Auctioning control and cash-flow rights separately*

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Abstract

We consider a classical auction setting in which an asset/project is sold to buyers who privately receive independently-distributed signals about expected payoffs, and payoffs are more sensitive to the signal of the bidder who controls the asset. We show that a seller can increase revenues by sometimes allocating cash-flow rights and control to different bidders. Allocating both cash-flow rights and control to the highest bidder only maximizes seller revenue when his signal sufficiently exceeds the next highest. When signals are closer, the seller optimally splits rights between the top two bidders, with the second-highest bidder receiving control if and only if the degree of common values in asset payoffs is high enough.

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1 Introduction

The starting point for our paper is a classical auction setting in which a seller seeks to sell a single asset/project to risk-neutral bidders who privately receive signals about the asset's expected future cash flows. When a bidder wins control, the asset's payoffs hinge on both his signal and those of rival bidders. The existing literature studies mechanisms in settings where only the bidder who controls the project receives cash flows (possibly splitting them with the seller), but *no* other bidder receives cash flows.

Our paper shows that a seller can do better by sometimes allocating control to one bidder and (some or all) cash flows to other bidders. We establish that as long as expected cash flows are more sensitive to the signal of the bidder who controls the project than to those of other bidders—so that project payoffs have both private value and common value components—expected seller revenues are strictly higher when the seller sometimes allocates control and cash-flow rights to *different* bidders, and that the bidder who should be assigned control may not be the one that would generate the highest project payoffs. We believe our paper is the first to propose a "separation" mechanism of this form, to establish its revenue advantages, and to identify the sources of those advantages.

To highlight how outcomes in our separation framework differ from those in the "no-separation" frameworks of existing studies, we focus on settings with ex-ante identical bidders. In the classical no-separation framework, given a standard monotone-hazard condition, it is optimal for a seller to always award both control and cash-flow rights to the highest bidder. This result reflects that (i) allocating control to the bidder with the highest signal maximizes social welfare, and (ii) allocating cash flows to the highest bidder reduces rents earned by bidders with lower signals, thereby minimizing bidders' total rents (reflecting the envelope theorem logic that rents earned by lower types cumulate to carry over to higher types). Our paper derives the surprising result that violating either (i) or (ii) can increase seller revenue. We show that separating control and cash-flow rights among different bidders facilitates rent extraction, the benefit of which will strictly outweigh the costs of not always assigning both cash flow rights and control to the highest bidder.

To see the benefit of separation, consider a simple example where two bidders receive independently-distributed signals t_1 and t_2. The asset generates cash flows of $v_i = \frac{1}{1+\rho}(t_i + \rho t_{-i})$ if $i \in \{1, 2\}$ has control, where $\rho \in (0, 1)$ means that cash flows are more sensitive to the controlling agent's signal than the rival bidder's signal. For simplicity, assume it is costless to run the project. First consider standard English auctions where cash-flow and control rights are not separated. The equilibrium bidding strategy is $\beta_i(t_i) = t_i$, and the bidder with the higher signal $t_h > t_s$ wins. The seller's revenue is t_s and the winning bidder's payoff is $\frac{t_h - t_s}{1+\rho}$.

Now consider the following two-stage separation mechanism. The first stage is an "always-separating" English auction in which the highest bidder pays the exit price of the second-highest bidder and receives cash-flow rights, but, unlike in a standard no-separation auction, the second-highest bidder receives control. In the second stage of the mechanism, the seller offers the first-stage winner an option to override the first-stage outcome: he can pay the seller a small fixed extra payment of p_{extra} to acquire control, while still retaining all cash flows.

One can show that bidding $\beta_i(t_i) = t_i$ still constitutes an equilibrium to the first stage of our separation mechanism. Thus, considering the first stage outcome alone, seller revenue is the same as in the no-separation auction, but the winning bidder's payoff is reduced by a factor of ρ to $\rho\frac{t_h - t_s}{1+\rho}$: only bidders bear the efficiency loss from assigning control to the lower-valuation bidder. Adding the second stage recovers some of this efficiency loss, leading to a Pareto improvement: as the asset generates more cash flows under the winning bidder's control, the winning bidder will pay to acquire control whenever the efficiency gain $(1-\rho)\frac{t_h - t_s}{1+\rho}$ exceeds the price p_{extra}. Both the seller and the winning bidder benefit, implying that expected seller revenue strictly exceeds that in the standard English auction.

These insights extend, holding for any number of bidders, general signal structures and valuation functions where cash flows strictly increase in the controller's signal and are weakly (strictly so for a positive measure) more sensitive to the controller's signal than those of the other bidders. We allow p_{extra} to depend on the exit prices of losing bidders and derive the form that maximizes seller revenues. We also show that the bidding equilibrium is *ex-post* incentive compatible. That is, our separation mechanism has the virtue that it is an ex-post equilibrium (Bergemann and Morris (2008))—ex post, no bidder regrets. Since it always generates (weakly and sometimes strictly) higher revenue than the optimal no-separation mechanism event by event, its advantages extend directly to risk-averse sellers.

The source for the benefits of separation is that the project's payoff is most sensitive to the information of the bidder who controls the project. As we know from standard auction theory, a bidder's information rent depends on the importance of his private information for project payoffs. Allocating both control and cash flow rights to the same bidder maximizes his informational advantage. Allocating cash flows, instead, to a bidder who does not control the project, reduces the sensitivity of project payoffs to this bidder's private information, lowering his informational advantage. When the two highest signals are sufficiently close, the inefficiency cost from allocating control to a lower signal bidder and the cost of increased bidder rents due to allocating cash flows to a lower signal bidder become arbitrarily small, leaving only the benefit from the reduced sensitivity of a bidder's payoff to his signal. Thus, when the difference in the two highest signals is small enough, separation dominates no-separation.

We then consider the possibility that the controller must receive a minimum share q_{min} of cash flows, for example to satisfy corporate regulations that mandate minimum equity stakes for control, or to assuage moral hazard concerns. We observe that one can split rights in two ways: instead of (1) always giving cash flow rights to the highest bidder and sometimes giving control to the second-highest bidder, one could (2) always give control to the highest bidder and sometimes give the second-highest bidder some cash flows. In each of these two separation mechanism designs, the highest bidder only receives all rights when his signal sufficiently exceeds the second highest. We show that for any $q_{min} \in (0, 1)$, at least one of these two types of separation mechanisms can be designed to (i) be ex-post incentive compatible, and (ii) generate strictly higher expected revenues than no-separation English auctions.

We extend our analysis to characterize when one of these two types of mechanisms is optimal among all incentive compatible separation mechanisms. To do this, we specialize to settings where cash flows are linear functions of signals that are independently-distributed across bidders and satisfy the standard monotone hazard condition that ensures global incentive compatibility of no-separation mechanisms. We show that with significant common values and q_{min} small, it is optimal to split by assigning control to the second-highest bidder and cash flow share $1-q_{min}$ to the highest bidder. When, instead, q_{min} is large enough relative to the extent of common values ($q_{\min} \geq 50\%$ suffices), it is optimal to reverse this split of rights.

Finally, we show our qualitative findings extend to a setting where bidders receive multi-dimensional signals. We assume the cash flows generated by a bidder's control are the sum of a bidder-specific component and a common component, and bidders receive signals about each component; in addition, a bidder privately observes his cost of running the project. Mechanism design is challenging with multi-dimensional signals.[1] We identify a class of separation mechanisms in which the three-dimensional signals can be reduced to a single dimension, rendering analysis tractable. In this mechanism class, the asset is always sold with the controller receiving a fixed share q of cash flows, and each rival bidder receiving share $\frac{1-q}{n-1}$. We derive closed-form solutions for bidding strategies, and show that this mechanism can generate both higher revenues and greater social welfare than no-separation mechanisms.

Lastly, we highlight a useful interim result in our analysis: we identify necessary and (sharp) sufficient conditions for direct-revelation separation mechanisms to be globally incentive compatible. This result is of independent interest, as it simplifies establishing ex post and interim global incentive compatibility in many settings. To understand this result, note that a standard way to think about incentive compatibility is to examine what happens if we fix an agent's true type but vary his reported type. We show that one can instead fix the reported type and vary the true type. Our approach eases analysis for two reasons: a bidder's payoff is typically continuous in his true type, but it may not be continuous in his reported type (as is often true off the equilibrium path); and reported types affect auction outcomes, but true types do not, so the derivative with respect to the true type takes a simpler form.

Other researchers have examined settings where a *single* bidder splits cash flows with the seller. Ekmekci, Kos and Vohra (2016) consider the problem of selling a firm to a single buyer who is privately informed about post-sale cash flows and the benefits of control. The seller can offer a menu of cash-equity mixtures, and the buyer must obtain a minimum equity claim to cash flows (with the seller retaining any residual cash flows) to gain control rights. They provide sufficient conditions for the optimal mechanism to take the form of a take-it-or-leave-it offer for either the minimum stake or for all shares. In contrast, we examine a multi-bidder auction setting in which the seller can allocate control and cash-flow rights to different bidders, showing that such separation *among bidders* increases seller revenues.

Mezzetti (2003; 2004) also studies two-stage mechanisms with interdependent valuations

[1] See Vohra (2011) and Ekmekci et al. (2016) for details on these challenges and ways to addressing them.

where first allocations are determined and then transfers are determined depending on the information revealed in the first stage. Mezzetti largely focuses on efficiency, showing that one can implement the ex-ante efficient allocation in an ex-post incentive compatible way using contingent transfers. He then adds an assumption that bidders at the first-stage *perfectly observe* their realized outcome-decision payoffs at the second-stage and shows that one can extract full rents. When signals perfectly reveal future *realized* cash flows—when realized cash flows are not subject to any noise or shocks—it allows for a "shoot the liar" design that asks bidders to report their types at the first stage and their realized payoffs at the second stage. With perfect observability, if a bidder misreports at the first stage, no one is subsequently fooled, so a designer can perfectly cross-check against bidders' reports, and thus detect and punish lying at the first stage.

The bankruptcy resolution, private equity/venture capital, and M&A settings that motivate our analysis do not feature deterministic relationships between signals and *realized* cash flows, rendering Mezzetti's cross-checking design infeasible. Instead, our mechanism exploits the feature that when the payoff of the auctioned asset depends more strongly on the information of the bidder who controls the asset, any cash flows that a bidder receives are less sensitive to his signal if he does not have control. Our design leverages this novel channel of lowered sensitivity to reduce a bidder's informational advantage by splitting control and cash flows, awarding control or cash flows to the second-highest bidder when signals are close, thereby raising seller revenues.

The literature has examined optimal designs of no-separation auctions with common valuations in many settings. McAfee, McMillan, and Reny (1989) derive conditions under which, with common values, the optimal no-separation selling procedure is implemented by a simple mechanism in which a seller solicits reports from one bidder and offers the asset to another. Bergemann, Brooks, and Morris (2016) and Brooks and Du (2018) identify robust auctions in pure common value settings that yield maximum revenue guarantees. Lauermann and Speit (2023) study bidding in common-value auctions with an unknown number of bidders.

Other researchers have examined the consequences of separating ownership and control in the market for corporate control in the context of agency issues, free riding problems and information aggregation (see, e.g., Bagnoli and Lipman 1988, Ekmekci and Kos 2016, Voss and Kulms 2022). Our paper contributes to this literature by identifying an advantage of the separation of ownership and control from the perspective of optimal auction design.

2 Base Model

There are $n > 1$ ex-ante identical bidders who bid for an asset/project.[2] The project can be controlled (run) by only one bidder who incurs a publicly-known opportunity cost $\tau \geq 0$ from running the project that then generates a stream of future cash flows. The bidders and the seller are risk-neutral. Bidders do not discount future cash flows, whereas the seller values only current cash payments from the auction, discounting future cash flows to zero.

Each bidder i receives a private signal t_i that is informative about the project's future cash flows. We sometimes refer to t_i as bidder i's type. We use $\mathbf{t} \equiv (t_1, t_2, ..., t_n)$ to denote the vector of all bidder types and $\mathbf{t}_{-i} \equiv (t_1, ..., t_{i-1}, t_{i+1}, ..., t_n)$ to denote the vector of bidder types other than i. We use $f(\mathbf{t})$ to denote the joint density of $\mathbf{t}$ with associated cdf $F(\mathbf{t})$. We assume that signals are weakly affiliated, nesting independently-distributed signals as a special case. We further assume that $f(\mathbf{t})$ is symmetric in its arguments, and uniformly continuous and strictly positive over its support $[\underline{t}, \bar{t}]^n$.

Valuations are interdependent: expected future cash flows from the project under bidder i's control, $v_i(t_1, ..., t_n)$, depend on the signals of all bidders. We assume v_i is nondecreasing in its arguments, twice continuously differentiable, and strictly increasing in t_i. Valuations are also symmetric:

$$v_i(t_1, ..., t_n) = u(t_i; \mathbf{t}_{-i}), \text{ for all } i, \tag{1}$$

where the function u is the same for each bidder and symmetric in its last $n-1$ components. Valuations satisfy a single-crossing condition: given any signal vector $\mathbf{t} \equiv (t_1, t_2, ..., t_n)$,

$$\frac{\partial v_i}{\partial t_i}(\mathbf{t}) \geq \frac{\partial v_j}{\partial t_i}(\mathbf{t}), \text{ for all } i \text{ and all } j \neq i. \tag{2}$$

The single-crossing condition implies that a bidder's signal has a greater influence on cash flows if he runs the project than if another bidder runs the project. Given the symmetry

[2] We focus on ex-ante identical bidders to emphasize that always assigning cash-flow rights and control to the same (highest) bidder does not maximize seller revenues. Our insights extend to heterogeneous bidders.

condition in (1), the single-crossing condition reduces to requiring that

$$u_1\left(t_1; t_2, ..., t_n\right) \geq u_2\left(t_2; t_1, ..., t_n\right), \tag{3}$$

where u_i is the derivative with respect to the i^{th} argument.

These assumptions on the signal distribution and valuations are standard in studies of auctions with interdependent values (e.g., they are identical to those in Krishna (2010), Chapter 6). We add a mild assumption that there exist signals $t_2 \geq t_3 \geq ... \geq t_n$ with $\bar{t} > t_2$ and $t_n > \underline{t}$, such that (3) holds as a strict inequality at $t_1 = t_2$:

$$u_1\left(t_2; t_2, ..., t_n\right) > u_2\left(t_2; t_2, ..., t_n\right). \tag{4}$$

We sometimes specialize to bidder valuations that are linear in the signals,

$$u\left(t_i, \mathbf{t_{-i}}\right) = A_n\big(t_i + \rho \sum_{j \neq i} t_j\big), \tag{5}$$

where $A_n \equiv \frac{1}{1+(n-1)\rho}$ is a normalizing parameter that sets $u(t, t, ..., t) = t$, and $\rho < 1$ implies that expected project payoffs are more sensitive to the controller's signal than to the signals of other bidders. We can rewrite this as $u(\mathbf{t}) = A_n\Big(\rho \sum_j t_j + (1-\rho)t_i\Big)$, i.e., a bidder's valuation is the sum of common value and private value components, where ρ measures the degree of common valuations: a higher ρ implies a higher degree of common valuations.

Our key departure from the literature is to consider settings in which a seller can allocate control and cash-flow rights to different bidders. That is, a bidder who does not control the project may nonetheless receive some or all of the future cash flows generated.

Formally, we consider direct-revelation mechanisms that allow for the separation of control from cash-flow rights. Let $R_j(\mathbf{t}) \in [0, 1]$ be the probability bidder j is assigned control when bidders report $\mathbf{t}$. Let $Q_{ji}(\mathbf{t}) \in [0, 1]$ be the fraction of the total cash flow that i gets when bidders report $\mathbf{t}$ and control is assigned to j.[3] Let $M_i(\mathbf{t})$ be i's expected cash payment to the seller when bidders report $\mathbf{t}$. We use $M_i(t_i'; \mathbf{t_{-i}}) = M_i(t_1, \ldots, t_{i-1}, t_i', t_{i+1}, \ldots, t_n)$ to denote i's expected cash payment from reporting t_i' when other bidders report truthfully; and we use $R_i(t_i'; \mathbf{t_{-i}})$ to denote the probability i receives control when i reports t_i' and other

[3] If given report $\mathbf{t}$, j is never assigned control, then the value of $Q_{ji}(\mathbf{t})$ is irrelevant.

bidders report truthfully. We require that

$$\sum_j R_j(\mathbf{t}) \leq 1, \quad \text{for all } \mathbf{t}, \tag{6}$$

and

$$\sum_i Q_{ji}(\mathbf{t}) = 1, \quad \text{for all } j \text{ and all } \mathbf{t}. \tag{7}$$

One can interpret $\sum_j R_j(\mathbf{t}) < 1$ as the seller retaining the project with some probability, in which case the project does not generate any cash flows.[4]

We also impose a minimum control stake requirement for the bidder who is given control:

$$Q_{jj}(\mathbf{t}) \geq q_{min} \text{ for all } j \text{ and } \mathbf{t}, \tag{8}$$

where $q_{min} \geq 0$. Our insights hold regardless of whether $q_{min} = 0$ or $q_{min} > 0$. We allow for $q_{min} > 0$ to capture settings in which a controller may need to retain a claim to cash flows, for example, to address moral hazard concerns (Ekmekci, Kos, and Vohra (2016)).

We term our mechanism a "separation mechanism" to contrast with the standard "no-separation" mechanisms that impose $Q_{jj}(\mathbf{t}) = 1$ for all j and $Q_{ji}(\mathbf{t}) = 0$ for all $i \neq j$, for all $\mathbf{t}$.

Define $U_i(t_i, t_i'; \mathbf{t}_{-i})$ to be bidder i's expected profit when he is type t_i and reports t_i', and all other bidders truthfully report $\mathbf{t}_{-i}$:

$$U_i(t_i, t_i'; \mathbf{t}_{-i}) \equiv \sum_j R_j(t_i'; \mathbf{t}_{-i}) Q_{ji}(t_i'; \mathbf{t}_{-i}) v_j(\mathbf{t}) - M_i(t_i'; \mathbf{t}_{-i}) - \tau R_i(t_i'; \mathbf{t}_{-i}). \tag{9}$$

The first term $\sum_j R_j Q_{ji} v_j$ on the right-hand side is the expected value of the cash flows awarded to bidder i, where the summation over j reflects that bidders other than i may run the project when i receives cash flows. The second term is the expected value of payments that i makes to the seller. The third term is the expected cost that i incurs from running the project, which is τ multiplied by the probability that i is assigned control.

[4] One may weaken (7) to $\sum_i Q_{ji}(\mathbf{t}) \leq 1$. $\sum_i Q_{ji}(\mathbf{t}) < 1$ corresponds to the seller receiving share $(1 - \sum_i Q_{ji}(\mathbf{t}))$ of future cash flows, which she fully discounts and hence does not value. Vis à vis this weaker version, (7) is without loss of generality: if $\sum_i Q_{ji}(\mathbf{t}) \in (0, 1)$ at some $\mathbf{t}$ and some j, then one can adjust the mechanism by multiplicatively scaling down $R_i(\mathbf{t})$ by $\sum_i Q_{ji}(\mathbf{t})$, and multiplicatively scaling up $Q_{ji}(\mathbf{t})$ by $1/\sum_i Q_{ji}(\mathbf{t})$ for all i, so that $\sum_i Q_{ji}(\mathbf{t}) = 1$. With this adjustment, bidder j runs the project with a reduced probability and hence incurs weakly less opportunity costs (since $\tau \geq 0$), so we can increase j's payment to the seller (weakly raising her revenues) while leaving j's payoff unchanged.

For any given bidder i with signal t_i, define

$$f_{-i}\left(\mathbf{t}_{-i}|t_i\right) \equiv \frac{f\left(t_i;\mathbf{t}_{-i}\right)}{\int_{\Omega_{n-1}} f\left(t_i;\mathbf{t}_{-i}\right) d\mathbf{t}_{-i}} \tag{10}$$

to be the conditional marginal density of $\mathbf{t}_{-i}$ given t_i, where $\Omega_{n-1} \equiv [\underline{t},\bar{t}]^{n-1}$ is the space of integration over the signals of bidders other than i. Let $\bar{U}_i(t_i,t_i')$ be i's expected profit when he has type t_i but reports t_i' and all other bidders report truthfully. Integrating (9) over $\mathbf{t}_{-i}$ yields

$$\begin{aligned}\bar{U}_i(t_i,t_i') \;&=\; \int_{\Omega_{n-1}} \sum_j R_j(t_i';\mathbf{t}_{-i})Q_{ji}(t_i';\mathbf{t}_{-i})v_j(t_i;\mathbf{t}_{-i})f_{-i}\left(\mathbf{t}_{-i}|t_i\right)d\mathbf{t}_{-i} \\ &\quad - \int_{\Omega_{n-1}} M_i\left(t_i';\mathbf{t}_{-i}\right)f_{-i}\left(\mathbf{t}_{-i}|t_i\right)d\mathbf{t}_{-i} - \tau\int_{\Omega_{n-1}} R_i(t_i';\mathbf{t}_{-i})f_{-i}(\mathbf{t}_{-i}|t_i)d\mathbf{t}_{-i}. \end{aligned} \tag{11}$$

The first term on the right-hand side of (11) is the expected value of cash flows awarded to bidder i when he reports t_i'. The second term (without the minus sign) is bidder i's expected cash payment when he reports t_i'. The third term (without the minus sign) is the expected opportunity cost to i of running the project when he reports t_i'.

The equilibrium expected profit for bidder i of type t_i is $\bar{U}_i(t_i,t_i)$. Equilibrium requires that both the (interim) incentive compatibility condition,

$$\bar{U}_i(t_i,t_i) = \max_{t_i'} \bar{U}_i\left(t_i,t_i'\right), \tag{12}$$

and the (interim) individual rationality condition,

$$\bar{U}_i(t_i,t_i) \geq 0, \tag{13}$$

hold for all i and t_i. We later characterize when the optimal auction design satisfies the stronger requirements of ex-post rationality and ex-post incentive compatibility.

The seller's expected revenue is the sum of the expected cash payments of all bidders, $\pi_s = \sum_{i=1}^{n}\int M_i(\mathbf{t})f(\mathbf{t})d\mathbf{t}$. Equivalently, seller revenue equals the total gain in social welfare less total bidder rents. To see this, let $g(t_i)$ be the marginal density distribution of t_i. Then

$$\pi_s = \int\sum_{j,i} R_j(\mathbf{t})Q_{ji}(\mathbf{t})v_j(\mathbf{t})f(\mathbf{t})dt - \sum_i\int_{\underline{t}}^{\bar{t}}\left(\bar{U}_i(t_i,t_i)\right)g(t_i)dt_i - \tau\sum_i\int_{\Omega_n} R_i(\mathbf{t})f(\mathbf{t})d\mathbf{t}. \tag{14}$$

Equation (14) is intuitive: expected revenue is the expected increase in social welfare gross of the costs of running the project (first term on the right-hand side) less the sum of bidders' expected rents (second term) less the expected costs of running the project (third term).

Our objective is to identify the mechanism that maximizes expected seller revenue (14) subject to the feasibility conditions (6) and (7), the incentive compatibility (12) and individual rationality (13) conditions, and the minimum control stake requirement (8).

Our characterizations hold regardless of whether bidder signals are correlated or independently distributed. With correlated signals, we know from Cremer and McLean (1988) that a seller can design a mechanism that exploits the correlation to achieve full extraction. However, such mechanisms require large side bets that may lead to large regrets, rendering an assumption of risk-neutral bidders problematic. This leads us to focus on separation mechanisms that either take an English-auction format or are direct-revelation mechanisms that are ex-post incentive compatible. We show they can always be designed to generate higher expected revenues than English no-separation auctions.[5]

2.1 Discussion

The intuition for the advantages of separation can be understood by applying the envelope theorem to (11) and (12), which yields

$$\begin{aligned}\frac{d\bar{U}_i(t_i,t_i)}{dt_i} &= \int_{\Omega_{n-1}} \sum_j R_j(\mathbf{t})Q_{ji}(\mathbf{t})\frac{\partial v_j(t_i;\mathbf{t}_{-i})}{\partial t_i} f_{-i}(\mathbf{t}_{-i}|t_i)d\mathbf{t}_{-i} \\ &\quad + \int_{\Omega_{n-1}} \Big[\sum_j R_j(\mathbf{t})Q_{ji}(\mathbf{t})v_j(t_i;\mathbf{t}_{-i}) - M_i(\mathbf{t}) - \tau R_i(\mathbf{t})\Big]\frac{df_{-i}(\mathbf{t}_{-i}|t_i)}{dt_i}d\mathbf{t}_{-i}. \quad (15)\end{aligned}$$

The first term is the contribution to a bidder's rents due to his private information regarding the value of the cash flows, while the second term is the contribution to bidder rents from any correlation in bidder signals. The first term is the key for understanding advantage of separation: as in the standard no-separation setting, allocating cash flows to a bidder i with signal t_i enables him to earn differential rents relative to when i has a lower signal, as reflected by the term $R_j(\mathbf{t})Q_{ji}(\mathbf{t})$; but, unlike the no-separation case, the differential rents are scaled by $\frac{\partial v_j(t_i;\mathbf{t}_{-i})}{\partial t_i}$. That is, bidder i's differential rents are weighted by the sensitivity of the value of his awarded cash flows to his signal when the project is run by bidder j. Because a bidder's signal has a greater influence on cash flows when he runs the project than if another bidder

[5] Lopomo (2000) and Chung and Ely (2007) provide general conditions under which the English no-separation auction yields the highest seller revenue among all ex-post incentive compatible no-separation mechanisms.

runs it, awarding bidder i cash flows when the project is run by a different bidder reduces i's overall rents, vis à vis awarding bidder i cash flows when i also runs the project himself.

Our separation mechanism extends the existing framework of mechanism design by incorporating the assignments of rights into the design consideration. In practice it loosely corresponds to settings in which, for example, an entrepreneur "sells" a project idea to a syndicated VC group, where the lead VC is directly involved in the project management, while the other VCs only contribute funding in return for claims to future cash flows. Private equity clubs (e.g., club deals) and limited partner frameworks feature a similar separation, where the limited partners provide capital and other inputs, while the general partner runs the business.

Our model considers an impatient seller who does not value retention of cash flows. This assumption is standard in the security design literature where a seller owns an asset that generates future cash flows, but has a higher discount rate than buyers, creating gains to trade (see, e.g., Biais and Mariotti 2005). In this setting, existing studies focus on revenue-maximizing mechanism that do not feature separation: given standard regularity conditions, the highest bidder receives both control and cash flow rights. Our paper shows that separation can improve seller revenue further. Our insights extend when the seller is as patient as bidders, but has to raise cash to cover an upfront investment or other liquidity need. Existing studies have examined such settings when cash-flow rights and control are split between the seller and a single bidder, with the seller awarding both control and a share of cash-flow rights to the (same) highest bidder in exchange for the cash needed for investment but no other bidder receives cash flows. Our insights apply here, too: a seller can do better by sometimes splitting a share of cash flows among different bidders—while retaining the remaining cash flows for herself, or awarding control to a bidder who is not the highest bidder.

Our framework assumes that cash flows are contractible and they are split in the form of equities. This assumption captures many settings—e.g., bankruptcy resolution, takeover auctions, private equity or venture capital—that fit our framework in many ways: (i) equity is commonly used, (ii) cash flows arrive over time and are subject to shocks, making bidder signals noisy indicators of future cash flows, (iii) sellers are impatient and must be paid before those cash flows arrive. In mergers and acquisitions, our mechanism can be implemented by having bidders take equity shares or dual class shares where only one class of shares has control rights, both of which are used in practice. In the market for corporate control, the use of equity is especially natural as regulations mandate a minimum equity share to gain control.[6]

2.2 Two-stage separation mechanism when $q_{min} = 0$

In a standard **no-separation English auction**, the auctioneer continuously increases price and the auction stops when second highest bidder exits, with the winner pays that exit price. As is well known (see equation 6.5 in Krishna 2010), bidding strategies in the symmetric equilibrium take the following form: if bidders $k+1, k+2, ..., N$ have dropped out, with their exit prices revealing their signals t_{k+1},t_{k+2}, ..., t_N (strategies are monotone) to the remaining k active bidders, then the strategy of a remaining bidder i with signal t_i is to drop out at the price

$$\beta^k(t_i, t_{k+1}, ..., t_N) = u(t_i; t_i, ..., t_i, t_{k+1}, ..., t_N) - \tau, \tag{16}$$

which is the expected value of the cash flows generated by bidder i when all k active bidders have signal t_i, while those bidders who exited have signals as revealed by their exit prices minus the opportunity cost for bidder i of running the project.

Next we describe our two-stage auction:

Definition 1 *(two-stage auction) The first stage is a standard English auction, i.e., the auctioneer continuously increases price and the auction stops when the next-to-last bidder exits.*

In the second stage, the seller offers the first-stage winner a choice of whether to receive cash-flow rights but give control rights to the highest losing bidder or to pay an additional fee to obtain both rights. If the winner only chooses cash flows then she pays the seller τ plus the exit price of the highest losing bidder, and control is assigned to the highest losing bidder who is paid τ in return for running the project. If, instead, the winner pays to obtain control then she pays the exit price of the highest losing bidder plus an extra payment of $p_{extra}(\cdot) \geq 0$ minus τ, where $p_{extra}(\cdot)$ can be any (symmetric) function of the exit prices of the losing bidders.

The auction rules, including $p_{extra}(\cdot)$, are public information before the first stage. One can interpret this two-stage mechanism as follows. The first stage corresponds to an always-separating English mechanism in which the second-highest bidder (who, in equilibrium, has

[6]It may also be practically difficult to pay bidders with other securities whose values depend non-linearly on a stream of future cash flows that arrives over time.

the second-highest valuation) always receives control and the highest bidder always receives cash-flow rights. In the second stage, the seller offers the winner the option to override the first stage outcome by paying the seller an additional $p_{extra}(\cdot)$ to acquire control.

Define $\Delta(t_1; t_2, t_3, ..., t_n)$ to be the difference in expected cash flow gains from giving control to a generic bidder 1 rather than a generic bidder 2, when the other signals are $t_3, ..., t_n$:

$$\Delta(t_1; t_2, t_3, ..., t_n) \equiv u(t_1; t_2, t_3, ..., t_n) - u(t_2; t_1, t_3, ..., t_n).$$

Obviously, $\Delta(t_1; t_2, ..., t_n)$ weakly increases in t_1 and is nonnegative given $t_1 \geq t_2$.[7] When bidder 1 has a higher signal than bidder 2, $\Delta(t_1; t_2, ..., t_n)$ is the "efficiency gain" from allocating control to the higher bidder 1 rather than bidder 2.

Proposition 1 *Part A: In the symmetric equilibrium of the two-stage auction:*

(i) In the first-stage, bidding strategies are given by (16), as in a no-separation English auction, regardless of the functional form of $p_{extra}(\cdot)$.

(ii) In the second stage, without loss of generality let t_1 *be the winner's type and let* t_2 *be the highest losing bidder's type as inferred from the exit prices.*[8] *The first-stage winner acquires control if and only if* $p_{extra} \leq \Delta(t_1; t_2, ..., t_n)$.

Part B: Given any $p_{extra}(\cdot)$, *this equilibrium is ex-post incentive compatible.*

Proof: Consider a generic bidder 1 with signal t_1 (not necessarily the highest) when all other bidders follow their posited equilibrium strategies. We show bidder 1 is weakly better off following his equilibrium strategy for any realization of his rivals' signals $t_2, ..., t_n$.

In stage 2 only the winner's strategy is relevant, so assume without loss of generality that bidder 1 won the first stage (but he need not have followed his equilibrium strategy in the first stage). The difference in bidder 1's profit from receiving both rights versus just receiving cash flow rights is $\Delta(t_1; t_2, ..., t_n) - p_{extra}$. This difference in profits is positive if and only if $p_{extra} \leq \Delta(t_1; t_2, ..., t_n)$. This establishes the optimality of the bidding strategy in stage 2.

In stage 1, we decompose analysis into two cases.

Case 1: $t_1 \geq t_2$. Then bidder 1 will win the first stage if he follows his equilibrium strategy. If he deviates and still wins the first stage, the deviation does not affect his profits because

[7] $\frac{d}{dt_1}\Delta(t_1; t_2...t_n) = \frac{d}{dt_1}v_1(t_1, t_2, ...t_n) - \frac{d}{dt_1}v_2(t_1, t_2, ...t_n) = u_1(t_1; t_2, ...t_n) - u_2(t_2; t_1, ...t_n) \geq 0.$

[8] Types can be inferred because the bidding strategy (16) is strictly increasing and hence invertible.

the winning price does not depend on his bid. If he deviates and loses the first stage, then his profit is zero, and hence deviation is not profitable (because his equilibrium profit is positive).

Case 2: $t_1 < t_2$. Bidder 1 will lose the first stage if he follows his equilibrium strategy. If he deviates and still loses the first stage, then his profit is unaffected. If he deviates and wins the first stage, then his profit is negative if he does not pay p_{extra} in stage 2 (so bidder 2 retains control and the value of the cash flows under bidder 2's control is less than bidder 1's payment), and his profit is even more negative if he pays p_{extra} to obtain control (since bidder 1 generates even lower cash flows than bidder 2 and $p_{extra} \geq 0$). ■

Proposition 1 implies that seller revenue is always weakly higher than in the standard English no-separation auction. In the first stage, the bidding strategy, and hence seller revenues, is the same as in the English auction where the outcome is efficient with the best bidder type receiving both control and cash flow rights. The efficiency loss in our mechanism from assigning control to the less productive bidder 2 is borne *entirely* by the winning bidder.

In the second stage, the seller offers the winner an opportunity to Pareto-improve on the first-stage outcomes, which benefits both the seller and the winner. The design extracts more rents from the winner when his type exceeds the second-highest type by enough to make the efficiency gain to the winner from running the project high enough that he would make an additional payment to achieve an efficient assignment of control. Thus, if the winning bidder chooses not to pay p_{extra} to obtain control then seller revenue is the same as in the no-separation auction; and if the winner pays to obtain control, then revenue is higher by p_{extra}.

Via a simple choice of p_{extra}, our mechanism can always generate strictly greater expected seller revenues than the standard English auction in a robust, detail free, way:

Result 1: Let $p_{extra} > 0$ be a constant. Then there exists a $p^* > 0$ such that for all $p_{extra} \in (0, p^*)$, the two-stage no-separation auction design generates strictly higher expected revenues than the no-separation English auction.

Proof: See the appendix. □

Next we derive the $p_{extra}(\cdot)$ that maximizes expected revenue in the two-stage mechanism. To do this, we write p_{extra} as a function of the losing bidders' types, which can be inferred from the exit prices by inverting (16). We use $p_{extra}(t_2, ..., t_n)$, where $t_2, ..., t_n$ denotes the (inferred) types of the losing bidders, with t_2 being the highest among them.

Result 2: The seller's optimal choice of $p_{extra}(t_2, ..., t_n)$ is

$$p_{extra}^{optimal}(t_2, ..., t_n) = \Delta\left(t^{opt}; t_2, ..., t_n\right),$$

where

$$t^{opt} \equiv \arg\max_t \Delta\left(t; t_2, ..., t_n\right) \int_t^{\bar{t}} f_1\left(x|\mathbf{t}_{-1}\right) dx$$

and $f_1\left(\cdot|\mathbf{t}_{-1}\right)$ is the conditional marginal density of t_1 given the losing signals $\mathbf{t}_{-1}$,

$$f_1(x|\mathbf{t}_{-1}) \equiv \frac{f(x; \mathbf{t}_{-1})}{\int_{t_2}^{\bar{t}} f(x; \mathbf{t}_{-1}) dt}.$$

That is, the optimal price $p_{extra}(t_2, ..., t_n)$ is the monopoly price conditional on the highest signal t_1 being at least t_2.

We next analyze mechanisms for settings where the agent controlling the project must receive a share of at least $q_{min} > 0$ of cash flows. The two-stage mechanism just analyzed does not satisfy this minimum stake requirement because when the auction winner does not pay $p_{extra}(t_2, ..., t_n)$ to gain control, the bidder who controls the project retains no cash flows. When $q_{min}>0$, we focus on direct-revelation separation mechanisms that are ex-post incentive compatible, and show they generate higher revenues than English no-separation auctions.

As a precursor to this analysis, we provide a useful result identifying necessary and sufficient conditions for direct-revelation mechanisms to be globally IC (incentive compatible). This result simplifies establishing ex post and interim global IC in both separation and no-separation settings, where a bidder's payoff is often not continuous in his reported type t'.

2.3 Conditions for Global Incentive Compatibility

Lemma 1 *Suppose $L(t, t')$ is a function of $t, t' \in [\underline{t}, \bar{t}]$ with the following properties:*

(a) L is continuous and twice differentiable with respect to t, i.e., $\frac{\partial}{\partial t} L(t, t')$ and $\frac{\partial^2}{\partial t^2} L(t, t')$ exist for all t and t';

(b) L is differentiable with respect to t' for all t' except possibly for a set S_{zero-m} of $t' \in [\underline{t}, \bar{t}]$ that has zero measure, i.e., $\frac{\partial}{\partial t'} L(t, t')$, $\frac{\partial}{\partial t'} \frac{\partial}{\partial t} L(t, t')$ and $\frac{\partial}{\partial t'} \frac{\partial^2}{\partial t^2} L(t, t')$ exist for all t and all $t' \in [\underline{t}, \bar{t}] \setminus S_{zero-m}$;

(c) L is continuous in t' at $t'=t$ for all t, and $\frac{\partial}{\partial t'}L(t,t')|_{t'=t}=0$ for all $t\notin S_{zero-m}$.

Then: ***1. Necessary condition for global IC.*** *If, for all t, $t'=t$ maximizes $L(t,t')$ over $t'\in[\underline{t},\bar{t}]$, then at any $t\in[\underline{t},\bar{t}]$ and $t'\notin S_{zero-m}$, $\frac{\partial}{\partial t'}\frac{\partial}{\partial t}L\left(t,t'\right)|_{t'=t}\geq 0$.*

2. Sufficient condition for local IC to imply global IC. *If, for all t, $\frac{\partial}{\partial t}L(t,t')$ weakly increases in t' over $t'\in[\underline{t},\bar{t}]$, then $t'=t$ maximizes $L(t,t')$ over $t'\in[\underline{t},\bar{t}]$ for all t.*

Proof: See the appendix. □

Here, one should interpret t and t' as a bidder's true type and reported type, respectively, and L as the bidder's payoff function. One can further interpret $L(t,t')$ in two ways. First, $L(t,t')$ can be the unconditional payoff of a bidder who has type t but reports t', i.e., integrating over all other bidders' types, assuming other bidders report truthfully. In this case, Lemma 1 provides conditions for an equilibrium to be interim incentive compatible. Alternatively, $L(t,t')$ can be the conditional payoff of a bidder who has type t but reports t', conditional on the other bidders' (truthfully-reported) types. In this case, Lemma 1 provides conditions for an equilibrium to be ex-post incentive compatible.

The lemma says that the necessary condition for IC is that $\frac{\partial}{\partial t}L(t,t')$ must weakly increase in t' for t' in the neighborhood of t. A sufficient condition for local IC to imply global IC is that $\frac{\partial}{\partial t}L(t,t')$ weakly increase in t' over the entire domain of t'.

To understand what is new in Lemma 1, note that a standard way to think about incentive compatibility is to examine what happens if we fix t but vary t'. Lemma 1 says that rather than fix t and vary t', one can instead fix t' and vary t. The necessary condition for incentive compatibility in the standard approach is that the second-order condition be negative, i.e., $\frac{\partial^2}{\partial^2 t'}L(t,t')=\frac{\partial}{\partial t'}\left(\frac{\partial}{\partial t'}L(t,t')\right)\leq 0$. Part (1) of Lemma 1 "replaces" this condition with the requirement that the cross-partial be positive, i.e., $\frac{\partial}{\partial t'}\left(\frac{\partial}{\partial t}L(t,t')\right)\geq 0$, when that derivative exists. That is, $\frac{\partial}{\partial t'}L(t,t')$ is replaced by $\frac{\partial}{\partial t}L(t,t')$, and the sign "$\leq 0$" is flipped to "$\geq 0$". The sufficient condition features a similar replacement: a standard sufficient condition for local IC to imply global IC is that $\frac{\partial}{\partial t'}L(t,t')$ weakly *decrease* in t' over $t'\in[\underline{t},\bar{t}]$; part (2) of Lemma 1 replaces this condition with the requirement that $\frac{\partial}{\partial t}L(t,t')$ weakly *increase* in t' over $t'\in[\underline{t},\bar{t}]$.

The approach in Lemma 1 of fixing t' and varying t is useful for two reasons. First, the payoff function is often not differentiable with respect to t' at all t and t'. Discontinuities arise even in simple no-separation mechanisms where a bidder's payoff may be discontinuous at the boundary between winning and losing, and the payoff depends on whether the bidder's reported type is the highest.[9] In addition, our separation mechanism has a second source of discontinuity: even when a bidder's report is the highest, there is a discontinuous change in allocations depending on whether his report exceeds the second highest by a sufficient amount. The second reason why the approach of fixing t' and varying t is useful is that for typical payoff functions $\frac{\partial}{\partial t} L(t, t')$ takes a simpler form than $\frac{\partial}{\partial t'} L(t, t')$ because bidders' reported types (t') affect auction outcomes (winning and allocations) and hence affect bidder payoffs, but bidders' true types (t) do not affect auction outcomes.[10]

2.4 Minimum control stake

With Lemma 1 in hand, we now analyze direct-revelation separation mechanisms that are ex-post incentive compatible, in settings where the agent controlling the project must receive a share of at least $q_{min} > 0$ of cash flows. For notational ease, when we say t_i is a "generic" signal, we mean that $t_i \in [\underline{t}, \bar{t}]$ but it is not necessarily the signal of bidder i.

Definition 2 *(Separation function and its inverse) For any $n-1$ "generic" signals $t_1, ..., t_{n-1}$, denote the highest signal by t_h and the second-highest by t_s. A "separation function" $S(t_1, ..., t_{n-1})$ is a symmetric function of $t_1, ..., t_{n-1}$ with $S(t_1, ..., t_{n-1}) \in [t_h, \bar{t}]$ that weakly increases in t_h.*

For $n = 2$, the "inverse" function $S^{-1}(t_1)$ is given by the minimum value of $t \in [\underline{t}, t_1]$ such that $S(t) \geq t_1$. For $n > 2$, $S^{-1}(t_1, ..., t_{n-1})$ is given by the minimum value of $t \in [t_s, t_h]$ such that $S(t_1, .., t_{h-1}, t, t_{h+1}, .., t_{n-1}) \geq t_h$, where $(t_1, .., t_{h-1}, t, t_{h+1}, .., t_{n-1})$ is the vector of $n-1$ signals formed by replacing t_h with t in $(t_1, ..., t_{n-1})$.

To illustrate, with two bidders who receive signals in $[0, 1]$, an example of a separation function and its inverse is $S(t_1) = wt_1 + 1 - w$ for some $w \in (0, 1)$ and $S^{-1}(t_1) =$

[9] E.g., in a second-price no-separation mechanism where bidders have private valuations and t_i is bidder i's valuation, consider a bidder 1 with valuation t_1 who reports t'_1 when everyone else reports truthfully. His conditional payoff (given other bidders' types) is $(t_1 - \max_{i \neq 1} \{t_i\}) \mathbf{1}_{\{t'_1 \geq \max_{i \neq 1}\{t_i\}\}}$, where $\mathbf{1}_{\{t'_1 \geq \max_{i \neq 1}\{t_i\}\}}$ is an indicator function: payoffs are discontinuous in t'_1 off the equilibrium path, i.e., for $t'_1 \neq t_1$.

[10] While we use Lemma 1 to establish incentive compatibility of separation mechanisms, it is also useful for no-separation mechanisms. With linear valuations, Lemma 1 is not needed for no-separation mechanisms because the necessary and sufficient condition for global (interim) incentive compatibility can be derived by noting that payoff functions are affine, and the maximum of a family of affine functions is convex. With non-linear valuations, payoff functions are not affine but Lemma 1 can be used to identify sharp sufficient conditions.

$\max\left(\frac{t_1+w-1}{w}, 0\right)$. We call S a "separation function" because it determines when assignment of cash flow and control rights are separated in our mechanisms. Separation occurs only when the reported types are "close enough", and S determines what comprises "close enough."

Our mechanisms have the property that a bidder i receives control and all cash flows (i.e., control and cash flows are not separated) if and only if his reported type $t'_i \geq S\left(\mathbf{t}'_{-i}\right)$, where $\mathbf{t}'_{-i}$ denotes the reported types of all bidders other than i. We focus on two classes of mechanisms, A and B, which reflect two ways to divide control and cash flow rights. In Mechanism A, when the two highest reported types are sufficiently close, the second-highest bidder receives control and a share q of cash flows, and the highest bidder receives share $1-q$ of cash flows. Thus, assignment of control is **inefficient**. Such a mechanism generalizes the two-stage mechanism detailed in Proposition 1, reducing to it in the limit as q goes to zero. Mechanism B has the opposite design: when the reported types are sufficiently close, the highest bidder receives control and a share q of cash flows, and the second-highest bidder receiving share $1-q$ of cash flows. Thus, assignment of control is **efficient**.

We show that for both classes of mechanisms, we can always design separation functions that generate strictly higher revenues than no-separation English auctions.

2.5 Inefficient splitting of cash flow and control rights

For notational ease, we use $\mathbf{t}'_{-i}$ to denote the reported types of all bidders other than i.

Definition 3 *(Mechanism A: inefficient splitting of rights) Suppose* $q_{\min} < 0.5$ *and consider* $q \in [q_{min}, 0.5)$. *Without loss of generality, let the reported types be* $t'_1 \geq t'_2 \geq t'_3 \geq ... \geq t'_n$.

If $t'_1 > S(\mathbf{t}'_{-1})$, *bidder 1 receives control and all cash flows and pays*

$$\begin{aligned} M_1 &= u\left(S(\mathbf{t}'_{-1}); t'_2, ..., t'_n\right) - (1-q)\, u\left(t'_2; S(\mathbf{t}'_{-1}), ..., t'_n\right) + (1-2q)\, u\left(t'_2; t'_2, ..., t'_n\right) \\ &\quad + qu\left(S^{-1}(\mathbf{t}'_{-1}), t'_2, ..., t'_n\right) - \tau. \end{aligned} \tag{17}$$

All other bidders receive nothing and pay nothing.

If $t'_1 \leq S(t'_2, ..., t'_n)$, *bidder 2 receives control and a fraction* q *of cash flows, and bidder 1 receives fraction* $1-q$ *of cash flows. Bidder 1 pays*

$$M_1 = (1-2q)\, u\left(t'_2; t'_2, ..., t'_n\right) + qu\left(S^{-1}(\mathbf{t}'_{-1}); t'_2, ..., t'_n\right) \tag{18}$$

and bidder 2 pays

$$M_2 = qu\left(S^{-1}(\mathbf{t}'_{-2}); t'_1, t'_3, ..., t'_n\right) - \tau. \tag{19}$$

All other bidders receive nothing and pay nothing.

We next identify conditions under which Mechanism A is *ex-post* incentive compatible and generates higher revenues than no-separation English auctions. Define

$$\rho_{\min} \equiv \min_{\mathbf{t}} \frac{\partial v_2(\mathbf{t})}{\partial t_1} / \frac{\partial v_1(\mathbf{t})}{\partial t_1}.$$

Here, $\frac{\partial v_2(\mathbf{t})}{\partial t_1} / \frac{\partial v_1(\mathbf{t})}{\partial t_1}$ is the ratio of the influence of a bidder's signal on cash flows if another bidder runs the project relative to when he runs the project. Thus, $\rho_{\min}$ is a measure of the minimum common-value component; when valuations are linear, $v_i(t_1, ..., t_n) = A_n\big(t_i + \rho \sum_{j \neq i} t_j\big)$, we have $\rho_{\min} = \rho$.

Proposition 2 *Suppose the common-value component in valuations is large enough that* $\rho_{\min} \geq \frac{q}{1-q}$*. Then for Mechanism A*

(i) truthful reporting is an ex post equilibrium given any separation function S.

(ii) there exist separation functions S for which Mechanism A generates strictly higher seller revenues than no-separation English auctions.

Proof: See the appendix. □

With Mechanism A, when the highest reported type t'_1 does not exceed the second highest reported type t'_2 by enough, i.e., when $t'_1 < S(\mathbf{t}'_{-1})$, control and cash flow rights are split with control being assigned inefficiently to bidder 2. When, instead, the highest reported type is high enough that $t'_1 \geq S(\mathbf{t}'_{-1})$, bidder 1 receives both control and all cash flows.

The logic for the ex-post incentive compatibility extends that for standard English no-separation auctions to settings where control and cash flow rights can be assigned to different bidders and more than one bidder may receive allocations. In the direct-revelation mechanism of the English no-separation auction, a bidder's payment and allocation only depend on which of *two* "report-regions" his reported type falls into—it only depends on whether it is the highest report or not. A bidder's payoff is non-zero only when his report is the highest.

Mechanism A retains the feature that a bidder i's payment and allocations do not depend on where his reported type is in a given report-region. However, it has *four* report-regions (reflecting the additional ways to separate control and cash flow allocations): region 1, t'_i exceeds the second-highest report by enough that $t'_i > S(\mathbf{t}'_{-i})$; region 2, t'_i is the highest report, but now $t'_i \in \left(t_2, S(\mathbf{t}'_{-i})\right)$; region 3, t'_i is the second-highest report but close enough to the highest, $t'_i \in \left(S^{-1}(\mathbf{t}'_{-i}), t_2\right)$; region 4, t'_i is lower yet with $t'_i < S^{-1}(\mathbf{t}'_{-i})$, i.e., t'_i is either lower than the second-highest or sufficiently lower than the highest. In regions 1 through 3, bidder i receives allocations and make payments, receiving a non-zero payoff.

We now explain how Mechanism A delivers ex-post IC. First, consider (19), which is bidder 2's payment in region 3,[11] where his reported type is the second highest but close enough to the highest reported type. Net of compensation for the cost of running the project, this payment equals the monetary value of the q shares of cash flows awarded to bidder 2 when his type is the minimum, $t_2 = S^{-1}\left(\mathbf{t}'_{-2}\right)$, that he can report in order to still receive q shares.[12] This leaves this boundary bidder type indifferent between reporting any type in regions 3 and 4 (the reports yield a payoff of zero), eliminating incentives for a local deviation.[13]

Next, consider bidder 1's payment in (18). This payment can be rewritten as

$$M_1 = (1-q)\, u\left(t'_2; t'_2, ..., t'_n\right) - \Delta_a, \tag{20}$$

where $\Delta_a \equiv q(u(t'_2; t'_2, ..., t'_n) - u(S^{-1}(\mathbf{t}'_{-1}); t'_2, ..., t'_n))$ is the differential rent of bidder 1 when he has type $t_1 = t'_2$ and reports truthfully vs. when he has the lower type $S^{-1}\left(\mathbf{t}'_{-1}\right)$ and reports truthfully. The first term in (20) is the value of the share $1-q$ of cash flows awarded to bidder 1 when he has type t'_2, which is the lowest type that he can report and still receive share $1-q$. Mechanism A reduces what the highest bidder has to pay by Δ_a to dis-incentivize him from reporting a lower type in hope of being the second-highest bidder. This payment reduction leaves a bidder 1 with boundary type $t_1 = t'_2$ indifferent between reporting any type in regions 2 and 3 (such reports yield payoff Δ_a), eliminating incentives for local deviations.

To understand (17), which is bidder 1's payment when his reported type sufficiently

[11] The numbering of regions is from the perspective of the bidder referred to in the context. Thus, region 3 is from bidder 2's perspective. When we discuss bidder 1's payments, the numbering reflects his perspective.

[12] The feature mirrors that in a English no-separation auction where the winner's payment is the value of the auctioned asset when the winner's type is replaced by the minimum that he can report and still win.

[13] Thus, our mechanism satisfies the premise in part (c) of Lemma 1 regarding the continuity of bidder 1's payoff at this boundary (in fact this premise is satisfied at all points).

exceeds the second-highest reported type so that $t'_1 \geq S(\mathbf{t}'_{-1})$, rewrite it as

$$M_1 = u\left(S(\mathbf{t}'_{-1}); t'_2, ..., t'_n\right) - \Delta_b - \Delta_a - \tau, \tag{21}$$

where $\Delta_b \equiv (1-q)(u(t'_2; S(\mathbf{t}'_{-1}), ..., t'_n) - u(t'_2; t'_2, ..., t'_n))$ is the differential rent of bidder 1 when he has type $S(\mathbf{t}'_{-1})$ and reports truthfully vis à vis when he has a lower type t'_2 and reports truthfully. The logic for (21) mirrors that for (18). The first term in (21) is the value to bidder 1 of receiving all cash flows when he has type $S(\mathbf{t}'_{-1})$, which is the lowest type that he can report and still receive all cash flows. The payment reduction of Δ_b leaves a bidder 1 with boundary type $t_1 = S(\mathbf{t}'_{-1})$ indifferent between reporting any type in regions 1 and 2 (such reports yield payoff $\Delta_a + \Delta_b$), again eliminating incentives for a local deviation.

Bidder 1's payment in (21) is reduced by two terms: Δ_a and Δ_b. When bidder 1's type sufficiently exceeds t'_2 he can deviate by reporting a lower type and still receive allocations in two ways. First, his reduced reported type can still exceed t'_2. Second, he can report an even lower type that falls slightly below t'_2. Each of these deviations yields differential rents (reflecting the allocation received for that type of deviation), and the differential rents earned at lower types add up and carry over to higher types reflecting the standard envelope theorem logic. This leads to the greater reduction in payments in (21) than (20) ($\Delta_a + \Delta_b$ vs. Δ_a).

The payments can alternatively be understood as follows. When bidder 1's report is not sufficiently above t'_2, (18) implies that bidder 1 pays for his awarded cash flows at two different unit prices. The second term means he pays for q shares at the lower unit price of $u(S^{-1}(\mathbf{t}'_{-1}); t'_2, ..., t'_n)$, and he pays for the remaining $(1-q) - q = 1 - 2q$ at the higher unit price of $u(t'_2; t'_2, ..., t'_n)$. This reflects that bidder 1 can deviate by bidding below t'_2, but above $S^{-1}(\mathbf{t}'_{-1})$. This would give him a share q at the lower unit price of $u(S^{-1}(\mathbf{t}'_{-1}); t'_2, ..., t'_n)$.[14] The payment in (18) rewards him for bidding truthfully by giving him the same benefit as if he had deviated (and hence would receive q shares at that lower price); he only pays the higher unit price for the remaining cash flows that he would not receive if he deviates.

The payment features detailed above make deviations unprofitable if they are within a given report-region, or if a bidder's type is at the boundary of two regions and he deviates locally. However, global ex-post IC further requires that given any realizations of the types

[14] Note that if bidder 1 deviates by bidding below t'_2, the highest among the n reported signals will be t'_2. This, by (19), yields bidder 1's (deviation) payment.

of other bidders, deviation be unprofitable for the remaining bidder when his true type falls in the interior of any of the four regions and he deviates to the interior of any other region.

To illustrate the added requirements needed for global IC, we revisit our example from the introduction where two bidders receive independently-distributed signals t_1 and t_2, $\tau = 0$, and the asset generates cash flows of $\frac{1}{1+\rho}(t_i + \rho t_{-i})$ if $i \in \{1, 2\}$ has control. Let bidder 2 with t_2 bids truthfully and consider bidder 1 with $t_1 \in (t_2, S(t_2))$. If bidder 1 reports truthfully, he receives $1-q$ shares and bidder 2 receives control. Thus, bidder 1's expected profit is the value of those cash flows, $(1-q)\frac{1}{1+\rho}(t_2 + \rho t_1)$, less his payment $(1-2q)t_2 + q\frac{1}{1+\rho}(S^{-1}(\mathbf{t}_2) + \rho t_2)$:

$$U_1(t_1, t_1' = t_1; t_2) = (1-q)\frac{1}{1+\rho}(t_2 + \rho t_1) - \left((1-2q)t_2 + q\frac{1}{1+\rho}\left(S^{-1}(\mathbf{t}_2) + \rho t_2\right)\right).$$

If bidder 1 deviates to report a type of $t_2 - \epsilon$, he receives control and q shares. His expected profit is the value of those cash flows, $q\frac{1}{1+\rho}(t_1 + \rho t_2)$, less his payment $q\frac{1}{1+\rho}(S^{-1}(\mathbf{t}_2) + \rho t_2)$:[15]

$$U_1(t_1, t_1' = t_2 - \epsilon; t_2) = q\frac{1}{1+\rho}(t_1 + \rho t_2) - q\frac{1}{1+\rho}\left(S^{-1}(\mathbf{t}_2) + \rho t_2\right).$$

His gain from deviation is the difference in these two expressions:

$$\frac{t_1 - t_2}{1+\rho}(q - (1-q)\rho).$$

This gain is non-positive under the premise of Proposition 2 that $\rho_{\min} \geq \frac{q}{1-q}$.

In principle, we have to consider all such deviations. In the appendix, we circumvent having to exhaustively rule out these many possibilities by exploiting Lemma 1 to simplify the establishment of IC. We show that $\rho_{\min} \geq \frac{q}{1-q}$ suffices to ensure the ex-post incentive compatibility of Mechanism A for any number of bidders and general (nonlinear) valuation functions.

When $q = 0$, the premise of Proposition 2 that $\rho_{\min} \geq \frac{q}{1-q}$ always holds. Proposition 2 extends the ex-post incentive compatibility of our two-stage mechanism in Definition 1, which corresponds to Mechanism A with $q = 0$, to settings where the minimum stake q is small enough relative to the minimum degree of common values ρ_{min}. Broadly, the intuition for why truthful reporting is an ex-post equilibrium is that when the high bidder does not receive control, (i) he still receives a sufficient share $1-q$ of cash flows, and (ii) the common value component ρ is large enough that the efficiency cost of having the bidder with the second-highest signal run the project is not too high. Obviously, the larger is $1-q$, the smaller is the requisite ρ.

[15]This follows from (19) after switching the index "1" and "2" as bidder 1's report is now the second highest.

The intuition for why separation functions exist that generate strictly higher revenues is the same as that for Result 1: when the two highest types are close enough, the seller strictly gains from separating cash-flow rights and control. A direct corollary of part (ii) is that for all $q_{\min} \in [0, 0.5)$, separation mechanisms exist that generate strictly higher expected revenues than no-separation English auctions as long as the common value component is large enough.

2.6 Efficient splitting of cash flow and control rights

We now construct mechanism B in which the highest reported type always receives control.

Definition 4 *(Mechanism B: efficient splitting of rights) Consider $q \in [q_{min}, 1)$. Without loss of generality let the reported types be $t'_1 \geq t'_2 \geq t'_3 ... \geq t'_n$. Then*

If $t'_1 > S(\mathbf{t}'_{-1})$, bidder 1 receives control and all cash flows, and pays

$$M_1 = (1-q)\, u\left(S(\mathbf{t}'_{-1}); t'_2, ..., t'_n\right) + (2q-1)\, u\left(t'_2; t'_2, ..., t'_n\right) + (1-q)\, u\left(t'_2; S^{-1}\left(\mathbf{t}'_{-1}\right), ..., t'_n\right) - \tau. \quad (22)$$

All other bidders receive nothing and pay nothing.

If $t'_1 \leq S(\mathbf{t}'_{-1})$, bidder 1 receives control and a share q of cash flows, and pays

$$M_1 = (2q-1)\, u\left(t'_2; t'_2, ..., t'_n\right) + (1-q)\left[u\left(t'_2; S^{-1}\left(\mathbf{t}'_{-1}\right), ..., t'_n\right)\right] - \tau, \quad (23)$$

while bidder 2 receives fraction $1-q$ of cash flows and pays

$$M_2 = (1-q)\, u\left(t'_1; S^{-1}\left(\mathbf{t}'_{-2}\right), t'_3, ..., t'_n\right). \quad (24)$$

All other bidders receive nothing and pay nothing.

With Mechanism B, when the highest reported type t'_1 does not exceed the second highest reported type t'_2 by enough, i.e., when $t'_1 < S(\mathbf{t}'_{-1})$, then the bidder 1 who reports the highest type receives control and share q of cash flows, and the second highest bidder 2 receives share $1-q$. When, instead, the highest reported type is sufficiently higher so that $t'_1 \geq S(\mathbf{t}'_{-1})$, then bidder 1 receives control and all cash flows.

Although the assignment of control is the opposite of Mechanism A, it shares key features:

- There are four report-regions; and a bidder i's payment and allocations do not depend on where his reported type is in a given report-region.

- When a bidder's type is at the boundary of any two adjacent regions, the payments leave the bidder indifferent to reporting any type in those regions. Thus, there is no incentive for local deviations, and when bidders bid truthfully, the differential rents earned at lower types add up and carry over to higher types.

- Truthful bidding is rewarded by a price discount. If a bidder would receive a share x of cash flows at a lower unit price by deviating to report a lower type, then when he bids his true type, he pays for the share x at that lower unit price; he only pays the higher unit price for the remaining cash flows that he would not have received if he deviated.[16]

These features deliver the local IC of Mechanism B. The additional requirements that ensure global IC are the opposite of those for Mechanism A, reflecting their opposite control assignments upon splitting. Mechanism A is ex-post IC when the cost of inefficient control assignment and share q for control are small enough. Mechanism B is ex-post IC in the opposite scenario where q is sufficiently large relative to the common-value component in valuations. We measure the maximum common-value component by:

$$\rho_{\max} \equiv \max_{\mathbf{t}} \frac{\partial v_2(\mathbf{t})}{\partial t_1} / \frac{\partial v_1(\mathbf{t})}{\partial t_1},$$

With linear valuations, i.e., with $v_i(t_1, ..., t_n) = A_n\big(t_i + \rho \sum_{j \neq i} t_j\big)$, we have $\rho_{\max} = \rho$. Routine modifications of the proof of Proposition 2 yield the ex-post IC of Mechanism B:

Proposition 3 *Suppose the common-value-component in valuations is sufficiently small and q is sufficiently large that $\rho_{\max} \leq \frac{q}{1-q}$. Then in Mechanism B*

(i) truthful reporting is an ex post equilibrium given any separation function S.

(ii) there exist separation functions S such that mechanism B generates strictly higher seller revenues than no-separation English auctions.

[16] If a bidder would receive more cash flows by deviating to reporting a lower type, the "remaining" cash flows would be negative. In this case, the bidder effectively "sells" the difference in the two cash flow amounts back to the seller at a higher unit price. Relatedly, unlike in no-separation mechanisms where global IC typically requires higher type bidders to receive greater allocations, in our separation mechanisms a higher type bidder may receive a reduced cash flow allocation if it is accompanied by a change of control.

When the highest bidder receives "enough" cash flows for control and the cash flow consequences of inefficiently assigning control matter enough, it reduces the value to the bidder with the highest signal of receiving cash flows but not control. This dis-incentivizes the bidder with the highest signal from reducing its bid to be the second-highest bidder, rendering Mechanism B ex-post incentive compatible.

The broad intuition for why separation functions exist that generate strictly higher revenues is the same as that for Mechanism A: when the two highest types are sufficiently close, the seller strictly gains from separating control and cash flows. Separation involves a cost-benefit tradeoff. Here, when cash flows (but not control) are assigned to the second-highest bidder, the seller benefits from bidders' reduced information rents due to cash flows being less sensitive to the signal of the bidder who receives the cash flows; while the costs are those of increased bidder rents due to assigning cash flows to a lower signal bidder. The key is that this cost *increases* from zero in the distance between the two highest signals, reflecting the standard envelope logic that rents earned by lower types cumulate to carry over to higher types. In contrast, the benefit due to reduced sensitivity is positive even when the difference in signals is zero. Thus, with sufficiently little separation, the benefit always outweighs the cost.

Proposition 3 is notably useful because a sufficiently large q (i) always satisfies the proposition's premise that $\rho_{\max} \leq \frac{q}{1-q}$ for Mechanism B to be ex-post incentive compatible;and (ii) it always satisfies the minimum-stake requirement that $q \geq q_{min}$ for any $q_{min} \in [0, 1)$. For example, if moral hazard is a sufficient concern that the bidder with control must always receive a majority of cash flow rights then $q \geq 1/2$ and hence $\frac{q}{1-q} \geq 1 \geq \rho_{max}$.

Thus, Mechanism B can always be designed to generate strictly higher seller revenues than no-separation English auctions by choosing q sufficiently high. If, instead, we fix q at any $q_{min} \in (0, 1)$, then observing that Proposition 2 holds for $\rho_{\min} \geq \frac{q}{1-q}$ and Proposition 3 holds for $\rho_{\max} \leq \frac{q}{1-q}$, we obtain:

Corollary 1 *Fix any $q = q_{min} \in [0, 1)$. Then there exist separation mechanisms that generate strictly higher seller revenues than no-separation English auctions.*

We next specialize to independently-distributed types to characterize optimal designs.

3 Optimal separation mechanisms: independent types

Assumption 1 *Bidder types are independently (and identically) distributed.*

Recall that $g(t_i)$ denotes the marginal density of t_i, which is continuous and strictly positive over $[\underline{t}, \bar{t}]$. Let $G(t_i)$ be the associated cdf. By independence, the joint density of $\mathbf{t}$ is $f(\mathbf{t}) = \Pi_{i=1}^{n} g(t_i)$ and the conditional marginal density of $\mathbf{t}_{-i}$ given t_i is $f_{-i}(\mathbf{t}_{-i}|t_i) = \Pi_{i\neq i}^{n} g(t_i)$.

With independent types, (15) reduces to

$$\frac{d\bar{U}_i(t_i, t_i)}{dt_i} = \int_{\Omega_{n-1}} \sum_j R_j(\mathbf{t}) Q_{ji}(\mathbf{t}) \frac{\partial v_j(t_i; \mathbf{t}_{-i})}{\partial t_i} \Pi_{j\neq i} g(t_j) d\mathbf{t}_{-i}. \tag{25}$$

Equation (25) conveys the advantages of separation in a more straightforward way than (15). With independent types, the terms in (15) related to signal correlations vanish, rendering the effects of separation more transparent: in (25) the differential rents are scaled by $\frac{\partial v_j(t_i; \mathbf{t}_{-i})}{\partial t_i}$. That is, bidder i's differential rents are weighted by the sensitivity of the value of his awarded cash flows to his signal when the project is run by bidder j.

Rewrite (25) as $\frac{d\bar{U}_i(t_i,t_i)}{dt_i} = J_i(t_i)$, where

$$J_i(t_i) \equiv \int_{\Omega_{n-1}} \sum_j R_j(\mathbf{t}) Q_{ji}(\mathbf{t}) \frac{\partial v_j(\mathbf{t})}{\partial t_i} \Pi_{j\neq i} g(t_j) d\mathbf{t}_{-i}. \tag{26}$$

Integration yields $\bar{U}_i(t_i, t_i) = \int_{\underline{t}}^{t_i} J_i(\widetilde{t}) d\widetilde{t} + \bar{U}_i(\underline{t}, \underline{t})$. Substituting this into seller revenues, and applying standard (integration by parts) mechanism design techniques yields:

Lemma 2 ***(Revenue Decomposition)*** *In any incentive-compatible mechanism, the seller's expected revenue (14) is given by*

$$\pi_s = \int_{\Omega_n} \hat{\pi}_s(\mathbf{t}) \Pi_{i=1}^{n} g(t_i) d\mathbf{t} - \sum_i \bar{U}_i(\underline{t}, \underline{t}), \tag{27}$$

where

$$\hat{\pi}_s(\mathbf{t}) \equiv \sum_{j,i} R_j(\mathbf{t}) Q_{ji}(\mathbf{t}) \phi_{ji}(\mathbf{t}, Q), \tag{28}$$

and

$$\phi_{ji}(\mathbf{t}) \equiv v_j(\mathbf{t}) - \frac{1 - G(t_i)}{g(t_i)} \frac{\partial v_j(\mathbf{t})}{\partial t_i} - \tau \tag{29}$$

is the matrix-form virtual valuation for cash flows generated by bidder j and assigned to i.

Proof: See the appendix. $\square$

Bidder rationality implies that $-\sum_i \bar{U}_i(\underline{t},\underline{t})$ is non-positive, and it reaches its maximum when $\bar{U}_i(\underline{t},\underline{t}) = 0$ for all i. The virtual valuations underlying $\hat{\pi}_s$ arise from the envelope theorem, which reflects local incentive compatibility. We next derive an upper bound on $\hat{\pi}_s$, and then explore which allocations achieve this upper bound and hence maximize $\hat{\pi}_s$, deferring the question of whether a mechanism with such allocations is globally incentive compatible.

Unlike standard settings where control and cash rights are not separated (i.e., where $Q_{jj}(\mathbf{t})$ is constrained to equal one), the virtual valuation in our separation framework takes a matrix form: it depends both on which bidders receive cash flows and which bidder generates them. Intuitively, ϕ_{ji} measures the rents a seller can extract per unit of cash flows generated by j and assigned to i. That is, the rents a seller can extract by assigning control to a bidder j depend on how she assigns cash-flow rights among bidders, conditional on j having control.

We now rewrite bidder valuations so as to account for the minimum stake requirement.

Definition 5 *For any $j = 1, ..., n$ and $i = 1, ..., n$, the minimum-stake-adjusted matrix-form valuation is*

$$\psi_{ji}(\mathbf{t}) \equiv v_j(\mathbf{t}) - q_{\min}\frac{1-G(t_j)}{g(t_j)}\frac{\partial v_j(\mathbf{t})}{\partial t_j} - (1-q_{\min})\,\frac{1-G(t_i)}{g(t_i)}\frac{\partial v_j(\mathbf{t})}{\partial t_i} - \tau. \tag{30}$$

Here, (30) is defined for both $i \neq j$ and $i = j$. When $i \neq j$, (30) represents the rents that the seller extracts from assigning control and share q_{min} of cash flows to j, and share $1-q_{min}$ of cash flows to another bidder i. When $i = j$, (30) reduces to $v_j(\mathbf{t}) - \frac{1-G(t_j)}{g(t_j)}\frac{\partial v_j(\mathbf{t})}{\partial t_j} - \tau$, which is the virtual valuation in a standard no-separation setting, and it represents the rents that the seller extracts from assigning control and all cash flows to a single bidder j.

Lemma 3 *Given any $\mathbf{t}$, among assignments that satisfy the feasibility conditions (6) and (7) and the minimum stake requirement (8), the following assignments of rights maximize $\hat{\pi}_s(\mathbf{t})$:*

(i) If $\max_{j,i}\psi_{ji}(\mathbf{t}) < 0$, then there is no sale (i.e., $R_j(\mathbf{t}) = 0$ for all j).

(ii) If $\max_{j,i}\psi_{ji}(\mathbf{t}) \geq 0$, then assign control and share q_{min} of cash flows to $\hat{\jmath}$, and assign share $1-q_{min}$ of cash flows to $\hat{\imath}$, where the pair $(\hat{\jmath},\hat{\imath})$ maximize $\psi_{ji}(\mathbf{t})$:

$$(\hat{\jmath},\hat{\imath}) \in \arg\max_{(j,i)} \psi_{ji}(\mathbf{t}). \tag{31}$$

Proof: See the appendix. $\square$

In essence, Lemma 3 says that for assignments that maximize $\hat{\pi}_s(\mathbf{t})$ under the minimum stake requirement it is without loss of generality to assume that either the constraint binds ($Q_{jj}(\mathbf{t}) = q_{min}$) or it is completely slack (i.e., no separation with $Q_{jj}(\mathbf{t}) = 1$).

We now specialize to linear valuations and identify conditions under which either Mechanism A or B is optimal. Qualitatively similar results obtain with nonlinear valuations.

3.1 Linear Valuations

Suppose valuations are linear as in (5). To ease notation, define the inverse hazard function,

$$L(t) \equiv \frac{1 - G(t)}{g(t)}.$$

Assumption 2 $L(t)$ *is strictly decreasing.*

Assumption 3 $\underline{t} - A_n L(\underline{t}) \geq \tau$.

Assumption 2 is a sufficient condition for the standard regularity condition that ensures the global IC (not just local IC) of the "no-separation" mechanism that always assigns control and cash flow rights to the bidder with the highest virtual valuation. Assumption 3 ensures that $\psi_{ji}(\mathbf{t},Q)$ (equation (30)) is nonnegative for any j and i, implying that always selling the asset is optimal (i.e., part (ii) of Lemma 3 holds). Under Assumptions 2 and 3, standard English auctions with no reserve price maximize seller revenues among all no-separation mechanisms. These assumptions simplify characterization of the optimal mechanism when control and cash flow rights can be separated.

By (5), the minimum-stake-adjusted matrix-form valuation (30) simplifies to

$$\psi_{ji}(\mathbf{t}) = \begin{cases} v_j(\mathbf{t}) - \tau - L(t_j) A_n & \text{if} \quad i = j \\ v_j(\mathbf{t}) - \tau - q_{\min} L(t_j) A_n - (1 - q_{\min}) L(t_i) A_n \rho & \text{if} \quad i \neq j. \end{cases} \tag{32}$$

We now explicitly characterize the $\hat{\pi}_s(\mathbf{t})$-maximizing allocations in Lemma 3. Let "h" and "s" index the highest and second-highest bidders, and let their associated signals be t_h and t_s.

Lemma 4 $\hat{\pi}_s(\mathbf{t})$*-maximizing allocations only allocate cash flow and control rights to h and s.*

Proof: See the appendix. □

Lemma 4 implies that the allocations that maximize $\hat{\pi}_s(\mathbf{t})$ take one of three forms: give control and all cash flows to the highest bidder; give control and cash-flow share q_{min} to the highest bidder and share $1-q_{min}$ to the second-highest bidder; or give control and cash-flow share q_{min} to the second-highest bidder and cash-flow share $1-q_m$ to the highest bidder. We now pin down when each of these allocations maximize $\hat{\pi}_s(\mathbf{t})$. From (32), the associated virtual valuations are given by

$$\psi_{hh}(\mathbf{t}) \quad = \quad v_h(\mathbf{t}) - \tau - L\left(t_h\right) A_n \tag{33}$$

$$\psi_{hs}(\mathbf{t}) = v_h(\mathbf{t}) - \tau - q_{\min} L\left(t_h\right) A_n - \left(1-q_{\min}\right) L\left(t_s\right) A_n \rho \tag{34}$$

and

$$\psi_{sh}(\mathbf{t}) = v_s(\mathbf{t}) - \tau - q_{\min} L\left(t_s\right) A_n - \left(1-q_{\min}\right) L\left(t_h\right) A_n \rho. \tag{35}$$

From Lemma 3, the $\hat{\pi}_s(\mathbf{t})$-maximizing allocation is associated with the highest virtual valuation. Allocating control and all cash flows to the highest bidder is optimal if and only if $\psi_{hh} \geq \max\{\psi_{hs}, \psi_{sh}\}$. We have

$$\psi_{hh} - \psi_{hs} = \left(1-q_m\right) A_n \left(\rho L\left(t_s\right) - L\left(t_h\right)\right), \tag{36}$$

which yields $\psi_{hh} \geq \psi_{hs}$ if and only if

$$t_h \geq K_B\left(t_s\right) \equiv L^{-1}\left(\rho L(t_s)\right), \tag{37}$$

where L^{-1} is the inverse function of L. L is decreasing by Assumption 2 and $L\left(\bar{t}\right) = 0$, so $K_A(t_s) \in [t_s, \bar{t}]$ is increasing in t_s. Similarly,

$$\psi_{hh} - \psi_{sh} = A_n\left(\left(1-\rho\right)\left(t_h - t_s\right) + q_{\min} L\left(t_s\right) - \left(1-\left(1-q_{\min}\right)\rho\right) L\left(t_h\right)\right) \equiv \hat{K}_A\left(t_s, t_h\right). \tag{38}$$

Let $K_A(t_s)$ be the value of t_h that sets $\hat{K}_A\left(t_s, t_h\right)$ to zero. $K_A(t_s)$ increases in t_s (since $\hat{K}_A(y,x)$ strictly increases in y, $\hat{K}_A(x,x) \leq 0$ and $\hat{K}_A(x,\bar{t}) \geq 0$). It follows that $\psi_{hh} \geq \psi_{sh}$ if and only if $t_h \geq K_A\left(t_s\right)$. Thus, allocating control and all cash flows to the highest bidder is optimal if and only if

$$t_h \geq \max\left\{K_A(t_s), K_B(t_s)\right\} \equiv K(t_s). \tag{39}$$

It remains to determine the optimal way to divide control and cash flow rights. We have

$$\psi_{hs} - \psi_{sh} = A_n \left((1-\rho)(t_h - t_s) - ((1-q_{\min})\rho - q_{\min})(L(t_s) - L(t_h)) \right). \tag{40}$$

The right-hand side of (40) strictly decreases in ρ, and it equals zero at

$$\bar{\rho} \equiv 1 - \frac{(1-2q_{\min})(L(t_s) - L(t_h))}{(t_h - t_s) + (1-q_{\min})(L(t_s) - L(t_h))} > 0. \tag{41}$$

That is, when splitting is optimal, control rights should be assigned to the bidder with the second-highest valuation if and only if the associated efficiency cost in terms of lost cash flows is sufficiently small, i.e., if and only if the common value component is large enough, $\rho \geq \bar{\rho}$.

Combining these results, we have established:

Proposition 4 *The following allocations maximize $\hat{\pi}_s$:*

(i) If $t_h \leq K(t_s)$, then cash-flow rights and control are split between bidders h and s.

If $\rho \leq \bar{\rho}$, then assign control and $q_{\min}$ cash flows to bidder h and $1 - q_{\min}$ cash flows to bidder s.

If $\rho \geq \bar{\rho}$, then assign control and $q_{\min}$ cash flows to bidder s and $1 - q_{\min}$ cash flows to bidder h.

(ii) If $t_h > K(t_s)$, then assign bidder h all cash flows and control.

Note that if $q_{\min} \geq 0.5$, then (41) yields $\bar{\rho} \geq 1$. Therefore, the condition $\rho \leq \bar{\rho}$ in part (i) of Proposition 4 always holds, so that if $t_h < K(t_s)$, then the $\hat{\pi}_s$-maximizing allocations are to assign control and $q_{\min}$ cash flows to bidder h and $1 - q_{\min}$ cash flows to bidder s.

We next construct optimal separation mechanisms. Define

$$\delta_1 \equiv \max_{t \in [\underline{t}, \bar{t}]} \frac{d}{dt} L(t) \quad \text{and} \quad \delta_2 \equiv \min_{t \in [\underline{t}, \bar{t}]} \frac{d}{dt} L(t). \tag{42}$$

By Assumption 2, $\delta_2 \leq \delta_1 < 0$. Observing that $K_A(t_s)$ and $K_B(t_s)$ correspond to the separation functions for Mechanisms A and B, we establish:

Proposition 5 *(i) If $q_{\min} \geq 0.5$, then Mechanism B with $q = q_{\min}$ and separation function K_B is the optimal separation mechanism.*

(ii) If $q_{\min} < 0.5$ and $\rho \leq \min\left\{1 + \frac{(1-2q_{\min})\delta_2}{1-(1-q_{\min})\delta_2}, \frac{q_{\min}}{1-q_{\min}}\right\}$, then again Mechanism B with $q = q_{\min}$ and separation function K_B is the optimal separation mechanism.

(iii) If $q_{\min} < 0.5$ *and* $\rho \geq \max\left\{1 + \frac{(1-2q_{\min})\delta_1}{1-(1-q_{\min})\delta_1}, \frac{q_{\min}}{1-q_{\min}}\right\}$, *then Mechanism A with* $q = q_{\min}$ *and separation function* K_A *is the optimal separation mechanism.*

Proof: See the appendix. □

The proposition says that if q_{min} is large, then it is optimal to always have the highest type control the project; but, if q_{min} is smaller, the optimal way to divide rights depends on the degree of common valuation. For example, with $q_{min} = 0.4$ and uniformly-distributed signals, Mechanism A is optimal for $\rho \geq 7/8$, and Mechanism B is optimal for $\rho \leq 2/3$.[17] Thus, either Mechanism A or B is the optimal no-separation mechanism unless ρ is in a small interval, $\rho \in (2/3, 7/8)$.

However, optimal mechanisms only require interim incentive compatibility, and the mechanisms identified in Proposition 5 are ex-post incentive compatible. We now use Lemma 1 to identify the weaker conditions needed for interim incentive compatibility, interpreting $L(t, t')$ as $U_i(t_i, t_i')$, i.e., $L(t, t')$ is i's expected profit when he has type t_i but reports t_i' and all other bidders report truthfully. With independent types, (11) simplifies to

$$\begin{aligned}\bar{U}_i(t_i, t_i') &= \int_{\Omega_{n-1}} \sum_j R_j(t_i'; \mathbf{t}_{-i}) Q_{ji}(t_i'; \mathbf{t}_{-i}) v_j(\mathbf{t}) \Pi_{j \neq i} g(t_j) d\mathbf{t}_{-i} \\ &\quad - \int_{\Omega_{n-1}} M_i(t_i'; \mathbf{t}_{-i}) \Pi_{j \neq i} g(t_j) d\mathbf{t}_{-i} - \tau \int_{\Omega_{n-1}} R_i(t_i'; \mathbf{t}_{-i}) \Pi_{j \neq i} g(t_j) d\mathbf{t}_{-i}, \quad (43)\end{aligned}$$

which yields

$$\frac{\partial}{\partial t_i} \bar{U}_i(t_i, t_i') = \int_{\Omega_{n-1}} \sum_j R_j(t_i'; \mathbf{t}_{-i}) Q_{ji}(t_i'; \mathbf{t}_{-i}) \frac{\partial v_j(\mathbf{t})}{\partial t_i} \Pi_{j \neq i} g(t_j) d\mathbf{t}_{-i}. \quad (44)$$

With linear valuations, $\frac{\partial v_j(\mathbf{t})}{\partial t_i} \equiv W_{ji}$, where $W_{ji} = A_n$ for $i = j$ and $W_{ji} = A_n \rho$ for $i \neq j$. Comparing (44) with (26) and noting that W_{ji} are constants, we have

$$\frac{\partial}{\partial t_i} \bar{U}_i(t_i, t_i') = J_i(t_i') = E_{-i}\left[\sum_j R_j(t_i'; \mathbf{t}_{-i}) Q_{ji}(t_i'; \mathbf{t}_{-i}) W_{ji}\right], \quad (45)$$

where E_{-i} denotes expectation over $\mathbf{t}_{-i}$. $\bar{U}_i(t_i, t_i')$ is the expected payoff after integrating over other bidders' type, so it is continuous and differentiable with respect to t_i'. Therefore,

[17] To see these results, note that $\frac{q_{\min}}{1-q_{\min}} = \frac{2}{3}$. Further, with a uniform distribution $\delta_1 = \delta_2 = -1$, and hence $1 + \frac{(1-2q_{\min})\delta_2}{1-(1-q_{\min})\delta_2} = 1 + \frac{(1-2q_{\min})\delta_1}{1-(1-q_{\min})\delta_1} = \frac{7}{8}$.

the associated S_{zero-m} in Lemma 1 is the empty set. Part 1 of Lemma 1 says that a necessary condition for global IC is that for all $t_i \in [\underline{t}, \bar{t}]$, $\frac{\partial}{\partial t'} J_i(t'_i)|_{t'_i=t_i} \geq 0$ for $t'_i \in [\underline{t}, \bar{t}]$, where $J_i \equiv \frac{d\bar{U}_i(t_i,t_i)}{dt_i}$ is given in (26). As $J_i(t'_i)$ does not depend on t_i (with linear valuations $\frac{\partial v_j(\mathbf{t})}{\partial t_i}$ does not depend on t_i), this necessary condition is equivalent to requiring $\frac{d}{dt'_i} J_i(t'_i) \geq 0$ for all $t'_i \in [\underline{t}, \bar{t}]$. Relabeling t'_i as t_i, this necessary condition is equivalent to requiring $\frac{d}{dt_i} J_i(t_i) \geq 0$ for all $t_i \in [\underline{t}, \bar{t}]$. This, by Part 2 of Lemma 1 and (45), is also the sufficient condition. Thus, with linear valuations, the necessary and sufficient conditions coincide:[18]

Lemma 5 *The condition that $J_i(t_i)$ be weakly increasing in t_i over $t_i \in [\underline{t}, \bar{t}]$ is both necessary and sufficient for local interim IC to imply global interim IC in the separation mechanism.*

In standard no-separation mechanisms with linear valuations, the necessary and sufficient condition for local interim IC to imply global interim IC is that the expected allocation be non-decreasing. In contrast, for our separation mechanism, the analogous condition is that the expected allocations weighted by W_{ji} (the sensitivity of cash flows under one bidder's control to the signals of other bidders) be non-decreasing. Thus, in separation mechanisms a bidder's expected allocation can *decrease* in his type if this decrease is compensated by an increased probability for control. Intuitively, separation mechanisms provide *two* distinct ways to incentivize a high type bidder not to deviate to reporting a lower type: (i) as in a no-separation mechanism, rewarding a high report by assigning more allocations; (ii) rewarding a high report by assigning a higher probability of control (conditional on given allocations). The increased probability of control is a reward for a high type because a higher type benefits more from running the project by himself rather than having it run by another bidder.

This latter channel,[19] which is novel, is closed in standard no-separation mechanisms because there, any cash flow allocation assigned to a bidder is always generated by that bidder's control—the probability of control conditional on being given cash flows is always one. Lemma 5 is more general than the standard result for IC in no-separation mechanisms, reducing to the latter if we set $Q_{jj} = 1$ and $Q_{ji} = 0$ for all $i \neq j$. Our separation mechanism allows the conditional probability of control to be less than one and vary with bidder type. This gives a seller more leeway in the mechanism design, facilitating rent extraction.

[18] With non-linear valuations, one can use Lemma 1 to derive necessary and sufficient conditions.

[19] This logic also underlies the conditions $\rho_{\min} \geq \frac{q}{1-q}$ in Proposition 2 and $\rho_{\max} \leq \frac{q}{1-q}$ in Proposition 3.

With this result, parts (ii) and (iii) of Proposition 5 generalize as follows:

Proposition 6 *(i) Suppose* $q_{\min} < 0.5$ *and* $\rho \leq 1 + \frac{(1-2q_{\min})\delta_2}{1-(1-q_{\min})\delta_2}$. *Then Mechanism B with* $q = q_{\min}$ *and separation function* K_B *is the optimal no-separation mechanism if and only if the associated* $J_i(t_i)$ *weakly increases in* $t_i \in [\underline{t}, \bar{t}]$. *(ii) Suppose* $q_{\min} < 0.5$ *and* $\rho \geq 1+\frac{(1-2q_{\min})\delta_1}{1-(1-q_{\min})\delta_1}$. *Then Mechanism A with* $q = q_{\min}$ *and separation function* K_A *is the optimal no-separation mechanism if and only if the associated* $J_i(t_i)$ *weakly increases in* $t_i \in [\underline{t}, \bar{t}]$.

Proof: See the Appendix. □

The parameter space identified in Proposition 6 is (weakly) larger than that identified in Proposition 5—where the extent to which it is "larger" varies with the number of bidders. This differs from standard no-separation mechanisms in which the regularity condition for global IC does not depend on the number of bidders.

4 Multi-dimensional signals

We now illustrate that our qualitative findings about the advantages of separation mechanisms do not hinge on the one-dimensional nature of signals. To show this, we consider two additional sources of bidder private information. First, the cash flows generated by a bidder i's control are given by $(1-\mu)\theta_i + \mu\big(\frac{1}{n}\sum_k \gamma_k\big)$, where $\theta_i \in \big[\underline{\theta}, \bar{\theta}\big]$ is a privately-observed, bidder-specific component that reflects i's skills, $\gamma_i \in \big[\underline{\gamma}, \bar{\gamma}\big]$ is a privately-observed common component, and μ measures the extent of common valuations.[20] Second, bidder i's opportunity cost $\tau_i \in [\underline{\tau}, \bar{\tau}]$ can be i's private information, and τ_i can take on negative values, i.e., the bidder may receive a benefit of control that is his private information (see, e.g., Ekmekci, Kos and Vohra (2016)). We assume that θ_i, γ_i and τ_i, are independently distributed.

When i has control and receives cash flow share q_i, his expected payoff is

$$v_i = q_i\Big[(1-\mu)\theta_i + \mu\Big(\frac{1}{n}\sum_k \gamma_k\Big)\Big] - \tau_i.$$

[20]Our base model essentially assumed that γ_i and θ_i were perfectly correlated. One can motivate this structure in a two-bidder setting where the project has two divisions (lines of business) with values of v_1 and v_2, and total value $v_1 + v_2$, where each bidder is an expert in one line of business. The expert in line i receives a private signal t_i about v_i, and he is more efficient at running line i than the other line. Specifically, the project's value is $(1+\alpha)t_i + t_{-i}$ when run by expert i rather than by $-i$, where where $\alpha > 0$ captures the feature that the expert for line i is more efficient at running i.

When, instead, a bidder $j \neq i$ has control, i's expected payoff is

$$v_i = q_i\Big[(1-\mu)\theta_j + \mu\Big(\frac{1}{n}\sum_k \gamma_k\Big)\Big].$$

With multi-dimensional signals, mechanism design is challenging. We next identify a class of separation mechanisms in which the three-dimensional signals reduce to a single dimension, rendering analysis tractable.

Restricted mechanism: The project is always sold with one bidder receiving control and a fixed share $q \geq q_{\min}$ of cash flows, and each other bidder receiving share $\frac{1-q}{n-1}$.

The no-separation mechanisms considered previously allowed a single bidder to receive control and all cash flows. The restriction to mechanisms in which all losing bidders receive a common share of cash flows obviously reduces a seller's ability to extract rents. Nonetheless, we identify conditions under which this restricted design still generates strictly higher expected revenues than the English no-separation mechanism.

We denote bidder i's type by $t_i = (\theta_i, \gamma_i, \tau_i)$; all other notation is unchanged. We have:

$$\begin{aligned} U_i(t_i, t_i'; \mathbf{t}_{-i}) = {} & qR_i(t_i'; \mathbf{t}_{-i})\Big((1-\mu)\theta_i + \frac{\mu}{n}\sum_k \gamma_k\Big) \\ & + \frac{1-q}{n-1}\sum_{j\neq i}\Big(R_j(t_i'; \mathbf{t}_{-i})\Big((1-\mu)\theta\gamma_j + \frac{\mu}{n}\sum_k \gamma_k\Big)\Big) - R_i(t_i'; \mathbf{t}_{-i})\tau_i - M_i(t_i'; \mathbf{t}_{-i}). \end{aligned} \tag{46}$$

Using (i) $\sum_k \gamma_k = \sum_{k\neq i}\gamma_k + \gamma_i$ and (ii) $\sum_{j\neq i} R_j(t_i'; \mathbf{t}_{-i}) = 1 - R_i(t_i'; \mathbf{t}_{-i})$ (because the asset is always sold in the restricted mechanism), we substitute

$$y_i \equiv q(1-\mu)\theta_i + \frac{nq-1}{n^2-n}\mu\gamma_i - \tau_i \tag{47}$$

to rewrite bidder i's expected payoff in (46) from reporting t_i' given t_i and $\mathbf{t}_{-i}$ as

$$U_i(t_i, t_i'; \mathbf{t}_{-i}) = y_i R_i(t_i'; \mathbf{t}_{-i}) - M_i(t_i'; \mathbf{t}_{-i}) + U_i^*,$$

where

$$U_i^* \equiv qR_i(t_i'; \mathbf{t}_{-i})\Big(\frac{\mu}{n}\sum_{k\neq i}\gamma_k\Big) + \frac{1-q}{n-1}\sum_{j\neq i}\Big(R_j(t_i'; \mathbf{t}_{-i})\Big((1-\mu)\theta_j + \frac{\mu}{n}\sum_{k\neq i}\gamma_k\Big)\Big) + \frac{1-q}{n-1}\frac{\mu}{n}\gamma_i.$$

This reveals that auction outcomes (R and M) only interact with i's private information via a single dimension y_i in bidder i's expected payoff. In particular, $\theta_i, \gamma_i, \tau_i$ and R do not interact in U_i^*. It follows that auction outcomes only depend on i's private information via y_i.[21] This dimension reduction preserves tractability. Note that this does *not* mean that i's payoff only depends on one dimension of private information; indeed, U_i^* also depends directly on γ_i.[22]

Proposition 7 *Truth telling is an equilibrium to the direct mechanism below in which each bidder i reports a single quantity y_i'. In this direct mechanism, when, without loss of generality, the reported types are $y_1' \geq y_2' \geq ... \geq y_n'$,*

(i) bidder 1 receives control and cash flow share q, and pays

$$M_1 = y_2' + q\frac{\mu}{n}\sum_{k\neq 1} E\left[\gamma | y = y_k'\right] + \frac{1-q}{n-1}\frac{\mu}{n}\underline{\gamma}, \tag{48}$$

where $y \equiv q(1-\mu)\theta + \frac{nq-1}{n^2-n}\mu\gamma - \tau$ is a function of θ, γ, τ as in (47).

(ii) each bidder $i > 1$ receives cash flow share $\frac{1-q}{n-1}$ and pays

$$M_{loser} = \frac{1-q}{n-1}\Big[(1-\mu)E\left[\theta | y = y_1'\right] + \frac{\mu}{n}\sum_{k\neq i} E\left[\gamma | y = y_k'\right] + \frac{\mu}{n}\underline{\gamma}\Big]. \tag{49}$$

Proof: We show that when bidders' true types y_i are such that $y_1 \geq y_2 \geq ... \geq y_n$, each bidder i is weakly better off reporting y_i if all other bidders truthfully report.

First consider $i = 1$. Then bidder 1 wins control by truthfully reporting. Bidder 1's equilibrium expected profit is the expected value of cash flows awarded to him less his opportunity cost and less the expected payment:

$$\begin{aligned}\pi_1 &= q\Big[(1-\mu)\theta_1 + \frac{\mu}{n}\sum_{k\neq 1} E\left[\gamma | y = y_k\right] + \frac{\mu}{n}\gamma_1\Big] - \tau_1 - \Big(y_2 + q\frac{\mu}{n}\sum_{k\neq 1} E\left[\gamma | y = y_k\right] + \frac{1-q}{n-1}\frac{\mu}{n}\underline{\gamma}\Big) \\ &= q\Big[(1-\mu)\theta_1 + \frac{\mu}{n}\gamma_1\Big] - \tau_1 - \Big(y_2 + \frac{1-q}{n-1}\frac{\mu}{n}\underline{\gamma}\Big).\end{aligned}$$

If bidder 1 deviates to $y_1' < y_2$, then he does not win control, obtaining expected profit

$$\pi_1^{deviate} = \frac{1-q}{n-1}\frac{\mu}{n}\left(\gamma_1 - \underline{\gamma}\right) \geq 0. \tag{50}$$

Algebra yields that such a deviation is unprofitable:

$$\begin{aligned}\pi_1^{deviate} - \pi_1 &= \frac{1-q}{n-1}\frac{\mu}{n}\left(\gamma_1 - \underline{\gamma}\right) - \Big(q\Big[(1-\mu)\theta_1 + \frac{\mu}{n}\gamma_1\Big] - \tau_1 - \Big(y_2 + \frac{1-q}{n-1}\frac{\mu}{n}\underline{\gamma}\Big)\Big) \\ &= y_2 - y_1 \leq 0.\end{aligned}$$

[21] More generally, this holds when for any two reported type profiles, t_i' and t_i'', the difference $U_i(t_i, t_i'; \mathbf{t}_{-i}) - U_i(t_i, t_i''; \mathbf{t}_{-i})$ depends on the components θ_i, γ_i, and τ_i of t_i only via y_i.

[22] Bidder i's payoff *net* of $\frac{1-q}{n-1}\mu\gamma_i$ depends only on one dimension of private information.

Finally, (50) yields $\pi_1^{deviate} \geq 0$. Thus, $\pi_1^{deviate} - \pi_1 \leq 0$ yields $\pi_1 \geq 0$, i.e., bidder 1's individual rationality constraint is satisfied.

Next consider $i > 1$. Then i does not win control by truthfully reporting. Bidder i's equilibrium expected profit is

$$\pi_i = \frac{1-q}{n-1}\frac{\mu}{n}\left(\gamma_i - \underline{\gamma}\right) \geq 0. \tag{51}$$

If i deviates by reporting $y_i' > y_1$ then i wins control. His expected deviation profit is the expected value of share q of cash flows less his opportunity cost and less the expected payment:

$$\begin{aligned}\pi_i^{deviate} &= q\Big[(1-\mu)\theta_i + \frac{\mu}{n}\sum_{k\neq i} E\left[\gamma | y = y_k\right] + \frac{\mu}{n}\gamma_i\Big] - \tau_i - \left(y_1 + q\frac{\mu}{n}\sum_{k\neq i} E\left[\gamma | y = y_k\right] + \frac{1-q}{n-1}\frac{\mu}{n}\underline{\gamma}\right) \\ &= q\Big[(1-\mu)\theta_i + \frac{\mu}{n}\gamma_i\Big] - \tau_i - \left(y_1 + \frac{1-q}{n-1}\frac{\mu}{n}\underline{\gamma}\right).\end{aligned}$$

Algebra yields that such a deviation is unprofitable:

$$\begin{aligned}\pi_i^{deviate} - \pi_i &= q\Big[(1-\mu)\theta_i + \frac{\mu}{n}\gamma_i\Big] - \tau_i - \left(y_1 + \frac{1-q}{n-1}\frac{\mu}{n}\underline{\gamma}\right) - \frac{1-q}{n-1}\frac{\mu}{n}\left(\gamma_i - \underline{\gamma}\right) \\ &= y_i - y_1 < 0.\end{aligned}$$

Finally, (51) yields $\pi_i \geq 0$, i.e., bidder i's individual rationality constraint is satisfied. □

For no-separation mechanisms, plugging $q = 1$ into (47) yields that outcomes only depend on bidder i's private information via

$$y_i^{no-sep} \equiv (1-\mu)\theta_i + \frac{\mu}{n}\gamma_i - \tau_i. \tag{52}$$

Assume without loss of generality that $y_1^{no-sep} \geq y_2^{no-sep} \geq ... \geq y_n^{no-sep}$. Then in the symmetric equilibrium of the no-separation English auction, bidder 1 wins control and all cash flows, and bidders $i > 1$ pay nothing. Bidder 1's payment to the seller is obtained by replacing y_i with y_i^{no-sep} and plugging $q = 1$ into the right-hand side of (48):

$$M_1^{no-sep} = y_2^{no-sep} + \frac{\mu}{n}\sum_{k\neq 1} E\left[\gamma | y = y_k^{no-sep}\right]. \tag{53}$$

One can show that for any q and n, the restricted-class separation mechanism generates greater revenues than no-separation English auctions for all μ sufficiently small and sufficiently limited private information about τ_i.[23]

We finish with an observation about social welfare. The point is clearest with no asymmetric information about τ, so we first set $\tau = 0$. Then (47) yields $y_i = q(1-\mu)(\theta_i + k^{sep}\gamma_i)$, where $k^{sep} \equiv \frac{1}{q}\frac{nq-1}{n^2-n}\frac{\mu}{1-\mu}$, and (52) yields $y_i^{no-sep} = (1-\mu)(\theta_i + k^{no-sep}\gamma_i)$, where $k^{no-sep} \equiv \frac{\mu}{n(1-\mu)}$. Thus, our separation mechanisms select the bidder with the highest $\theta_i + k^{sep}\gamma_i$ to run the project, whereas no-separation mechanisms select the one with the highest $\theta_i + k^{no-sep}\gamma_i$. Social welfare maximization requires the bidder with the highest θ_i run the project: welfare is higher when the coefficient k on γ_i is smaller. Crucially, $k^{no-sep} > k^{sep} \geq 0$ for all $q \in [\frac{1}{n}, 1)$. This means that controller-selection in the separation mechanism yields higher welfare than no-separation mechanisms.[24] In particular, $k^{sep} = 0$ when $q = \frac{1}{n}$. It follows that social welfare is maximized by the separation mechanism that gives each bidder the same cash flow share, and that no-separation mechanisms do strictly worse. When the extent of information asymmetry on τ is not zero but small, continuity of the separation mechanism in Proposition 7 implies that it still generate higher social welfare than no-separation mechanisms.

This has implications for bankruptcy resolution. As Hart (2023) points out, there are different approaches to bankruptcy resolution reflecting the conflicting possible objectives (e.g., welfare maximization versus revenue maximization) and hence there is no one-size-fits-all approach. In this regard, our analysis of separation mechanisms suggests an advance. In bankruptcy resolution settings, agents likely have multiple dimensions of private information, the court cares about both revenues and efficiencies, and the court's strong bargaining power gives it more leeway than a typical seller in structuring allocations. We identify when separation mechanisms can improve outcomes, leading both to higher revenues (presumably reducing the probability of bankruptcy and any resulting deadweight loss) and to higher social efficiency in terms of assigning control to the agent who generates the highest cash flows.

[23] For example, solving equations (48), (49) and (53) with two bidders and $\theta \sim uniform[\underline{\theta}, \underline{\theta}+1]$, $\gamma \sim uniform[\underline{\gamma}, \underline{\gamma}+1]$, and $\tau \sim uniform[\underline{\tau}, \underline{\tau}+\Delta\tau]$ with $\Delta\tau \geq 0$, yields that when $\mu = 0.4$ and $q = 0.8$, the restricted-class separation mechanism generates higher expected revenues than no-separation mechanisms as long as $\Delta_\tau \leq 0.36$; when $\mu = 0.1$, it does so for all $\Delta_\tau \leq 4.3$; and a larger q raises the range for which the restricted-class mechanism generates higher expected revenues. Note also that the values of $\underline{\theta}$, $\underline{\gamma}$, and $\underline{\tau}$ do not affect revenue comparisons of the two mechanisms.

[24] Thus when bidders have multiple-dimensional private information, standard no-separation mechanisms typically do not lead to efficient allocations even if bidders are ex ante identical.

5 Conclusions

Our paper revisits the classical auction setting in which a seller seeks to sell a single asset/project to potential bidders who privately receive independently-distributed signals about the asset's future cash flows. The asset's payoffs hinge on both the signal of the bidder who controls the asset and those of rival bidders. The literature characterizes optimal mechanisms when bidders who do not receive control receive no cash flows. We show that a seller can increase expected revenues by sometimes allocating cash flows to a bidder who does not control the project. The qualitative nature of these findings extend when bidders have private information over both private and common values of cash flows and private information about their opportunity costs: we identify when a simple separation mechanism generates both higher seller revenues and higher social welfare than no-separation mechanisms.

Separating control and cash-flow rights helps rent extraction because a bidder's valuation is more sensitive to his private information than to that of other bidders. This means that the expected value of cash-flow rights is less sensitive to a bidder's signal when he does not control the project, reducing his informational rents. When the two highest signals are sufficiently close, this benefit of separation outweighs its costs. We prove that a seller should award both rights to the bidder with the highest signal *only* when his signal sufficiently exceeds the second-highest signal. When the two highest signals are close, we characterize how optimal divisions of cash flow rights and control hinge on the private vs. common value composition of bidder signals and the minimum share that the controller must have to ameliorate moral hazard.

A final contribution is to identify necessary and (sharp) sufficient conditions for direct-revelation mechanisms to be globally incentive compatible. These conditions simplify establishing both ex-post and interim global incentive compatibility in many common settings where a bidder's payoff is not continuous in his reported type.

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7 Appendix

Proof of Result 1: Recall there exist signals $\bar{t} > t_2 \geq t_3 \geq ... \geq t_n > \underline{t}$ such that the strict single-crossing condition (4) holds. Denote one such signal vector by $t_2^*, ..., t_n^*$.

For any $\epsilon \in (0, \min\{\bar{t} - t_2^*, t_n^* - \underline{t}\})$, define the set

$$H_\epsilon \equiv \left\{ \begin{array}{c} (t_3, ..., t_n) : \exists \ (x_3^*, ..., x_n^*) \text{ that is a permutation of } (t_3^*, ..., t_n^*) \text{ such that} \\ t_i \in [x_i^* - \epsilon, x_i^*] \text{ for all } i = 3, ..., n \end{array} \right\}.$$

H_ϵ includes all points $(t_3, ..., t_n)$ in an ϵ-neighborhood of $(t_3^*, ..., t_n^*)$ and their permutations.

From the continuity of $u(t_1; ..., t_n)$ and its derivatives, there exists an $\omega > 0$ and an $\epsilon \in (0, \min\{\bar{t} - t_2^*, t_n^* - \underline{t}\})$ such that for all $t_2 \in [t_2^*, t_2^* + \epsilon/2]$, $t_1 \in [t_2, t_2^* + \epsilon]$, and $t_3, ..., t_n \in H_\epsilon$, inequality (4) holds with:

$$u_1(t_1; t_2, t_3, ..., t_n) - u_2(t_2; t_1, t_3, ..., t_n) \geq \omega. \tag{54}$$

We now show that a price offer of $p^* \equiv \frac{\epsilon}{2}\omega > 0$ will be accepted with strictly positive probability, which establishes the result. To proceed, consider n signals $t_1, ..., t_n$, where (wlog) t_1 and t_2 are the highest and second-highest signals. Suppose t_2 is in the interval $\left[t_2^*, t_2^* + \frac{\epsilon}{2}\right]$, and $t_3, t_4, ..., t_n$ are in H_ϵ. When $t_1 \in [t_2^* + \epsilon, \bar{t}]$, we have

$$\begin{aligned} u(t_1; t_2, ..., t_n) - u(t_2; t_1, ..., t_n) &= \int_{t_2}^{t_1} (u_1(t; t_2, ..., t_n) - u_2(t_2; t, ..., t_n))\, dt \\ &\geq \omega(t_1 - t_2) \geq \frac{\epsilon}{2}\omega, \end{aligned}$$

Thus, for all $t_1 \in [t_2^* + \epsilon, \bar{t}]$, a price offer of $p^* \equiv \frac{\epsilon}{2}\omega > 0$ is accepted. $\square$

Proof of Lemma 1: Define

$$D(t, t') \equiv L(t, t) - L(t, t'). \tag{55}$$

Claim 1: $t' = t$ maximizes $L(t, t')$ over $t' \in [\underline{t}, \bar{t}]$ for all given t if and only if

$$D(t, t') \geq 0 \quad \text{for all } t \text{ and } t'.$$

Proof: Immediate. By construction $D(t, t) = 0$ at all t.

Below, to economize on language, whenever we mention t and t', we assume that $t, t' \in [\underline{t}, \bar{t}]$. For any $t' \notin S_{zero-m}$ and t, we have

$$\begin{aligned} \frac{\partial}{\partial t} D(t, t') &= \frac{d}{dt} L(t, t) - \frac{\partial}{\partial t} L(t, t') = \frac{\partial}{\partial t} L(t, t')|_{t'=t} + \frac{\partial}{\partial t'} L(t, t')|_{t'=t} - \frac{\partial}{\partial t} L(t, t') \\ &= \frac{\partial}{\partial t} L(t, t')|_{t'=t} - \frac{\partial}{\partial t} L(t, t') \end{aligned} \tag{56}$$

where we have used (i) for $t' \notin S_{zero-m}$, $\frac{\partial}{\partial t'}\frac{\partial}{\partial t} L(t, t')$ exists, and hence $\frac{d}{dt} L(t, t) = \frac{\partial}{\partial t} L(t, t')|_{t'=t} + \frac{\partial}{\partial t'} L(t, t')|_{t'=t}$; and (ii) the lemma's premise that $\frac{\partial}{\partial t'} L(t, t')|_{t'=t} = 0$.

Differentiating both sides of (56) with respect to t yields

$$\frac{\partial^2}{\partial^2 t} D(t, t')|_{t'=t} = \frac{\partial^2}{\partial t^2} L(t, t')|_{t'=t} + \frac{\partial^2}{\partial t \partial t'} L(t, t')|_{t'=t} - \frac{\partial^2}{\partial^2 t} L(t, t') \text{ for } t' \notin S_{zero-m}. \tag{57}$$

At $t = t' \notin S_{zero-m}$, the first and third terms in (57) cancel out, so (57) reduces to

$$\frac{\partial^2}{\partial^2 t} D(t, t')|_{t'=t} = \frac{\partial^2}{\partial t \partial t'} L(t, t')|_{t'=t}. \tag{58}$$

Now we prove necessity. The premise that "for all given t, $t' = t$ maximizes $L(t, t')$ over $t' \in [\underline{t}, \bar{t}]$" and the "only if" part of Claim 1 together imply that for all given t', $t = t'$ minimizes $L(t, t')$. Further, (58) holds for $t \notin S_{zero-m}$. The second-order condition for minimization, $\frac{\partial^2}{\partial^2 t} D(t, t')|_{t'=t} \geq 0$, reduces to $\frac{\partial^2}{\partial t \partial t'} L(t, t')|_{t'=t} \geq 0$ via (58). This establishes necessity.

Next we establish the sufficient condition for Global IC: for all t, $\frac{\partial}{\partial t} L(t, t')$ weakly increases in t' over $t' \in [\underline{t}, \bar{t}]$).

Claim 2: $D(t, t') \geq 0$ for any $t' \notin S_{zero-m}$ and any t.

Proof: Suppose $t' \notin S_{zero-m}$. Then (56) holds. By the premise in part 2, if $t \geq t'$, then $\frac{\partial}{\partial t} L(t, t')|_{t'=t} \geq \frac{\partial}{\partial t} L(t, t')$. Via this, (56) yields $\frac{\partial}{\partial t} D(t, t') \geq 0$ for $t \geq t'$. By the same logic, if

$t \leq t'$, then $\frac{\partial}{\partial t} L(t, t')|_{t'=t} \leq \frac{\partial}{\partial t} L(t, t')$, and hence $\frac{\partial}{\partial t} D(t, t') \leq 0$ by (56). Thus, fixing t' and varying t, $D(t, t')$ is minimized at $t = t'$. Therefore, $D(t, t') \geq D(t', t') = 0$ for all t.

Claim 3: $D(t, t') \geq 0$ for any $t' \in S_{zero-m}$ and any t.

Proof: By contradiction. Suppose instead that $D(t^*, t^{**}) < 0$, for some $t^* \in [\underline{t}, \bar{t}]$ and $t^{**} \in S_{zero-m}$, where, throughout the proof, we use the short-hand notation $D(t^*, t^{**})$ to refer to $D(t = t^*, t' = t^{**})$. For any $t^{***} \in (\underline{t}, \bar{t})$, the definition of D in (55) yields

$$D(t^*, t^{***}) - D(t^*, t^{**}) = L(t^*, t^{**}) - L(t^*, t^{***}),$$

which we rewrite as

$$D(t^*, t^{***}) = L(t^*, t^{**}) - L(t^*, t^{***}) + D(t^*, t^{**}). \tag{59}$$

By the premise that $D(t^*, t^{**}) < 0$, we have $t^* \neq t^{**}$. Now we prove Claim 3 for the following two collectively exhaustive scenarios.

Scenario 1: $t^* > t^{**}$. Because $t^* \leq \bar{t}$, we have $t^{**} < \bar{t}$. Consider $t^{***} \in (t^{**}, \bar{t})$. By the premises that (i) $\frac{\partial}{\partial t} L(t, t')$ weakly increases in t', (ii) $t^* > t^{**}$ and, (iii) $t^{***} > t^{**}$, we have

$$L(t^*, t^{**}) - L(t^{**}, t^{**}) \leq L(t^*, t^{***}) - L(t^{**}, t^{***}). \tag{60}$$

Rearranging terms yields

$$L(t^*, t^{**}) \leq L(t^{**}, t^{**}) + L(t^*, t^{***}) - L(t^{**}, t^{***}).$$

Substituting this into the right-hand side of (59) yields

$$D(t^*, t^{***}) \leq L(t^{**}, t^{**}) - L(t^{**}, t^{***}) + D(t^*, t^{**}). \tag{61}$$

Because $L(t, t')$ is continuous in t' at $t' = t$, there exists $\epsilon > 0$ such that $L(t^{**}, t^{**}) - L(t^{**}, t^{***}) < -D(t^*, t^{**})$ for all $t^{***} \in (t^{**}, \min(\epsilon + t^{**}, \bar{t}))$. Here, note that $-D(t^*, t^{**}) > 0$ because $D(t^*, t^{**}) < 0$ by the contradiction premise. Next, note that because the points in S_{zero-m} are of zero measure, there exists a $t^{***} \in (t^{**}, \min(\epsilon + t^{**}, \bar{t}))$ such that $t^{***} \notin S_{zero-m}$. Then (61) yields $D(t^*, t^{***}) < 0$, which contradicts Claim 2, proving Claim 3 in Scenario 1.

Scenario 2: $t^* < t^{**}$. Because $t^* \geq \underline{t}$, we have $t^{**} > \underline{t}$. Consider $t^{***} \in (\underline{t}, t^{**})$. By the premises that (i) $\frac{\partial}{\partial t} L(t, t')$ weakly increases in t', (ii) $t^* < t^{**}$, and (iii) $t^{***} < t^{**}$, equation (59) from Scenario 1 also holds in Scenario 2. Similarly, (61) also holds in Scenario 2.

Because $L(t, t')$ is continuous in t' at $t' = t$, there exists $\epsilon > 0$ such that $L(t^{**}, t^{**}) - L(t^{**}, t^{***}) < -D(t^{*}, t^{**})$ for all $t^{***} \in (\max(t^{**} - \epsilon, \underline{t})\, t^{**})$. Further, because the points in S_{zero-m} are of zero measure, there must exist some $t^{***} \in (\max(t^{**} - \epsilon, \underline{t}), t^{**})$ such that $t^{***} \notin S_{zero-m}$. Then (61) yields $D(t^{*}, t^{***}) < 0$, which contradicts Claim 2. This proves Claim 3 for Scenario 2.

Claims 2 and 3 yield that $D(t, t') \geq 0$ for all t' and t. This and the "only if" part of Claim 1 establish the sufficiency of the conditions in 2. □

Proof of Proposition 2 (i): We use Lemma 1 to prove that mechanism A is ex-post incentive compatible, interpreting $L(t, t')$ as $U_i(t_i, t_i'; \mathbf{t}_{-i})$ in (9), and taking $\mathbf{t}_{-i}$ as given. Without loss of generality, consider bidder $i = 1$ and assume $t_2 \geq t_3 ... \geq t_n$.

We now show that Mechanism A satisfies the premises "a" – "c" of Lemma 1. Premise (a) is satisfied since $U_1(t_1, t_1'; \mathbf{t}_{-1})$ is continuous and differentiable with respect to t_1. Premise (b) is also satisfied: S_{zero-m} consists of three points, i.e., $S_{zero-m} = \{S^{-1}(t_2, t_3, ..., t_n), t_2, S(t_2, ..., t_n)\}$, and $U_1(t_1, t_1'; \mathbf{t}_{-1})$ is continuous and differentiable with respect to t_1' for all $t_1' \notin S_{zero-m}$. Premise (c) is also satisfied: $U_1(t_1, t_1'; \mathbf{t}_{-1})$ is independent of t_1' when t_1' is in any of the four report-regions defined in the text, which are partitioned by the elements in S_{zero-m}. Hence, for all $t_1 \notin S_{zero-m}$, $U_1(t_1, t_1'; \mathbf{t}_{-1})$ is continuous in t_1' at $t_1' = t_1$, and $\frac{\partial}{\partial t_1'} U_1(t_1, t_1'; \mathbf{t}_{-1})_{t_1' = t_1} = 0$. It is also easy to show that if $t_1 \in S_{zero-m}$, then $U_1(t_1, t_1'; \mathbf{t}_{-1})$ is continuous in t_1' at $t_1' = t_1$.

Next, refer to (9). For given $\mathbf{t}_{-1}$, we compute the derivative

$$\frac{\partial}{\partial t_1} U_1(t_1, t_1'; \mathbf{t}_{-1}) \equiv \sum_j R_j(t_1'; \mathbf{t}_{-1}) Q_{j1}(t_1'; \mathbf{t}_{-1}) \frac{\partial v_j(\mathbf{t})}{\partial t_1}. \tag{62}$$

Mechanism A has the feature that a bidder receives neither control nor cash flows if his reported type is not among the two highest reported types. Thus, in (62) we only have to sum over $j = 1, 2$, because (a) bidders 2 through n report truthfully and $t_2 \geq t_3 ... \geq t_n$, so t_2 must be one of the two highest reported types, and (b) if bidder 1 is awarded any cash flows, i.e., if $Q_{j1}(t_1'; \mathbf{t}_{-1}) \neq 0$ for any j, then t_1' must be among the two highest reported types. Thus, t_3 through t_n are outside the two highest reported types, so $R_j(t_1'; \mathbf{t}_{-1}) = 0$ for $j \geq 3$.

Therefore, (62) reduces to

$$\frac{\partial}{\partial t_1} U_1(t_1, t_1'; \mathbf{t}_{-1}) = R_1(t_1'; \mathbf{t}_{-1}) Q_{11}(t_1'; \mathbf{t}_{-1}) \frac{\partial v_1(\mathbf{t})}{\partial t_1} + R_2(t_1'; \mathbf{t}_{-1}) Q_{21}(t_1'; \mathbf{t}_{-1}) \frac{\partial v_2(\mathbf{t})}{\partial t_1}. \tag{63}$$

We use (63) to show that Mechanism A satisfies the premise in part B of Lemma 1. In our context this means that for all t_1, $\frac{\partial}{\partial t_1}U_1(t_1, t_1'; \mathbf{t}_{-1})$ weakly increases in t_1' for $t_1' \in [\underline{t}, \bar{t}]$. For Mechanism A, recall that $U_1(t_1, t_1'; \mathbf{t}_{-1})$ is independent of t_1' for t_1' in any of the four report-regions. Hence, for all $t_1' \notin S_{zero-m}$, $\frac{\partial}{\partial t_1}U_1(t_1, t_1'; \mathbf{t}_{-1})$ is independent of t_1'—and hence is weakly increasing in t_1'. Thus, we only need to show that for any given $\mathbf{t}_{-1}$, the right-hand side of (63) is nondecreasing in t_1' when $t_1' \in S_{zero-m}$. Consider, in turn, the three such realizations of t_1'.

Scenario 1: $t_1' = S^{-1}(t_2, t_3, ..., t_n)$. If $t_1' < S^{-1}(t_2, t_3, ..., t_n)$, then bidder 1 would receive no cash flows and hence $\frac{\partial}{\partial t_1}U_1(t_1, t_1'; \mathbf{t}_{-1}) = 0$; and if $t_1' > S^{-1}(t_2, t_3, ..., t_n)$, then the right-hand side of (63) is nonnegative. Thus, $\frac{\partial}{\partial t_1}U_1(t_1, t_1'; \mathbf{t}_{-1})$ weakly increases when t_1' increases from below $S^{-1}(t_2, t_3, ..., t_n)$ to above.

Scenario 2: $t_1' = t_2$. Refer to the right-hand side of (63). When $t_1' \to t_2$ from below, $R_1(t_1'; \mathbf{t}_{-1})Q_{11}(t_1'; \mathbf{t}_{-1}) = 1 \times q = q$, and $R_2(t_1'; \mathbf{t}_{-1}) = 0$, so $R_2(t_1'; \mathbf{t}_{-1})Q_{21}(t_1'; \mathbf{t}_{-1}) = 0$. Thus,

$$\lim_{t_1' \to t_2^-} \frac{\partial}{\partial t_1}U_1(t_1, t_1'; \mathbf{t}_{-1}) = q\frac{\partial v_1(\mathbf{t})}{\partial t_1},$$

where $\lim_{t_1' \to t_2^-}$ denotes the left limit. When $t_1' \to t_2$ from above, $R_2(t_1'; \mathbf{t}_{-1})Q_{21}(t_1'; \mathbf{t}_{-1}) = 1 - q$, and $R_1(t_1'; \mathbf{t}_{-1}) = 0$ so $R_1(t_1'; \mathbf{t}_{-1})Q_{11}(t_1'; \mathbf{t}_{-1}) = 0$. Thus,

$$\lim_{t_1' \to t_2^+} \frac{\partial}{\partial t_1}U_1(t_1, t_1'; \mathbf{t}_{-1}) = (1-q)\frac{\partial v_2(\mathbf{t})}{\partial t_1},$$

where $\lim_{t_1' \to t_2^+}$ denotes the right limit. Hence,

$$\begin{aligned}\lim_{t_1' \to t_2^+} \frac{\partial}{\partial t_1}U_1(t_1, t_1'; \mathbf{t}_{-1}) - \lim_{t_1' \to t_2^-} \frac{\partial}{\partial t_1}U_1(t_1, t_1'; \mathbf{t}_{-1}) &= (1-q)\frac{\partial v_2(\mathbf{t})}{\partial t_1} - q\frac{\partial v_1(\mathbf{t})}{\partial t_1} \\ &= q\frac{\partial v_1(\mathbf{t})}{\partial t_1}\left(\frac{1-q}{q}\frac{\partial v_2(\mathbf{t})}{\partial t_1}/\frac{\partial v_1(\mathbf{t})}{\partial t_1} - 1\right),\end{aligned}$$

where the inequality follows from the premise that $\rho_{\min} \geq \frac{q}{1-q}$. Hence, $\frac{\partial}{\partial t_1}U_1(t_1, t_1'; \mathbf{t}_{-1})$ weakly increases when t_1' increases from below t_2 to above t_2.

Scenario 3: $t_1' > S(t_2, ..., t_n)$. When $t_1' \to S(t_2, ..., t_n)$ from below, $R_2(t_1'; \mathbf{t}_{-1})Q_{21}(t_1'; \mathbf{t}_{-1}) = 1-q$, and $R_1(t_1'; \mathbf{t}_{-1}) = 0$ so $R_1(t_1'; \mathbf{t}_{-1})Q_{11}(t_1'; \mathbf{t}_{-1}) = 0$. Thus, $\frac{\partial}{\partial t_1}U_1(t_1, t_1'; \mathbf{t}_{-1}) \to (1-q)\frac{\partial v_2(\mathbf{t})}{\partial t_1}$. When $t_1' \to S(t_2, ..., t_n)$ from above, $R_2(t_1'; \mathbf{t}_{-1})Q_{21}(t_1'; \mathbf{t}_{-1}) = 0$, and $R_1(t_1'; \mathbf{t}_{-1})Q_{11}(t_1'; \mathbf{t}_{-1}) = 1$. Thus, $\frac{\partial}{\partial t_1}U_1(t_1, t_1'; \mathbf{t}_{-1}) \to \frac{\partial v_1(\mathbf{t})}{\partial t_1}$. Clearly, $\frac{\partial v_1(\mathbf{t})}{\partial t_1} \geq (1-q)\frac{\partial v_2(\mathbf{t})}{\partial t_1}$. Hence, $\frac{\partial}{\partial t_1}U_1(t_1, t_1'; \mathbf{t}_{-1})$ weakly increases when t_1' increases from below $S(t_2, ..., t_n)$ to above. □

Proof of Proposition 2 (ii): Recall there exist $n-1$ signals such that the strict single-crossing condition (4) holds. Denote one such signal vector by $t_2^*,...,t_n^*$, where $\bar{t} > t_2^* \geq t_3^* \geq ... \geq t_n^* > \underline{t}$.

For any $\epsilon \in (0, \min\{\bar{t}-t_2^*, t_n^* - \underline{t}\})$, define the set

$$H_\epsilon \equiv \left\{ \begin{array}{c} (t_3,...,t_n) : \exists \ (x_3^*,...,x_n^*) \text{ that is a permutation of } (t_3^*,...,t_n^*) \text{ such that} \\ t_i \in [x_i^* - \epsilon, x_i^*] \text{ for all } i = 3,...,n \end{array} \right\}.$$

H_ϵ includes all points $(t_3,...,t_n)$ in an ϵ-neighborhood of $(t_3^*,...,t_n^*)$ and their permutations.

From the continuity of $u(t_1;...,t_n)$ and its derivatives, there exists an $\omega > 0$ and an $\epsilon \in (0, \min\{\bar{t}-t_2^*, t_n^* - \underline{t}\})$ such that for all $t_2 \in [t_2^*, t_2^* + \epsilon]$, $t_1 \in [t_2, t_2^* + \epsilon]$ and $t_3,...,t_n \in H_\epsilon$, inequality (4) holds with:

$$u_1(t_1; t_2, t_3, ..., t_n) - u_2(t_2; t_1, t_3, ..., t_n) > \omega. \tag{64}$$

Fix such an ϵ. For any $\delta \in (0, \epsilon]$ and $t_2 \geq t_3 \geq ... \geq t_n$ define the "δ-separation function" $S_\delta(t_2,...,t_n)$ by:

$$\begin{aligned} S_\delta(t_2,...,t_n) &= t_2^* + \delta \text{ if } t_2 \in [t_2^*, t_2^* + \delta] \text{ and } t_3,...,t_n \in H_\epsilon; \\ S_\delta(t_2,...,t_n) &= t_2 \text{ otherwise.} \end{aligned}$$

This δ-separation function has the feature that separation occurs if and only if the highest of the $n-1$ signals is in $[t_2^*, t_2^* + \delta]$ and the other $n-2$ signals are in H_ϵ. Its inverse is given by

$$\begin{aligned} S_\delta^{-1}(t_2,...,t_n) &= t_2^* \text{ if } t_2 \in [t_2^*, t_2^* + \delta] \text{ and } t_3,...,t_n \in H_\epsilon; \\ S_\delta^{-1}(t_2,...,t_n) &= t_2 \text{ otherwise.} \end{aligned}$$

We show that for all δ sufficiently small, the Mechanism A that uses the δ-separation function generates strictly higher seller revenues than no-separation English auctions. Given any n signals $t_1,...,t_n$, denote seller revenue in the δ-separation mechanism minus that in the no-separation English auction by $D(t_1, t_2; t_3, ..., t_n)$. Note that $D(t_1, t_2; t_3, ..., t_n) \neq 0$ only when the second-highest signal among $t_1,...,t_n$ is in $[t_2^*, t_2^* + \delta]$ and all lower signals are in H_ϵ.

To fix ideas, consider n signals $t_1,...,t_n$ where t_1 and t_2 are the highest and second-highest signals, with $t_2 \in [t_2^*, t_2^* + \delta]$ and $(t_3,...,t_n) \in H_\epsilon$. Consider two cases:

Case 1: $t_1 \geq t_2^* + \delta$. Seller revenue in the δ-separation mechanism is bidder 1's payment in (17):

$$u(t_2^* + \delta; t_2, ..., t_n) - (1-q)\, u(t_2; t_2^* + \delta, ..., t_n) + (1-2q)\, u(t_2; t_2, ..., t_n)$$
$$+ qu(t_2^*, t_2, ..., t_n) - \tau.$$

Revenue in a no-separation English auction is $u(t_2; t_2, ..., t_n) - \tau$. Algebra yields the difference:

$$D(t_1, t_2; t_3, ..., t_n) = D_1(t_1, t_2; t_3, ..., t_n) + D_2(t_1, t_2; t_3, ..., t_n),$$

where

$$D_1 \equiv (1-q)\left[u(t_2^* + \delta; t_2, ..., t_n) - u(t_2; t_2^* + \delta, ..., t_n)\right] \tag{65}$$

$$D_2 \equiv q\left[u(t_2^* + \delta; t_2, ..., t_n) - 2u(t_2; t_2, ..., t_n) + u(t_2^*; t_2, ..., t_n)\right]. \tag{66}$$

Case 2: $t_1 \in [t_2, t_2^* + \delta)$. Seller revenue in the δ-separation mechanism is the sum of bidder 1's payment in (18) and bidder 2's payment in (19):

$$(1-2q)\, u(t_2, t_2, ..., t_n) + qu(t_2^*, t_2, ..., t_n) + qu(t_2^*, t_1, t_3, ..., t_n) - \tau.$$

Seller revenue in the no-separation English auction is $u(t_2, t_2, ..., t_n) - \tau$. The difference is

$$D(t_1, t_2; t_3, ..., t_n) = q\left[u(t_2^*, t_2, ..., t_n) + u(t_2^*, t_1, t_3, ..., t_n) - 2u(t_2, t_2, ..., t_n)\right] \equiv D_3(t_1, t_2; t_3, ..., t_n) \tag{67}$$

We can write the expected revenue difference as:

$$E[D] = n(n-1) \int_{\Omega_{n-2}} \int_{t_2^*}^{t_2^*+\delta} \int_{t_2}^{\bar{t}} D(t_1, t_2; t_3, ..., t_n) f(\mathbf{t})\, \mathbf{1}_{t_3,...,t_n \in H_\epsilon} dt_1 dt_2 dt_3 ... dt_n,$$

where $\Omega_{n-2} \equiv [\underline{t}, \bar{t}]^{n-2}$ is the space of integration for $t_3, ..., t_n$, and $\mathbf{1}_{t_3,...,t_n \in H_\epsilon}$ is an indicator function that equals 1 if $(t_3, ..., t_n) \in H_\epsilon$, and zero if $t_3, ..., t_n \notin H_\epsilon$ (recall $D = 0$ if $(t_3, ..., t_n) \notin H_\epsilon$). The factor n reflects that any of the n signals, not necessarily t_1, can be the highest signal, and the factor $n-1$ reflects that any of the remaining $n-1$ signals, not necessarily t_2, can be the second-highest.

Breaking up the integration of t_1 from t_2 to $\bar{t}$ into the sum of integrations from t_2 to $t_2^* + \delta$ and from $t_2^* + \delta$ to $\bar{t}$, and using (65), (66) and (67), we have

$$E[D] = n(n-1)(ED_1 + ED_2 + ED_3),$$

where

$$ED_i \equiv \int_{\Omega_{n-2}} \int_{t_2^*}^{t_2^*+\delta} \int_{t_2^*+\delta}^{\bar{t}} D_i\left(t_1, t_2; t_3, ..., t_n\right) f\left(\mathbf{t}\right) \mathbf{1}_{t_3,...,t_n \in H_\epsilon} dt_1 dt_2 dt_3 ... dt_n, \quad \text{for } i = 1, 2.$$

$$ED_3 \equiv \int_{\Omega_{n-2}} \int_{t_2^*}^{t_2^*+\delta} \int_{t_2}^{t_2^*+\delta} D_3\left(t_1, t_2; t_3, ..., t_n\right) f\left(\mathbf{t}\right) \mathbf{1}_{t_3,...,t_n \in H_\epsilon} dt_1 dt_2 dt_3 ... dt_n.$$

Next, we show that for δ sufficiently small, ED_1 exceeds a term that is positive and quadratic in δ, and ED_2 and ED_3 each exceed a term that goes to zero at a rate faster than δ^2.

Step 1: By (65) we have

$$ED_1 \equiv (1-q) \int_{\Omega_{n-2}} \int_{t_2^*}^{t_2^*+\delta} \int_{t_2^*+\delta}^{\bar{t}} (u(t_2^*+\delta; t_2, ..., t_n) - u(t_2; t_2^*+\delta, ..., t_n)) f(\mathbf{t}) \mathbf{1}_{t_3,...,t_n \in H_\epsilon} dt_1 dt_2 dt_3 ... dt_n.$$

Note that

$$\begin{aligned} u\left(t_2^*+\delta; t_2, ..., t_n\right) - u\left(t_2; t_2^*+\delta, ..., t_n\right) &= \int_{t_2}^{t_2^*+\delta} \left(u_1\left(t; t_2, ..., t_n\right) - u_2\left(t_2; t, ..., t_n\right)\right) dt \\ &> \omega\left(t_2^*+\delta-t_2\right), \end{aligned}$$

where the inequality follows from (64). Defining $f_{\min} \equiv \min_{t_1,...,t_n} f(\mathbf{t})$, we have

$$\begin{aligned} ED_1 &> (1-q)\,\omega \int_{\Omega_{n-2}} \int_{t_2^*}^{t_2^*+\delta} \int_{t_2^*+\delta}^{\bar{t}} \left(t_2^*+\delta-t_2\right) f\left(\mathbf{t}\right) \mathbf{1}_{t_3,...,t_n \in H_\epsilon} dt_1 dt_2 dt_3 ... dt_n \\ &\geq (1-q)\,\omega f_{\min} \int_{\Omega_{n-2}} \int_{t_2^*}^{t_2^*+\delta} \int_{t_2^*+\delta}^{\bar{t}} \left(t_2^*+\delta-t_2\right) \mathbf{1}_{t_3,...,t_n \in H_\epsilon} dt_1 dt_2 dt_3 ... dt_n \\ &= (1-q)\,\omega f_{\min} \left(\bar{t}-t_2^*-\delta\right) \int_{\Omega_{n-2}} \int_{t_2^*}^{t_2^*+\delta} \left(t_2^*+\delta-t_2\right) \mathbf{1}_{t_3,...,t_n \in H_\epsilon} dt_2 dt_3 ... dt_n \\ &= \frac{1}{2}(1-q)\,\omega f_{\min} \left(\bar{t}-t_2^*-\delta\right) \delta^2 \int_{\Omega_{n-2}} \mathbf{1}_{t_3,...,t_n \in H_\epsilon} dt_3 ... dt_n \end{aligned}$$

where we use $\int_{t_2^*}^{t_2^*+\delta} \left(t_2 - t_2^*\right) dt_2 = \frac{1}{2}\delta^2$. For $\delta < \frac{\bar{t}-t_2^*}{2}$, we have $\left(\bar{t}-t_2^*-\delta\right) > \frac{1}{2}\left(\bar{t}-t_2^*\right)$. Thus, for $\delta < \frac{\bar{t}-t_2^*}{2}$, the above yields

$$ED_1 > \left[\frac{1}{4}(1-q)\,\omega f_{\min}\left(\bar{t}-t_2^*\right) \int_{\Omega_{n-2}} \mathbf{1}_{t_3,...,t_n \in H_\epsilon} dt_3 ... dt_n\right] \delta^2.$$

Step 2: Define

$$k_{\max}(t_3, t_4, ...t_n; \delta) \equiv \max_{t_1, t_2 \in \left[t_2^*, t_2^*+\delta\right]} u_1(t_1; t_2, ...t_n) \quad \text{and} \quad k_{\min}(t_3, t_4, ...t_n; \delta) \equiv \min_{t_1, t_2 \in \left[t_2^*, t_2^*+\delta\right]} u_1(t_1; t_2, ...t_n). \tag{68}$$

Note that $k_{\max}$ and $k_{\min}$ are functions of $\delta, t_3, ..., t_n$, and $k_{\max} \geq k_{\min} > 0$. Using the Taylor series expansions

$$u(t_2^* + \delta, t_2, ..., t_n) \geq u(t_2^*, t_2, ..., t_n) + k_{\min}\delta \quad \text{and} \quad u(t_2, t_2, ..., t_n) \leq u(t_2^*, t_2, ..., t_n) + k_{\max}(t_2 - t_2^*).$$

for the right-hand side of (66), we have

$$u(t_2^* + \delta, t_2, ..., t_n) - 2u(t_2, t_2, ..., t_n) + u(t_2^*, t_2, ..., t_n) \geq q(k_{\min}\delta - 2k_{\max}(t_2 - t_2^*)).$$

Thus,

$$\begin{aligned} ED_2 &\geq q\int_{\Omega_{n-2}}\int_{t_2^*}^{t_2^*+\delta}\int_{t_2^*+\delta}^{\bar{t}}(k_{\min}\delta - 2k_{\max}(t_2 - t_2^*)) f(\mathbf{t})\mathbf{1}_{t_3,...,t_n \in H_\epsilon} dt_1 dt_2 dt_3 ... dt_n \\ &= q\int_{\Omega_{n-2}}\int_{t_2^*+\delta}^{\bar{t}}\int_{t_2^*}^{t_2^*+\delta}(k_{\min}\delta - 2k_{\max}(t_2 - t_2^*)) f(\mathbf{t})\mathbf{1}_{t_3,...,t_n \in H_\epsilon} dt_2 dt_1 dt_3 ... dt_n, \end{aligned}$$

where the second equation switches the order of integration. Define

$$f_a(t_1, t_3, t_4, ...t_n; \delta) \equiv \min_{t_2 \in [t_2^*, t_2^*+\delta]} f(t_1, t_2, ...t_n) \quad \text{and} \quad f_b(t_1, t_3, t_4, ...t_n; \delta) \equiv \max_{t_2 \in [t_2^*, t_2^*+\delta]} f(t_1, t_2, ...t_n).$$

Note that f_a and f_b are functions of $t_1, t_3, ..., t_n$, and $f_b \geq f_a > 0$. Integrating over t_2 yields:

$$\int_{t_2^*}^{t_2^*+\delta}\left(k_{\min}^\delta\delta - 2k_{\max}^\delta(t_2 - t_2^*)\right) f(\mathbf{t}) dt_2 \;\geq\; \left(f_a k_{\min}^\delta - f_b k_{\max}^\delta\right)\delta^2,$$

where we use $\int_{t_2^*}^{t_2^*+\delta}(t_2 - t_2^*)\, dt_2 = \frac{1}{2}\delta^2$.

Claim 1: For any constant $\kappa > 0$, there exists a $\delta(\kappa) > 0$ such that $f_b k_{\max}^\delta - f_a k_{\min}^\delta \in [0, \kappa)$ for all $\delta < \delta(\kappa)$ and all $t_1, t_3, ..., t_n$.

Proof of Claim 1: Define $c_1 \equiv \max_{t_1,...,t_n} |\frac{d}{dt_1}u_1(\mathbf{t})|$ and $c_2 \equiv \max_{t_1,...,t_n} |\frac{d}{dt_2}u_1(\mathbf{t})|$. Refer to (68). It follows from the Taylor series expansion that $k_{\max}^\delta - k_{\min}^\delta \leq (c_1 + c_2)\delta$.[25] Thus,

$$\begin{aligned} f_b k_{\max}^\delta - f_a k_{\min}^\delta &= (f_b - f_a) k_{\max}^\delta + f_a(k_{\max}^\delta - k_{\min}^\delta) \\ &\leq (f_b - f_a) k_{\max}^\delta + f_a(c_1 + c_2)\delta. \end{aligned}$$

[25] To see this, assume $k_{\max}^\delta$ is obtained at $(t_1, t_2) = (t_1^{***}, t_2^{***})$ so that $k_{\max}^\delta \equiv u_1(t_1^{***}; t_2^{***}, t_3, ..., t_n)$, and assume that $k_{\min}^\delta$ is obtained at $(t_1, t_2) = (t_1^{**}, t_2^{**})$ so that $k_{\min}^\delta \equiv u_1(t_1^{**}; t_2^{**}, t_3, ..., t_n)$. Then

$$\begin{aligned} k_{\max}^\delta - k_{\min}^\delta &= u_1(t_1^{***}; t_2^{***}, t_3, ..., t_n) - u_1(t_1^{**}; t_2^{**}, t_3, ..., t_n) \\ &= (u_1(t_1^{***}; t_2^{***}, t_3, ...) - u_1(t_1^{**}; t_2^{***}, t_3, ...)) + (u_1(t_1^{**}; t_2^{***}, t_3, ...) - u_1(t_1^{**}; t_2^{**}, t_3, ...)) \\ &\leq c_1|t_1^{***} - t_1^{**}| + c_2|t_2^{***} - t_2^{**}|. \end{aligned}$$

Because $f_a \leq f_{\max}$ and hence is bounded, the second term $f_a(c_1 + c_2)\delta$ goes to zero as $\delta \to 0$. For the first term, uniform continuity of f yields that $f_b - f_a \to 0$ uniformly as $\delta \to 0$. Since $k^{\delta}_{\max} \leq \max_{t_1,...,t_n} u_1(t_1; t_2, ..., t_n)$ and hence is bounded, $(f_b - f_a) k^{\delta}_{\max} \to 0$ uniformly as $\delta \to 0$. This establishes Claim 1.

By Claim 1, for any constant $\kappa > 0$, there exists a $\delta(\kappa) > 0$ such that for all $\delta < \delta(\kappa)$:

$$ED_2 \geq -q\kappa\delta^2 \int_{\Omega_{n-2}} \int_{t_2^*+\delta}^{\bar{t}} \mathbf{1}_{t_3,...,t_n \in H_\epsilon} dt_1 dt_3 ... dt_n.$$

Step 3: By (67) and $t_1 \geq t_2$, we have:

$$D_3 > 2q\left[u\left(t_2^*, t_2^*, t_3, ..., t_n\right) - u\left(t_2^* + \delta, t_2^* + \delta, ..., t_n\right)\right] \geq -2qk^*\delta,$$

where

$$k^* \equiv \max_{t_1 \in [t_2, t_2^*+\delta], t_3, ..., t_n \in H_\epsilon} \frac{d}{dt_1} u\left(t_1, t_1, t_3, ..., t_n\right).$$

Define $f_{\max} \equiv \max_{t_1,...,t_n} f(\mathbf{t})$. Then we have

$$\begin{aligned} ED_3 &\geq -2qk^*\delta \int_{\Omega_{n-2}} \int_{t_2^*}^{t_2^*+\delta} \int_{t_2}^{t_2^*+\delta} f(\mathbf{t})\, \mathbf{1}_{t_3,...,t_n \in H_\epsilon} dt_1 dt_2 dt_3 ... dt_n \\ &\geq -\left(2qk^* f_{\max} \int_{\Omega_{n-2}} dt_3 ... dt_n\right) \delta^3, \end{aligned}$$

where we use $\int_{t_2^*}^{t_2^*+\delta} \int_{t_2}^{t_2^*+\delta} f(\mathbf{t}) dt_1 dt_2 \leq f_{\max}\delta^2$.

Step 4: Steps 2 and 3 show that when δ is small, ED_2 and ED_3 exceed a term that approaches zero faster than δ^2. Hence, for δ sufficiently small, $ED_1 > |ED_2 + ED_3|$. Thus, the expected revenue difference is strictly positive, i.e., the δ-separation mechanism generates strictly higher expected revenues. □

Proof of Proposition 3: The proof for part (i) follows from the same logic as that for Proposition 2 (i). The proof for part (ii) also uses a similar logic to that for Proposition 2 (ii). More specifically, we define ϵ, the set H_ϵ, and the δ-separation function in the same way. So, too, we consider n signals $t_1, ..., t_n$ where t_1 and t_2 are the highest and second-highest signals, with $t_2 \in [t_2^*, t_2^* + \delta]$ and $(t_3, ..., t_n) \in H_\epsilon$. There are two cases:

Case 1: $t_1 \geq t_2^* + \delta$. Seller revenue in the δ-separation mechanism is bidder 1's payment in (22):

$$(1-q)\, u\left(t_2^* + \delta; t_2, ..., t_n\right) + (2q-1)\, u\left(t_2; t_2, ..., t_n\right) + (1-q)\, u\left(t_2; t_2^*, ..., t_n\right) - \tau.$$

Revenue in a no-separation English auction is: $u(t_2; t_2, ..., t_n) - \tau$. The revenue difference is

$$\begin{aligned} D(t_1, t_2; t_3, ..., t_n) &= (1-q)\left[u\left(t_2^* + \delta; t_2, ..., t_n\right) - 2u\left(t_2; t_2, ..., t_n\right) + u\left(t_2; t_2^*, ..., t_n\right)\right] \\ &= D_1\left(t_1, t_2; t_3, ..., t_n\right) + D_2\left(t_1, t_2; t_3, ..., t_n\right), \end{aligned}$$

where

$$D_1 \equiv (1-q)\left[u\left(t_2; t_2^*, ..., t_n\right) - u\left(t_2^*; t_2, ..., t_n\right)\right]$$

$$D_2 \equiv (1-q)\left[u\left(t_2^* + \delta; t_2, ..., t_n\right) - 2u\left(t_2; t_2, ..., t_n\right) + u\left(t_2^*; t_2, ..., t_n\right)\right].$$

Case 2: $t_1 \in [t_2, t_2^* + \delta)$. Seller revenue in the δ-separation mechanism is the sum of bidder 1's payment in (23) and bidder 2's payment in (24):

$$(2q-1)\, u\left(t_2; t_2, ..., t_n\right) + (1-q)\left[u\left(t_2; t_2^*, ..., t_n\right)\right] - \tau + (1-q)\, u\left(t_1; t_2^*, t_3, ..., t_n\right)$$

Seller revenue in the no-separation English auction is $u\left(t_2, t_2, ..., t_n\right) - \tau$. Thus, the difference in revenues is

$$\begin{aligned} D(t_1, t_2; t_3, ..., t_n) &= (1-q)\left(u\left(t_2; t_2^*, ..., t_n\right) + u\left(t_1; t_2^*, t_3, ..., t_n\right) - 2u\left(t_2; t_2, ..., t_n\right)\right) \\ &\geq 2(1-q)\left(u\left(t_2; t_2^*, ..., t_n\right) - u\left(t_2; t_2, ..., t_n\right)\right) \equiv D_3\left(t_1, t_2; t_3, ..., t_n\right), \end{aligned}$$

where the inequality follows from $t_1 \geq t_2$ and that u weakly increases in its arguments.

Using a similar logic to that in the proof for Proposition 2 (i), one can show that for δ sufficiently small, the contribution to the expected revenue difference from D_1 exceeds a term that is positive and quadratic in δ, while the (possibly negative) contributions from D_2 and D_3 exceed terms that go to zero at rates faster than δ^2. Hence, for δ sufficiently small, the expected revenue difference is strictly positive, i.e., the δ-separation mechanism generates strictly higher expected revenues. □

Proof of Lemma 2:

$$\pi_s = \int \sum_{j,i} R_j(\mathbf{t}) Q_{ji}(\mathbf{t}) v_j(\mathbf{t}) \Pi_{i=1}^n g(t_i) dt - \sum_i \int_{\underline{t}}^{\bar{t}} \left(\int_{\underline{t}}^{t_i} J_i(\widetilde{t}) d\widetilde{t} + \bar{U}_i(\underline{t}, \underline{t}) \right) g(t_i) dt_i - \tau \sum_i \int_{\Omega_n} R_i(\mathbf{t}) \Pi_{i=1}^n g(t_i) d\mathbf{t},$$

where $\Omega_n \equiv [\underline{t}, \bar{t}]^n$ denotes the space of integration for all n bidders. That is, expected seller revenue equals the expected increase in social welfare gross of the costs of running the project (first term on the right-hand side) less the sum of bidders' expected rents (second term) less the expected costs of running the project (third term). Applying integration by parts to the second term on the right-hand side (without the summation) yields:

$$\begin{aligned}\int_{\underline{t}}^{\bar{t}}\Big(\int_{\underline{t}}^{t_i} J_i(\widetilde{t})d\widetilde{t}+\bar{U}_i(\underline{t},\underline{t})\Big)g(t_i)dt_i &= -\int_{\underline{t}}^{\bar{t}}\left(\int_{\underline{t}}^{t_i} J_i(\widetilde{t})d\widetilde{t}+\bar{U}_i(\underline{t},\underline{t})\right)d\left(1-G(t_i)\right)\\ &= \bar{U}_i(\underline{t},\underline{t})+\int_{\underline{t}}^{\bar{t}}\left(1-G(t_i)\right)d\left(\int_{\underline{t}}^{t_i} J_i(\widetilde{t})d\widetilde{t}+\bar{U}_i(\underline{t},\underline{t})\right)\\ &= \bar{U}_i(\underline{t},\underline{t})+\int_{\underline{t}}^{\bar{t}}\frac{1-G(t_i)}{g(t_i)}J_i(t_i)g(t_i)dt_i. \qquad (69)\end{aligned}$$

Substituting (26) for $J_i(t_i)$ into the right-hand side of (69) yields

$$\begin{aligned}\int_{\underline{t}}^{\bar{t}}\Big(\int_{\underline{t}}^{t_i} J_i(\widetilde{t})d\widetilde{t}+\bar{U}_i(\underline{t},\underline{t})\Big)g(t_i)dt_i &= \bar{U}_i(\underline{t},\underline{t})+\int_{\underline{t}}^{\bar{t}}\frac{1-G(t_i)}{g(t_i)}\Big(\int_{\Omega_{n-1}}\sum_j R_j(\mathbf{t})Q_{ji}(\mathbf{t})\frac{\partial v_j(\mathbf{t})}{\partial t_i}f_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i}\Big)g(t_i)dt_i\\ &= \bar{U}_i(\underline{t},\underline{t})+\int_{\underline{t}}^{\bar{t}}\int_{\Omega_{n-1}}\frac{1-G(t_i)}{g(t_i)}\Big(\sum_j R_j(\mathbf{t})Q_{ji}(\mathbf{t})\frac{\partial v_j(\mathbf{t})}{\partial t_i}\Big)f_{-i}(\mathbf{t}_{-i})g(t_i)d\mathbf{t}_{-i}dt_i\\ &= \bar{U}_i(\underline{t},\underline{t})+\int_{\Omega_n}\frac{1-G(t_i)}{g(t_i)}\Big(\sum_j R_j(\mathbf{t})Q_{ji}(\mathbf{t})\frac{\partial v_j(\mathbf{t})}{\partial t_i}\Big)\Pi_{i=1}^n g(t_i)d\mathbf{t}.\end{aligned}$$

Hence, expected seller revenue is:

$$\begin{aligned}\pi_s &= \int_{\Omega_n}\sum_{ji} R_j(\mathbf{t})Q_{ji}(\mathbf{t})v_j(\mathbf{t})\Pi_{i=1}^n g(t_i)d\mathbf{t}-\sum_i\int_{\Omega_n}\frac{1-G(t_i)}{g(t_i)}\Big(\sum_j R_j(\mathbf{t})Q_{ji}(\mathbf{t})\frac{\partial v_j(\mathbf{t})}{\partial t_i}\Big)\Pi_{i=1}^n g(t_i)d\mathbf{t}\\ &\quad -\tau\sum_j\int_{\Omega_n}R_j(\mathbf{t})\Pi_{i=1}^n g(t_i)d\mathbf{t}-\sum_i\bar{U}_i(\underline{t},\underline{t}). \quad \square\end{aligned}$$

Proof of Lemma 3: Claim 1: For any given j^*, consider any vector $Q_{j^*i}(\mathbf{t})$ (where i runs from 1 through n) that satisfies (7) and (8) (i.e., $\sum_{i=1}^n Q_{j^*i}(\mathbf{t})=1$ and $Q_{j^*j^*}(\mathbf{t})\geq q_{min}$). Then, $\sum_{i=1}^n Q_{j^*i}(\mathbf{t})\phi_{j^*i}(\mathbf{t},Q)\leq\psi_{\hat{j}\hat{i}}(\mathbf{t})$.

To prove the Claim, note that (29) and (7) yield

$$\begin{aligned}\sum_i Q_{j^*i}(\mathbf{t})\phi_{j^*i}(\mathbf{t},Q) &= v_{j^*}(\mathbf{t})-\tau-\sum_i Q_{j^*i}(\mathbf{t})\frac{1-G(t_i)}{g(t_i)}\frac{\partial v_{j^*}(\mathbf{t})}{\partial t_i}\\ &= v_{j^*}(\mathbf{t})-\tau-q_{min}\frac{1-G(t_{j^*})}{g(t_{j^*})}\frac{\partial v_{j^*}(\mathbf{t})}{\partial t_{j^*}}+\left(Q_{j^*j^*}(\mathbf{t})-q_{min}\right)\left(\frac{1-G(t_{j^*})}{g(t_{j^*})}\frac{\partial v_{j^*}(\mathbf{t})}{\partial t_{j^*}}\right)\\ &\quad -\sum_{i\neq j^*}Q_{j^*i}(\mathbf{t})\left(\frac{1-G(t_i)}{g(t_i)}\frac{\partial v_{j^*}(\mathbf{t})}{\partial t_i}\right). \qquad (70)\end{aligned}$$

Define

$$i^*\equiv\arg\min_i\frac{1-G(t_i)}{g(t_i)}\frac{\partial v_{j^*}(\mathbf{t})}{\partial t_i}. \qquad (71)$$

Then given the constraints $Q_{j^*j^*}(\mathbf{t}) \geq q_{min}$ and $Q_{j^*i} \geq 0$ for all i, (70) yields

$$\begin{aligned}
\sum_i Q_{j^*i}(\mathbf{t})\phi_{j^*i}(\mathbf{t},Q) &\leq v_{j^*}(\mathbf{t}) - \tau - q_{min}\frac{1-G(t_{j^*})}{g(t_{j^*})}\frac{\partial v_{j^*}(\mathbf{t})}{\partial t_{j^*}} \\
&\quad - \left((Q_{j^*j^*}(\mathbf{t}) - q_{min}) + \sum_{i \neq j^*} Q_{j^*i}(\mathbf{t})\right)\left(\frac{1-G(t_{i^*})}{g(t_{i^*})}\frac{\partial v_{j^*}(\mathbf{t})}{\partial t_{i^*}}\right) \\
&= v_{j^*}(\mathbf{t}) - \tau - q_{min}\frac{1-G(t_{j^*})}{g(t_{j^*})}\frac{\partial v_{j^*}(\mathbf{t})}{\partial t_{j^*}} - (1-q_{\min})\left(\frac{1-G(t_{i^*})}{g(t_{i^*})}\frac{\partial v_{j^*}(\mathbf{t})}{\partial t_{i^*}}\right) \\
&= \psi_{j^*i^*}(\mathbf{t}) \leq \psi_{\hat{j}\hat{\imath}}(\mathbf{t}),
\end{aligned}$$

where the first inequality follows from (71) that $\frac{1-G(t_{i^*})}{g(t_{i^*})}\frac{\partial v_{j^*}(\mathbf{t})}{\partial t_{i^*}} \leq \frac{1-G(t_i)}{g(t_i)}\frac{\partial v_{j^*}(\mathbf{t})}{\partial t_i}$ for all i (including when $i = j^*$), and the first equality follows from $\sum_i Q_{j^*i} = 1$. This yields Claim 1.

We now prove Part (i) of the lemma by contradiction. Suppose to the contrary that some vector of control rights $R(\mathbf{t})$ and cash flow rights matrix $Q(\mathbf{t})$ satisfy $\sum_{j,i} R_j(\mathbf{t})Q_{ji}(\mathbf{t})\phi_{ji}(\mathbf{t},Q) \geq 0$. Then there must exist a j such that $\sum_i Q_{ji}(\mathbf{t})\phi_{ji}(\mathbf{t},Q) \geq 0$. Then Claim 1 yields that $\psi_{\hat{j}\hat{\imath}}(\mathbf{t}) \geq \sum_i Q_{ji}(\mathbf{t})\phi_{ji}(\mathbf{t},Q) \geq 0$, contradicting the premise of Part (i) that $\max_{j,i}\psi_{ji}(\mathbf{t}) < 0$.

To prove Part (ii), consider any vector $R(\mathbf{t})$ and any matrix $Q(\mathbf{t})$ such that (6), (7) and (8) hold. Then

$$\sum_{j,i} R_j(\mathbf{t})Q_{ji}(\mathbf{t})\phi_{ji}(\mathbf{t},Q) \leq \sum_j R_j(\mathbf{t})\psi_{\hat{j}\hat{\imath}}(\mathbf{t}) \leq \psi_{\hat{j}\hat{\imath}}(\mathbf{t}).$$

The first inequality follows from Claim 1, and the second follows from (6) and the premise of Part (ii) that $\psi_{\hat{j}\hat{\imath}}(\mathbf{t}) \geq 0$. This means that no assignment can achieve a higher value of $\hat{\pi}_s(\mathbf{t})$ than assigning control and q_{min} cash flows to $\hat{j}$ and assigning $1 - q_{min}$ cash flows to $\hat{\imath}$. □

Proof of Lemma 4. Claim 1: Assigning all cash flow rights and control to h achieves a higher $\hat{\pi}_s(\mathbf{t})$ than assigning all cash flow rights and control to any other bidder.

Proof: Consider any bidder $k \neq h$. Then (32) yields

$$\psi_{hh}(\mathbf{t}) - \psi_{kk}(\mathbf{t}) = A_n(1-\rho)(t_h - t_k) + A_n(L(t_k) - L(t_h)) > 0,$$

where the inequality follows from $t_h > t_k$ and $L(t_k) > L(t_h)$ from Assumption 2. □

Claim 2: In the $\hat{\pi}_s(\mathbf{t})$-maximizing allocation characterized in Lemma 3, a bidder who is not among the two highest receives neither cash flow rights nor control.

Proof: Let k be a bidder who is not among the two highest. We must show that $k \neq \hat{j}$ and $k \neq \hat{\imath}$, where $(\hat{j}, \hat{\imath})$ maximize $\psi_{ji}(\mathbf{t})$ as in (31).

Case 1: Suppose instead that $k = \hat{\jmath}$ (i.e., k receives control in the $\hat{\pi}_s(\mathbf{t})$-maximizing allocation). Then $\hat{\imath} \neq k$ by Claim 1. Then $\psi_{k\hat{\imath}}(\mathbf{t}) = v_k(\mathbf{t}) - q_{\min} L(t_k) A_n \rho - (1 - q_{\min}) L(t_{\hat{\imath}}) A_n \rho - \tau$. Clearly, either $\hat{\imath} \neq s$ or $\hat{\imath} \neq h$ (or both) must hold. If $\hat{\imath} \neq s$, then $\psi_{s\hat{\imath}}(\mathbf{t}) - \psi_{k\hat{\imath}}(\mathbf{t}) = A_n(1-\rho)(t_s - t_k) + q_{\min} A_n \rho (L(t_k) - L(t_s)) > 0$ (because $t_s > t_k$ and $L(t_k) > L(t_s)$). Thus, assigning control to k does not maximize $\hat{\pi}_s(\mathbf{t})$, because assigning control to s (leaving $\hat{\imath}$ unchanged) would do better. An analogous contradiction arises if $\hat{\imath} \neq h$.

Case 2: Suppose instead that $k = \hat{\imath}$. We next assume $\hat{\jmath} \neq \hat{\imath}$, which is without loss of generality (since the proof in Case 1 applies if $\hat{\jmath} = \hat{\imath}$). Then $\hat{\jmath} \neq k$ by Claim 1. Clearly, either $\hat{\imath} \neq s$ or $\hat{\imath} \neq h$ (or both) must hold. If $\hat{\jmath} \neq s$, then $\psi_{\hat{\jmath}s}(\mathbf{t}) - \psi_{\hat{\jmath}k}(\mathbf{t}) = (L(t_k) - L(t_s)) A_n \rho > 0$. This implies that $\hat{\imath} = k$ does not maximize $\hat{\pi}_s(\mathbf{t})$, because $\hat{\imath} = s$ (leaving $\hat{\jmath}$ unchanged) would do better. An analogous contradiction arises if $\hat{\jmath} \neq h$.

This completes the proof. $\square$

Proof of Proposition 5: We have shown that if $q_{\min} \geq 0.5$, then $\bar{\rho} \geq 1 \geq \rho$ for all $t_h > t_s$, where $\bar{\rho}$ is defined in (41) and used in Proposition 4.

Claim 1: If $q_{\min} < 0.5$, then for all $t_h > t_s$,

$$1 - \frac{(1 - 2q_{\min})}{(1 - q_{\min}) - \delta_2} \leq \bar{\rho} \leq 1 - \frac{(1 - 2q_{\min})}{(1 - q_{\min}) - \delta_1}.$$

Proof: Rewrite (41) as

$$\bar{\rho} = 1 - \frac{(1 - 2q_{\min})}{(1 - q_{\min}) - \frac{1}{\frac{L(t_h) - L(t_s)}{t_h - t_s}}},$$

which increases in $\frac{L(t_h) - L(t_s)}{t_h - t_s}$ under the premise $q_{\min} < 0.5$. This and (42) yield Claim 1.

Claim 2: (i) If $\bar{\rho} \geq \rho$ for all $t_h > t_s$, then $K_B(t_s) \geq K_A(t_s)$ so that $K(t_s) = K_B(t_s)$, where K is defined in (39). (ii) If $\bar{\rho} \leq \rho$ for all $t_h > t_s$, then $K_B(t_s) \leq K_A(t_s)$ so that $K(t_s) = K_A(t_s)$.

Proof: The right-hand side of (40) strictly decreases in ρ, and it equals zero at $\rho \leq \bar{\rho}$. Thus, for any $t_h > t_s$,

$$\psi_{hs} \geq \psi_{sh} \text{ if } \bar{\rho} \geq \rho \text{ and } \psi_{hs} \leq \psi_{sh} \text{ if } \bar{\rho} \leq \rho. \tag{72}$$

Next, fix a given t_s and consider $t_h = K_B(t_s)$. Then (36) and (37) yield $\psi_{hh} = \psi_{hs}$. If $\bar{\rho} \geq \rho$ for all $t_h > t_s$, then (72) yields $\psi_{hs} \geq \psi_{sh}$, hence we have (by $\psi_{hh} = \psi_{hs}$) $\psi_{hh} \geq \psi_{sh}$. Hence (38) yields $t_h \geq K_A(t_s)$, which yields (by our assumption that $t_h = K_B(t_s)$) $K_B(t_s) \geq K_A(t_s)$. This establishes part (i) of Claim 2. A similar argument establishes part (ii) of Claim 2.

Next refer to (27). Note that the second term of the right-hand side of (27) is zero for both Mechanisms A and B. Further, Mechanism A is incentive compatible if $\rho \geq \frac{q}{1-q}$ (by Proposition 2) and Mechanism B is incentive compatible if $\rho \leq \frac{q}{1-q}$ (by Proposition 3). These properties, combined with Claims 1 and 2 and Proposition 4, establishes Proposition 5. $\square$

Proof of Proposition 6: From the proof of Proposition 2, Mechanism A satisfies ex post local IC, regardless of whether $\rho_{\min} \geq \frac{q}{1-q}$ holds (i.e., $\rho_{\min} \geq \frac{q}{1-q}$ is only needed to ensure *global ex post* IC). Hence, Mechanism A satisfies local interim IC with $q = q_{\min}$, regardless of the value of $q_{\min}$. Similarly, Mechanism B satisfies local interim IC with $q = q_{\min}$. With these results and Lemma 5, the same logic as that for parts (ii) and (iii) of Proposition 5 implies the result. $\square$