}\tau - \frac{1}{3} < \frac{1}{6}t_2 - \frac{1}{6}\tau - \frac{1}{3} \leq 0,$$

where the first inequality follows from the premise that $t_1 < t_2$, and the second inequality follows from $t_2 \leq \tau + 2$. Thus, bidder 1's deviation profit is strictly negative.

(ii) Bidder 1 receives cash flow rights but not control. By Claim 3, his profit is $\frac{1}{3}(t_1 - t_2)$, which is strictly negative by the premise that $t_1 < t_2$.

Case 4: t_2 is such that $t_1 = t_2$. Then bidder 1's equilibrium profit is zero, no matter how the tie is broken. Similar logic as in Case 3 shows his deviation profit is non-positive. □

3 Analysis

Applying the envelope theorem to (3) and (5) yields

$$\frac{d\bar{U}_i(t_i,t_i)}{dt_i} = \int_{\Omega_{n-1}} \sum_j R_j(\mathbf{t})Q_{ji}(\mathbf{t}) \frac{\partial v_j\left(t_i;\mathbf{t}_{-i}\right)}{\partial t_i} f_{-i}\left(\mathbf{t}_{-i}\right) d\mathbf{t}_{-i}. \tag{12}$$

Equation (12) conveys our insight on the advantages of separation: as in the standard no-separation setting, allocating cash flows to a bidder i with signal t_i enables him to earn differential rents relative to when i has a lower signal, as reflected by the term $R_j(\mathbf{t})Q_{ji}(\mathbf{t})$; but, unlike the no-separation case, the differential rents are scaled by $\frac{\partial v_j(t_i;\mathbf{t}_{-i})}{\partial t_i}$. That is, bidder i's differential rents are weighted by the sensitivity of the value of his awarded cash flows to his signal when the project is run by bidder j. Because the project's value is more sensitive to the signal of the bidder who runs the project, awarding bidder i cash flows when the project is run by a different bidder reduces i's overall rents, vis à vis awarding bidder i cash flows when i also runs the project himself.

With the linear valuation structure, $v_j = \sum_k W_{jk}t_k$, we have $\frac{\partial v_j}{\partial t_i} = W_{ji}$. Substituting this into (12) yields

$$\frac{d\bar{U}_i(t_i,t_i)}{dt_i} = J_i(t_i), \tag{13}$$

where

$$J_i(t_i) \equiv \int_{\Omega_{n-1}} \sum_j R_j(\mathbf{t})Q_{ji}(\mathbf{t})W_{ji} f_{-i}\left(\mathbf{t}_{-i}\right) d\mathbf{t}_{-i}. \tag{14}$$

Integrating (13) yields,

$$\bar{U}_i(t_i,t_i) = \int_{\underline{t}}^{t_i} J_i(\widetilde{t})d\widetilde{t} + \bar{U}_i(\underline{t},\underline{t}). \tag{15}$$

Rearranging (3) yields bidder i's equilibrium expected cash payment to the seller:

$$\begin{aligned} m_i(t_i) \quad = \quad & \int_{\Omega_{n-1}} \sum_j R_j(t_i;\mathbf{t}_{-i})Q_{ji}\left(t_i;\mathbf{t}_{-i}\right) v_j(t_i;\mathbf{t}_{-i}) f_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i} - \bar{U}_i(t_i,t_i) \\ & -\tau \int_{\Omega_{n-1}} R_i(t_i;\mathbf{t}_{-i}) f_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i}. \end{aligned} \tag{16}$$

Substituting for $m_i(t_i)$ and $\bar{U}_i(t_i,t_i)$ into seller revenue (7) yields

$$\pi_s = \int \sum_{j,i} R_j(\mathbf{t})Q_{ji}(\mathbf{t})v_j(\mathbf{t})f(\mathbf{t})d\mathbf{t} - \sum_i \int_{\underline{t}}^{\bar{t}} \left(\int_{\underline{t}}^{t_i} J_i(\widetilde{t})d\widetilde{t} + \bar{U}_i(\underline{t},\underline{t}) \right) f(t_i)dt_i - \tau \sum_i \int_{\Omega_n} R_i(\mathbf{t})f(\mathbf{t})d\mathbf{t}, \tag{17}$$

where $\Omega_n \equiv [\underline{t}, \bar{t}]^n$ denotes the space of integration for all n bidders. Equation (17) is intuitive: expected revenue is the expected increase in social welfare gross of the costs of running the project (first term on the right-hand side) less the sum of bidders' expected rents (second term) less the expected costs of running the project (third term). Applying integration by parts to the second term on the right-hand side (without the summation) yields:

$$\begin{aligned}
\int_{\underline{t}}^{\bar{t}} \Big(\int_{\underline{t}}^{t_i} J_i(\widetilde{t}) d\widetilde{t} + \bar{U}_i(\underline{t}, \underline{t}) \Big) f(t_i) dt_i &= -\int_{\underline{t}}^{\bar{t}} \left(\int_{\underline{t}}^{t_i} J_i(\widetilde{t}) d\widetilde{t} + \bar{U}_i(\underline{t}, \underline{t}) \right) d\left(1 - F(t_i)\right) \\
&= \bar{U}_i(\underline{t}, \underline{t}) + \int_{\underline{t}}^{\bar{t}} (1 - F(t_i))\, d \left(\int_{\underline{t}}^{t_i} J_i(\widetilde{t}) d\widetilde{t} + \bar{U}_i(\underline{t}, \underline{t}) \right) \\
&= \bar{U}_i(\underline{t}, \underline{t}) + \int_{\underline{t}}^{\bar{t}} (1 - F(t_i))\, J_i(t_i) dt_i \\
&= \bar{U}_i(\underline{t}, \underline{t}) + \int_{\underline{t}}^{\bar{t}} \frac{1 - F(t_i)}{f(t_i)} J_i(t_i) f(t_i) dt_i. \qquad (18)
\end{aligned}$$

Substituting (14) for $J_i(t_i)$ into the right-hand side of (18) yields

$$\begin{aligned}
\int_{\underline{t}}^{\bar{t}} \Big(\int_{\underline{t}}^{t_i} J_i(\widetilde{t}) d\widetilde{t} + \bar{U}_i(\underline{t}, \underline{t}) \Big) f(t_i) dt_i &= \bar{U}_i(\underline{t}, \underline{t}) + \int_{\underline{t}}^{\bar{t}} \frac{1 - F(t_i)}{f_i(t_i)} \Big(\int_{\Omega_{n-1}} \sum_j R_j(\mathbf{t}) Q_{ji}(\mathbf{t}) W_{ji} f_{-i}(\mathbf{t}_{-i}) d\mathbf{t}_{-i} \Big) f(t_i) dt_i \\
&= \bar{U}_i(\underline{t}, \underline{t}) + \int_{\underline{t}}^{\bar{t}} \int_{\Omega_{n-1}} \frac{1 - F(t_i)}{f(t_i)} \Big(\sum_j R_j(\mathbf{t}) Q_{ji}(\mathbf{t}) W_{ji} \Big) f_{-i}(\mathbf{t}_{-i}) f(t_i) d\mathbf{t}_{-i} dt_i \\
&= \bar{U}_i(\underline{t}, \underline{t}) + \int_{\Omega_n} \frac{1 - F(t_i)}{f(t_i)} \Big(\sum_j R_j(\mathbf{t}) Q_{ji}(\mathbf{t}) W_{ji} \Big) f(\mathbf{t}) d\mathbf{t}.
\end{aligned}$$

Hence, the seller's expected revenue (equation (17)) is:

$$\begin{aligned}
\pi_s &= \int_{\Omega_n} \sum_{ji} R_j(\mathbf{t}) Q_{ji}(\mathbf{t}) v_j(\mathbf{t}) f(\mathbf{t}) d\mathbf{t} \\
&\quad - \sum_i \int_{\Omega_n} \frac{1 - F(t_i)}{f(t_i)} \left(\sum_j R_j(\mathbf{t}) Q_{ji}(\mathbf{t}) W_{ji} \right) f(\mathbf{t}) d\mathbf{t} - \tau \sum_j \int_{\Omega_n} R_j(\mathbf{t}) f(\mathbf{t}) d\mathbf{t} - \sum_i \bar{U}_i(\underline{t}, \underline{t}).
\end{aligned}$$

To ease presentation of seller revenues, we follow the literature and define the virtual valuation of bidder j, $\phi_j(\mathbf{t},Q)$, which depends on both $\mathbf{t}$ and Q:

Definition 1 *Bidder j's virtual valuation is given by*

$$\phi_j(\mathbf{t},Q) \equiv \sum_i Q_{ji}(\mathbf{t}) \left(v_j(\mathbf{t}) - \frac{1 - F(t_i)}{f(t_i)} W_{ji} \right) - \tau. \qquad (19)$$

Theorem 1 ***(Revenue decomposition)*** *In any incentive-compatible mechanism, the seller's expected revenue (7) is given by*

$$\pi_s = \pi_{s,a} + \pi_{s,b}, \tag{20}$$

where

$$\pi_{s,a} \equiv \int_{\Omega_n} \Big[\sum_j R_j(\mathbf{t}) \phi_j(\mathbf{t},Q) \Big] f(\mathbf{t}) d\mathbf{t} \qquad \textit{and} \qquad \pi_{s,b} = -\sum_i \bar{U}_i(\underline{t}, \underline{t}). \tag{21}$$

Bidder rationality implies that $\pi_{s,b}$ is non-positive, and it reaches its maximum when $\bar{U}_i(\underline{t}, \underline{t}) = 0$ for all i. The virtual valuations underlying $\pi_{s,a}$ arise from the envelope theorem, which reflects local incentive compatibility. We next derive an upper bound on $\pi_{s,a}$, and then explore which allocations achieve this upper bound and hence maximize $\pi_{s,a}$, deferring the question of whether a mechanism with such allocations is globally incentive compatible.

Define $\phi_j^{\max}(\mathbf{t})$ to be the maximum virtual valuation $\phi_j(\mathbf{t},Q)$ achievable by a feasible cash-flow rights allocation Q:

$$\phi_j^{\max}(\mathbf{t}) \equiv \max_{\sum_i Q_{ji}(\mathbf{t}) \leq 1} \phi_j(\mathbf{t},Q).$$

Here $\phi_j^{\max}(\mathbf{t})$ represents the (maximum) rents a seller can extract by assigning control to a bidder j. Inspection of equations (1) and (21) reveals an upper bound on $\pi_{s,a}$:

$$\pi_{s,a} \leq \int \left[\max \left\{ \phi_1^{\max}(\mathbf{t}), ..., \phi_n^{\max}(\mathbf{t}), 0 \right\} \right] f(\mathbf{t}) d\mathbf{t}. \tag{22}$$

The right-hand side of (22) is achieved by allocating control to the bidder with the highest $\phi_j^{\max}(\mathbf{t})$ provided that it is nonnegative; and by no sale when it is not.

In sharp contrast to standard settings where, by assumption, control and cash rights are not separated, bidder j's virtual valuation in our separation framework depends on the values of Q_{ji} for all i. That is, the rents a seller can extract by assigning control to a bidder j depends on how she assigns cash-flow rights among bidders, conditional on j having control. Thus, the joint assignments of the two rights among bidders creates an externality (or interaction).

We next write bidder valuations in a matrix form. This form eases determinaion of the joint allocations (combinations) of control-rights and cash-flow-rights that maximize $\pi_{s,a}$.

Definition 2 *Bidder j's matrix-form valuation is*

$$\psi_{ji}(\mathbf{t}) \equiv v_j(\mathbf{t}) - \frac{1 - F(t_i)}{f(t_i)} W_{ji}. \tag{23}$$

Lemma 1 *The following joint assignments of rights achieve the upper bound of $\pi_{s,a}$ on the right-hand side of (22):*

(i) If $\max_{j,i} \psi_{ji}(\mathbf{t}) < \tau$, then there is no sale.

(ii) If $\max_{j,i} \psi_{ji}(\mathbf{t}) \geq \tau$, then assign control to $\hat{\jmath}$, and assign cash rights to $\hat{\imath}$, where the pair $(\hat{\jmath}, \hat{\imath})$ are maximizers of $\psi_{ji}(\mathbf{t})$: $(\hat{\jmath}, \hat{\imath}) \in \arg\max_{(j,i)} \psi_{ji}(\mathbf{t})$.

Proof of Lemma 1: Rewrite (19) as:

$$\phi_j(\mathbf{t},Q) \equiv \sum_i Q_{ji}(\mathbf{t})\psi_{ji}(\mathbf{t},Q) - \tau. \tag{24}$$

$\max_{j,i} \psi_{ji}(\mathbf{t}) < \tau$ implies that $\psi_{ji}(\mathbf{t}) < \tau$ for all j and i. It follows that $\phi_j(\mathbf{t},Q) < 0$ for any $Q_{ji}(\mathbf{t})$ satisfying the constraint $\sum_i Q_{ji}(\mathbf{t}) \leq 1$, and hence $\phi_j^{\max}(\mathbf{t}) < 0$, establishing part (i).

To prove part (ii), note that $\psi_{\hat{\jmath}\hat{\imath}}(\mathbf{t}) \geq \psi_{ji}(\mathbf{t})$ for all j and i. Thus, for all j and Q, (24) yields $\phi_j(\mathbf{t},Q) \leq \left(\sum_i Q_{ji}(\mathbf{t})\right) \psi_{\hat{\jmath}\hat{\imath}}(\mathbf{t}) - \tau \leq \psi_{\hat{\jmath}\hat{\imath}}(\mathbf{t}) - \tau$. Thus,

$$\phi_j^{\max}(\mathbf{t}) \leq \psi_{\hat{\jmath}\hat{\imath}}(\mathbf{t}) - \tau \tag{25}$$

for all j. Further, setting $Q_{\hat{\jmath}\hat{\imath}} = 1$ and $Q_{\hat{\jmath}i} = 0$ for $i \neq \hat{\imath}$ results in $\phi_{\hat{\jmath}}(\mathbf{t},Q) = \psi_{\hat{\jmath}\hat{\imath}}(\mathbf{t}) - \tau$. Thus, $\phi_{\hat{\jmath}}^{\max}(\mathbf{t}) = \psi_{\hat{\jmath}\hat{\imath}}(\mathbf{t}) - \tau$, and giving control to $\hat{\jmath}$ and cash flows to $\hat{\imath}$ achieves this. By (25), $\phi_{\hat{\jmath}}^{\max}(\mathbf{t}) \geq \phi_j^{\max}(\mathbf{t})$ for all j, and $\phi_{\hat{\jmath}}^{\max}(\mathbf{t}) \geq 0$. Thus, assigning control to $\hat{\jmath}$ achieves the upper bound of $\pi_{s,a}$ on the right-hand side of (22). We have already established that assigning cash flows to $\hat{\imath}$ achieves $\phi_{\hat{\jmath}}^{\max}(\mathbf{t})$, so part (ii) follows.[7] ■

To ease notation, we define the inverse hazard function:

$$L(t) \equiv \frac{1 - F(t)}{f(t)}.$$

Assumption 1 *$L(t)$ is strictly decreasing.*

Assumption 2 *$\underline{t} - A_n L(\underline{t}) \geq \tau$.*

Assumption 1 is a sufficient condition for the standard regularity condition that ensures global incentive compatibility (and not just local incentive compatibility) holds for "no-separation" mechanisms. However, we will show that Assumption 1 is still not enough to

[7] One can alternatively conduct our analysis via the matrix-form virtual valuation (23), without defining (19). We introduce (19) to facilitate comparison of our results with the standard Myersonian framework.

ensure that regularity always holds in our separation mechanism. Assumption 2 ensures that $\psi_{ji}(\mathbf{t},Q)$ (equation (23)) is at least τ for any j and i. Thus, it is optimal to always sell the asset (by Lemma 1). These assumptions simplify characterization of the optimal mechanism and facilitate comparison with the no-separation benchmark.

We now characterize the allocations that maximize $\pi_{s,a}$ while delegating to later the question of whether such allocations are consistent with the global incentive compatibility requirement of the mechanism. Let "h" and "s" index the highest and second-highest bidders, and denote their signals by t_h and t_s respectively. Define

$$K(t_h) \equiv \begin{cases} \min\left\{t_h - L(t_h), L^{-1}\left(\frac{1}{\rho}L(t_h)\right)\right\} & \text{if } \frac{1}{\rho}L(t_h) \leq L\left(\underline{t}\right) \\ \underline{t} & \frac{1}{\rho}L(t_h) > L\left(\underline{t}\right), \end{cases} \tag{26}$$

where L^{-1} is the inverse function of L. Note that $K(t_h)$ weakly increases in t_h and $K(t_h) < t_h$.

Proposition 1 *The following allocations maximize $\pi_{s,a}$:*

(i) If $t_s > K(t_h)$, then cash-flow rights and control are split between bidders h and s. The cutoff

$$\bar{\rho} \equiv \frac{t_h - t_s}{t_h - t_s + L(t_s) - L(t_h)} \tag{27}$$

on ρ determines how cash-flow rights and control are split between bidders.

If $\rho \leq \bar{\rho}$, then assign control to bidder h and cash-flow rights to bidder s.

If $\rho \geq \bar{\rho}$, then assign cash-flow rights to bidder h and control to bidder s.[8]

(ii) If $t_s < K(t_h)$, then assign bidder h both cash-flow rights and control.[9]

Proof of Proposition 1:

Claim 1: Assigning both cash flow rights and control to h achieves a higher $\pi_{s,a}$ than assigning both cash flow rights and control to any other bidder.

Proof: Consider any bidder $k \neq h$. By (23) and $W_{hh} = W_{kk} = A_n$,

$$\psi_{hh}(\mathbf{t}) - \psi_{kk}(\mathbf{t}) = A_n\left(1-\rho\right)\left(t_h - t_k\right) + A_n\left(L\left(t_k\right) - L\left(t_h\right)\right),$$

[8] If $\rho = \bar{\rho}$, then either way of splitting maximizes $\pi_{s,a}$.

[9] In the measure-zero event that $t_s = K(t_h)$, assign bidder h both cash-flow rights and control (i.e., break indifference in favor of the no-separation mechanism), except in the special case where $t_s = \underline{t}$ and $\frac{1}{\rho}L(t_h) > L\left(\underline{t}\right)$, in which case part (i) of the proposition applies.

which is strictly positive because $t_h > t_k$ and $L(t_k) > L(t_h)$. This and part (ii) of Lemma 1 establishes Claim 1. $\square$

Claim 2: In the optimal ($\pi_{s,a}$ maximizing) allocation, a bidder who is not among the two highest receives neither cash flow rights nor control.

Proof: Denote this bidder by k. There are two cases.

Case 1: Suppose instead that k receives control in the optimal allocation. Let i be the bidder who receives cash flow rights. By Claim 1, $i \neq k$. Then $\psi_{ki}(\mathbf{t}) \equiv v_k(\mathbf{t}) - L(t_i) A_n \rho$. Clearly, i cannot be both h and s. If $i \neq s$, then $\psi_{si}(\mathbf{t}) - \psi_{ki}(\mathbf{t}) = A_n (1-\rho)(t_s - t_k) > 0$, implying that assigning control to s and cash flow rights to i results in a higher $\pi_{s,a}$ than assigning control to k and cash flow rights to i. Thus, assigning control to k and cash flow rights to i is not optimal. An analogous contradiction arises if $i \neq h$.

Case 2: Suppose instead that k receives cash flow rights in the optimal allocation. Let i be the bidder who receives control. By Claim 1, $i \neq k$. Then $\psi_{ik}(\mathbf{t}) \equiv v_i(\mathbf{t}) - L(t_k) A_n \rho$. Clearly, i cannot be both h and s. If $i \neq s$, then $\psi_{is}(\mathbf{t}) - \psi_{ik}(\mathbf{t}) = (L(t_k) - L(t_s)) A_n \rho > 0$, implying that allocating control to i and cash flow rights to s results in a higher $\pi_{s,a}$ than allocating control to i and cash flow rights to s. Thus, allocating control to i and cash flow rights to s is not optimal. An analogous contradiction arises if $i \neq h$. $\square$

By Claims 1 and 2, the allocations that maximize $\pi_{s,a}$ can only take one of three forms: give both control and cash flows to the highest bidder; give cash flows to the highest bidder and control to the second-highest bidder; or give control to the highest bidder and cash flows to second-highest bidder. We next pin down when each of these allocations arises. We have

$$\psi_{hh} - \psi_{hs} = A_n (\rho L(t_s) - L(t_h)),$$

which, since $L(\cdot)$ is decreasing, yields $\psi_{hh} \geq \psi_{hs}$ if and only if $t_s \leq L^{-1}\left(\frac{1}{\rho} L(t_h)\right)$. Next, use

$$\begin{aligned} \psi_{hh} - \psi_{sh} &= A_n(1-\rho)(t_h - t_s) - A_n(1-\rho)L(t_h) \\ &= A_n(1-\rho)(t_h - t_s - L(t_h)), \end{aligned}$$

which yields $\psi_{hh} \geq \psi_{sh}$ if and only if $t_s \leq t_h - L(t_h)$.

The above and Lemma 1 yield part (ii) of Proposition 1 and the first portion of part (i) that splitting cash flow rights and control between bidders h and s maximizes $\pi_{s,a}$ when $t_s > K(t_h)$. To establish the remaining portion of part (i) concerning $\bar{\rho}$, note that

$$\psi_{hs} - \psi_{sh} = (1-\rho)(t_h - t_s) - \rho\left(L(t_s) - L(t_h)\right). \tag{28}$$

The right-hand side of (28) strictly decreases in ρ, and it equals zero when $\rho = \bar{\rho}$, where $\bar{\rho}$ is defined in (27). This and Lemma 1 establish part (i). ■

To further pin down the allocations, define

$$\delta_1 \equiv \max_{t\in[\underline{t},\bar{t}]} \frac{d}{dt} L(t) \quad \text{and} \quad \delta_2 \equiv \min_{t\in[\underline{t},\bar{t}]} \frac{d}{dt} L(t).$$

By Assumption 1, $\delta_2 \leq \delta_1 < 0$. The optimal allocation of cash flow rights and control depends on the size ρ of the common value component of project payoffs (relative to δ_1 and δ_2) and the closeness of bidder signals:

Corollary 1 *(i) If $\rho \geq \frac{1}{1-\delta_1}$, then assigning cash-flow rights to bidder h maximizes $\pi_{s,a}$. Control is assigned to bidder h if $t^s < t^h - L\left(t^h\right)$ and to bidder s if $t^s > t^h - L\left(t^h\right)$.*[10]

(ii) If $\rho \leq \frac{1}{1-\delta_2}$, then assigning control to bidder h maximizes $\pi_{s,a}$. Cash flow rights are assigned to bidder h if $t^s < L^{-1}\left(\frac{1}{\rho}L\left(t^h\right)\right)$ and to bidder s if $t^s > L^{-1}\left(\frac{1}{\rho}L\left(t^h\right)\right)$.

Proof of Corollary 1: Define $x \equiv L^{-1}\left(\frac{1}{\rho}L(t_h)\right)$ and

$$\begin{aligned} \Delta \equiv t_h - x - L(t_h) &= \frac{L(t_h) - L(x)}{\frac{L(t_h)-L(x)}{t_h - x}} - L(t_h) \\ &= \frac{L(t_h) - \frac{1}{\rho}L(t_h)}{\frac{L(t_h)-L(x)}{t_h - x}} - L(t_h). \end{aligned}$$

By $\delta_2 \leq \frac{L(t_h)-L(x)}{t_h-x} \leq \delta_1 < 0$ and $L(t_h) - \frac{1}{\rho}L(t_h) \leq 0$, we have

$$\frac{L(t_h) - \frac{1}{\rho}L(t_h)}{\delta_2} - L(t_h) \leq \frac{L(t_h) - \frac{1}{\rho}L(t_h)}{\frac{L(t_h)-L(x)}{t_h-x}} - L(t_h) = \Delta \leq \frac{L(t_h) - \frac{1}{\rho}L(t_h)}{\delta_1} - L(t_h). \tag{29}$$

[10] If $t^s = t^h - L\left(t^h\right)$, assigning control to either bidder maximizes $\pi_{s,a}$. In this case we adopt the convention of breaking the indifference in favor of the no-separation mechanism, and hence assign control to bidder h. A similar statement holds for part (ii) of the corollary.

When $\rho \geq \frac{1}{1-\delta_1}$, the second inequality in (29) yields

$$\Delta \leq \frac{L(t_h) - (1-\delta_1)\, L\,(t_h)}{\delta_1} - L(t_h) = 0.$$

Thus, $\min\left\{t_h - L(t_h), L^{-1}\left(\frac{1}{\rho}L\,(t_h)\right)\right\}$ in (26) equals $t_h - L(t_h)$.

When $\rho \leq \frac{1}{1-\delta_2}$, the first inequality in (29) yields

$$\Delta \geq \frac{L(t_h) - (1-\delta_2)\, L\,(t_h)}{\delta_2} - L(t_h) = 0.$$

Thus, $\min\left\{t_h - L(t_h), L^{-1}\left(\frac{1}{\rho}L\,(t_h)\right)\right\} = L^{-1}\left(\frac{1}{\rho}L\,(t_h)\right)$. In addition, note that $\frac{1}{1-\delta_2} \leq \bar{\rho} \leq \frac{1}{1-\delta_1}$, where $\bar{\rho}$ is defined in Proposition 1. This establishes the corollary via Proposition 1. ∎

The intuition for these results reflects the tradeoffs between the benefit and costs of separation. The benefit of separation, as conveyed by equation (12), is that bidder i's differential rents are weighted by W_{ji}, the sensitivity of the value of his awarded cash flows to his signal when the project is run by bidder j. Because the value of the project run by a bidder is more sensitive to his own signal than to his rivals' signals, separating cash-flow rights from control reduces bidders' overall rents. However, separating the two rights gives rise to one of two costs: (i) the efficiency loss from not assigning control to the highest bidder, or (2) the increase in bidders' overall rents from not assigning cash flows to the highest bidder.

When the second-highest signal is sufficiently close to the highest signal, the benefit of separation exceeds the smaller of the two costs, making separation optimal. To see the advantages of separation most clearly, suppose s_i and s_j tie for the highest signal. In the classical no-separation framework, assigning both rights to a single bidder, either i or j, is optimal. By contrast, in our framework, separation is strictly better: splitting the two rights between the two bidders strictly dominates assigning both rights to a single bidder: with a tie, the two costs of separation are zero, leaving only the benefit.

When the difference in the two highest signals is not too large, separation continues to dominate no-separation, but the optimal way of splitting depends on which cost of splitting is larger. In turn, this depends on the importance of common values relative to private values—the larger is ρ, the lower is the efficiency cost of allocating control to the second-highest bidder rather than the highest. Thus, when ρ is large enough, it is optimal to assign control to the second-highest bidder and cash flows to the highest bidder. When, instead, ρ is smaller, efficiency considerations make it optimal to reverse the allocations: assign control to the highest bidder and cash-flow rights to the second-highest bidder.

3.1 Restrictions imposed by global incentive compatibility

Our analysis so far characterizes allocations that maximize $\pi_{s,a}$. The virtual valuations underlying $\pi_{s,a}$ arise from the envelope theorem, which reflects local incentive compatibility. Thus, if local incentive compatibility implies global incentive compatibility then such allocations would be optimal for the seller (assuming a bidder with the lowest possible signal $\underline{t}$ earns zero rents). That is, if the global incentive compatibility requirement does not bind, a seller optimally assigns cash-flow rights and control to the bidder pair with the highest matrix-form valuation; and the seller needs only consider assignments to highest and second-highest bidders. We next derive the restrictions imposed by global incentive compatibility.

Rewrite bidder j's valuation as $v_j = W_{ji}t_i + \sum_{k\neq i} W_{jk}t_k$ and substitute it into (3). The incentive compatibility condition (5) becomes

$$\bar{U}_i(t_i, t_i) = \max_{t_i'} \int \sum_j R_j\left(t_i'; \mathbf{t}_{-i}\right) Q_{ji}\left(t_i'; \mathbf{t}_{-i}\right) \Big(W_{ji}t_i + \sum_{k\neq i} W_{jk}t_k\Big) f_{-i}\left(\mathbf{t}_{-i}\right) d\mathbf{t}_{-i} - M_i(t_i').$$

This shows $\bar{U}_i(t_i, t_i)$ is the maximum of a family of affine functions and hence is convex. Thus, a necessary condition for global IC is that $J_i(t_i) = \int_{\Omega_{n-1}} \sum_j R_j(\mathbf{t}) Q_{ji}(\mathbf{t}) W_{ji} f_{-i}\left(\mathbf{t}_{-i}\right) d\mathbf{t}_{-i}$ is weakly increasing in t_i for all i.

We next impose this condition as a regularity condition and show that it is sufficient for local incentive compatibility to imply global incentive compatibility. This simplifies construction of the optimal mechanism.

Definition 3 *The design problem is regular if, given the "$\pi_{s,a}$-maximizing allocations" in Lemma 1, $J_i(t_i)$ is weakly increasing in t_i over $[\underline{t}, \bar{t}]$ for all i.*

Theorem 2 *If the design problem is regular, then the following mechanism is incentive compatible and individually rational, delivering an expected revenue equal to the right-hand side of (22), which is maximal among all incentive compatible, individually rational mechanisms:*

(i) Control and cash-flow rights allocations, R and Q, are given by the $\pi_{s,a}$-maximizing allocation in Lemma 1.

(ii) Cash payments by each bidder i are given by

$$M_i(\mathbf{t}) = \sum_j R_j(\mathbf{t})Q_{ji}(\mathbf{t})v_j(\mathbf{t}) - \int_{\underline{t}}^{t_i} \Big(\sum_j R_j(\widetilde{t};\mathbf{t}_{-i})Q_{ji}(\widetilde{t};\mathbf{t}_{-i})W_{ji}\Big)d\widetilde{t} - \tau R_i(\mathbf{t}), \tag{30}$$

where $(\widetilde{t};\mathbf{t}_{-i})$ is the vector of types constructed by replacing bidder i's type t_i with $\widetilde{t}$ in $\mathbf{t}$.

Proof of Theorem 2: We first show that the mechanism is individually rational and incentive compatible. Substituting (30) into (4) yields[11]

$$\begin{aligned} m_i(t_i) &= \int_{\Omega_{n-1}} \sum_j R_j(t_i;\mathbf{t}_{-i})Q_{ji}\left(t_i;\mathbf{t}_{-i}\right) v_j(t_i;\mathbf{t}_{-i})f_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i} - \int_{\underline{t}}^{t_i} J_i(\widetilde{t})d\widetilde{t} \\ &\quad -\tau \int_{\Omega_{n-1}} R_i(t_i;\mathbf{t}_{-i})f_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i}, \end{aligned} \tag{31}$$

where $J_i(\cdot)$ is given by (14). Substituting (31) into (3) and setting $t_i' = t_i$ yields

$$\bar{U}_i(t_i,t_i) = \int_{\underline{t}}^{t_i} J_i(\widetilde{t})d\widetilde{t}. \tag{32}$$

Thus, the mechanism is individually rational and $\bar{U}_i(\underline{t},\underline{t}) = 0$ for all i. Next, by (3), we have:

$$\begin{aligned} \bar{U}_i(t_i,t_i) &- \bar{U}_i\left(t_i,t_i'\right) \\ &= \bar{U}_i(t_i,t_i) - \bar{U}_i\left(t_i',t_i'\right) - \left(\bar{U}_i\left(t_i,t_i'\right) - \bar{U}_i(t_i',t_i')\right) \\ &= \bar{U}_i(t_i,t_i) - \bar{U}_i(t_i',t_i') - \int \Sigma_j R_j(t_i';\mathbf{t}_{-i})Q_{ji}(t_i';\mathbf{t}_{-i})(v_j(t_i;\mathbf{t}_{-i}) - v_j(t_i';\mathbf{t}_{-i}))f_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i} \\ &= \bar{U}_i(t_i,t_i) - \bar{U}_i(t_i',t_i') - (t_i - t_i') \int \Sigma_j R_j\left(t_i';\mathbf{t}_{-i}\right) Q_{ji}(t_i';\mathbf{t}_{-i})W_{ji}f_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i} \\ &= \bar{U}_i(t_i,t_i) - \bar{U}_i\left(t_i',t_i'\right) - \left(t_i - t_i'\right) J_i\left(t_i'\right). \end{aligned} \tag{33}$$

The third equation follows from $v_j\left(t_i;\mathbf{t}_{-i}\right) - v_j\left(t_i';\mathbf{t}_{-i}\right) = W_{ji}\left(t_i - t_i'\right)$. By (32), (33) becomes

$$\bar{U}_i(t_i,t_i) - \bar{U}_i\left(t_i,t_i'\right) = \int_{t_i'}^{t_i} J_i(\widetilde{t})d\widetilde{t} - \left(t_i - t_i'\right) J_i(t_i'). \tag{34}$$

By assumption, the design problem is regular—J_i is weakly increasing—so the right-hand side of (34) is nonnegative for all $t_i' \neq t_i$. This means there are no gains to deviation and hence the mechanism is incentive compatible. Because the mechanism maximizes both terms $\pi_{s,a}$ and $\pi_{s,b}$ in (20), it maximizes seller revenues. ∎

[11] (31) is what is needed to ensure individual rationality and incentive compatibility, and (30) is a natural way to implement (31).

Proposition 2 *When* $\rho \geq \frac{1}{1-\delta_1}$ *(where* $\delta_1 \equiv \max_{t \in [\underline{t}, \bar{t}]} \frac{d}{dt} L(t)$*), the design problem is regular (in the sense of Definition 3) for any number of bidders* n*, and, hence, the mechanism specified by parts (i) and (ii) of Theorem 2 is the optimal mechanism for the seller.*

Proof of Proposition 2: Consider a generic bidder i. Let $t^* \in (\underline{t}, t)$ be the unique solution to $t^* - L(t^*) = \underline{t}$. Let $\widetilde{t}$ be the highest signal of all bidders other than i. By Corollary 1, to receive the cash flow allocation, bidder i must have the highest signal; i.e., $\widetilde{t} \leq t_i$.

- If $t_i \leq t^*$, then i never receives control. Then $J_i(t_i) = \rho F^{n-1}(t_i)$, where $F^{n-1}(t_i)$ is the probability that $\widetilde{t} < t_i$. By inspection, $J_i(t_i)$ is increasing in t_i.

- If $t_i > t^*$, then i will receive control if $\widetilde{t} \in (\underline{t}, t_i - L(t_i))$, which occurs with probability $F^{n-1}(t_i - L(t_i))$; and i will not receive control if $\widetilde{t} \in (t_i - L(t_i), t_i]$, which occurs with probability $F^{n-1}(t_i) - F^{n-1}(t_i - L(t_i))$. Summing these two parts yields

$$\begin{aligned} J_i(t_i) &= F^{n-1}(t_i - L(t_i)) + \rho \left(F^{n-1}(t_i) - F^{n-1}(t_i - L(t_i)) \right) \\ &= \rho F^{n-1}(t_i) + (1-\rho) F^{n-1}(t_i - L(t_i)). \end{aligned}$$

  Because

$$\frac{d}{dt_i} \left(F^{n-1}(t_i - L(t_i)) \right) = (n-1) F^{n-2}(t_i - L(t_i)) f(t_i - L(t_i)) \left(1 - \frac{d}{dt_i} L(t_i) \right) \geq 0, \quad (35)$$

  $J_i(t_i)$ is weakly increasing in t_i, establishing the proposition. ■

When ρ is smaller, the regularity condition may not hold, and whether it holds depends on the values of both ρ and n. We investigate this in Section 3.3 for uniformly distributed signals.

At this point, it is useful to compare our results with the extensively-studied no-separation benchmark. To apply our analysis to this benchmark, we just add the restrictions that $Q_{jj} = 1$ for all j and $Q_{ji} = 0$ for all $i \neq j$. Then seller revenue in (20) holds with $\pi_{s,a}$ replaced by:

$$\pi_{s,a}^{no-sep} \equiv \int_{\Omega_n} \left[\sum_j R_j(\mathbf{t}) \phi_j^{no-sep}(\mathbf{t}) \right] f(\mathbf{t}) d\mathbf{t},$$

where the virtual valuation is

$$\phi_j^{no-sep}(\mathbf{t}) \equiv v_j(\mathbf{t}) - \frac{1 - F(t_j)}{f(t_j)} A_n - \tau. \quad (36)$$

Under a regularity condition that $\frac{\partial}{\partial t_j}\phi_j^{no-sep}(\mathbf{t}) \geq 0$, or $1 - \frac{d}{dt_j}\left(\frac{1-F(t_j)}{f(t_j)}\right) \geq 0$, a no-separation mechanism is optimal if it allocates both cash-flow rights and control to the bidder j with the highest signal (who also has the highest $\phi_j^{no-sep}(\mathbf{t})$), provided that $\phi_j^{no-sep}(\mathbf{t})$ is nonnegative. The regularity condition in the "no-separation" benchmark is guaranteed by Assumption 1, and Assumption 2 ensures that $\phi_j^{no-sep}(\mathbf{t}) \geq 0$ for all $\mathbf{t}$. Hence, the optimal mechanism allocates both cash-flow rights and control to the bidder with the highest signal (with no reserve prices), delivering an expected revenue of:

$$\pi_{opt}^{no-sep} = \int \left[\max\left\{\phi_1^{no-sep}(\mathbf{t}), ..., \phi_n^{no-sep}(\mathbf{t})\right\}\right] f(\mathbf{t})d\mathbf{t}.$$

The regularity condition (Definition 3) for our separation mechanism is more demanding. In the no-separation framework where ex-ante identical bidders have independent signals, the regularity condition depends only on the signal distribution F and is independent of the number of bidders. However, whether the regularity condition holds in our separation framework for a given F will depend on the number of bidders (see Section 3.3).

It is beyond the scope of our paper to solve for the optimal mechanism when regularity does not hold. However, importantly, we can design a separation mechanism that always generates higher revenues than the optimal no-separation mechanism, regardless of whether regularity holds. We use "h" and "s" to index the highest and second-highest bidders.

Proposition 3 *The mechanism below is incentive compatible and individually rational, yielding strictly higher expected revenues than the optimal no-separation mechanism:*

- *The highest bidder always receives cash-flow rights and is assigned control if* $\psi_{hh}(\mathbf{t}) \geq \psi_{sh}(\mathbf{t})$. *The second-highest bidder is assigned control if* $\psi_{hh}(\mathbf{t}) < \psi_{sh}(\mathbf{t})$.
- *Cash payments are given by part (ii) of Theorem 2.*

Proof of Proposition 3: The same logic as in the proof of Proposition 2 yields that for all ρ, the proposed allocations in Proposition 3 have the property that $J_i(t_i)$ is increasing in t_i (see equation 35).[12] Hence, by the same logic as in the proof of Theorem 2 (see equation

[12] Although Proposition 2 is stated only for large ρ, this part of its proof applies to all ρ.

34), the proposed mechanism in Proposition 3 is incentive compatible. The same logic as that which led to (32) shows that the mechanism is individually rational.

Expected revenue in the optimal no-separation mechanism is:

$$\pi_{opt}^{no-sep} = \int \left(\psi_{hh}(\mathbf{t}) - \tau\right) f(\mathbf{t}) d\mathbf{t}. \tag{37}$$

Expected revenue in the separation mechanism proposed in Proposition 3 is:

$$\pi^{sep} = \int \max\left\{\psi_{hh}(\mathbf{t}) - \tau, \psi_{sh}(\mathbf{t}) - \tau\right\} f(\mathbf{t}) d\mathbf{t}. \tag{38}$$

Because $\max\left\{\psi_{hh}(\mathbf{t}) - \tau, \psi_{sh}(\mathbf{t}) - \tau\right\} \geq \psi_{hh}(\mathbf{t}) - \tau$, we have $\pi^{sep} \geq \pi_{opt}^{no-sep}$. Moreover, there exists a strictly positive measure of signal realizations such that $\psi_{sh}(\mathbf{t}) > \psi_{hh}(\mathbf{t})$—to see this, note that $\psi_{hh} - \psi_{sh} = A_n (1-\rho) \left(t^h - t^s - L(t_h)\right)$. Because $L(\bar{t}) = 0$, there exists an $\epsilon > 0$ such that for all $t^h \in (\bar{t} - \epsilon, \bar{t})$, $t^h - L\left(t^h\right) > \underline{t}$. Then for all $t^s \in \left(t^h - L\left(t^h\right), t^h\right)$, $\psi_{sh}(\mathbf{t}) > \psi_{hh}(\mathbf{t})$. Thus, a strict inequality holds: $\pi^{sep} > \pi_{opt}^{no-sep}$. ■

This mechanism is the optimal separation mechanism when the common value component is sufficiently high, $\rho \geq \frac{1}{1-\delta_1}$, but it is not the optimal separation mechanism for $\rho < \frac{1}{1-\delta_1}$. Because the optimal separation mechanism can only generate higher revenues, we have:

Corollary 2 *The optimal separation mechanism generates strictly higher revenues than what would obtain in any no-separation mechanism.*

3.2 Cash payments and implementation of separation mechanism

We next characterize cash payments (equation (30)) in the optimal separation mechanism when the design is regular. The first term on the right-hand side of (30), $\sum_j R_j(\mathbf{t}) Q_{ji}(\mathbf{t}) v_j(\mathbf{t})$, is zero for bidders who do not receive cash-flow rights, and it is the ex-post value of the cash flows for the bidder who receives cash flows. The third term is $-\tau$ for the bidder who is assigned control (for whom $R_i = 1$), i.e., the seller pays the bidder his opportunity cost; and it is zero for all other bidders. Thus, these two terms represent "fair" payments/compensations that yield zero rents for bidders: a bidder who receives cash flows pays for the actual value of the cash flows, and a bidder who runs the project is exactly compensated for his costs.

The second term, with its minus sign, is non-positive. This term is what gives a bidder possible rents. We will show that this term is zero for bidders who receive neither cash-flow rights nor control, implying that they do not receive rents. In contrast, this term is strictly negative (with probability one) for a bidder who receives cash flow rights, implying that this bidder receives rents.

We first establish properties of our mechanism concerning ex-post bidder payment, bidder profit and seller revenue.

Proposition 4 *Suppose the design problem is regular. Then in the optimal mechanism as specified by parts (i) and (ii) of Theorem 2,*

(i) A bidder who does not receive cash flow rights or control makes zero payment.

(ii) The mechanism is ex-post individually rational for all bidders: given any vector of bidder types $\mathbf{t}$*, the profit of each bidder is nonnegative.*

(iii) Given any vector of bidder types $\mathbf{t}$*, the total cash payment to the seller is positive.*

Proof of Proposition 4: For notational ease, for any given $\mathbf{t}$, we denote the second term of the right-hand side of (30) without its minus sign by $B_i(\mathbf{t})$:

$$B_i(\mathbf{t}) \equiv \int_{\underline{t}}^{t_i} \Big(\sum_j R_j \left(\widetilde{t}; \mathbf{t}_{-i}\right) Q_{ji}(\widetilde{t}; \mathbf{t}_{-i}) W_{ji} \Big) d\widetilde{t}.$$

Claim 1: If bidder i receives neither cash flow rights nor control, then $B_i(\mathbf{t}) = 0$. We first note that t_i cannot be the highest signal, because if it were the highest signal, then by Proposition 1, i would have received at least one of the two rights, contradicting the premise that bidder i does not receives cash flow rights or control. There are two possible cases.

Case 1: t_i is not one of the two highest signals. Then when i's signal is replaced by $\widetilde{t} < t_i$, bidder i remains outside of the two highest bidders. Thus, by Proposition 1, bidder i receives no cash flows, so $\sum_j R_j \left(\widetilde{t}; \mathbf{t}_{-i}\right) Q_{ji}(\widetilde{t}; \mathbf{t}_{-i}) = 0$. Thus, $B_i \left(\mathbf{t}\right) = 0$.

Case 2: t_i is the second-highest signal. Then it must be that the highest signal bidder, denoted by $j \neq i$, receives both control and cash flow rights. By part (ii) of Proposition 1, when i's signal is replaced by $\widetilde{t} < t_i$, it remains optimal for j to receive both rights, and hence for i to receive no rights. Thus, $\sum_j R_j \left(\widetilde{t}; \mathbf{t}_{-i}\right) Q_{ji}(\widetilde{t}; t_{-i}) = 0$ and hence again $B_i \left(\mathbf{t}\right) = 0$.

Part (i) of Proposition 4 follows from Claim 1 and the properties that the first and third terms on the right-hand side of (30) are zero for a bidder who receives neither right.

To establish part (ii), note that for a given $\mathbf{t}$, bidder i's ex post profit $\pi_i(\mathbf{t})$ is the ex post value of cash flows assigned to him, minus his cost of running the project, minus his payment. The ex post value of cash flows assigned to bidder i is $\sum_j R_j(\mathbf{t})Q_{ji}(\mathbf{t})v_j(\mathbf{t})$, which precisely cancels the first term in his payment in (30); his (expected) cost of running the project is $\tau R_i(\mathbf{t})$, which cancels the third term in his payment. Thus, bidder i's net profit is simply the second term, $B_i(\mathbf{t})$, which is nonnegative for any $\mathbf{t}$.

To prove (iii), note that by Proposition 1 and part (i), all bidders other than the two highest make zero payments. Thus, a seller's revenue can only come from the highest two bidders. Without loss of generality, let bidder 1 have the highest signal and let bidder 2 have the second highest. By Proposition 1, the optimal assignment of rights takes one of three forms.

Case 1: Bidder 1 gets both cash flow rights and control. Then bidder 2 pays zero (by part (i) of this proposition) and the seller's ex post revenue is simply $M_1(\mathbf{t})$, which, by (30), equals

$$M_1(\mathbf{t}) = v_1(\mathbf{t}) - \int_{\underline{t}}^{t_1} \Big(\sum_j R_j(\widetilde{t};\mathbf{t}_{-1})Q_{j1}(\widetilde{t};\mathbf{t}_{-1})W_{j1}\Big)d\widetilde{t} - \tau, \tag{39}$$

Because $W_{j1} \leq A_n$ for all j, by (1) and (2), we have

$$\sum_j R_j(\widetilde{t};\mathbf{t}_{-1})Q_{j1}(\widetilde{t};\mathbf{t}_{-1})W_{j1} \leq A_n \sum_j R_j(\widetilde{t};\mathbf{t}_{-1})Q_{j1}(\widetilde{t};\mathbf{t}_{-1}) \leq A_n. \tag{40}$$

Thus (39) yields

$$\begin{aligned} M_1(\mathbf{t}) \ \geq \ v_1(\mathbf{t}) - (t_1 - \underline{t})\, A_n - \tau \ &= \ A_n\underline{t} + \rho A_n \Sigma_{i\neq 1} t_i - \tau \\ &\geq \ A_n\underline{t} + \rho A_n \Sigma_{i\neq 1} \underline{t} - \tau \\ &= \ \underline{t} - \tau. \end{aligned}$$

By Assumption 2 and $L(\underline{t}) > 0$, we have $\underline{t} > \tau$. Thus, seller revenue, $M_1(\mathbf{t})$, is positive.

Case 2: Bidder 1 receives control and bidder 2 receives cash flows rights. Then

$$M_1(\mathbf{t}) = \int_{\underline{t}}^{t_1} \Big(\sum_j R_j(\widetilde{t};\mathbf{t}_{-1})Q_{j1}(\widetilde{t};\mathbf{t}_{-1})W_{j1}\Big)d\widetilde{t} - \tau \ \geq \ (t_1 - \underline{t})\, A_n - \tau,$$

where the inequality follows from (40). We also have

$$M_2(\mathbf{t}) = v_1(\mathbf{t}) - \int_{\underline{t}}^{t_2} \Big(\sum_j R_j(\widetilde{t};\mathbf{t}_{-2})Q_{j2}(\widetilde{t};\mathbf{t}_{-2})W_{j2}\Big)d\widetilde{t} \;\geq\; v_1(\mathbf{t}) - (t_2 - \underline{t})\, A_n\rho.$$

Note that when bidder 2's signal is replaced by $\widetilde{t} \leq t_2$, we have $t_1 > \widetilde{t}$. Therefore, bidder 2 cannot receive both control and cash-flow rights. Hence, if bidder 2 receives any cash flows, the project must be run by a different bidder $j \neq 2$, so $W_{j2} = A_n\rho$. Thus,

$$\sum_j R_j(\widetilde{t};\mathbf{t}_{-2})Q_{j2}(\widetilde{t};\mathbf{t}_{-2})W_{j2} = A_n\rho \sum_j R_j(\widetilde{t};\mathbf{t}_{-2})Q_{j2}(\widetilde{t};\mathbf{t}_{-2}) \leq A_n\rho.$$

Hence,

$$M_2(\mathbf{t}) \geq v_1(\mathbf{t}) - (t_2 - \underline{t})\, A_n\rho.$$

The seller's revenue is the sum of bidders 1 and 2's payments, and

$$\begin{aligned} M_1(\mathbf{t}) + M_2(\mathbf{t}) \;\geq\; (t_1 - \underline{t})\, A_n - \tau + v_1(\mathbf{t}) - (t_2 - \underline{t})\, A_n\rho \;&=\; A_n\underline{t} + \rho A_n \Sigma_{i\neq 1} t_i - \tau \\ &\geq\; A_n\underline{t} + \rho A_n \Sigma_{i\neq 1}\underline{t} - \tau \\ &=\; \underline{t} - \tau \;>\; 0. \end{aligned}$$

Case 3: Bidder 1 receives cash-flow rights and bidder 2 receives control. Then

$$M_1(\mathbf{t}) = v_2(\mathbf{t}) - \int_{\underline{t}}^{t_1} \Big(\sum_j R_j(\widetilde{t};\mathbf{t}_{-1})Q_{j1}(\widetilde{t};\mathbf{t}_{-1})W_{j1}\Big)d\widetilde{t}. \tag{41}$$

The two rights are split, so Proposition 1 implies that $t_2 < K(t_1)$. When bidder 1's signal is replaced by $\widetilde{t} \leq t_1$ but t_2 remains unchanged, by Proposition 1 and the property that $K(\cdot)$ is weakly increasing, bidder 1 cannot receive both control and cash-flow rights. This means that if bidder 1 receives any cash flows, the project must be run by a different bidder $j \neq 1$, and hence $W_{j1} = A_n\rho$. Therefore,

$$\sum_j R_j(\widetilde{t};\mathbf{t}_{-1})Q_{j1}(\widetilde{t};\mathbf{t}_{-1})W_{j1} = A_n\rho \sum_j R_j(\widetilde{t};\mathbf{t}_{-1})Q_{j1}(\widetilde{t};\mathbf{t}_{-1}) \leq A_n\rho.$$

Hence

$$M_1(\mathbf{t}) \geq v_2(\mathbf{t}) - (t_1 - \underline{t})\, A_n\rho.$$

We also have:

$$M_2(\mathbf{t}) = -\int_{\underline{t}}^{t_2} \Big(\sum_j R_j(\widetilde{t};\mathbf{t}_{-2})Q_{j2}(\widetilde{t};\mathbf{t}_{-2})W_{j2}\Big)d\widetilde{t} - \tau \tag{42}$$
$$\geq -\left(t_2 - \underline{t}\right)A_n - \tau,$$

where the inequality follows from (40) upon replacing the index "1" by "2". The seller's revenue is the sum of bidders 1 and 2's payments:

$$\begin{aligned} M_1(\mathbf{t}) + M_2(\mathbf{t}) \geq v_2(\mathbf{t}) - \left(t_1 - \underline{t}\right)A_n\rho - \left(t_2 - \underline{t}\right)A_n - \tau &= A_n\underline{t} + \rho A_n\underline{t} + \rho A_n\Sigma_{i\neq 1,2}t_i - \tau \\ &\geq A_n\underline{t} + \rho A_n\Sigma_{i\neq 2}\underline{t} - \tau \\ &= \underline{t} - \tau > 0. \qquad \blacksquare \end{aligned}$$

We now focus on the mechanism detailed in Proposition 3, which generates higher revenues than no-separation mechanisms for all ρ and is the optimal mechanism for $\rho \geq \frac{1}{1-\delta_1}$. Its allocations take one of two forms: either the highest bidder receives cash flow rights and the second-highest bidder receives control, or the highest bidder gets both rights. Without loss of generality, let bidder 1 have the highest signal and bidder 2 have the second highest.

Lemma 2 *For the mechanism specified in Proposition 3,*

(i) If $t_2 > t_1 - L(t_1)$, then bidder 1 receives cash-flow rights and bidder 2 receives control. Bidder 1's cash payment is

$$M_1 = v_2(t_2, t_2, t_3...t_n) = A_n t_2 + A_n\rho\Sigma_{j\neq 1}t_j, \tag{43}$$

where $v_2(t_2, t_2, t_3...t_n)$ is the value of the project when it is controlled by bidder 2 and bidder 1's signal is t_2 (i.e., $t_1 = t_2$). Bidder 2's cash payment is $M_2 = -\tau$ and no other bidders pay.

(ii) If $t_2 \leq t_1 - L(t_1)$, then bidder 1 receives both cash flow rights and control, and his cash payment is

$$\begin{aligned} M_1 &= v_2(t_2, t_2, t_3...t_n) + m_{extra} - \tau \\ &= A_n t_2 + A_n\rho\Sigma_{j\neq 1}t_j + m_{extra} - \tau, \end{aligned} \tag{44}$$

where

$$m_{extra} \equiv A_n(1-\rho)\left(\hat{t}(t_2) - t_2\right) \tag{45}$$

is an extra payment that a bidder who receives both rights pays beyond the amount paid by a bidder who receives only cash-flow rights, and $\hat{t}\left(t_2\right) \in [t_1, t_2]$ *solves*

$$\hat{t} - L\left(\hat{t}\right) = t_2. \tag{46}$$

No other bidders pay.

Proof of Lemma 2: We first prove part (i). Given the premise that bidder 1 receives cash-flow rights and bidder 2 is assigned control, bidder 1's payment is given by (41). When bidder 1's signal is replaced by $\widetilde{t} \leq t_1$ and t_2 is unchanged, Proposition 1 implies that bidder 1 receives cash flows and bidder 2 gets control for all $\widetilde{t} \in (t_2, t_1]$, and that bidder 1 receives no cash flows for all $\widetilde{t} < t_2$. Thus, the second term on the right-hand side of (41) reduces to

$$\int_{\underline{t}}^{t_1} \Big(\sum_j R_j(\widetilde{t}; \mathbf{t}_{-1}) Q_{j1}(\widetilde{t}; \mathbf{t}_{-1}) W_{j1}\Big) d\widetilde{t} = \left(t_1 - t_2\right) A_n \rho. \tag{47}$$

Hence bidder 1's payment, (41), becomes

$$M_1(\mathbf{t}) = v_2(\mathbf{t}) - \left(t_1 - t_2\right) A_n \rho,$$

which gives (43). Bidder 2's payment is given by (42). When bidder 1's signal is replaced by $\widetilde{t} \leq t_2$ but t_1 remains unchanged, Proposition 1 implies that bidder 1 receives no cash flows (since $\widetilde{t} < t_1$), so the first term on the right-hand side of (42), $\int_{\underline{t}}^{t_2} \Big(\sum_j R_j(\widetilde{t}; \mathbf{t}_{-2}) Q_{j2}(\widetilde{t}; \mathbf{t}_{-2}) W_{j2}\Big) d\widetilde{t}$, is zero. Thus, bidder 2's payment, (41), becomes $M_2 = -\tau$.

We now prove part (ii). Given the premise that bidder 1 receives both rights, his payment is given by (39). The condition $t_2 \leq t_1 - L\left(t_1\right)$ is equivalent to $t_1 \geq \hat{t}\left(t_2\right)$, where $\hat{t}\left(\cdot\right)$ is defined in (46). Rewrite the second term on the right-hand side of (39) as the sum of two integrals:

$$\begin{aligned} \int_{\underline{t}}^{t_1} \Big(\sum_j R_j(\widetilde{t}; \mathbf{t}_{-1}) Q_{j1}(\widetilde{t}; \mathbf{t}_{-1}) W_{j1}\Big) d\widetilde{t} \;&=\; \int_{\hat{t}(t_2)}^{t_1} \Big(\sum_j R_j(\widetilde{t}; \mathbf{t}_{-1}) Q_{j1}(\widetilde{t}; \mathbf{t}_{-1}) W_{j1}\Big) d\widetilde{t} \\ &\quad + \int_{\underline{t}}^{\hat{t}(t_2)} \Big(\sum_j R_j(\widetilde{t}; \mathbf{t}_{-1}) Q_{j1}(\widetilde{t}; \mathbf{t}_{-1}) W_{j1}\Big) d\widetilde{t}. \end{aligned} \tag{48}$$

When bidder 1's signal is replaced by $\widetilde{t} \in [\hat{t}(t_2), t_1]$ but t_2 remains unchanged, Proposition 1 implies that bidder 1 receives both cash flow rights and control. Thus, the first term on the right-hand side of (48) becomes

$$\int_{\hat{t}(t_2)}^{t_1} \Big(\sum_j R_j(\widetilde{t}; \mathbf{t}_{-1}) Q_{j1}(\widetilde{t}; \mathbf{t}_{-1}) W_{j1}\Big) d\widetilde{t} = A_n(t_1 - \hat{t}(t_2)).$$

The second term on the right-hand side of (48) is

$$\int_{\underline{t}}^{\hat{t}(t_2)} \Big(\sum_j R_j(\widetilde{t};\mathbf{t}_{-1})Q_{j1}(\widetilde{t};\mathbf{t}_{-1})W_{j1}\Big)d\widetilde{t} = A_n\rho\left(\hat{t}\left(t_2\right)-t_2\right),$$

which follows from (47) by replacing t_1 there with $\hat{t}\left(t_2\right)$.

Plugging the above into (48) yields

$$\int_{\underline{t}}^{t_1} \Big(\sum_j R_j(\widetilde{t};\mathbf{t}_{-1})Q_{j1}(\widetilde{t};\mathbf{t}_{-1})W_{j1}\Big)d\widetilde{t} = A_n\left(t_1-\hat{t}\left(t_2\right)\right)+A_n\rho\left(\hat{t}\left(t_2\right)-t_2\right),$$

which, upon substituting into (39), yields bidder 1's payment:

$$\begin{aligned} M_1(\mathbf{t}) &= v_1(\mathbf{t}) - A_n\left(t_1-\hat{t}\left(t_2\right)\right) - A_n\rho\left(\hat{t}\left(t_2\right)-t_2\right) - \tau \\ &= v_1(\mathbf{t}) - A_n\left(t_1-t_2\right) + A_n\left(1-\rho\right)\left(\hat{t}\left(t_2\right)-t_2\right) - \tau, \end{aligned}$$

which gives (44). This completes the proof. ■

The seller's revenue is the sum of all bidders' payments, which equals $v_2\left(t_2,t_2,t_3...t_n\right)-\tau$ when the two rights are split between the two highest bidders, and $v_2\left(t_2,t_2,t_3...t_n\right)-\tau+m_{extra}$ when the highest bidder receives both rights, where m_{extra} is given in (45)). Note that $v_2\left(t_2,t_2,t_3...t_n\right)-\tau$ is precisely a seller's revenue in the symmetric equilibrium of standard no-separation English auction.[13] Hence, our separation mechanism generates the same revenue as the no-separation mechanism when the two highest signals are sufficiently close, and it generates strictly higher revenue when the highest signal sufficiently exceeds the second highest.

Note that, counterintuitively, revenues in the separation mechanism strictly exceed those in the no-separation mechanism only following signal realizations where the two rights are not separated. We discuss this further when examining the term m_{extra} in (45).

Proposition 5 *The mechanism specified in Proposition 3 is ex-post incentive compatible: for any signal realization (type profile)* $\mathbf{t}$*, each bidder is weakly better off reporting his true type if the other bidders report their true types.*

Proof of Proposition 5: To prove the result, consider a generic bidder i who has signal t_i but reports t_i', and assume that all other bidders report their true types. We show bidder i is weakly better off reporting his true type, for any realization of the other signals $\mathbf{t}_{-i}$.

[13]See equation (6.5) in Krishna (2003), which has no τ term, as the project is assumed to be costless to run.

Without loss of generality denote the highest and second-highest signals among the $n-1$ bidders *other than* i by "h" and "s". In what follows we prove Proposition 5 only for the case where $t_h < t_i$; the proof for $t_h \geq t_i$ is similar (and its logic mirrors that in the proof for the ex-post incentive compatibility of the mechanism in Example 1) and hence omitted.

As in the proof of Lemma 2, we rewrite the condition $t_h \leq t_i' - L(t_i')$, which is necessary and sufficient for bidder i to receive both rights, as $t_i' \geq \hat{t}(t_h)$, where $\hat{t}$ is defined in (46). We first establish some useful results in two scenarios which will aid our proof:

Scenario 1: Bidder i receives both rights. This occurs if $t_i' \geq \hat{t}(t_h)$. Then the value of the cash flows he receives is $A_n t_i + A_n \rho \Sigma_{j \neq i} t_j$, and his payment is

$$A_n t_h + A_n \rho \Sigma_{j \neq i} t_j + A_n (1-\rho)\left(\hat{t}(t_h) - t_h\right) - \tau,$$

which is given by (44) when we replace "1" with "i" and "2" with "h", because here bidder i bids highest and bidder h bids second-highest. Bidder i's profit is the value of the cash flows minus his payment and minus his cost of running the project, which yields

$$A_n (t_i - t_h) - A_n (1-\rho)(\hat{t}(t_h) - t_h). \tag{49}$$

Scenario 2: Bidder i only receives cash flow rights. This implies $t_i' \in [t_h, \hat{t}(t_h))$. Then the value of the cash flows that i receives is $A_n t_h + A_n \rho \Sigma_{j \neq h} t_j$, and his payment is $A_n t_h + A_n \rho t_h + A_n \rho \Sigma_{j \neq i,h} t_j$. The difference is bidder i's profit, which is

$$A_n \rho (t_i - t_h). \tag{50}$$

We now show bidder i does not profit from deviation for any realization of other bidders' signals. Given that $t_h < t_i$, there are two possible cases.

Case 1: $t_i \in (t_h, \hat{t}(t_h))$. Then, in equilibrium, bidder i receives cash flow rights and bidder h receives control. In equilibrium (i reports truthfully), i's profit is $A_n \rho (t_i - t_h) > 0$. We now examine i's deviation profit from bidding t_i'. There are four subcases. (i) If $t_i' > \hat{t}(t_h)$, then by deviating, i receives both control and cash flow rights. By (49), i's deviation profit is $A_n (t_i - t_h) - A_n (1-\rho)(\hat{t}(t_h) - t_h)$. Subtracting i's equilibrium profit from his deviation profit reveals that deviation is unprofitable:

$$A_n(t_i - t_h - (1-\rho)(\hat{t}(t_h) - t_h)) - A_n \rho (t_i - t_h) = A_n (1-\rho)(t_i - \hat{t}(t_h)) < 0. \tag{51}$$

The inequality follows from our premise that $t_i < \hat{t}\,(t_h)$.

(ii) If $t_i' \in (t_h, \hat{t}(t_h)]$, then deviation leads to the same profit as bidding truthfully.

(iii) If $t_i' < t_h$, bidder i receives no cash flow rights so the deviation profit is zero.

(iv) If $t_i' = t_h$, we can similarly show that deviation is not profitable, no matter how the tie is resolved.

Case 2: $t_i \geq \hat{t}(t_h)$. Then bidder i receives both cash flow rights and control. Bidder i's equilibrium profit is

$$A_n\,(t_i - t_h) - A_n(1-\rho)(\hat{t}(t_h) - t_h) > 0.$$

We next examine i's deviation profit from bidding t_i' in four subcases.

(i) If $t_i' > \hat{t}(t_h)$, then deviation leads to same profit as bidding truthfully.

(ii) If $t_i' \in (t_h, \hat{t}(t_h))$, then bidder i upon deviating receives cash flow rights and bidder h receives control. Thus, i's deviation profit is $A_n\rho\,(t_i - t_h)$. The difference between this and i's equilibrium profit reveals no gain from deviation:

$$A_n\,(1-\rho)\left(\hat{t}(t_h) - t_i\right) \;<\; 0, \tag{52}$$

where the inequality follows from the premise that $t_i > \hat{t}\,(t_h)$.

(iii) If $t_i' < t_h$, then i receives no cash flow rights so his deviation profit is zero.

(iv) If $t_i' = t_h$, we can similarly show that deviation is not profitable, regardless of how the tie is resolved. ∎

The ex-post incentive compatibility established in Proposition 5 is stronger than (5), which only requires a bidder to be better off bidding truthfully in an *expected* sense after taking expectations over signal realizations of other bidders. That is, although we impose the weaker interim incentive compatibility condition, (5), the resulting mechanism has the stronger property of being ex-post incentive compatible, and, as established earlier, it always delivers (weakly and sometimes strictly) higher revenue than the optimal no-separation mechanism event by event, and it achieves maximal seller revenues whenever $\rho \geq \frac{1}{1-\delta_1}$.

This ex-post incentive compatibility reflects the feature of our mechanism that the highest bidder's payment only depends on his reported type in a coarse way: whether his reported type is sufficiently higher than that of the second-highest bidder (see (43) and (44)). Moreover, as the proof of Proposition 5 shows, when the highest bidder's signal sufficiently exceeds the second-highest signal ($t_i \geq \hat{t}(t_h)$), the highest bidder would be worse off if he under-reported his type (to be $t_i' < \hat{t}(t_h)$; see (52)); whereas when the two highest signals are closer, ($t_i \in (t_h, \hat{t}(t_h))$), the highest bidder would be worse off if he over-reported his type (see (51)).

The extra payment term m_{extra} in (45) is crucial for ensuring the incentive compatibility and optimality of our mechanism. To see this, consider the special case of $t_i = \hat{t}\,(t_h)$. In this case, a seller is indifferent between bidder i reporting a high type $t_i' \geq \hat{t}(t_h)$ in which case i receives both cash flow rights and control, and reporting a slightly lower type $t_i' \in (t_h, \hat{t}\,(t_h))$, in which case i receives cash flow rights but not control—both outcomes maximize $\pi_{s,a}$. Interestingly, when $t_i = \hat{t}(t_h)$, the value of m_{extra} also leaves bidder i indifferent between reporting a high type of $t_i' > \hat{t}(t_h)$ and reporting a slightly lower type of $t_i' \in (t_h, \hat{t}(t_h))$. The value of the cash flows awarded to i when he over-reports his type to be $t_i' > \hat{t}(t_h)$ exceeds the value of cash flows when i under-reports his type to be $t_i' \in (t_h, \hat{t}\,(t_h))$ by $A_n\,(1-\rho)\,(t_i - t_h)$. This reflects that i runs the project when he reports $t_i' > \hat{t}(t_h)$ but the second-highest bidder h—whose signal is lower than i's—runs the project when i reports $t_i' \in (t_h, \hat{t}\,(t_h))$. However, when i over-reports his type, he incurs the extra payment m_{extra} (vis-à-vis what i would pay by slightly under-reporting). When $t_i' = \hat{t}(t_h)$, the gains equal the costs, resulting in indifference.[14]

When $t_i > \hat{t}(t_h)$, similar algebra shows that bidder i would be strictly worse off if he under-reports (because the value of the cash flows when i runs the project exceeds their value when the second-highest bidder runs the project by more)—and so would be the seller; and when $t_i \in (t_h, \hat{t}(t_h))$, bidder i would be strictly worse off if he over-reports (because the value of the cash flows when he runs the project exceeds their value when the second-highest bidder runs the project by less)—and so would be the seller. Both results reflect the extra payment m_{extra} given in (45), which induces the highest bidder to report in an optimal way for the seller.

This logic suggests the mechanism detailed in Proposition 3 can be implemented in a two-stage auction:

[14]To see why they are equal, in (45) make the notational adjustment that bidder h has the second-highest signal, and then use the premise that $t_i = \hat{t}(t_h)$. An added inconsequential difference is that when i over-reports, he also receives control and hence incurs cost τ; but i's payment is also reduced by τ.

Definition 4 *(two-stage auction) The first stage is a standard English auction. In the second stage, the seller offers the first-stage winner a choice: the winner chooses whether to receive cash-flow rights but not control or to receive both rights. If the winner only chooses cash flows then he pays the seller the price at which the highest losing bidder exited and control is assigned to the highest losing bidder who receives a payment of τ to compensate for running the project. If, instead, the winner chooses to retain control, he pays the seller the highest losing bidder's exit price plus an extra payment of m_{extra} (equation (45)) minus τ.*

Proposition 6 *The mechanism specified in Proposition 3 is implemented by the symmetric equilibrium in the two-stage auction in Definition 4. The bidding strategies are as follows:*

(i) In the first-stage, if bidders $k+1, k+2, ..., N$ have dropped out, thereby revealing their signals t_{k+1}, t_{k+1}, ..., t_n to the remaining k active bidders, then the strategy of a remaining bidder i with signal t_i is to drop out at a price that equals

$$v_i \left(t_1 = t_i, t_2 = t_i, ...t_i = t_i ...t_{k=t_i}, t_{k+1}, t_{k+1}, ..., t_n\right). \tag{53}$$

(ii) In the second stage, the winner of the first stage (wlog bidder 1) accepts both cash-flow rights and control if and only if $t_1 \geq \hat{t}\left(t_2\right)$, where t_2 is the signal of the highest losing bidder.

Proof of Proposition 6: Claim 1: In the second stage it is optimal for the winning bidder 1 of the first stage to follow the strategy given in part (ii) of Proposition 6, regardless of whether bidder 1 followed his equilibrium strategy in the first stage. The proof of Claim 1 mirrors that for Proposition 5. The difference in the winner's profit from receiving both rights versus just receiving cash flow rights is

$$(1-\rho)\left(t_1 - t_2\right) - m_{extra},$$

where t_2 is the highest losing bidder's signal, and $(1-\rho)\left(t_1 - t_2\right)$ is the difference in the value of the cash flows. This difference in profits is positive if and only if $t_1 \geq \hat{t}\left(t_2\right)$. This establishes Claim 1.

To show bidding according to (53) constitutes an equilibrium in the first stage, consider a generic bidder 1 (not necessarily the auction winner) with signal t_1 and suppose all other bidders follow the equilibrium strategy. There are two relevant cases.

Case 1: Bidder 1 follows his equilibrium strategy and wins the first stage. Then $t_1 > t_i$ for all $i \neq 1$ and bidder 1 earns a positive profit from following his equilibrium strategy in the second stage. Now suppose bidder 1 deviates in the first stage. There are two sub-cases.

(i) Bidder 1 still wins the first stage. By Claim 1 and noting that the winning price in the first stage is determined by the signals of the losing bidders, it follows that his profit is unaffected by his deviation.

(ii) Bidder 1 loses the first stage. Then his profit is zero, so deviation is unprofitable.

Case 2: Bidder 1 follows his equilibrium strategy and loses the first stage, so his equilibrium profit is zero. Denoting the winner by bidder 2, we have $t_1 < t_2$. If bidder 1 deviates in the first stage and still loses, his profit remains zero. If he wins when he deviates, then by Claim 1 and $t_1 < t_2$, in the second stage his best response is to receive only the cash flow rights. The value of the cash flow to him is $v_2(t_1, t_2, t_3...t_n)$, and his payment is $v_2(t_2, t_2, t_3...t_n)$. By $t_1 < t_2$, his profit is strictly negative.

It is routine to show that the allocations and payments in the equilibrium of this two-stage auction correspond to parts (i) and (ii) of Theorem 2. Using the same arguments as in Krishna (2003), one can show this is the only symmetric equilibrium in the two-stage auction. ■

The bidding strategy in the first-stage, (53), is the same as in the symmetric equilibrium strategy in a standard no-separation English auction, and is given by (6.5) in Krishna (2003) for $\tau = 0$. When $t_1 \in \left(t_2, \hat{t}(t_2)\right)$, the highest signal is close enough to the second-highest that the winner chooses to receive only the cash-flow rights, and net seller revenue is the highest losing bidder's bid minus τ, which equals seller revenue in a no-separation English auction (allowing for $\tau \neq 0$). When $t_1 \geq \hat{t}(t_2)$, the highest signal exceeds the second-highest by enough that the winner chooses to receive both cash-flow rights and control, and net seller revenue exceeds revenue in a no-separation English auction by m_{extra}.[15]

Thus, the two-stage auction can implement the mechanisms in Proposition 3 for all ρ

[15] Proposition 6 suggests an alternative way to understand m_{extra}. Consider a *generalized* class of two-stage auctions of Definition 4, where the extra payment in the second stage is $x(\cdot)$, which can be an arbitrary function of the losing bidders' signals (revealed by their exit prices), rather than hardwired to be m_{extra}, and the highest bidder chooses either to receive both rights and make this extra payment, or receive cash flow rights only without having to pay the extra amount. One can show (proof omitted) that the form of $x(\cdot)$ that maximizes seller revenue within this class of two-stage auctions is precisely m_{extra} as given in (45).

(even $\rho < \frac{1}{1-\delta_1}$), it is ex-post incentive compatible,[16] and it generates (weakly and sometimes strictly) higher revenue, event by event, than the no-separation English equilibrium.

3.3 Uniform Distribution

Proposition 2 shows that the regularity condition holds when ρ is sufficiently large, and hence Theorem 2 identifies the optimal mechanism. We next examine the regularity condition for all $\rho \in (0,1)$ when signals are drawn from a uniform distribution on $[\bar{t}-1, \bar{t}]$.

With a uniform distribution, $L(t) = \bar{t} - t$, $\frac{d}{dt}L(t) = -1$, and $\delta_1 = \delta_2 = -1$. Thus, by Proposition 2, the regularity condition holds for all n when $\rho \geq 0.5$. Now consider $\rho < 0.5$. For simplicity we use a change of variables, defining $x_i \equiv t_i - \bar{t} + 1$ for all i, so that x_i is uniformly distributed over $[0,1]$. Let h and s denote the highest and second highest bidders. From Corollary 1, the allocations that maximize $\pi_{s,a}$ are for h to receive control and cash-flow rights if $x_h \geq 1 - \rho(1 - x_s)$ and for h to receive control and s to receive cash flows if $x_h < 1 - \rho(1 - x_s)$. To see when regularity holds, consider a generic bidder 1 and compute $J_1(x_1)$:

Case 1: $x_1 \leq 1 - \rho$. Then for bidder 1 to receive cash flows, x_1 must be the second-highest signal and not too far below the highest signal. This means that the highest signal is in the interval $[x_1, (1 - \rho(1 - x_1))]$, which has length $(1 - x_1)(1 - \rho)$, and the other $n-2$ signals are less than x_1. This occurs with probability $(n-1)(1-x_1)(1-\rho)x_1^{n-2}$, where the factor $(n-1)$ reflects that there are $n-1$ ways to single out the one bidder who is in the interval $[(1 - \rho(1 - x_1)), x_1])$ from the $n-1$ bidders. Thus,

$$J_1(x_1) = (n-1)\rho(1-x_1)(1-\rho)x_1^{n-2} \text{ for } x_1 \leq 1-\rho, \tag{54}$$

where the factor ρ reflects $W_{ji} = \rho$ for $j \neq i$.

Case 2: $x_1 > 1 - \rho$. Then bidder 1 receives cash-flow rights in two scenarios. First, x_1 may be the second-highest, and the contribution to $J_1(x_1)$ in this scenario is given by the right-hand side of (54). Second, if x_1 is highest, then for bidder 1 to receive cash flows, it must be that $x_1 > 1 - \rho$ and the signals of the $n-1$ other bidders are below

[16] The proof of Proposition 6 provides an alternative way to establish the ex-post incentive compatibility stated in Proposition 5.

$1-\frac{1-x_1}{\rho}$. This occurs with probability $\left(1-\frac{1-x_1}{\rho}\right)^{n-1}$. The contribution to $J_1(x_1)$ in this scenario is $\left(1-\frac{1-x_1}{\rho}\right)^{n-1}$. Adding these two contributions yields

$$J_1(x_1)=(n-1)\rho(1-x_1)(1-\rho)x_1^{n-2}+\left(1-\frac{1-x_1}{\rho}\right)^{n-1},\ \text{for } x_1>1-\rho. \quad (55)$$

We now examine the regularity condition for $\rho\in(0,0.5)$. The right-hand side of (54) increases in x_1 for $x_1\in\left[0,\frac{n-2}{n-1}\right]$, but strictly decreases in x_1 for $x_1\in\left(\frac{n-2}{n-1},1\right)$. Because (54) holds for $x_1\leq 1-\rho$, regularity is violated when $\frac{n-2}{n-1}<1-\rho$ or equivalently when $\rho<\frac{1}{n-1}$. For $n=2$ or 3, $\frac{1}{n-1}\geq 0.5$, so regularity is violated for all $\rho\in(0,0.5)$.

When $n\geq 4$ and $\rho\geq\frac{1}{n-1}$, $J_1(x_1)$ increases in x_1 for $x_1\leq 1-\rho$, by (54). However, regularity requires $J_1(x_1)$ to increase in x_1 for all $x_1\in[0,1]$, so we also need to see if the right-hand side of (55) increases in x_1 for $x_1>1-\rho$. When $n=4$, the right-hand side of (55) is a third degree polynomial, so its derivative is a quadratic. Algebra yields that the regularity condition holds for $\rho\in[\mu_4,0.5)$ but not for $\rho\in(0,\mu_4)$, where $\mu_4=$0.338. One can prove that regularity holds for all $n>4$, when ρ is not too far below 0.5. For $n=5$, 6, 7, 8, numerical calculations show that the regularity condition holds for $\rho\in[\mu_n,0.5)$, but not for $\rho\in(0,\mu_n)$, where $\mu_5=$0.269, $\mu_6=$0.232, $\mu_7=$0.210, and $\mu_8=$0.195.

Regularity is harder to satisfy when ρ is smaller. Regularity requires a bidder's expected cash flow allocation (weighted by W) weakly increase in his signal, and when $\rho\geq 0.5$, the "$\pi_{s,a}$–maximizing" allocations always assign cash flows to the highest bidder, making it easier to satisfy regularity. By contrast when ρ is lower, the "$\pi_{s,a}$–maximizing" allocations sometimes assign cash flows to the second-highest bidder, rather than the highest, making it harder to satisfy regularity, especially when n is small. Concretely, when $n=2$, the second-highest bidder is the lowest bidder and assigning cash flow to lowest bidder goes strongly against regularity. In contrast, when n is large, the second-highest bidder tends to have a relatively high value, making it easier to satisfy regularity.

3.4 Controller needs minimum cash-flow rights

In some settings, a controller may need to receive some minimum fraction of cash flows, as in Ekmekci, Kos, and Vohra (2016). Our findings extend to such settings. Letting $q_{\min}\in(0,1)$

denote this minimum fraction, the feasibility condition (2) for cash flow allocations becomes

$$Q_{jj}(\mathbf{t}) \geq q_{\min} \text{ and } \sum_i Q_{ji}(\mathbf{t}) \leq 1 \text{ for all } j. \tag{56}$$

Our earlier analysis corresponds to $q_{\min} = 0$, and can be extended to this setting using similar approaches. One can show that allocations that maximize $\pi_{s,a}$ fall into one of three possibilities: (1) give both control and all cash flows to the highest bidder; (2) give the highest bidder control and share $q_{\min}$ of cash flow rights and give the second-highest bidder $(1 - q_{\min})$ of cash flow rights; or (3) give the second-highest bidder control and share $q_{\min}$ of cash flow rights, and give the highest bidder share $(1 - q_{\min})$ of cash flow rights.

Qualitatively, introducing $q_{\min}$ has the following effects:

- A bidder who receives control but not all cash flows has to pay the seller (rather than the seller pay him) if $q_{\min}$ is sufficiently large that the value of the $q_{\min}$ portion of cash flows is high relative to his cost of running the project.
- The larger is q_{min}, the lower are revenues in the optimal mechanism, but they still strictly exceed those in any no-separation mechanism.
- The larger is q_{min}, the easier it is to satisfy regularity. This means that the allocations that maximize $\pi_{s,a}$ are optimal for a larger parameter space.

4 Conclusions

Our paper revisits the classical auction setting of Myerson (1981), in which a seller seeks to sell a single asset/project to potential bidders who privately receive independently-distributed signals about the asset's future cash flows. The asset's payoffs hinge on both the signal of the bidder who controls the asset and those of rival bidders. The literature characterizes optimal mechanisms when only the bidder who controls the project receives cash flows. We show that the seller can increase revenues by sometimes allocating (a portion of) cash flows to a bidder who does not control the project.

When local incentive compatibility implies global incentive compatibility we prove that the seller should award both rights to the bidder with the highest signal *only* when the highest signal sufficiently exceeds the second-highest signal. When, instead, the two highest signals are closer, the seller optimally splits the two rights between these bidders. The optimal nature of the division hinges on the private vs. common value composition of bidder signals: when common values comprise a sufficiently high share of the total project value, the seller optimally awards control to the second-highest bidder and cash-flow rights to the highest bidder; but when common values comprise a smaller share, the seller should reverse this allocation, awarding control to the highest bidder and cash-flow rights to the second-highest bidder. Separating control and cash-flow rights helps rent extraction from bidders because a bidder's valuation is more sensitive to his own signal than to those of other bidders. When this is so, the expected value of cash-flow rights is less sensitive to the bidder's signal when he does not control the project, reducing his informational rents, and when the two highest signals are sufficiently close, this benefit outweighs the associated costs.

When the common value component in bidder valuations is sufficiently high, the mechanism above is globally incentive compatible and hence is the optimal separation mechanism. When, instead, the common value component is lower, seller revenues from the mechanism that always assigns the highest bidder cash flow rights, but gives the second-highest bidder control rights whenever their signals are close enough still exceed revenues from no-separation mechanisms. This mechanism has a simple implementation via a two-stage auction: the first stage is a standard English auction and then the seller offers the auction winner a choice of whether to have both rights, or just to retain cash flow rights and have the second-highest bidder run the project in return for a reduced cash payment. This separation mechanism is ex-post incentive compatible with no bidder regrets and it always generates (weakly and sometimes strictly) higher revenue than the optimal no-separation mechanism event by event.

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