1.0 Plan

- The axioms of preference, existence and construction of a utility function (Block 1.1 🎤)
- Utility maximization, Marshallian demands and the indirect utility function (Block 1.2 🎤)
- Duality: Expenditure minimisation, Hicksian demands and the expenditure function (Block 1.3 🎤)
- Testing consumer theory (Block 1.4 🎤)
- The Slutsky equation and the “law of demand” (Block 1.5 🎤)
- Utility maximization with endowments, with an application to labour supply (Block 1.6 🎤)
- 😁 Indicates live session
Consumer theory explains how a rational decision-maker chooses to allocate her income between different goods and services, taking as fixed the prices for these goods and services.

Basic consumer theory can easily be extended to study how a decision-maker makes her labour supply and savings decisions.

The consumer: a rational decision-maker who has a fixed income at her disposal, and who values goods and services.

[Note that in practice, expenditure decisions are usually made by groups of people living together (households), although there are increasing numbers of one-person households.]
1.2 Consumption Bundles and the Consumption Set

- The consumer values \( n \) different commodities for \( i = 1, \ldots, n \).

- \( x_i \) - the amount of commodity \( i \) purchased and consumed by the consumer.

- \( x_i \) is a non-negative real number (commodities infinitely divisible).

- \( x = (x_1, x_2, \ldots, x_n) \) – a vector of consumption levels, the consumption bundle.

- We will denote specific bundles by letters (\( x, y, z, \) etc.).

In the case of two goods: 

\[ x = (x_1, x_2) \]
1.2 Consumption Bundles and the Consumption Set

- The *consumption set* $X$ is the set of all possible consumption bundles.
- Without any further restrictions, $X$ is simply the set of all *non-negative* consumption bundles:

However, in practice, certain minimum levels of consumption of goods may be needed e.g. to keep the consumer alive and so the consumption set is modified to take these constraints into account. [Exercise: draw the consumption set in the case of $n = 2$, if the consumer needs at least amount $y_1$ of good 1, and $y_2$ of good 2]
1.3 Consumer Preference Relations

- Modern theory only assumes that consumers can rank pairs of consumption bundles, and that these rankings satisfy certain assumptions.

- It can then be shown that given these assumptions on rankings, consumers behave as if they maximise utility

\[ x \gneq y \text{ - the consumer judges bundle } x \text{ to be at least as good as bundle } y. \]

- \( \gneq \) is called the weak preference relation.

- Note that \( \gneq \) does not in itself tell us whether the consumer strictly prefers \( x \) to \( y \) or whether the consumer is indifferent between \( x \) and \( y \).

- However, once we know \( \gneq \), we can easily calculate which of these is the case.
1.3 Consumer Preference Relations

• The strict preference relation \( \succ \) on \( X \) is defined as follows:
  \[
  x \succ y \text{ if and only if } x \succeq y \text{ and not } y \succeq x .
  \]

• The indifference relation on \( X \) is defined as follows:
  \[
  x \sim y \text{ if and only if } x \succeq y \text{ and } y \succeq x .
  \]

• In consumer theory, we make some assumptions about the weak preference relation \( (\succeq) \).

• These assumptions are called axioms, but are really hypotheses about consumer behaviour that are in principle, testable.
Axiom 1: Completeness For all $x$ and $y$ in $X$, either $x \succeq y$ or $y \succeq x$.
This says that the consumer can rank any pair of consumption bundles.
The completeness axiom obviously implies that either $x \succ y$ or $y \succ x$ or $y \sim x$

Axiom 2: Transitivity. For any three bundles $x, y, z$ in $X$, if $x \succeq y$ and $y \succeq z$ then $x \succeq z$.
This assumes that a consumer’s choices be consistent.

Without the transitivity axiom, it would be possible that a consumer would prefer $x$ to $y$, and $y$ to $z$, and finally $z$ to $x$. So, she may not be able to identify any feasible alternative as “best”, and so could not make a rational choice.

Note that transitivity of $\succeq$ implies transitivity of $\succ$ and transitivity of $\sim$ (Q1, problem set 1).

The next axiom is the continuity axiom.
Consider a sequence of consumption bundles \( \{x^n\} = (x^1, x^2, x^3, \ldots x^n \ldots) \) which converges to a limit bundle \( y \).

Also, let \( z \) be some other consumption bundle.

**Axiom 3: Continuity**. If \( \{x^n\} \) converges to \( y \), and \( x^n \succeq z \), all \( n = 1, 2, \ldots \) then \( y \succeq z \) also.

In words: if all consumption bundles “close” to \( y \) (in \( n \)-dimensional space) are preferred to \( z \), bundle \( y \) is preferred to bundle \( z \) also.
Example where Axiom 3 is not satisfied. (lexicographic preferences)

Assume \( n = 2 \), and that good 1 is beer, and good 2 is cheese.

Given any two bundles \( x, y \), the consumer strictly prefers \( x \) as long as it contains more beer than \( y \). If \( x, y \) have the same amount of beer, the consumer strictly prefers the bundle that has more cheese. [So, the consumer ranks beer as “infinitely” better than cheese].

Now consider the sequence \( \{x^n\} \), where \( x^n = (1+1/n, 0) \), and the bundle \( z = (1,1) \).

Note that:

• \( x^n \) converges to \( y = (1,0) \) as \( n \to \infty \)

• The consumer strictly prefers any \( x^n \) to \( z \) as \( x^n \) has more beer (i.e. \( x^n \succ z \), all \( n \))

• But, the consumer strictly prefers \( z \) to \( y \) as \( z \) has more cheese (i.e. \( z \succ y \))

• So, axiom 3 is not satisfied as \( x^n \not\succ z \), all \( n = 1,2,\ldots \), but \( z \succ y \).
1.3 Consumer Preference Relations

Axiom 4: Strict Monotonicity. Assume that bundle $x$ contains at least as much of all goods as bundle $y$ (i.e. $x_i \geq y_i$, all $i = 1,..n$) and strictly more of at least one good as $y$ (i.e. $x_i > y_i$, some $i = 1,..n$). Then, $x \succ y$.

This says that consumers are always greedy; no matter how much they have of any commodity, they always want more.
1.4 Representation of Preferences by a Utility Function

Under assumptions 1-4, the preference relation $\succeq$ can be represented, or described, by a utility function $u$ mapping consumption bundles to real numbers.

Formally, a utility function is a function $u: X \rightarrow \mathbb{R}$ where $u(x)$ is the number assigned to consumption bundle $x$. ($\mathbb{R} = \text{the set of real numbers}$)

The utility function $u: X \rightarrow \mathbb{R}$ represents consumer preferences if: $u(x) \geq u(y)$ if and only if $x \succeq y$.

1. $u(x)$ is a function of $n$ variables i.e. $u(x) = u(x_1, x_2 \ldots x_n)$ where $x_i$ is the level of consumption of the $i^{th}$ good.
2. By definition, if $u(.)$ represents a particular consumer’s preferences, then so does any strictly increasing function of $u$ e.g. $u^2$, $\ln u$, $u+1$, etc. This is because if $f(.) : \mathbb{R} \rightarrow \mathbb{R}$ is any strictly increasing function, then
   
   $u(x) \geq u(y)$ if and only if $f(u(x)) \geq f(u(y))$
• That is, \( f \) preserves the utility ranking of alternatives

• So, if a preference relation is represented by any utility function, it is always represented by a whole family of utility functions that are strictly increasing transformations of each other.

**Theorem.** Suppose that the consumer preference relation is complete, transitive, continuous, and strictly monotonic (i.e. satisfies Axioms 1-4). Then there exists a continuous utility function which represents those preferences. Moreover, \( u(x) = u(x_1, x_2, ..., x_n) \) is strictly increasing in each of its arguments.
1.4 Representation of Preferences by a Utility Function

**Step 1: Construction of a Utility Number**

Consider some bundle $x = (x_1, x_2)$ in $X$.

Then, given A1-A4, we can find a bundle $z = (u(x), u(x))$ i.e. with equal amounts of goods 1 and 2 which is indifferent to $x$, i.e.

$$(u(x), u(x)) \sim x$$

$u(x)$ - the utility number of $x$.

It can be proved that for any $x \in X$, there exists a unique utility number $u(x)$ – see Appendix to notes.

So, as $x$ varies, we can think of $u(x)$ as a function mapping all $x$ in $X$ to real numbers. We call this function the utility function.
Step 2: Proof that the Candidate Utility Function Represents Preferences

Consider two bundles $x$ and $y$. We need to show that $x \succeq y$ if and only if $u(x) \geq u(y)$. But by definition:

$$(u(x),u(x)) \sim x, \quad (u(y),u(y)) \sim y$$

But then given (*),

$x \succeq y \iff (u(x),u(x)) \sim x \text{ and } x \succeq y \text{ and } y \sim (u(y),u(y))$

$\iff (u(x),u(x)) \succeq (u(y),u(y)) \quad \text{(by A2, transitivity)}$

$\iff u(x) \geq u(y) \quad \text{(by A4, strict monotonicity)}$

[Last step: if $u(x) \geq u(y)$, the bundle $(u(x),u(x))$ contains at least as much of both goods as the bundle $(u(y),u(y))$, so must be weakly preferred to $(u(y),u(y))$ by A4].
1.5 Indifference Curves

• For any reference bundle \( y \), define the *indifference set* to be the set of consumption bundles ranked as indifferent to \( y \):

\[
I(y) = \{ x \mid u(x) = u(y) \}
\]

• From the strict monotonicity axiom A4, we can see that the indifference sets must be “thin” and downward-sloping, as shown below.

Due to this shape, indifference sets are usually called *indifference curves*, and we use this terminology from now on.
The final axiom is the convexity axiom. It is not required for the existence of a utility function. Rather, it is needed to make sure that the demand functions generated by utility maximisation are “well-behaved” i.e. single-valued and continuous.

Let \( x = (x_1, x_2, ..x_n) \), \( y = (y_1, y_2, ..y_n) \). Then a *convex combination* of \( x \) and \( y \) is

\[
\lambda x + (1-\lambda)y = (\lambda x_1 + (1-\lambda)y_1, \lambda x_2 + (1-\lambda)y_2, ..., \lambda x_n + (1-\lambda)y_n), \text{ for some } 0 \leq \lambda \leq 1
\]

For example, if \( x = (1,2) \), and \( y = (2, 1) \), then a convex combination is:

\[
\lambda x + (1-\lambda)y = (\lambda 1 + (1-\lambda)2, \lambda 2 + (1-\lambda)1) \text{ for some } 0 \leq \lambda \leq 1.
\]
1.6 Convexity of Preferences

Geometrically, a convex combination of two consumption bundles lies on a straight line between the two bundles.

\[ \lambda x + (1-\lambda)y \] for some \( 0 \leq \lambda \leq 1 \)
Axiom 5: Strict Convexity. Given consumption bundles \( x, y, z \) with \( x \neq y \), then \( \lambda x + (1-\lambda)y > z \) for all \( 0 < \lambda < 1 \).

A5 implies indifference curves convex to the origin.

In this example, \( x, y, z \) lie on the same IC, so \( x \gtrsim z, y \gtrsim z \).

So, by A5, we have \( \lambda x + (1-\lambda)y > z \), all \( 0 < \lambda < 1 \).

So, all points on the straight line between \( x \) and \( y \) are preferred to \( z \).

So, \( z \) must lie below the straight line connecting \( x \) and \( y \) by A4.
**Theorem.** The utility function is strictly quasi-concave if and only if preferences are strictly convex i.e. A5 holds.

Recall strict quasi-concavity of $u$ is:

For any $x,y$ and $0 < \lambda < 1$, \[ u(\lambda x + (1-\lambda)y) > \min\{u(x), u(y)\} \]

**Proof of “if”**

Recall A5: Given consumption bundles $x$, $y$, $z$ with $x \succsim z$, $y \succsim z$ and $x \neq y$,

$\lambda x + (1-\lambda)y > z$ for all $0 < \lambda < 1$

Assume w.l.o.g $x \succsim z$ and also that $z = y$. Then A5 implies then $\lambda x + (1-\lambda)y > y$ for all $0 < \lambda < 1$, or

$u(\lambda x + (1-\lambda)y) = \min\{u(x), u(y)\}$  \hspace{1cm} (Proof of “only if” left as an exercise)
1.7 Utility Maximisation

The consumer is assumed to choose the consumption bundle that is the most preferred one in her consumption set, given also the financial constraints that she faces. We assume A1-A5 throughout.

The budget set

We assume \( p_i > 0, m > 0 \)

\( p_i \) – price of one unit of good \( i \)

\( m \) – total income of the consumer

Expenditure on goods is \( p_1 x_1 + p_2 x_2 + \ldots p_n x_n = \sum_{i=1}^{n} p_i x_i \)

The budget constraint: expenditure be no more than income i.e \( \sum_{i=1}^{n} p_i x_i \leq m \)

\( m \) and \( p_i \) are assumed exogenous to the consumer, and are strictly positive real numbers.
The budget set $B$ - the set of consumption bundles that are physically feasible and satisfy the budget constraint.

Generally, the budget set is defined as: $B = \{x \mid x_i \geq 0, \ i = 1,..,n, \ \Sigma p_i x_i \leq m \}$. Note that it is closed and bounded.
1.7 Utility Maximisation

The Consumer’s Utility Maximisation Problem (UMP)

To choose \((x_1, x_2, \ldots, x_n)\) to maximise utility \(u(x_1, x_2, \ldots, x_n)\) subject to the constraint that \((x_1, x_2, \ldots, x_n)\) lies in the budget set \(B\).

Note:

- As \(B\) is compact (closed and bounded) and \(u(x_1, x_2, \ldots, x_n)\) is continuous, this problem has at least one solution (Extreme value theorem).

- From Axiom 4, the budget constraint will hold with equality at any solution: \(\sum_{i=1}^{n} p_i x_i = m\).

- From Axiom 5, strict convexity, the solution to this problem is unique.

We assume from now on that \(u(x_1, \ldots, x_n)\) is differentiable i.e. no kinks in indifference curves.
1.7 Utility Maximisation

Then, as $u(.)$ is strictly quasi-concave, by the pre-sessional maths notes, the solution to the UMP is fully described by the Kuhn-Tucker conditions (Lagrangian Approach).

There then two possibilities:

A. **Interior Solution**: Indifference curve tangent to the budget line, strictly positive amounts of both goods consumed.

B. **Corner solution**: Indifference curve at a “corner” of the budget set, only one good consumed.

We assume an **interior** solution in what follows.
1.7 Utility Maximisation

**Interior**

**Corner**
1.7 Utility Maximisation

Solving the UMP

Then, the UMP can be stated as:
choose \((x_1, x_2, \ldots, x_n)\) to maximise \(u(x_1, x_2, \ldots, x_n)\) subject to

\[
\sum_{i=1}^{n} p_i x_i = m
\]

To solve, write the Lagrangian

\[
L = u(x_1, x_2, \ldots, x_n) + \lambda [m - p_1 x_1 - p_2 x_2 - \cdots - p_n x_n]
\]

The FOC for a solution to UMP are

\[
\frac{\partial L}{\partial x_i} = \frac{\partial u}{\partial x_i} - \lambda p_i = 0, \quad i = 1, \ldots, n \quad (1)
\]

\[
\frac{\partial L}{\partial \lambda} = m - p_1 x_1 - p_2 x_2 - \cdots - p_n x_n = 0 \quad (2)
\]

Equations (1) and (2) are \(n+1\) equations that can be solved for the \(n+1\) unknowns \(x_1, x_2, \ldots, x_n\) and \(\lambda\).
1.7 Utility Maximisation

Example: Cobb-Douglas utility

Assume \( u = (x_1)^{\alpha_1} (x_2)^{\alpha_2} \cdots (x_n)^{\alpha_n}, \quad \alpha_i > 0. \)

We take the log of \( u \) before solving UMP (this is legitimate) to get

\[ \ln u = \sum_i \alpha_i \ln x_i \]

Then, Lagrangean is:

\[ L = \sum_i \alpha_i \ln x_i + \lambda (m - \sum_i p_i x_i) \]

So, the \( n+1 \) equations are:

\[ \frac{\partial L}{\partial x_i} = \frac{\alpha_i}{x_i} - \lambda p_i = 0, \quad i = 1, \ldots, n \quad (1) \]

\[ \frac{\partial L}{\partial \lambda} = m - p_1 x_1 - p_2 x_2 - \cdots - p_n x_n = 0 \quad (2) \]

Unknowns are \( x_1, x_2, \ldots, x_n, \lambda \), and parameters are \( m, p_1, p_2, \ldots, p_n, \alpha_1, \ldots, \alpha_n \)
1.7 Utility Maximisation

Step I: Find formula for $\lambda$ in terms of parameters

\[ \alpha_i = \lambda p_i x_i, \ i = 1,..n \]  

Now sum over $i=1,..n$:

\[ \sum_{i=1}^{n} \alpha_i = \lambda \sum_{i=1}^{n} p_i x_i, \ i = 1,..n \]

\[ \Rightarrow \sum_{i=1}^{n} \alpha_i = \lambda m \quad \text{(using (2))} \]

\[ \Rightarrow \lambda = \frac{\sum_{i=1}^{n} \alpha_i}{m} = \frac{\beta}{m} \quad \text{(3)} \]

NB: This method will also work in more complex problems.
Step 2: Find formulae for $x_1, \ldots, x_n$ in terms of parameters, using (1), (2), (3):

$$x_i = \frac{\alpha_i}{\lambda p_i} = \frac{\alpha_i}{\beta p_i} m, \quad i = 1, \ldots, n$$

Special case $n = 2$, (used below):

$$\alpha_1 = \alpha, \quad \alpha_2 = 1 - \alpha, \quad \Rightarrow \quad x_1 = \frac{\alpha m}{p_1}, \quad x_2 = \frac{(1 - \alpha)m}{p_2}$$
Marshallian Demands
The solutions $x_1,..x_n$ to UMP can be written as functions of parameters $p_1,..p_n, m$:

$$ x_i = x_i(p_1,..p_n, m) $$

(Marshallian demand functions)

Key Properties of Marshallian Demands:

1. **Continuity.** $x_i(p_1,..p_n, m)$ is continuous in $(p_1,..p_n, m)$.

2. **Homogeneity.** If all prices and incomes double (or change by some common multiple) then as the budget set is unchanged, Marshallian demands do not change. Mathematically, $x_i(p_1,..p_n, m)$ is homogenous of degree zero in $(p_1,..p_n, m)$.

3. **Differentiability.** Under certain technical conditions (J-R, Theorem 1.5, assumed satisfied), $x_i(p_1,..p_n, m)$ is differentiable in $(p_1,..p_n, m)$

4. **Budget balance.** $\sum_{i=1}^{n} p_i x_i = m$
1.7 Utility Maximisation

Checking Homogeneity of Marshallian Demand

Generally, \( x_i(p_1, p_2, \ldots, p_n, m) \) is HOD0 if

\[
x_i(\mu p_1, \mu p_2, \ldots, \mu p_n, \mu m) = x_i(p_1, p_2, \ldots, p_n, m) \quad \text{all} \quad \mu > 0
\]

In Cobb-Douglas case,

\[
x_i(\mu p_1, \mu p_2, \ldots, \mu p_n, \mu m) = \frac{\alpha_i}{\beta} \frac{\mu m}{\mu p_i} = \frac{\alpha_i}{\beta} \frac{m}{p_i} = x_i(p_1, p_2, \ldots, p_n, m)
\]

as required.

This method obviously works for all HOD0 functions.
1.7 Utility Maximisation

Indirect Utility

The *indirect utility function* measures the maximum utility that the consumer can achieve by choice of a consumption bundle, given prices $p_1,..,p_n$ and income $m$.

$$v(p_1,..,p_n,m) \equiv u(x_1(p_1,..,p_n,m), x_2(p_1,..,p_n,m),.., x_n(p_1,..,p_n,m))$$

Example: two goods ($n = 2$), and Cobb-Douglas utility $u = \left(x_1\right)^\alpha \left(x_2\right)^{1-\alpha}$

In this special case, $\alpha_1 = \alpha, \alpha_2 = 1 - \alpha$ so the Marshallian demands are

$$x_1(p_1, p_2, m) = \frac{\alpha m}{p_1}, \quad x_2(p_1, p_2, m) = \frac{(1 - \alpha)m}{p_2}$$

Substituting back into the utility function:

$$v(p_1, p_2, m) = \left(\frac{\alpha m}{p_1}\right)^\alpha \left(\frac{(1 - \alpha)m}{p_2}\right)^{1-\alpha} = \alpha^\alpha (1 - \alpha)^{1-\alpha} m(p_1)^{-\alpha} (p_2)^{\alpha-1}$$
Properties of the Indirect Utility Function

1. *Continuous* in \((p_1, \ldots, p_n, m)\)

2. *Strictly increasing* in \(m\), and *strictly decreasing* in \(p_i\).

3. \(v(p_1, \ldots, p_n, m)\) is *homogenous of degree zero* in \((p_1, \ldots, p_n, m)\). If all prices and incomes double (or change by some common multiple) then as the budget set is unchanged, indirect utility is unchanged.

4. If \(v(p_1, \ldots, p_n, m)\) is *differentiable*, Roy’s identity

\[
- \frac{\partial v}{\partial p_i} \frac{\partial p_i}{\partial v} = x_i(p_1, \ldots, p_n, m)
\]

This says that the Marshallian demand function can be “recovered” from the indirect utility function.
1.7 Utility Maximisation

Proof of Roy’s Identity (n=2)

We can write:

\[ v(p_1,p_2,m) \equiv u(x_1(p_1,p_2,m), x_2(p_1,p_2,m)) \]

\[ \equiv u(x_1(p_1,p_2,m), x_2(p_1,p_2,m)) + \lambda(p_1,p_2,m)[m - p_1x_1(p_1,p_2,m) - p_2x_2(p_1,p_2,m)] \]

So, differentiating both sides with respect to \( m \):

\[ \frac{\partial v}{\partial m} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial m} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial m} + \lambda \left[ 1 - p_1 \frac{\partial x_1}{\partial m} - p_2 \frac{\partial x_2}{\partial m} \right] + \frac{\partial \lambda}{\partial m} \left[ m - p_1x_1 - p_2x_2 \right] \]

\[ = \left( \frac{\partial u}{\partial x_1} - \lambda p_1 \right) \frac{\partial x_1}{\partial m} + \left( \frac{\partial u}{\partial x_2} - \lambda p_2 \right) \frac{\partial x_2}{\partial m} + \lambda \]

\[ = \lambda \]

By a similar argument, \[ \frac{\partial v}{\partial p_i} = -\lambda x_i \]  Combining \[ \frac{\partial v}{\partial p_i} = -\lambda x_i \]  \[ \frac{\partial v}{\partial m} = \lambda \] gives Roy’s identity.

In the last line, we have used the fact that the terms in the brackets are zero when the consumer is choosing consumption optimally.
The Expenditure Minimisation Problem

An alternative way of thinking about consumer behaviour is to observe that the consumer can be thought of as taking a *utility level* (rather than income, m) as given. In this case, a rational consumer will minimise the level of expenditure on goods needed to achieve this utility level.

Iso-expenditure line – combinations of bundles of goods that can be bought given expenditure e
All iso-expenditure lines have slope \(-\frac{p_1}{p_2}\)
1.8 Duality

The expenditure minimisation problem (EMP) of the consumer is then to find the minimum level of expenditure required to obtain a “target” level of utility $u$. 

The resulting consumption levels are known as **Hicksian demands** $(h_1, h_2)$. 

**I I – fixed indifference curve**

The consumer can achieve the target utility level at least cost by spending $e_2$ ($e_1$ is too much, and $e_3$ not enough).
1.8 Duality

Formally, the EMP is:

Choose \((x_1, x_2, \ldots, x_n)\) to minimise \(e = p_1 x_1 + p_2 x_2 + \cdots + p_n x_n\) subject to \(u(x_1, \ldots, x_n) \geq u\)

where \(u\) - the “target” level of utility.

Note:

• The constraint set is closed (but not bounded) and

  \(p_1 x_1 + p_2 x_2 + \cdots + p_n x_n\)

  is continuous in the \(x_i\). The extreme value theorem does not apply but a different argument can be made that EMP has at least one solution (Jehle and Reny, p35).

• From Axiom 4, the utility constraint will hold with equality at any solution: \(u(x_1, \ldots, x_n) = u\)

• From Axiom 5, strict convexity, the solution to this problem is unique.
1.8 Duality

Take the Lagrangean approach:

\[ L = p_1 x_1 + p_2 x_2 \ldots + p_n x_n + \mu [u - u(x_1, x_2, \ldots x_n)] \]

The FOC for a solution to EMP are:

\[ \frac{\partial L}{\partial x_i} - \frac{\partial u}{\partial x_i} = 0, \quad i = 1, \ldots, n \] (1)

\[ \frac{\partial L}{\partial \mu} = u - u(x_1, \ldots, x_n) = 0 \] (2)

The solution to this problem gives the Hicksian demands \( h_i(p_1, \ldots, p_n, u) \)

which can then be plugged back into the expenditure formula to get the expenditure function:
1.8 Duality

\[ e(p_1, \ldots, p_n, u) = \sum_{i=1}^{n} p_i h_i(p_1, \ldots, p_n, u) \]

1. **Direct/brute force method.** Lagrangean method above to find \( h_i(p_1, \ldots, p_n, u) \) and then plug back into expenditure formula to get the expenditure function.

2. **Indirect/inversion method.** Suppose that at given prices \( p_1, \ldots, p_n, u \) is the maximum utility that is attainable with income \( m \) i.e.

\[ u = v(p_1, \ldots, p_n, m) \quad (1) \]

Then, the minimum expenditure required to attain \( u \) must be \( m \) i.e.

\[ m = e(p_1, \ldots, p_n, u) \quad (2) \]

We can obtain the expenditure function by solving (1) for \( m \) as a function of \( p_1, \ldots, p_n, u \). This is often convenient as (i) the UMP is often easier to solve than the EMP; (ii) it avoids the need to solve both problems.
1.8 Duality

Example

Assume n = 2 and Cobb-Douglas preferences. Then we know that

\[ v(p_1, p_2, m) = \alpha^\alpha (1 - \alpha)^{1 - \alpha} m(p_1)^{-\alpha} (p_2)^{\alpha - 1} \]

So, equation (1) becomes

\[ u = \alpha^\alpha (1 - \alpha)^{1 - \alpha} (p_1)^{-\alpha} (p_2)^{\alpha - 1} m \]

So, solving this equation for m, we get

\[ m = \alpha^{-\alpha} (1 - \alpha)^{-(1 - \alpha)} (p_1)^{\alpha} (p_2)^{1 - \alpha} u \]

On the right-hand side, we have the formula for the expenditure function, as required i.e.

\[ e(p_1, p_2, u) = \alpha^{-\alpha} (1 - \alpha)^{-(1 - \alpha)} (p_1)^{\alpha} (p_2)^{1 - \alpha} u \]
1.8 Duality

Properties of the Expenditure Function

1. **Continuous** in \((p_1,..p_n, u)\)

2. **Strictly increasing** in \(u\), and in \(p_i\).

3. \(e(p_1,..p_n, u)\) is *homogenous of degree one* in \((p_1,..p_n)\). E.g. if all prices double then twice as much as expenditure is needed to achieve \(u\).

4. It is **concave** in \(p=(p_1,..p_n)\) i.e. \[e(\lambda p + (1-\lambda)p', u) \geq \lambda e(p, u) + (1-\lambda)e(p', u) \quad 0 \leq \lambda \leq 1\]

5. If \(e(p_1,..p_n, u)\) is differentiable, **Shepard’s Lemma**:
\[
\frac{\partial e}{\partial p_i} = h_i(p_1,..p_n, u)
\]

This says that the Hicksian demand function can be “recovered” from the indirect utility function.

Properties 4 and 5 can be used to derive several interesting properties of the Hicksian demands.
Shepard’s Lemma: An Example

Recall in the two-good Cobb-Douglas case, 

\[ e(p_1, p_2, u) = \kappa(p_1)^\alpha (p_2)^{1-\alpha} u \]

where \( \kappa = \alpha^{-\alpha} (1 - \alpha)^{(1-\alpha)} \)

Then by Shepard’s Lemma,

\[ \frac{\partial e}{\partial p_1} = \alpha \kappa \left( \frac{p_2}{p_1} \right)^{1-\alpha} u = h_1(p_1, p_2, u) \]

\[ \frac{\partial e}{\partial p_2} = (1 - \alpha) \kappa \left( \frac{p_1}{p_2} \right)^\alpha u = h_2(p_1, p_2, u) \]
1.8 Duality

Concavity of the Expenditure Function in Prices: Proof

Notation. Let $p, x$ be quantity and price vectors, denote inner products as

$$p \cdot x = \sum_{i=1}^{n} p_i x_i$$

Suppose:

$x^1$ achieves $u$ at minimum cost when prices are $p^1$

$x^2$ achieves $u$ at minimum cost when prices are $p^2$

$x^*$ achieves $u$ at minimum cost when prices are $p^*$

Then by definition:

$$p^1 x^1 \leq p^1 x^* \quad p^2 x^2 \leq p^2 x^*$$

$$p^* = \lambda p^1 + (1 - \lambda) p^2, \quad 0 \leq \lambda \leq 1$$
1.8 Duality

Here are the inequalities again:

\[ p^1 x^1 \leq p^1 x^* \quad \text{and} \quad p^2 x^2 \leq p^2 x^* \]

Multiply first inequality by \( \lambda \) second by \( 1 - \lambda \), and add:

\[ \lambda p^1 x^1 + (1 - \lambda) p^2 x^2 \leq \lambda p^1 x^* + (1 - \lambda) p^2 x^* = (\lambda p^1 + (1 - \lambda) p^2) x^* = p^* x^* \]

But by definition:

\[ e(p^1, u) = p^1 x^1 \quad e(p^2, u) = p^2 x^2 \quad e(p^*, u) = p^* x^* \]

\[ \Rightarrow \lambda e(p^1, u) + (1 - \lambda) e(p^2, u) \leq e(p^*, u) = e(\lambda p^1 + (1 - \lambda) p^2, u) \]

As required. QED.
Intuitive Explanation for \( p^1x^1 \leq p^2x^* \), \( p^2x^2 \leq p^2x^* \)

Bundle \( x^* \) is more expensive than \( x^1 \) at prices \( p^1 \)

Bundle \( x^* \) is more expensive than \( x^2 \) at prices \( p^2 \)
1.8 Duality

Proof of Shepard’s Lemma (n=2)

We can write:

\[ e(p_1, p_2, u) = p_1 h_1(p_1, p_2, u) + p_2 h_2(p_1, p_2, u) \]

So, differentiating both sides with respect to (say) \( p_1 \):

\[
\frac{\partial e}{\partial p_1} = h_1 + p_1 \frac{\partial h_1}{\partial p_1} + p_2 \frac{\partial h_2}{\partial p_1} + \mu \left[ - \frac{\partial u}{\partial x_1} \frac{\partial h_1}{\partial p_1} - \frac{\partial u}{\partial x_2} \frac{\partial h_2}{\partial p_1} \right] + \frac{\partial \mu}{\partial p_1} [u - u(h_1, h_2)]
\]

In the last line, we have used the fact that the terms in the brackets are zero when the consumer is choosing consumption optimally in the EMP.
1.8 Duality

Interpreting Hicksian Demands

Hicksian demands measure the change in demand due to changes in prices, given that the consumer is compensated by changes to her income $m$ so that her utility is unchanged i.e. she stays on the same indifference curve.

- **AA** – old budget line with slope $-\frac{p_1^A}{p_2}$
- **BB** – new budget line with slope $-\frac{p_1^B}{p_2}$

**Note** $p_1^B > p_1^A$
1.8 Duality

Properties of Hicksian Demands

From now on, define
\[ h_{ij} = \frac{\partial h_i(p_1, \ldots, p_n, u)}{\partial p_j} \]
to be the substitution effect of a change in the price of good j on demand for good i.

As Hicksian demands are related to the expenditure function via Shepard’s Lemma, the properties of the expenditure function imply certain properties of the Hicksian demands and therefore substitution effects.

1. \( h_i(p_1, \ldots, p_n, u) \) is homogenous of degree zero in \( (p_1, \ldots, p_n) \). That is, if all prices double (or change by some common multiple) then \( h_i \) does not change.

Reason: relative prices do not change i.e. the slope of the budget line is unchanged, so the same consumption bundle is required to achieve the target level of utility at minimum cost. [this result follows mathematically from Property 1 of the expenditure function: as e is HOD1, all its partial derivatives are HODO].
1.8 Duality

2. Symmetry of Hicksian demands:

This says that the marginal effect of the price of good \( j \) on the Hicksian demand for good \( i \) is equal to the marginal effect of the price of good \( i \) on the Hicksian demand for good \( j \).

Reason: from Shepard’s Lemma,

\[
\begin{align*}
    h_i &\equiv \frac{\partial e}{\partial p_i} \Rightarrow \frac{\partial h_i}{\partial p_j} = \frac{\partial^2 e}{\partial p_i \partial p_j} & \left(1\right) \\
    h_j &\equiv \frac{\partial e}{\partial p_j} \Rightarrow \frac{\partial h_j}{\partial p_i} = \frac{\partial^2 e}{\partial p_j \partial p_i}
\end{align*}
\]

But Young’s Theorem says \( \frac{\partial^2 e}{\partial p_i \partial p_j} = \frac{\partial^2 e}{\partial p_j \partial p_i} \) which implies \( \frac{\partial h_i}{\partial p_j} = \frac{\partial h_j}{\partial p_i} \) from \(1\).
The matrix of substitution effects is

\[
S = \begin{pmatrix}
  h_{11} & h_{12} & \ldots & h_{1n} \\
  h_{21} & h_{22} & \ddots & \vdots \\
  \vdots & \ddots & \ddots & \vdots \\
  h_{n1} & \ldots & h_{n2} & h_{nn}
\end{pmatrix}
\]

The matrix of substitution effects \( S \) is symmetric and negative semi-definite (NSD).

To interpret NSD, consider the two-good case:

\[
S = \begin{pmatrix}
  h_{11} & h_{12} \\
  h_{21} & h_{22}
\end{pmatrix}
\]

From the *principal minor test* for negative semi-definiteness, NSD of \( S \) requires:

(i) \( h_{11}, h_{22} \leq 0 \) (i.e. own-price substitution effects negative)

(ii) \( h_{11}h_{22} - h_{12}h_{21} = h_{11}h_{22} - (h_{12})^2 \geq 0 \) (own-price substitution effects dominate cross-price substitution effects)
1.8 Duality

**Reason:** as $e(p_1,..p_n,u)$ is concave in $p_1,..p_n$, the $n \times n$ matrix of second derivatives of $e$ with respect to prices (the Hessian matrix of $e$) is negative semi-definite. But as shown above, the second derivative $e_{ij}$ of $e$ is equal to the substitution effect $h_{ij}$.

In fact, under our assumptions (A1-A5 plus interior solutions and differentiable utility), the own-price substitution effect is strictly negative: $h_{ii} < 0, i = 1,..n$
The Parametric Approach

• Start from a utility or expenditure function
• Derive budget share equations for goods $i=1,..n$

- Budget share for good $i$ is $s_i = \frac{p_i x_i}{m}$
- Generally $s_i$ will vary with prices and income
- Budget shares needed as the data come in this form, typically from expenditure surveys

• Estimate the parameters of the share equations

• There will be parameter restrictions within and across share equations imposed by utility maximization/expenditure minimization)

• Test to see whether the restrictions imposed by the theory are rejected by the data or not
1.9 Testing Consumer Theory

Example - Cobb-Douglas

- We know Marshallian demands are \( x_i = \frac{\alpha_i m}{p_i} \).
- So, Cobb-Douglas share equations are \( s_i = \frac{p_i x_i}{m} = \alpha_i \).
- This is way too simple to be of any use; budget shares are not constant as prices and incomes change.

The Almost Ideal Demand System

1.9 Testing Consumer Theory

The share equations of the Almost Ideal Demand System are

\[ s_i = \alpha_i + \sum_j \gamma_{ij} \ln p_j + \beta_i \ln \left( \frac{m}{P} \right) \]

\( P \) is a price index, HODO in prices, can be specific to household characteristics (D&M, equation (9))

Interpretation: \( \gamma_{ij}, \beta_i \) capture the effects of relative price and real income changes respectively on budget shares

• Restrictions are:

- \( s_i \) must sum to one; \( \sum_{i=1}^{n} \alpha_i = 1, \sum_{i=1}^{n} \gamma_{ij} = 0, \text{all } j, \sum_{i=1}^{n} \beta_i = 0 \)
- \( s_i \) must be HODO in prices and income; \( \sum_{j=1}^{n} \gamma_{ij} = 0 \)

Slutsky symmetry: \( \gamma_{ij} = \gamma_{ji} \)
1.9 Testing Consumer Theory

D&M estimate the model using annual British data from 1954 to 1974 inclusive on eight nondurable groups of consumers' expenditure; food, clothing, housing services, fuel, drink and tobacco, transport and communication services, other goods, and other services.

Main findings:
• $s_i$ must sum to one by data construction, so these restrictions cannot be tested
• HODO in prices and income i.e. $\sum_{j=1}^{n} y_{ij} = 0$ can be rejected for 3 out of 8 categories of goods
• Slutsky symmetry: $y_{ij} = y_{ji}$ is rejected

Many subsequent studies find similar results
BUT: rejection of homogeneity/symmetry may be due either to wrong functional form for expenditure function or non-rational behavior

SO: rejection may not imply failure of utility maximization

More recent approach is non-parametric testing (no assumptions about form of the utility function).
1.9 Testing Consumer Theory

Non-Parametric Tests

Basic idea: suppose an individual is faced with a series of different budget constraints $i=1,..K$ where each budget constraint is characterized by a price vector $p^i$, and chooses a consumption bundle $x^i$ at each price $p^i$. The $(p^i,x^i)$, $i=1,..,K$ are data on choice behavior.

Definition. When $x^i$ is chosen and $p^ix^i \geq p^ix$, so that $x$ is an available choice, we say that $x^i$ is directly revealed preferred to $x$. This is written $x^i \text{ DR } x$.

Definition. A utility function $u(x)$ rationalizes a set of data $(p^i,x^i)$, $i=1,..,K$, if $u(x^i) \geq u(x)$ for all $x$ such that $x^i \text{ DR } x$, $i=1,..,K$.

Theorem. (Afriat, Varian). If the data satisfy a condition called Generalized Axiom of Revealed Preference (GARP), there exists a non-satiated, continuous, concave, monotonic utility function that rationalizes the data.

Given data $(p_i,x_i)$, $i=1,..,K$, GARP can be checked via algorithms, no econometrics needed! Where do the data come from?

Real choice data (expenditure surveys) and increasingly, experimental data
1.9 Testing Consumer Theory

The WARP
Recall when $x^i$ is chosen and $p^i x^i \geq p^i x$ so that $x$ is an available choice, we say that $x^i$ is *directly revealed preferred* to $x$. This is written $x^i \text{ DR } x$.

Weak Axiom of Revealed Preference. If $x^i \text{ DR } x^j$ and $x^i$ is not equal to $x^j$, then it is not the case that $x^j \text{ DR } x^i$.

The pair $(x^i, x^i)$ pass WARP, but the pair $(y^i, x^i)$ fail WARP.
1.9 Testing Consumer Theory

The GARP

If \( x^1 \) DR \( x^2 \), \( x^2 \) DR \( x^3 \), and so on until \( x^{n-1} \) DR \( x^n \), then we say that \( x^1 \) is (indirectly) \textit{revealed preferred} to \( x^n \).

If \( p^i x^i > p^i x \) and \( x^i \) is the chosen consumption level, then \( x^i \) is \textit{strictly directly revealed preferred} to \( x \).

GARP: If \( x^i \) is (indirectly) revealed preferred to \( x^i \), then \( x^i \) cannot be strictly directly revealed preferred to \( x^i \).
Large number of participants (N=1,182), taken from the CentERpanel, a panel study of a large representative sample of households in the Netherlands that collects a wide range of individual socio-demographic and economic information about its members.

“Each decision problem started with the computer selecting a budget line randomly from the set of budget lines…. The budget lines selected for each subject in different decision problems were independent of each other and of the sets selected for any of the other subjects in their decision problems. Choices were restricted to allocations on the budget constraint. Choices were made using the computer mouse to move the pointer on the computer screen to the desired point and then clicking the mouse or hitting the enter key.”

“We assess how nearly individual choice behavior complies with GARP by using the Afriat (1972) Critical Cost Efficiency Index (CCEI), which measures the fraction by which all budget constraints must be shifted in order to remove all violations of GARP.”

CCEI is between zero and one, higher numbers indicate “more rational” decision-makers

Results?
1.9 Testing Consumer Theory

Figure 3: Mean CCEI Scores
1.10 The Slutsky equation

Hicksian Substitutes and Complements

\[ h_{ij} < 0 \quad \text{goods } i,j \text{ Hicksian complements, } \quad h_{ij} > 0 \quad \text{goods } i,j \text{ Hicksian substitutes} \]

**Theorem** For any good \( i \), \( h_{ij} > 0 \) for some \( j \neq i \); that is, every good is a Hicksian substitute with at least one other good.

Implies if \( n=2 \), goods 1,2 are Hicksian substitutes.

**Proof.** By property 1 above, \( h_i(p_1,..p_n,u) \) is HOD0 in prices. But then using Euler’s homogenous function theorem, we get:

\[
h_{i1}p_1 + h_{i2}p_2 + ..h_{ii}p_i + ..h_{in}p_n = 0
\]

But then, as \( h_{ii}<0 \), and \( p_i>0 \), all \( i \), it must be that \( h_{ij}>0 \), some \( j \). QED.
1.10 The Slutsky equation

### Hicksian vs. Marshallian Substitutes

Define

\[
\frac{\partial x_i(p_1, \ldots, p_n, m)}{\partial p_j} \equiv x_{ij}
\]

\[x_{ij} < 0: \text{goods } i,j \text{ Marshallian complements}, \quad x_{ij} > 0: \text{goods } i,j \text{ Marshallian substitutes}\]

- No simple relationship between Hicksian and Marshallian substitutes

For example, in the Cobb-Douglas case, it is possible to show that \( h_{ij} > 0 \) all i,j i.e. all goods are Hicksian substitutes

But, as Marshallian demands are \( x_i = \frac{\alpha_i}{\beta} \frac{m}{p_i}, i = 1, \ldots n \) in the Cobb-Douglas case, we have \( x_{ij} = 0, \ all i, j \) i.e. Marshallian demands are independent i.e. neither complements nor substitutes.
1.10 The Slutsky equation

The substitution effect on good \( i \) of a rise in \( p_i \) is always negative. The income effect is negative as long as \( i \) is not an inferior good.

OA – old budget line with slope \(-\frac{p_1^*}{p_2}\)
OB – new budget line with slope \(-\frac{p_1^{**}}{p_2}\)
CC - hypothetical budget line resulting from increase in price from \( p_1^* \) to \( p_1^{**} \)
1.10 The Slutsky equation

Using duality theory, we can derive a mathematical expression for this decomposition.

By definition: \( h_i(p_1, \ldots, p_n, u) \equiv x_i(p_1, \ldots, p_n, m) \), where \( m \equiv e(p_1, \ldots, p_n, u) \)

By substitution: \( h_i(p_1, \ldots, p_n, u) \equiv x_i(p_1, \ldots, p_n, e(p_1, \ldots, p_n, u)) \)

As this is an identity, we can differentiate both sides with respect to \( p_j \) to get:

\[
\frac{\partial h_i}{\partial p_j} = \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial m} \frac{\partial e}{\partial p_j} \quad \text{(S1)}
\]

But now \( \frac{\partial e}{\partial p_j} = h_j \) : by Shepard’s Lemma, and \( h_j = x_j \) by definition. So:

\[
\frac{\partial h_i}{\partial p_j} = \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial m} x_j \quad \text{(S2)}
\]
1.10 The Slutsky equation

Rearranging (S2), we get the Slutsky equation:

\[
\frac{\partial x_i}{\partial p_j} = \frac{\partial h_i}{\partial p_j} - \frac{\partial x_i}{\partial m} x_j
\]

substitution income(−?)

Note: normal good has. \( \frac{\partial x_i}{\partial m} > 0 \)
so in this case income effect is negative overall

In the special case of \( j=i \), we have:

\[
\frac{\partial x_i}{\partial p_i} = \frac{\partial h_i}{\partial p_i} - \frac{\partial x_i}{\partial m} x_i
\]

substitution (−) income (−?)

General implication: the law of demand i.e. the demand curve slopes down if the good is normal
1.10 The Slutsky equation

Utility maximization with Endowments

So far, we have treated income m as exogenous and independent of prices

But now suppose that \( m = \sum_{i=1}^{n} p_i e_i \) where \( e_1, \ldots, e_n \) are non-negative endowments of goods (applications: labour supply, savings).

Marshallian demand for good i is then

\[
x_i (p_1, \ldots, p_n, m) = x_i (p_1, \ldots, p_n, p_1 e_1 + p_2 e_2 + \ldots p_n e_n)
\]

Excess demand = Marshallian demand – endowment:

\[
z_i (p_1, p_2, \ldots, p_n) = x_i ((p_1, \ldots, p_n, p_1 e_1 + p_2 e_2 + \ldots p_n e_n) - e_i
\]
1.10 The Slutsky equation

Example

\[ e = (0.5, 1) \quad p = (p_1, p_2) \quad \rightarrow \quad m = 0.5p_1 + p_2 \quad u = (x_1)^\alpha (x_2)^{1-\alpha} \]

So, \[ x_1 = \frac{\alpha m}{p_1} = \frac{\alpha(0.5p_1 + p_2)}{p_1} \quad \text{and} \quad x_2 = \frac{(1-\alpha)m}{p_2} = \frac{(1-\alpha)(0.5p_1 + p_2)}{p_2} \]

So, \[ z_1 = x_1 - e_1 = \frac{\alpha(0.5p_1 + p_2)}{p_1} - 0.5 \quad \text{and} \quad z_2 = x_2 - e_2 = \frac{(1-\alpha)(0.5p_1 + p_2)}{p_2} - 1 \]
1.10 Utility maximization with endowments

The Slutsky Equation with Endowments
With endowments, the Slutsky equation is modified because \( m \) depends on prices:

\[
x_i \left( p_1, \ldots, p_n, \frac{1}{m} \left( p_1 e_1 + p_2 e_2 + \ldots + p_n e_n \right) \right) =
\]

\[
\frac{\partial x_i}{\partial p_j} = \left. \frac{\partial x_i}{\partial p_j} \right|_{m \text{ const}} + \frac{\partial x_i}{\partial m} e_j = \frac{\partial h_i}{\partial p_j} - \frac{\partial x_i}{\partial m} x_j + \frac{\partial x_i}{\partial m} e_j = \frac{\partial h_i}{\partial p_j} - \frac{\partial x_i}{\partial m} (x_j - e_j) = \frac{\partial h_i}{\partial p_j} - \frac{\partial x_i}{\partial m} z_j
\]

Now, income effect depends on excess demand \( z_i \), not \( x_i \)

If \( z_i > 0 \), and good \( i \) is normal, \( -\frac{\partial x_i}{\partial m} z_i < 0 \) and so the “law of demand” holds

If \( z_i < 0 \), and good \( i \) is normal, \( -\frac{\partial x_i}{\partial m} z_i > 0 \). So, then, if the income effect is large enough, possible that \( \frac{\partial x_i}{\partial p_i} > 0 \) i.e. the law of demand need no longer hold when excess demands are negative, and income effect is large
1.10 Utility maximization with endowments

Intuition: 🍏Apple buyer vs. 🍏Apple Grower

Apple buyer; excess demand for apples $z>0$; so, an increase in the price of apples makes him worse off, implies negative income effect on his demand for apples (or anything else he buys).

Apple Grower: excess demand for apples $z<0$, so, an increase in the price of apples makes her better off, implies positive income effect on her demand for apples (or anything else she buys).
1.10 Utility maximization with endowments

Application: *Labour supply*

Good 1 is consumption, $c$, good 2 is leisure, $l$.

The household has utility function $u(c, l)$

The household has endowment vector: $e = (y, H)$ where $y$ is non-wage income in consumption units, $H$ is the number of hours available for leisure or work.

Prices are $p_1 = 1$, $p_2 = w$ (note: the wage is the price of leisure)

So, value of endowment is $y + wH = y + wH$ (Note: sometimes $y + wH$ known as full income)

So, budget constraint is $c + wl = y + wH$
Application: Labour supply

Endowment of leisure $H$ is greater than consumption of leisure $l$, so excess demand for leisure is negative ($l - H = z < 0$). So, if leisure is a normal good, the income effect of an increase in the wage (from $w^*$ to $w^{**}$) is positive i.e. increases the demand for leisure. Thus, if the income effect dominates the substitution effect, the leisure demand curve can be upward-sloping.

As shown, the income effect ($l'$ to $l^{**}$) dominates the substitution effect ($l^*$ to $l'$)}
1.10 Utility maximization with endowments

Cobb-Douglas Example

Utility function is \( u(x, l) = c^\alpha l^{1-\alpha} \)

The budget constraint is \( c + wl = y + wH \)

The solution to the household maximization problem is

\[
\begin{align*}
c &= \alpha(wH + y), \\
l &= (1 - \alpha) \frac{(wH + y)}{w} = (1 - \alpha)(H + y / w)
\end{align*}
\]

If \( y=0 \), \( l = (1 - \alpha)H \) and is thus independent of \( w \) i.e. income effect exactly offsets the substitution effect.

If \( y>0 \), \( l \) is decreasing in \( w \) and thus income effect is smaller than the substitution effect (implies “normal” upward-sloping labour supply).
Overview of consumer theory

Utility maximisation

Choose \( x_1 \ldots x_n \) to maximise \( u(x_1 \ldots x_n) \)
subject to \( \sum_{i=1}^{n} p_i x_i = m \)

solve

Marshallian demands
\( x_i(p_1, \ldots, p_n, m) \)

Substitute into \( u \)

Roy's Identity

Indirect utility function
\( v(p_1, \ldots, p_n, m) \)

Expenditure Minimisation

Choose \( x_1 \ldots x_n \) to minimise

solve

Hicksian demands
\( h_i(p_1, \ldots, p_n, u) \)

Substitute into \( e \)

Shepard Lemma

Expenditure function
\( e(p_1, \ldots, p_n, u) \)

Substitute into \( e \)

Inversion
1.10 Utility maximization with endowments

END OF TOPIC 1

Are Your Ready for the mini QUIZ 1?
Do some practice questions here
Once you feel confident go to Moodle

Available 1-5 pm on Tuesday Week 4.
QUIZ 1 Counts for 2 % of your overall mark on EC901
You have 30 minutes to complete the test once you start, so you should start by 4:30 pm!