Partition Dependent Expected Utility[*]

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Abstract

In this paper, we study choice under objective risk where the primitive is enriched to include an exogenous equivalence relation on the space of lotteries. We seek conditions on this enlarged primitive ensuring the existence of an expected utility representation in which the Bernoulli utility index may depend on the partition generated by the equivalence relation. We term this model the Partition Dependent Expected Utility (PDEU) and show examples of recent choice models in the literature on non-expected utility that fall into this class. We prove representation theorems characterizing PDEU preferences when the partition generates convex cells, and under different continuity assumptions. Our theorems address partitions with both countable and uncountable elements, with cells that can be lower-dimensional, fully dimensional, or a combination of both. We show that for fully dimensional cells, the parameters of the representation are suitably unique, but this is not the case for lower-dimensional cells. We conclude with a discussion of the technical challenges that may arise when studying partitions with non-convex cells.

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1 Introduction

In this paper we study preferences over objective lotteries for which the primitive is enriched to include not only the usual preference relation $\succsim$, but also an equivalence relation $\odot$. We seek conditions on the primitive $\langle \succsim, \odot \rangle$ that guarantee the existence of an expected utility representation for which the Bernoulli utility index might depend on the partition generated by $\odot$. We call any such primitive *Partition Dependent Expected Utility* (PDEU) preferences.

For concreteness, assume Z is any compact metric space and denote by $\Delta(Z)$ the space of all finitely supported lotteries on Z. Denote by $\mathscr{P}_{\odot}$ the partition generated by $\odot$, with general cell $[p] = \{q \in \Delta(Z) : p \odot q\}$. PDEU boils down to the existence of a family of utility indices $\langle u_{[p]} \rangle_{[p] \in \mathscr{P}_{\odot}}$, such that for any two lotteries $p \in [p]$ and $q \in [q]$

$$p \succsim q \Leftrightarrow \sum_{z \in \operatorname{supp}(p)} p(z)\, u(z, [p]) \geq \sum_{z \in \operatorname{supp}(q)} q(z)\, u(z, [q]) \quad \text{for all } p, q \in \Delta(Z) \tag{1}$$

Observe that under (1), when restricted to a fixed cell $[p]$, preferences are regular vNM expected utility, but if, for a second cell, $[q]$, it is true that $u_{[p]} \neq u_{[q]}$ then preferences need not be linear in probabilities. A rather trivial case arises when the partition contains only singletons, $\mathscr{P}_{\odot} = \{\{p\} : p \in \Delta(Z)\}$. In this case, if V represents $\succsim$, we can simply set $u(z, [p]) = V(p)$ for all $z \in \operatorname{supp}(p)$. Clearly in this case the model has no testable implications over and above the preference being a representable weak order. There are various models studied in the literature of non-expected utility that have a PDEU structure.

- Dillenberger and Raymond (2020)'s *Additive-Belief-Based (ABB) preferences*: The representation takes $Z = \Delta(X)$ and an equivalence relation that is represented by a function $\varphi : \Delta(\Delta(X)) \to \Delta(X)$ that maps each two-stage lottery to its reduced probability distribution over X. The intuition of the model is that, when ranking a two-stage lottery $P \in \Delta(\Delta(X))$, the decision maker (DM) uses the criterion (1) in which $[P] = \{Q \in \Delta(\Delta(X)) : \varphi(Q) = \varphi(P)\}$.

- Gilboa et al. (2024)'s *Zero-Risk preferences*: Within the space of consequences Z, a subset Z_0 of *"acceptable"* outcomes is identified. The partition of $\Delta(Z)$ in their setting has two elements, $\Delta(Z) = \Delta(Z_0) \cup \Delta(Z \backslash Z_0)$. The intuition being that lotteries belonging to $\Delta(Z_0)$ present *"zero-risk"* to the DM of getting a non-acceptable outcome. The resulting representation takes the form of (1) where $p \odot q$ iff $\operatorname{supp}(p) \cup \operatorname{supp}(q) \subseteq Z_0$ or $\operatorname{supp}(p) \cup \operatorname{supp}(q) \subseteq Z \backslash Z_0$.

- Troccoli Moretti (2024)'s *Disappointment-Dependent Self-Control preferences*: In this setting the domain is dynamic: $Z = C \times \mathscr{M}$, where C is a space of consumption and $\mathscr{M}$ the space of all non-empty compact menus over $\Delta(C)$, hence each $p \in \Delta(Z)$ delivers a consumption for today and a choice problem for tomorrow. If we let p^1 denote the marginal of p over C, the equivalence relation is defined so that $p \odot q$ iff $p^1 = q^1$. The intuition is that first-period disappointments affect second-period self-control and hence the value of a particular menu is history-dependent, where histories record not only past consumption but also p^1, the distribution from which said consumption came from. The representation takes the form of (1) with $[p] = \{q \in \Delta(C \times \mathscr{M}) : p^1 = q^1\}$.

- Dekel (1986)'s *Betweenness Preferences:* This is a rather trivial, but nonetheless special case of PDEU. By letting $\sim = \odot$, we get that $p \odot q$ iff $p \sim q$. This example also shows some of the limits of PDEU, as it applies no restrictions to cells whose elements are all in the same indifference class.

While all the papers above share a common PDEU structure, they differ in the properties of the partitions they work with. For instance, Dekel (1986), Dillenberger and Raymond (2020) and Troccoli Moretti (2024) work with partitions with an uncountable number of cells, each of which is convex and lower-dimensional, that is, they are *"slices"* of the simplex. On the other hand, Gilboa et al. (2024) work with a finite number of convex cells, some of which are fully dimensional, while others are not.

These models differ also on their continuity requirements. For example, discontinuities between cells is at the heart of the intuition behind Gilboa et al. (2024) zero risk model, that want to capture the idea that including certain elements in the support of a lottery is unacceptable according to some underlying principle; while Troccoli Moretti (2024) insist in keeping preferences continuous on the entire space of lotteries. We present a unifying theorem characterising PDEU preferences covering many cases of the structure that the primitive $\langle \succsim, \odot \rangle$ can take. To our surprise, many technical issues that, *prima facie*, might appear to require separate arguments, can be proven by using the same technique. For concreteness, after identifying the common axioms underlying the aforementioned models, we show that, whenever two cells (A and B) of the partition (the $\odot-$equivalence classes) share multiple indifference classes (the $\succsim$-equivalence classes), a representation that is expected utility for cell A also has to be expected utility for the part of cell B that overlaps with cell A in the indifference space. This allows us to find a PDEU representation by going along a list of all cells and augmenting the representation at each step to make it expected utility for the cell. The same argument can be applied to finite, countable, and uncountable partitions, provided that the cells are convex. As for the distinction between lower and fully dimensional cells, the representation result is the same, but the uniqueness properties differ.

Related Literature. A curiosity of the above mentioned literature is that, with the exception of Troccoli Moretti (2024), none of the papers we present here as belonging the the PDEU class cite each other, despite generally being contemporaneous. This may be surprising because all of these papers, except Dekel (1986), use a similar axiomatic structure. Particularly, the axiom we call *parallel classes* (Axiom 6) shows up in each of the papers in a very similar form. These papers prove their representation theorems in different ways, which of course is to be expected since they impose different additional restrictions on preferences.

The current paper serves to not only characterise existing models into a common class, but it also provides a proof technique that is applicable to any PDEU structure with convex cells. In particular, it sheds light on what characteristics of the primitive (e.g the cardinality of the partition, or convexity/dimensionality of it's cells) are essential (in that they need a different proof technique), and what implications they have for the uniqueness properties of the model.

We are not the first to consider a non-expected utility model with intuitions similar to PDEU. Becker and Sarin (1987) introduce the *lottery-dependent expected utility* model (LDU), in which the agent values a lottery by taking the expectation of a utility index that might depend on a linear real-valued function of the lottery itself. Later, Schmidt (2001) presents a re-examination of LDU by providing axiomatic foundations for a more general version in which the real valued function considered by Becker and Sarin (1987) need not be linear, with the only restriction that the equivalence classes it induces are convex. Schmidt (2001)'s characterisation works with axioms similar to ours, and his proof has a similar spirit. However, he does not provide a proof that Axiom 6 (presented in Section 3) implies that the utility indexes in each cell can be made suitably "compatible", as he perceives this step to be obvious. As we will see, this is far from being the case. In fact, showing how to use Axiom 6 is the main intermediate step in proving the PDEU representation theorem, and is the one in which most care has to be taken (this is our Lemma 6). In some sense, the present paper re-examines Schmidt (2001)'s re-examination of Becker and Sarin (1987) LDU model, by working in a more flexible framework, and elaborating on some important steps and details that were previously omitted. Hopefully, the treatment of PDEU preferences in this paper is rigorous and exhaustive enough that no further re-examinations will be necessary of the axiomatic foundations of this class of models.

The rest of the paper is organised as follows. Section 2 sets up the mathematical framework. Section 3 gives a formal definition of PDEU preferences and introduces the different axioms that will be used in the rest of the paper. In Section 4 we introduce the main theorem of the paper. The results so far require continuity to hold only within each cell, and only deal with finitely supported lotteries. In Section 5 we deal with the case of general probability measures and strengthen continuity to hold on the entire simplex. The characterisation of PDEU for this case appears in Corollary **??**. Section 6 discussed the uniqueness properties of PDEU preferences, in which the dimensionality of the cells plays a crucial role. Section 7 concludes.

2 Preliminaries

Let Z be a compact metric space of outcomes. Write $\Delta^m(Z)$ for the space of all Borel probability measures on Z, endowed with the topology of weak convergence, under which it is compact and metrizable. Its subset of finitely supported elements will be denoted by $\Delta(Z)$, the set of *simple* lotteries. For $p \in \Delta^m(Z)$ the support is $\text{supp}(p) := \{z \in Z : p(z) > 0\}$. For $p, q \in \Delta(Z)$ and $\alpha \in [0,1]$, their mixture $\alpha p + (1-\alpha)q \in \Delta(Z)$ is defined by $(\alpha p + (1-\alpha)q)(z) = \alpha p(z) + (1-\alpha)q(z), z \in \text{supp}(p) \cup \text{supp}(q)$, and the degenerate lottery assigning probability one to $z \in Z$ is written δ_z.

The decision maker (DM) has a preference relation $\succsim$ on $\Delta^m(Z)$, with strict preference and indifference written $\succ$ and $\sim$. Alongside preferences, the DM also considers an equivalence relation $\odot$ on $\Delta^m(Z)$, which induces a partition into cells $[p] := \{q \in \Delta^m(Z) : p \odot q\}$. The quotient space is $\mathscr{P}_\odot = \Delta^m(Z)/\odot$, and the primitive of our model is the pair $\langle \succsim, \odot \rangle$. For any $p \in \Delta^m(Z)$, its indifference class is $I(p) := \{q \in \Delta^m(Z) : p \sim q\}$, and for each $[p] \in \mathscr{P}_\odot$ we write $\mathcal{I}([p]) = \bigcup_{p \in [p]} I(p)$ for the set of indifference classes it covers.

3 Functional Form and Axioms

In this section we define a PDEU representation and introduce the different axioms that will be used during the analysis.

Definition 1. The primitive $\langle \succsim, \odot \rangle$ admits a *Partition-Dependent Expected Utility (PDEU)* representation if, for every $[p] \in \mathscr{P}_\odot$, there exists a utility function

$$u(\cdot, [p]) : \text{dom}([p]) \to \mathbb{R}, \qquad \text{where } \text{dom}([p]) := \bigcup_{p \in [p]} \text{supp}(p),$$

such that for any $p \in [p]$, $q \in [q]$, and $[p], [q] \in \mathscr{P}_\odot$,

$$p \succsim q \iff \int_Z u(z, [p]) \, \mathrm{d}p \geq \int_Z u(z, [q]) \, \mathrm{d}q. \tag{2}$$

We now proceed to introduce and discuss the different axioms that will be used throughout. First, we impose the basic requirement that the preference relation is complete and transitive.

Axiom 1 (Weak Order). *The preference relation $\succsim$ on $\Delta^m(Z)$ is complete and transitive.*

Next we introduce two forms of continuity. First we impose a classical continuity requirement: the upper and lower contour sets of every lottery must be closed. In conjunction with Axiom 1, the theorem by Debreu (1954) guarantees that $\succsim$ can be represented by a continuous utility function.

Axiom 2 (Global Continuity). *For every $p \in \Delta^m(Z)$ the sets $\{q \in \Delta^m(Z) : q \succsim p\}$ and $\{q \in \Delta^m(Z) : p \succsim q\}$ are closed in the weak convergence topology.*

However, requiring continuity on the entire space might be too restrictive for some applications of PDEU. For instance Gilboa et al. (2024) require continuity to hold only within cells. For this reasons, we introduce the following weaker version of continuity:

Axiom 3 (Local Mixture-Continuity). *For every $[p] \in \mathscr{P}_\odot$ and $p, q, r \in [p]$ such that $p \succ q \succ r$, each of the sets*

$$\{\alpha \in [0,1] : \alpha p + (1-\alpha) r \succsim q\} \quad \textit{and} \quad \{\alpha \in [0,1] : q \succsim \alpha p + (1-\alpha) r\}$$

are closed.

Without global continuity there is not enough structure to guarantee the existence of a utility function on the set $\Delta^m(Z)$. In this case we will impose the following separability condition, which, in conjunction with Axiom 1, is necessary and sufficient for the preference $\succsim$ to admit a real-valued utility representation.

Axiom 4 (Separability). *There exists a countable subset S of $\Delta^m(Z)$ such that, for every $p, q \in \Delta^m(Z)$, if $p \succ q$, then there exists an element $s \in S$ such that $p \succsim s \succsim q$.*

Finally, we will impose two weakened versions of the independence axiom. As discussed in the introduction, when comparing lotteries belonging to the same class, PDEU reduces to expected utility, and we should hence expect the independence axiom to hold.

Axiom 5 (Partition Independence). *For every $[p] \in \mathscr{P}_\odot$, $p, q, r \in [p]$ and $\alpha \in (0,1]$ we have*

$$p \succsim q \Leftrightarrow \alpha p + (1-\alpha) r \succsim \alpha q + (1-\alpha) r$$

While, in addition to the previous axioms, Partition Independence guarantees that $\succsim$ restricted to a particular class admits an expected utility representation, so far we have no axiom relating rankings across cells. The next axiom carries the intuition that the *"scale"* that we use to measure utils in one cell should be the same across cells. Alternatively, one may think of it as fixing "utility exchange rates" across cells.

Axiom 6 (Parallel Classes). *For every* $[p], [q] \in \mathscr{P}_{\odot}$ *with elements* $p, p' \in [p]$ *and* $q, q' \in [q]$ *if* $p \sim q$ *and* $p' \sim q'$*, then for every* $\alpha \in [0,1]$

$$\alpha p + (1-\alpha)p' \sim \alpha q + (1-\alpha)q'$$

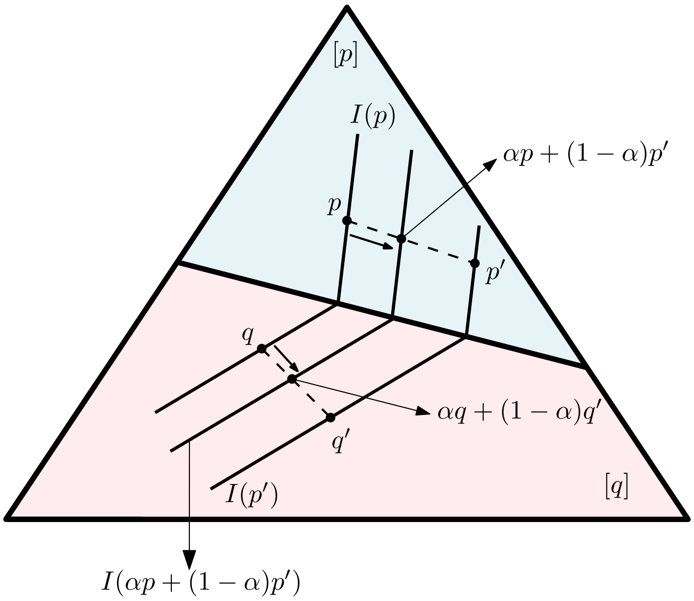

Figure 1: A Marschak–Machina triangle illustrating Axiom 6.

Figure 1 may help to understand the intuition behind Parallel Classes. The figure depicts a partition of $\Delta^m(Z)$ into two cells, $[p]$ and $[q]$, and two indifference curves $I(p)$ and $I(p')$, noting that $q \in I(p)$ and $q' \in I(p')$. Assume, without loss of generality that $p' \succ p$. Hence, since $p \sim q$ and $p' \sim q'$, Axiom 6 requires that every distance $\alpha \in [0,1]$ that is travelled in the segment of increasing preference in both cells, then indifference must be preserved: $\alpha p + (1-\alpha)p' \sim \alpha q + (1-\alpha)q'$.

4 Main Result

In this section we present the most general result available. We do not impose global continuity (Axiom 2), but instead require only continuity within each cell, as in Axiom 3. Since global continuity is absent, we additionally impose Axiom 4, which, together with Axiom 1, guarantees the existence of a utility representation for the preference relation $\succsim$.

Theorem 4.1. $\succsim$ *satisfies Axioms 1, 3, 4, 5, and 6 if and only if it admits a PDEU representation of the form*

$$p \succsim q \iff \sum_{z \in \text{supp}(p)} p(z)\, u(z,[p]) \geq \sum_{z \in \text{supp}(q)} q(z)\, u(z,[q]), \qquad p \in [p],\ q \in [q].$$

A detailed proof of Theorem 4.1 can be found in the Appendix. Here we simply present an intuitive explanation of the proof. Axioms 1 and 4 guarantee the existence of a global real-valued representation $V : \Delta^m(Z) \to \mathbb{R}$, while Axioms 1, 3, and 5 guarantee that the restriction of $\succsim$ to each cell $[p]$ admits an expected utility representation $U_{[p]} : [p] \to \mathbb{R}$. The task is therefore to find a strictly increasing transformation $f : \mathbb{R} \to \mathbb{R}$ such that $f \circ V(p) = a_{[p]}U_{[p]}(p) + b_{[p]}$ for some $(a_{[p]}, b_{[p]}) \in \mathbb{R}_{++} \times \mathbb{R}$, simultaneously for every cell. We fix an arbitrary representation, V, and use it to derive the PDEU representation, V^*, by defining the transformation f stepwise. Starting with the image of an arbitrary cell, $V([p])$, we expand its domain cell-image by cell-image to $\mathrm{Im}(V)$.
The key insight is that Axiom 6 constrains f wherever two cells' images overlap on a non-degenerate interval: If $p \sim \ell$ and $p' \sim \ell'$ for $p, p' \in [r]$ and $\ell, \ell' \in [q]$, then Axiom 6 requires $\alpha p + (1-\alpha)p' \sim \alpha\ell + (1-\alpha)\ell'$ for all $\alpha \in [0, 1]$, which, combined with mixture-linearity of $U_{[q]}$, uniquely pins down the slope $a_{[r]}$ and intercept $b_{[r]}$ from the already-known parameters of $[q]$. The transformation f is then constructed by transfinite induction over a well-ordering, chosen so that each new cell's image overlaps on a non-degenerate interval with the already-covered region whenever possible. This also ensures that every pair of previously processed cells that both overlap with the new cell are connected by a chain of pairwise-overlapping cells within the covered region. This chain property ensures that the derived affine parameters are independent of the reference cell used to construct them, guaranteeing consistency throughout the induction.

5 Global Continuity

In some applications of PDEU it is natural to require a representation that is continuous on the entire space of probability measures, so that it can rank not only finitely supported lotteries but also general distributions.

The key observation is that the argument of Theorem 4.1 carries over unchanged: the transformation functions used to calibrate scales across cells are strictly increasing and therefore preserve continuity. In fact, working with continuous preferences simplifies some steps of the proof, as the existence of a continuous utility representation follows directly from Debreu's classical theorem. Therefore, we derive the following corollary.

Corollary 1. *Take a primitive $\langle \succsim, \odot \rangle$ that generates a partition $\mathscr{P}_\odot$ of $\Delta^m(Z)$. Then $\succsim$ satisfies axioms A1, A2, A5 and A6 if and only if it admits a PDEU representation: For each cell $[p] \in \mathscr{P}_\odot$ there exists a continuous function $u(\cdot, [p]) : \mathrm{dom}([p]) \to \mathbb{R}$ such that for all $[p], [q] \in \mathscr{P}_\odot$ and $p \in [p], q \in [q]$,*

$$p \succsim q \iff \int u(z, [p])\mathrm{d}p \geq \int u(z, [q])\mathrm{d}q.$$

By Debreu (1954), Axioms 1 and 2 yield a continuous utility $V : \Delta^m(Z) \to \mathbb{R}$. Within any cell $[p]$, Axiom 5 gives a vNM representation on $[p]^s = [p] \cap \Delta(Z)$. By the same as in the proof of Theorem 4.1 and density of $\Delta(Z)$ in $\Delta^m(Z)$, this yields a continuous Bernoulli index $u(\cdot, [p])$ on $\mathrm{dom}([p])$ with $U_{[p]}(r) = \int u(z, [p])\mathrm{d}r$ for all $r \in [p]$. The assemble across cells uses Axiom 6 in the same way as in the proof of Theorem 4.1. This produces a strictly increasing map that is affine on each V-range and continuous on $V(\Delta^m(Z))$.

6 Uniqueness

We first fix the ambient face used to assess dimension. For any cell $[p] \in \mathscr{P}_{\odot}$, recall $\mathrm{dom}([p]) := \bigcup_{r \in [p]} \mathrm{supp}(r)$ and take $\Delta\big(\mathrm{dom}([p])\big) \subseteq \Delta^m(Z)$ as the relevant face. Unless stated otherwise, interior and dimension are always evaluated *relative to* $\Delta\big(\mathrm{dom}([p])\big)$.

Definition 2. A cell $[p]$ is *full-dimensional* if it has non-empty relative interior in $\Delta\big(\mathrm{dom}([p])\big)$. Equivalently, there exists a relative open neighbourhood $O \subseteq \Delta\big(\mathrm{dom}([p])\big)$ with $O \subseteq [p]$. A cell is *lower-dimensional* if it is not full-dimensional.

Note that if $\mathrm{dom}([p])$ is finite and contains k points, then the simplex $\Delta\big(\mathrm{dom}([p])\big)$ has dimension $k-1$. The cell $[p]$ is full dimensional if and only if the affine hull of $[p]$ has dimension $k-1$. Otherwise $[p]$ is lower dimensional.

For example, in Troccoli Moretti (2024) cells fix the first-period marginal over consumption. Fixing a marginal imposes linear constraints, so $[p]$ has empty relative interior in $\Delta(\mathrm{dom}([p]))$ and is therefore lower dimensional. In Dillenberger and Raymond (2020) cells are fibres $[P] = \{Q : \varphi(Q) = \varphi(P)\}$ of the reduction map, so each such fibre lies in a proper affine subspace of $\Delta(\mathrm{dom}([P]))$, hence $[P]$ is lower dimensional. By contrast, in Gilboa et al. (2024) the partition is $\Delta(Z) = \Delta(Z_0) \cup \Delta(Z \setminus Z_0)$ and each cell coincides with its ambient face $\Delta(\mathrm{dom}([p]))$, so every cell is full dimensional.[1]

6.1 Uniqueness for Fully-Dimensional Cells

A PDEU representation is determined by the family of cell-dependent indices $\mathbf{u} := \langle u_{[p]} \rangle_{[p] \in \mathscr{P}_{\odot}}$. When every cell is full dimensional, we obtain the standard cardinal uniqueness: if $\mathbf{u} = \langle u_{[p]} \rangle$ and $\mathbf{u}' = \langle u'_{[p]} \rangle$ are two PDEU representations of the same primitive $\langle \succsim, \odot \rangle$, then for each cell $[p]$ there exist $\alpha_{[p]} > 0$ and $\beta_{[p]} \in \mathbb{R}$ such that $u'_{[p]} = \alpha_{[p]} u_{[p]} + \beta_{[p]}$. The intuition is simple: full dimensionality guarantees that the affine extension used to define the Bernoulli index within each cell is unique. Consequently, there is no difference between treating $u_{[p]}$ as defined on $\mathrm{dom}([p])$ or on Z. The unique affine extension picks out the same function.

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[1] If, instead, one evaluates dimension relative to the whole simplex $\Delta^m(Z)$, then whenever Z_0 is a strict subset of Z each cell is a proper face of $\Delta^m(Z)$ and would be classified as lower dimensional under that convention.

We call a PDEU primitive $\langle \succsim, \odot \rangle$ *nontrivial* if there exist $p, q \in \Delta(Z)$ with $p \succ q$. We say it satisfies the *richness* condition if no block of cells is disconnected from the rest in the preference order. We formalise this below.

Definition 3. The primitive $\langle \succsim, \odot \rangle$ satisfies *richness* if, for every lottery $q \in \Delta(Z)$ that is neither $\succsim$-maximal nor $\succsim$-minimal, there exists a cell $[p] \in \mathscr{P}_\odot$ and elements $r, s \in [p]$ such that $r \succ q \succ s$.[2]

Intuitively, fix any lottery q that is neither maximal nor minimal. Richness requires that some cell intersects *both* the strict upper and the strict lower contour sets at q. Equivalently, within a single cell there exist $r \succ q$ and $s \prec q$. Thus the partition cannot be split into disjoint blocks that can be calibrated independently, because at every non-extreme q at least one cell links the two sides of the preference order.

Theorem 6.1. *Let $\langle \succsim, \odot \rangle$ be nontrivial and satisfy* richness, *and suppose every cell in $\mathscr{P}_\odot$ is full dimensional. Then two PDEU representations $\mathbf{u} = \langle u_{[p]} \rangle$ and $\mathbf{u}' = \langle u'_{[p]} \rangle$ represent the same primitive if and only if there exist constants $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}$ such that, for all cells $[p] \in \mathscr{P}_\odot$,*

$$u'_{[p]} \;=\; \alpha\, u_{[p]} + \beta, \qquad \textit{and hence} \qquad V' \;=\; \alpha\, V + \beta.$$

As discussed above, full dimensionality pins down, within each cell, a unique cardinal class for the Bernoulli index—any two indices on the same cell differ by a positive affine transformation $(\alpha_{[p]}, \beta_{[p]})$. Richness forces these cellwise affine parameters to coincide across cells, so the entire family is unique up to a single common positive affine transformation.

Uniqueness without Richness

To understand the identification properties of PDEU it is useful to examine the case *without* the richness assumption. Within any full–dimensional cell $[p]$, standard vNM uniqueness gives that any other per–cell index representing the same within–cell preference must be of the form $v_{[p]} = a_{[p]} u_{[p]} + b_{[p]}$ with $a_{[p]} > 0$. The primitive $\succsim$ is defined on the whole $\Delta(Z)$, so a PDEU representation is the entire family $\mathfrak{u} = \langle u_{[p]} \rangle_{[p] \in \mathscr{P}_\odot}$. It follows that *even without richness* one cannot choose the coefficients $\{a_{[p]}, b_{[p]}\}$ arbitrarily if $\mathfrak{u}' := \langle v_{[p]} \rangle$ is to represent the *same* $\succsim$. A simple counterexample makes the constraint vivid. Fix a cell $[q]$ and set $v_{[q]} := u_{[q]} + M$ with M very large, while keeping $v_{[r]} := u_{[r]}$ on all other cells $[r] \neq [q]$. Then $V'(\cdot) = V(\cdot) + M$ on $[q]$ and $V'(\cdot) = V(\cdot)$ off $[q]$. This pushes every lottery in $[q]$ above every lottery outside $[q]$ provided M is large enough, which generally reverses some cross–cell

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[2] Equivalently, for each such q define $A(q) := \{[p] \in \mathscr{P}_\odot : \exists r \in [p] \text{ with } r \succ q\}$ and $B(q) := \{[p] \in \mathscr{P}_\odot : \exists s \in [p] \text{ with } s \prec q\}$. Then richness is $A(q) \cap B(q) \neq \varnothing$. Equivalently again, there exists a cell $[p]$ with $[p] \cap U(q) \neq \varnothing$ and $[p] \cap L(q) \neq \varnothing$, where $U(q) := \{r : r \succ q\}$ and $L(q) := \{s : s \prec q\}$.

comparisons dictated by $\succsim$. The next result gives the exact compatibility conditions that characterise when cellwise affine changes still yield the same global preference.

Proposition 1 (Uniqueness without richness: full-dimensional cells). *Let $\mathfrak{u} = \langle u_{[p]} \rangle_{[p] \in \mathscr{P}_{\odot}}$ be a PDEU representation with induced functional V. Let $\mathfrak{u}' = \langle v_{[p]} \rangle$ be another family with $v_{[p]} = a_{[p]} u_{[p]} + b_{[p]}$ on each full–dimensional cell $[p]$, which induces V'. Write $J_{[p]} := \{V(p) : p \in [p]\} \subset \mathbb{R}$.*

Then $\mathfrak{u}'$ represents the same $\succsim$ if and only if the following compatibility *conditions hold:*

1. *For every pair of cells $[p], [q]$ with $J_{[p]} \cap J_{[q]}$ containing at least two distinct values,*

$$a_{[p]} = a_{[q]} \quad \textit{and} \quad b_{[p]} = b_{[q]}.$$

2. *For every pair with $J_{[p]} \cap J_{[q]} = \{v_0\}$ for $v_0 \in \mathbb{R}$,*

$$a_{[p]} v_0 + b_{[p]} \;=\; a_{[q]} v_0 + b_{[q]}.$$

3. *For every pair with $J_{[p]}$ and $J_{[q]}$ disjoint and ordered such that $\sup J_{[p]} < \inf J_{[q]}$,*

$$\sup\{a_{[p]} v + b_{[p]} : v \in J_{[p]}\} \;<\; \inf\{a_{[q]} v + b_{[q]} : v \in J_{[q]}\},$$

   constraining one of the intercepts.

Equivalently, there exists a strictly increasing h on $V(\Delta(Z))$ such that $V' = h \circ V$, and on each set $J_{[p]}$ the restriction of h is the affine map $v \mapsto a_{[p]} v + b_{[p]}$.

Richness links cells through chains of nondegenerate overlaps of the value sets $J_{[p]}$. In Proposition 1, item 1 forces equality of coefficients whenever two cells have a nondegenerate overlap. Under richness one can connect any two cells by a finite chain of such overlaps, so equality propagates along the chain and yields a *common* pair (α, β) across all cells. This is the collapse from cell-wise affine freedom to a single global co–affine transform.

6.2 Uniqueness for Lower-Dimensional Cells

When some cells are lower dimensional, identification changes in a specific way. Within a cell we still have vNM cardinal uniqueness, but there is extra slack because the linear extension that produces a Bernoulli index $u(\cdot, [p])$ off the affine hull of $[p]$ need not be unique. The behavioural content of the representation on a cell is captured by integrals.

This leaves room for adding functions that vanish once averaged over the cell. The right object is the set of *null functions*

$$\mathcal{N}([p]) \ := \ \Big\{ w \in \mathbb{R}^{\mathrm{dom}([p])} : \int w \mathrm{d}r = 0 \text{ for every } r \in [p] \Big\}.$$

Such w are behaviourally irrelevant on $[p]$. On every cell the PDEU functional is linear in probabilities, hence the image $J_{[p]} := V([p])$ is an interval in $\mathbb{R}$.

Theorem 6.2 (Uniqueness with lower-dimensional cells, no richness)**.** *Let $\langle \succsim, \odot \rangle$ be nontrivial, and suppose the partition $\mathscr{P}_\odot$ may contain lower-dimensional cells. Two PDEU representations $\mathfrak{u} = \langle u_{[p]} \rangle$ and $\mathfrak{u}' = \langle u'_{[p]} \rangle$ represent the same primitive if and only if the following hold.*

(i) *For every cell $[p]$ there exist $a_{[p]} > 0$, $b_{[p]} \in \mathbb{R}$, and $w_{[p]} \in \mathcal{N}([p])$ such that*

$$u'_{[p]} \ = \ a_{[p]}\, u_{[p]} \ + \ b_{[p]} \ + \ w_{[p]} \quad \textit{on } \mathrm{dom}([p]), \tag{3}$$

hence

$$V'(r) \ = \ a_{[p]} V(r) + b_{[p]} \quad \textit{for every } r \in [p]. \tag{4}$$

(ii) *There exists a strictly increasing function*

$$h : \ V(\Delta^m(Z)) \to \mathbb{R} \quad \textit{such that} \quad V' \ = \ h \circ V, \tag{5}$$

and on each interval $J_{[p]}$ the restriction of h is the affine map $v \mapsto a_{[p]} v + b_{[p]}$. Equivalently, for any two cells $[p], [q]$:

- *if $J_{[p]} \cap J_{[q]}$ contains two distinct values, then $a_{[p]} = a_{[q]}$ and $b_{[p]} = b_{[q]}$,*
- *if $J_{[p]} \cap J_{[q]} = \{v_0\}$, then $a_{[p]} v_0 + b_{[p]} = a_{[q]} v_0 + b_{[q]}$,*
- *if $J_{[p]}$ and $J_{[q]}$ are disjoint with $\sup J_{[p]} < \inf J_{[q]}$, then*

$$\sup\{a_{[p]} v + b_{[p]} : v \in J_{[p]}\} \ < \ \inf\{a_{[q]} v + b_{[q]} : v \in J_{[q]}\}.$$

Example. Let $Z = \{a, b, c\}$ and write lotteries as $r = (r_a, r_b, r_c) \in \Delta(Z)$. Define the partition by fixing the probability of a: $r \odot s \iff r_a = s_a$. For each $t \in [0,1]$ the cell is

$$[t] \ := \ \{ r \in \Delta(Z) : r_a = t \}.$$

Each $[t]$ is an affine line segment in the simplex $\Delta(Z)$, hence *lower dimensional* relative to $\Delta\big(\mathrm{dom}([t])\big) = \Delta(Z)$ whenever $t \in (0,1)$. Fix any $t \in (0,1)$. Define a function $w_{[t]}$ on

$\mathrm{dom}([t]) = \{a, b, c\}$ by

$$w_{[t]}(b) = 1, \qquad w_{[t]}(c) = 1, \qquad w_{[t]}(a) \;=\; -\frac{1-t}{t}.$$

This $w_{[t]}$ is not constant on $\{a, b, c\}$. For any $r \in [t]$ we have $r_a = t$ and $r_b + r_c = 1 - t$, so

$$\int w_{[t]}\, dr \;=\; t\cdot\Big(-\frac{1-t}{t}\Big) \;+\; r_b\cdot 1 \;+\; r_c\cdot 1 \;=\; -(1-t) + (r_b + r_c) \;=\; 0.$$

Hence $w_{[t]} \in \mathcal{N}([t])$. Let $\mathfrak{u} = \langle u_{[p]}\rangle$ be any PDEU parameter family with induced functional $V(r) \;=\; \int u(z, [r])\, dr$. Define a new family $\mathfrak{u}' = \langle u'_{[p]}\rangle$ by

$$u'(\cdot, [t]) \;:=\; u(\cdot, [t]) + w_{[t]}, \qquad u'(\cdot, [s]) \;:=\; u(\cdot, [s]) \text{ for every } s \neq t.$$

For any $r \in \Delta(Z)$, let $s := r_a$ be the cell coordinate so that $r \in [s]$. Then

$$V'(r) \;=\; \int u'(z, [r])\, dr \;=\; \begin{cases} \displaystyle\int (u + w_{[t]})(z, [t])\, dr = \int u(z, [t])\, dr + \underbrace{\int w_{[t]}\, dr}_{=0} = V(r), & \text{if } s = t, \\ \displaystyle\int u(z, [s])\, dr \;=\; V(r), & \text{if } s \neq t. \end{cases}$$

Thus $V' \equiv V$ on $\Delta(Z)$. The global weak order $\succsim$ is unchanged, and the within–cell order on $[t]$ is unchanged as well. Since $w_{[t]}$ is nonconstant, this shows explicitly that lower dimensionality permits nonconstant $w_{[p]} \in \mathcal{N}([p])$ to be added to $u(\cdot, [p])$ without affecting behavior.[3] ♣

Corollary 2 (Uniqueness with lower-dimensional cells under richness). *Let $\langle \succsim, \odot\rangle$ be nontrivial and satisfy richness. Suppose the partition $\mathscr{P}_\odot$ may contain lower-dimensional cells, and assume that for every cell $[p]$ the restriction of V to $[p]$ is nonconstant, hence $J_{[p]} := V([p])$ contains a nondegenerate interval. Then two PDEU families $\mathfrak{u} = \langle u_{[p]}\rangle$ and $\mathfrak{u}' = \langle u'_{[p]}\rangle$ represent the same primitive if and only if there exist constants $\alpha > 0$ and $\beta \in \mathbb{R}$ such that for every cell $[p]$*

$$u'_{[p]} \;=\; \alpha\, u_{[p]} \;+\; \beta \;+\; w_{[p]} \quad \text{with } w_{[p]} \in \mathcal{N}([p]), \tag{6}$$

and therefore

$$V' \;=\; \alpha V \;+\; \beta \quad \text{on } \Delta^m(Z). \tag{7}$$

To see why the corollary follows, we reuse Lemma 8 from the fully dimensional case. In its proof the only role of full dimensionality is to guarantee that $J_{[c]} = V([c])$ has nonempty interior for each cell, which we assume here by requiring $V|_{[c]}$ to be nonconstant. Indeed,

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[3] One may choose a family $\{w_{[t]}\}_{t\in(0,1)}$ with each $w_{[t]} \in \mathcal{N}([t])$ and set $u'(\cdot, [t]) = u(\cdot, [t]) + w_{[t]}$ for all t. For every r the integral is taken over its cell $[r_a]$, where $w_{[r_a]}$ averages to zero, so $V' \equiv V$ still holds.

if $V(r) \neq V(s)$ for some $r, s \in [c]$, then for $\lambda \in [0,1]$,

$$V\big((1-\lambda)r + \lambda s\big) = (1-\lambda)V(r) + \lambda V(s),$$

so $J_{[c]}$ contains the nondegenerate interval $[V(r), V(s)]$. Under richness, Lemma 8 thus applies without change. Any two cells can be linked by a finite chain of cells whose images overlap on nondegenerate intervals. On each overlap the two affine pieces in (4) must agree pointwise on an interval, hence their slopes and intercepts coincide. Propagating equalities along the chain yields global (α, β) as in (7). The null parts $w_{[p]}$ remain behaviourally irrelevant on each cell.

Example (continued). We continue with the partition $r \odot s \Leftrightarrow r_a = s_a$ and write $[t] = \{r \in \Delta(Z) : r_a = t\}$. Fix a baseline PDEU family $\mathfrak{u} = \langle u_{[t]} \rangle$ with

$$u(a, [t]) = 0, \qquad u(b, [t]) = 1, \qquad u(c, [t]) = 0 \quad \text{for all } t \in [0,1]. \tag{8}$$

For $r = (t, r_b, 1 - t - r_b) \in [t]$ the induced value is

$$V(r) \;=\; t \cdot 0 \;+\; r_b \cdot 1 \;+\; (1 - t - r_b) \cdot 0 \;=\; r_b. \tag{9}$$

Since $r_b \in [0, 1-t]$, the image on the cell is the interval $J_{[t]} \;=\; V([t]) \;=\; [\,0,\; 1-t\,]$.

We now show that two overlapping cells force alignment of coefficients. Pick $t_0 = \frac{1}{2}$ and $t_1 = \frac{1}{3}$. Then

$$J_{[t_0]} = [0, 1/2], \qquad J_{[t_1]} = [0, 2/3],$$

so $J_{[t_0]} \cap J_{[t_1]} = [0, 1/2]$ is a nondegenerate interval. Consider a second family $\mathfrak{u}' = \langle u'_{[t]} \rangle$ defined by *cellwise* affine rescalings

$$u'_{[t]} \;=\; a_{[t]}\, u_{[t]} + b_{[t]} \qquad \text{with } a_{[t]} > 0, \tag{10}$$

and let V' be the induced functional. On $[t]$ we have $V'|_{[t]} = a_{[t]} V|_{[t]} + b_{[t]}$. If $\mathfrak{u}'$ represents the *same* primitive, there exists a strictly increasing h on $V(\Delta(Z))$ with $V' = h \circ V$. Hence on the overlap $[0, 1/2]$,

$$h(v) \;=\; a_{[t_0]} v + b_{[t_0]} \;=\; a_{[t_1]} v + b_{[t_1]} \qquad \text{for all } v \in [0, 1/2]. \tag{11}$$

Evaluating (11) at two distinct points, say $v = 0$ and $v = 1/3$, gives

$$b_{[t_0]} = b_{[t_1]} \quad \text{and} \quad a_{[t_0]} \cdot \tfrac{1}{3} + b_{[t_0]} = a_{[t_1]} \cdot \tfrac{1}{3} + b_{[t_1]},$$

hence

$$a_{[t_0]} = a_{[t_1]} \quad \text{and} \quad b_{[t_0]} = b_{[t_1]}. \tag{12}$$

Thus in this simple overlapping case the coefficients on $[t_0]$ and $[t_1]$ must in fact be equal.

On the other hand, disjoint cells impose order–preserving inequalities. To see what happens when images are disjoint, modify the baseline (8) on the cell $[s]$ with $s = \frac{1}{2}$ by taking

$$u(a, [s]) = 2, \qquad u(b, [s]) = 3, \qquad u(c, [s]) = 2. \tag{13}$$

Then, for $r \in [s]$,

$$V(r) = 2 \cdot s + 3 \cdot r_b + 2 \cdot (1 - s - r_b) = 2 + r_b,$$

hence $J_{[s]} = [2, 2 + (1 - s)] = [2, 2.5]$, which is disjoint from $J_{[t_0]} = [0, 0.5]$. If $\mathfrak{u}'$ preserves the global order, its coefficients must obey

$$\sup\{a_{[t_0]}v + b_{[t_0]} : v \in [0, 0.5]\} \ < \ \inf\{a_{[s]}v + b_{[s]} : v \in [2, 2.5]\}. \tag{14}$$

Otherwise h could not be strictly increasing on the union $J_{[t_0]} \cup J_{[s]}$.

Finally, we explore what richness adds. If richness holds and $V|_{[t]}$ is nonconstant on every cell, Lemma 8 gives a finite chain of cells between any two cells whose images overlap on nondegenerate intervals. Equalities like (12) then propagate along the chain, which forces a *common* pair (α, β) across all cells. This is exactly the conclusion of Corollary 2.[4] ♣

7 Conclusion

We have presented a theoretical framework to examine preferences over lotteries in a general setting, where the decision maker (DM) values a lottery by taking the expectation of a Bernoulli utility index that may depend on an exogenous partition of the space of lotteries. We discussed how many recent models in the non-expected utility literature fall into this category. We proved representation theorems characterising PDEU preferences under the condition that the partition's cells are convex. The most surprising result is that there is no difference in treatment between countable and uncountable partitions. Moreover, we emphasised the importance of the dimensionality of the cells for the identification of the parameters of the representation.

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[4] As in the example after Theorem 6.2, one may also add a nonconstant $w_{[t]} \in \mathcal{N}([t])$ to $u(\cdot, [t])$. This leaves V unchanged on $[t]$ and does not affect the alignment constraints, since $\int w_{[t]}\, dr = 0$ for every $r \in [t]$.

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Appendix A Proofs

Proof of Theorem 4.1

Necessity of the axioms is straightforward and hence omitted. We now proof sufficiency. We start by showing that the preference relation $\succsim$ admits a (non-necessarily continuous) utility representation.

Lemma 1. *The preference relation* $\succsim$ *on* $\Delta(Z)$ *satisfies axioms A1 and A4 if and only if there exists a real valued function* $V : \Delta(Z) \to \mathbb{R}$ *such that* $p \succsim q$ *iff* $V(p) \geq V(q)$ *for all* $p, q \in \Delta(Z)$*. Furthermore,* V *is unique up to a strictly monotone transformation.*

Proof. This is the classical representation theorem for weak orders with a countable dense subset, going back to Cantor (1915). Standard references include Gilboa (2009, Ch. 6) and Bridges and Mehta (2013, Ch. 1, pp. 14-16), where full proofs can be found. □

Next we derive an expected utility representation within each cell $[p] \in \mathscr{P}_\odot$. For any such cell, let $\succsim_{[p]}$ denote the induced preference relation obtained by restricting $\succsim$ to $[p]$; that is, for $p', p'' \in [p]$, we write $p' \succsim_{[p]} p''$ iff $p' \succsim p''$. Since cells are convex, they are closed under mixtures, so each $[p]$ is itself a mixture space and $\succsim_p$ is well defined on it.

Lemma 2. *For every* $[p] \in \mathscr{P}_\odot$*, the induced preference relation* $\succsim_p$ *admits an expected utility representation; that is, there exists a Bernoulli utility function* $u(\cdot, [p]) : \mathrm{dom}([p]) \to \mathbb{R}$ *and a utility function* $U_{[p]} : [p] \to \mathbb{R}$ *representing* $\succsim_p$*, that is,* $p, q \in [p]$, $p \succsim_p q \Leftrightarrow U_{[p]}(p) \geq U_{[p]}(q)$*, where*

$$U_{[p]}(q) = \sum_{z \in \mathrm{supp}(q)} q(z)\, u(z, [p]) \qquad \textit{for all } q \in [p]$$

Proof. Since each cell $[p] \in \mathscr{P}_\odot$ is convex, it is closed under mixtures and hence inherits the structure of a mixture space from $\Delta(Z)$ (see Mongin, 2001). By restriction, Axiom 1 implies that $\succsim_{[p]}$ is complete and transitive; Axiom 3 gives mixture continuity on $[p]$; and Axiom 5 is exactly the independence axiom on $[p]$. Therefore, by the Mixture Space Theorem (Herstein and Milnor, 1953), there exists an affine (mixture-linear) utility $U_{[p]} : [p] \to \mathbb{R}$:

$$U_{[p]}(\alpha p + (1-\alpha)q) = \alpha U_{[p]}(p) + (1-\alpha)U_{[p]}(q) \quad \text{for all } p, q \in [p],\ \alpha \in [0,1].$$

It remains to find Bernoulli weights $\{u(z, [p])\}_{z \in \mathrm{dom}([p])}$ realising this representation in expected-utility form. Since every lottery in $[p]$ is finitely supported, the domain $\mathrm{dom}([p]) := \bigcup_{r \in [p]} \mathrm{supp}(r) = \{z_1, \ldots, z_n\}$ is finite, and the simplex

$$\Delta(\mathrm{dom}([p])) = \mathrm{conv}\{\delta_{z_i} : i = 1, \ldots, n\}$$

lies in the finite-dimensional affine subspace $H_{[p]} := \mathrm{aff}(\Delta(\mathrm{dom}([p]))) \subset \Delta(Z)$.

Let $L_p := \operatorname{aff}([p])$ denote the affine hull of the cell within $H_{[p]}$; note that $[p] \subseteq \Delta(\operatorname{dom}([p]))$ so $L_{[p]} \subseteq H_{[p]}$. The functional $U_{[p]}$ is affine on $[p]$, hence it extends uniquely to an affine functional on $L_{[p]}$. Since $H_{[p]}$ is a finite-dimensional real affine space and $L_{[p]}$ is an affine subspace of $H_{[p]}$, the affine functional $U_{[p]}|_{L_{[p]}}$ admits an affine extension to all of $H_{[p]}$ – this is a direct consequence of of the fact (implied e.g. by the finite dimensional Hahn-Banach theorem) that every linear functional on a subspace of a finite-dimensional vector space extends to the whole space. We fix any such extension and denote it $A_{[p]} : H_{[p]} \to \mathbb{R}$; we do not require it to be unique.

Define the Bernoulli utility index by

$$u(z_i, [p]) \;:=\; A_{[p]}(\delta_{z_i}), \qquad i = 1, \ldots, n.$$

For any $q \in \Delta(\operatorname{dom}([p]))$ we write $q = \sum_{i=1}^{n} q(z_i)\, \delta_{z_i}$. Then by affinity of $U_{[p]}$,

$$U_{[p]}(q) \;=\; \sum_{i=1}^{n} q(z_i)\, A_{[p]}(\delta_{z_i}) \;=\; \sum_{i=1}^{n} q(z_i)\, u(z_i, [p]),$$

which is the desired expected-utility representation within the cell.

□

So far we have established two representations: a global utility function V for $\succsim$ on $\Delta(Z)$, and, for each cell $[p]$, an expected utility function $U_{[p]}$ representing $\succsim_{[p]}$. By construction, V and $U_{[p]}$ are ordinally equivalent on $[p]$: there exists a strictly increasing function $f_p : \operatorname{Im}(U_{[p]}) \to \mathbb{R}$ such that $V(q) = f_p\big(U_{[p]}(q)\big), \forall q \in [p]$. In the remainder of the proof we show that there exists a global representation, V, such that each f_p is affine. That is, for every $[p] \in \mathscr{P}_{\odot}$ there exist constants $(a_p, b_p) \in \mathbb{R}_{>0} \times \mathbb{R}$ such that $f_p(t) = a_p\, t + b_p$ for all $t \in \operatorname{Im}(U_{[p]})$.

We will now introduce some further tools to drive intuition and help in proving the representation theorem. First, while not used directly in the proof, it is useful to understand that at most a countable number of cells will be used in the definition of the final representation. That is, there exists at most a countable number of cells, such that they cover all non-degenerate cells, where we call a cell degenerate if its image is a point; $V(p) = V(p')\ \forall p, p' \in [p]$. Importantly any representation is trivially expected utility on each degenerate cell, so that they can be excluded without loss of generality in the representation theorem.

The proof of the existence of such a cover follows from axiom 3, which ensures that $V([p])$ is an interval, and the well known fact, stated as lemma 3 below, that there are at most a countable number of non-overlapping intervals in $\mathbb{R}$.

Lemma 3. *Let $\mathfrak{X}$ be an arbitrary collection of (open) intervals in $\mathbb{R}$. If, for every $(a_i, b_i), (a_j, b_j) \in \mathfrak{X}$,*

$$(a_i, b_i) \neq (a_j, b_j) \implies (a_i, b_i) \cap (a_j, b_j) = \emptyset$$

then $\mathfrak{X}$ is at most countably infinite.

Proof. Since, for every distinct $(a_i, b_i), (a_j, b_j) \in \mathfrak{X}$, $(a_i, b_i) \cap (a_j, b_j) = \emptyset$ and each open interval contains a rational number, the result follows from the countability of $\mathbb{Q}$. $\square$

Lemma 4. *Let $\mathcal{N} = \{[p] : V(p) = x \text{ for some } x \in \mathbb{R}\}$ be the set of degenerate cells.*[5] *If Axioms 1 and 3-4 hold, there exists a countable set of cells, $\mathcal{S} \subseteq \mathcal{P}_\odot \backslash \mathcal{N}$ such that*

$$\bigcup_{[s]\in\mathcal{S}} V([s]) = \bigcup_{[p]\in\mathcal{P}_\odot\backslash\mathcal{N}} V([p]) \text{ for any } V(\cdot) \text{ that represents the preferences.}$$

That is, $\mathcal{S}$ is a countable cover of all the indifference classes that intersect any non-degenerate cell.

Proof. By Axioms 1 and 3–4, there exists a utility representation $V\colon \Delta^m(Z) \to \mathbb{R}$ of $\succsim$. For each $[p] \in \mathcal{P}_\odot \setminus \mathcal{N}$, Axiom 3 implies that V is continuous along mixing paths within $[p]$, so $V([p])$ is a non-degenerate interval in $\mathbb{R}$ whose interior $(a_{[p]}, b_{[p]})$ is non-empty.

For each pair $r, s \in \mathbb{Q}$ with $r < s$, say that a cell $[p] \in \mathcal{P}_\odot \backslash \mathcal{N}$ *covers* (r, s) if $(r, s) \subseteq V([p])$. Whenever at least one such cell exists, choose one and call it $[p_{r,s}]$. The resulting collection

$$\mathcal{S} := \Big\{[p_{r,s}] : r, s \in \mathbb{Q},\ r < s,\ \text{and } (r, s) \subseteq V([p]) \text{ for some } [p] \in \mathcal{P}_\odot \setminus \mathcal{N}\Big\}$$

is countable, since it is indexed by a subset of $\mathbb{Q}^2$.

It remains to verify that $\mathcal{S}$ is a cover, i.e. that

$$\bigcup_{[s]\in\mathcal{S}} V([s]) = \bigcup_{[p]\in\mathcal{P}_\odot\setminus\mathcal{N}} V([p]).$$

The inclusion $\subseteq$ is immediate since $\mathcal{S} \subseteq \mathcal{P}_\odot \setminus \mathcal{N}$. For the reverse inclusion, take any $v \in \bigcup_{[p]\in\mathcal{P}_\odot\setminus\mathcal{N}} V([p])$. Then $v \in V([p])$ for some non-degenerate cell $[p]$. Since $(a_{[p]}, b_{[p]}) \subseteq V([p])$ is a non-empty open interval and $\mathbb{Q}$ is dense in $\mathbb{R}$, there exist $r, s \in \mathbb{Q}$ with $v \in (r, s) \subseteq (a_{[p]}, b_{[p]}) \subseteq V([p])$. In particular, $[p]$ covers (r, s), so a representative $[p_{r,s}] \in \mathcal{S}$ was chosen. Since $(r, s) \subseteq V([p_{r,s}])$, we have $v \in V([p_{r,s}]) \subseteq \bigcup_{[s]\in\mathcal{S}} V([s])$. $\square$

We will also introduce some new terminology.

Definition 4. Say that a set $I \subseteq \mathbb{R}$ is *almost connected* if there exists a set of countable points, $\mathcal{X}$, s.t. $I \cup \mathcal{X}$ is connected.

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[5] Ie., all the cells for which each element in the cell is indifferent to every other element in the cell.

Without loss of generality, assume that $\cup_{[p]\in\mathscr{P}_\odot\setminus\mathscr{N}} V([p])$ is almost connected.,[6]

Definition 5. For intervals $I, J \subseteq \mathbb{R}$, write $I \pitchfork J$ if $I \cap J$ contains a non-degenerate sub-interval, i.e. if $I \cap J \supset (a, b)$ for some $a < b \in \mathbb{R}$.

The following lemma allows us to rely on the special nature of our ordering to establish a finite chain of cells that we will use to fix the representation in later steps of the induction, showing consistency of the affine parameters.

Lemma 5. *Let $I \subseteq \mathbb{R}$ be a non-degenerate interval and let $\mathcal{F}$ be a family of intervals such that, every point in the interior of I lies in the interior of some $F \in \mathcal{F}$:*

$$\textit{for every } x \in \operatorname{Int}(I)\ \exists\ J \in \mathscr{J} \textit{ s.t. } x \in \operatorname{Int}(J).$$

Then for any $J, J' \in \mathscr{J}$ with $J \pitchfork I$ and $J' \pitchfork I$ there exists a finite sequence $J = J_0, J_1, \ldots, J_L = J'$ in $\mathcal{F}$ with $J_{i-1} \pitchfork J_i$ and $J_i \pitchfork I$ for each i.

Proof. For any $J \in \mathscr{J}$ such that $J \not\supseteq I$ there exists $J'' \in \mathscr{J}$ such that $J \pitchfork J''$ since at least one of $\sup J, \inf J$ is in $\operatorname{Int}(I)$ and therefore in the interior of J'' for some $J'' \in \mathscr{J}$. Therefore, naturally some chain has to exist between any two J, J'. Since, for any i, i' $J_{i-1} \cap J_i$ and $J_{i'-1} \cap J_{i'}$ are non-overlapping open intervals there can be at most a countable number of them.
Assume there are a countable number necessary to bridge the cells. Then it has to be true that $\lim_{i\to\infty} \sup J_i = \inf J' + \delta$ for some $\delta > 0$. However, by the definition of limits, for every $\epsilon > 0$ there has to be a finite n_ϵ such that, $\forall i \geq n_\epsilon$, $\sup J_i \geq \inf J' + \delta - \epsilon$. Choosing $\epsilon < \delta$ proves that we only need a finite chain. ☐

Definition 6. We will call $C \subseteq \mathscr{P}_\odot \setminus \mathscr{N}$ a connected component of V if $V(C)$ is almost connected and, for any $[p] \in \mathscr{P}_\odot \setminus \{\mathscr{N}, C\}$, $V([p]) \cup V(C)$ fails to be almost connected.

We are now ready to prove the main lemma. Let $g_{[p],V}$, implicitly defined by $V(p) = g_{[p],V} \circ U_{[p]}(p)$ for all $p \in [p]$, be the function that maps the expected utility representation on the cell, $U_{[p]}$, to the representation V.

Lemma 6. *For any connected component, C, of V, there exists a global representation, $V_C^* : C \mapsto \mathbb{R}$, such that, for every $[p] \in C$, the function $g_{[p],V_C^*}$ is affine for all $[p]$.*

Proof. **Step 0: Constructing the well-ordering.**

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[6] Defining this condition using the closures of images means that we can preserve continuity and have achieved an appropriate representation when the image of two cells in $\mathscr{P}_\odot \setminus \mathscr{N}$ is separated by a single cell in $\mathscr{N}$.

Using the axiom of choice, fix an arbitrary well-ordering $\leq_S$ of C. Using transfinite construction, we will build a well-ordering of C, $\leq_P$, that we will use in the following steps to construct the desired representation. To achieve this, we will ensure that $\leq_P$ satisfies a key property.

Key structural property. For every $[r]$, let $[q]$ be the $\leq_P$-minimal cell such that $V([r]) \pitchfork V([q])$. For every cell $[q'] <_P [r]$ with $V([q']) \pitchfork V([r])$ there exists a sequence of cells $\{[q_i]\}_{i=0}^{n}$ with $[q_0] = [q]$ and $[q_n] = [q']$ such that $[q_{i-1}] <_P [q_i]$ and $V([q_{i-1}]) \pitchfork V([q_i])$.
This ensures that, at the stage when $[r]$ is processed in the construction of the utility representation, all previously processed cells overlapping with $[r]$ also have a chain of overlapping cells to the bridge cell, $[q]$, which is the key to the uniqueness argument in Steps 1 and 1′ below.

Well-ordering Let ω be an ordinal. Let $\leq_\omega$ be a partial order and $\mathfrak{X}_\omega \subseteq C$ be the set of already covered cells. Furthermore, let $D_\omega := \cup_{[p]\in\mathfrak{X}_\omega} V([p])$.
To initialise the construction let $[r_0]$ be the $\leq_S$-minimal element. Let $\leq_0 = \{([r_0], [r_0])\}$ and $D_0 := V([r_0])$.
At a successor stage, ω,

1. If it exists, let $[r_\omega] \notin \mathfrak{X}_{\omega-1}$ be the $\leq_S$-minimal element not yet covered such that $V([r_\omega]) \pitchfork D_{\omega-1}$.
2. Else let $[r] \notin \mathfrak{X}_{\omega-1}$ be the $\leq_S$-minimal element such that $\overline{V([r])} \cap \overline{D_{\omega-1}} \in \mathbb{R}$, where $\overline{V([r])}$ and $\overline{D_{\omega-1}}$ denote the closure of $V([r])$ and $D_{\omega-1}$ respectively.

Then set $\leq_\omega = \leq_{\omega-1} \cup \{([r_\omega], [p]) \mid [p] \in \mathfrak{X}_{\omega-1} \cup [r_\omega]\}$. Step 2 will allows us to keep continuity and have D_ω be an almost connected interval.

At a limit stage, λ,

$$D_\lambda := \cup_{\omega<\lambda} D_\omega,\ \mathfrak{X}_\lambda := \cup_{\omega<\lambda} \mathfrak{X}_\omega \text{ and } \leq_\lambda := \cup_{\omega<\lambda} \leq_\omega,$$

thereby preserving all the desired properties.

Key property proof. To see that the key property holds, note that, if there existed $[q] <_P [q'] <_P [r]$ with $V([r]) \pitchfork [q]$ and $V([r]) \pitchfork [q']$ with no chain, $\{[q_i]\}_{i=0}^{N}$, connecting $[q]$ and $[q']$ or no chain connecting $[p]$ and $[q]$ that satisfied $[q_{i-1}] <_P [q_i]$. But then there has to exist a cell, $[q''] >_P [q]$ that was chosen in step ω while $V([q'']) \cap D_{\omega-1}$ is at most a single point. However, that would violate the construction of $\leq_P$ since $[r]$ could have been chosen at step ω and satisfied $V([r]) \pitchfork D_{\omega-1}$ since $V([q]) \subseteq D_{\omega-1}$, proving the key property.

Initialization. Let $[r_0]$ be the $\leq_P$-minimal cell. Set $f_0 := g^{-1}_{[r_0],V}$, $D_0 := V([r_0])$, $a_{[r_0]} := 1$, and $b_{[r_0]} := 0$. Then $f_0 \circ V(p) = U_{[r_0]}(p)$ for all $p \in [r_0]$, satisfying the inductive hypothesis.

Successor step. Let ω be an ordinal. Suppose that the inductive hypothesis holds at stage ω, and let $[r] := [r_{\omega+1}]$ be the next cell according to $\leq_P$.

Step 1 (Two anchors force affinity on the overlap). If $\overline{D_\omega} \cap \overline{V([r])} \in \mathbb{R}$ the overlap with previously processed cells is at most a point and we proceed directly to Step 2. Otherwise, choose an arbitrary $[q] <_P [r]$ with $V([q]) \pitchfork V([r])$. Choose $p, p' \in [r]$ and $\ell, \ell' \in [q]$ with $p \sim \ell$, $p' \sim \ell'$, and $p \nsim p'$. By Axiom 6, for all $\alpha \in [0,1]$,

$$\begin{aligned}
&g_{[r],f_\omega \circ V}(\alpha U_{[r_{\omega+1}]}(p) + (1-\alpha)U_{[r]}(p')) \\
&= f_\omega\big(V(\alpha p + (1-\alpha)p')\big) && \text{(by definition of } V\text{)} \\
&= f_\omega\big(V\big(\alpha \ell + (1-\alpha)\ell'\big)\big) && \text{(by axiom 6)} \\
&= a_{[q]} U_{[q]}\big(\alpha \ell + (1-\alpha)\ell'\big) + b_{[q]} && \text{(by the inductive assumption)} \\
&= a_{[q]}\big(\alpha U_{[q]}(\ell) + (1-\alpha)U_{[q]}(\ell')\big) + b_{[q]} && \text{(by mixture linearity of } U_{[q]}\text{)} \\
&= \alpha f_\omega(V(\ell)) + (1-\alpha) f_\omega(V(\ell')) && \text{(by the inductive assumption)} \\
&= \alpha f_\omega(V(p)) + (1-\alpha) f_\omega(V(p')) && \text{(since } p \sim \ell \text{ and } p' \sim \ell'\text{)} \\
&= \alpha g_{[r],f_\omega \circ V}(U_{[r]}(p)) + (1-\alpha) g_{[r],f_\omega \circ V}(U_{[r_{\omega+1}]}(p')) && \text{(by definition of V).}
\end{aligned}$$

Let $u := U_{[r]}(p)$, $u' := U_{[r]}(p')$, $s := a_{[q]}U_{[q]}(\ell) + b_{[q]}$, and $s' := a_{[q]}U_{[q]}(\ell') + b_{[q]}$. As α runs over $[0,1]$, the argument $\alpha u + (1-\alpha)u'$ runs over the closed interval $[u', u] \subset \operatorname{Im}(U_{[r_{\omega+1}]})$, and the identity above yields, for all $t \in [u', u]$,

$$g_{[r],f_\omega \circ V}(t) \;=\; a_{[r]}\, t + b_{[r]}, \qquad a_{[r]} \;:=\; \frac{s - s'}{u - u'} \;>\; 0, \quad b_{[r]} \;:=\; s - a_{[r]}u.$$

Hence $f_\omega(V(\cdot))$ is an affine transformation of $U_{[r]}$ on the anchored interval $[u', u] \subset \operatorname{Im}(U_{[r]})$.

Step 1′ (Uniqueness of the affine parameters). We show that $a_{[r]}$ and $b_{[r]}$ are independent of the choice of anchor pair and of the previously processed cell used to construct them.

Independence from the anchor pair. Axiom 3 implies that any cell with two anchors has uncountably many. For any three distinct anchors $(p_i, \ell_i) \in [r] \times [q]$ with $p_i \sim \ell_i$, set $u_i := U_{[r]}(p_i)$ and $s_i := a_{[q]}U_{[q]}(\ell_i) + b_{[q]}$. Without loss of generality $u_1 > u_2 > u_3$, so $s_1 > s_2 > s_3$. Consider $\alpha^\star := \frac{s_2 - s_3}{s_1 - s_3} \in (0,1)$, so that $\alpha^\star s_1 + (1-\alpha^\star)s_3 = s_2$. By Axiom 6, for all $\alpha \in [0,1]$,

$$\alpha p_1 + (1-\alpha)p_3 \sim \alpha \ell_1 + (1-\alpha)\ell_3.$$

Applying V and using mixture linearity on $[r]$ and on $[q]$ gives

$$\begin{aligned}
a_{[r]}u_2 + b_{[r]} &= g_{[r],f_\omega \circ V}(u_2) && \text{(by Step 1)}\\
&= s_2 && \text{(by assumption)}\\
&= \alpha^\star s_1 + (1-\alpha^\star)s_3 && \text{(by definition)}\\
&= \alpha^* g_{[r],f_\omega \circ V}(u_1) + (1-\alpha^*) g_{[r],f_\omega \circ V}(u_3) && \text{(by assumption)}\\
&= \alpha^* a_{[r]}u_1 + (1-\alpha^*)a_{[r]}u_3 + b_{[r]} && \text{(by Step 1)}
\end{aligned}$$

Hence

$$\frac{s_1 - s_2}{u_1 - u_2} = \frac{s_2 - s_3}{u_2 - u_3} = \frac{s_1 - s_3}{u_1 - u_3}.$$

Hence any two distinct anchors in $[r] \times [q]$ yield the same slope $a_{[r]}$ and, consequently, the same intercept $b_{[r]}$.

Independence from the choice of previously processed cell. By the key structural property of $\leq_P$, for any other previously processed cell $[q']$ with $V([q']) \pitchfork V([r])$, there exists a chain, $\{[q_i]\}_{i=0}^N$ between $[q^*]$ and $[q']$ with $V([q_{i-1}]) \pitchfork V([q_i])$ and $V([q_i]) \pitchfork V([r])$ $\forall i$. On this common overlap, the inductive hypothesis gives $f_\omega \circ V = a_{[q^*]}U_{[q^*]} + b_{[q^*]}$ on $[q^*]$ and $f_\omega \circ V = a_{[q_i]}U_{[q_i]} + b_{[q_i]}$ on $[q_i]$. These two affine representations of the same function on a common interval uniquely pin the relationship between $(a_{[q_{i-1}]}, b_{[q_{i-1}]})$ and $(a_{[q_i]}, b_{[q_i]})$. More concretely, using Step 1, we get that $a_{[r]}$ and $b_{[r]}$ have to be the same on $V([r]) \cap V([q_{i-1}]) \cap V([q_i])$ regardless of whether we use $[q_{i-1}]$ or $[q_i]$ as the reference cell. Furthermore, by Step 1′, $a_{[r]}, b_{[r]}$ are the same regardless of what anchors in the reference cell we use. Since this holds for each i, the parameters have to be the same along the chain, proving that they are independent of the reference cell.

Step 2 (Extending the transformation). Define $D_{\omega+1} := D_\omega \cup V([r])$ and:

$$f_{\omega+1}(x) := \begin{cases} f_\omega(x) & \text{if } x \in D_\omega,\\ a_{[r]}\, g^{-1}_{[r],V}(x) + b_{[r]} & \text{if } x \in V([r]) \setminus D_\omega, \end{cases}$$

where $a_{[r]}$ and $b_{[r]}$ are as determined in Steps 1 and 1′ if $V([r]) \pitchfork D_\omega$. If $\inf V_{[r]} = \sup D_\omega$, set $a_{[r]} := 1$ and

$$b_{[r]} := \sup f_\omega(D_\omega) - \inf_{p \in [r]} U_{[r]}(p)$$

If instead $\sup V_{[r]} = \inf D_\omega$, set $a_{[r]} := 1$[7] and

$$b_{[r]} := \inf f_\omega(D_\omega) - \sup_{p\in[r]} U_{[r]}(p).$$

This ensures $f_{\omega+1}$ is strictly increasing and preserves continuity at the boundary. Furthermore, the inductive hypothesis is satisfied for all cells processed up to and including $[r]$.

Limit step. At a limit ordinal λ, define $f_\lambda(x) := f_\omega(x)$ for any $\omega < \lambda$ such that $x \in D_\omega$. This is well-defined since each $f_{\omega+1}$ extends f_ω without alteration on D_ω, so all stages agree wherever they are jointly defined. The domain of f_λ is $D_\lambda := \bigcup_{\omega<\lambda} D_\omega$ and the representation is affine for all cells that have been covered, as $[r_\omega]$ will have been covered in step $\omega < \lambda$ and $f_\lambda V(p) = f_\omega V(p) = a_{[r_\omega]} U_{[r_\omega]}(p) + b_{[r_\omega]}$ by construction. That is, the inductive hypothesis holds at λ.

Conclusion. Once the well-ordering $\leq_P$ is exhausted at its terminal ordinal, every cell in C has been processed, so $\bigcup_\omega D_\omega = \bigcup_{[p]\in C} V([p])$The limiting transformation f is strictly increasing on $\mathrm{Im}(V)$ and satisfies $f \circ V(p) = a_{[p]} U_{[p]}(p) + b_{[p]}$ for every $[p] \in C$ and every $p \in [p]$. Hence $g_{[p]}(t) = a_{[p]} t + b_{[p]}$ is affine for every non-degenerate cell.

□

Definition 7. We call a representation V of $\succsim$ maximally continuous if V is continuous whenever $\succsim$ is continuous.

Lemma 7. *Let V be a maximally continuous representation of $\succsim$ and let $\{C_i\}_{i=1}^\infty$ be its connected components. Furthermore, let $V^*_{C_i}$ be a representation of $\succsim$ restricted to C_i recovered using Lemma 6.*

If $\exists x \in \Delta^m(Z)$ such that $V(x) > \sup V(C_i)$, then $\sup V^*(C_i) < \infty$. Similarly, if $\exists x \in \Delta^m(Z)$ such that $V(x) < \inf V(C_i)$, then $\inf V^*(C_i) > -\infty$.

Proof. Let C_j be a connected component that is better than C_i, that is $p \succ q$ for any $p \in [p] \in C_j$ and $q \in [q] \in C_i$. Then we can create a new preference relation, $\succsim_{exp}$ that is defined over a subset of an expanded choice space, $\Delta^m(Z \cup w)$, with an expanded partition $\mathcal{P}_\odot \cup [w]$, where $[w] := \{\alpha w + (1-\alpha)p,\ \alpha \in [0,1)\}$ for some $p \in Z$. Define this preference relation so that there exist distinct $w, w', w'', w'' \in [w]$, $p, p' \in C_i$ and $q, q' \in C_j$ with $w \sim_{ext} p,\ w' \sim_{ext} p',\ w'' \sim_{ext} q,\ w''' \sim_{ext} q'$ and so that $\succsim_{ext}$ satisfies 3, 6, and 5 and is a weak order. Also, without loss of generality, assume that $(\alpha w + (1-\alpha)p) \succ_{ext} (\alpha' w + (1-\alpha')p)$ whenever $\alpha > \alpha'$.

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[7] Since there is no overlap with previously covered cells, $a_{[r]}$ is unconstrained. Any $a_{[r]} > 0$ would work. We have chosen $a_{[r]} = 1$ for simplicity.

If this extended preference relation is also separable it satisfies all conditions necessary to apply Lemma 6 with the connected component $C_{ext} \supseteq \{C_i, C_j, [w]\}$, which immediately implies that $\sup V^*(C_i) < \infty$ and $\inf V^*(C_j) > -\infty$.

Let S denote that countable subset of $\Delta^m(Z)$ such that, for every $p, q \in \Delta^m(Z)$, if $p \succ q$, then there exists $s \in S$ such that $p \succsim s \succsim q$. To establish separability note that, for any $w \in [w]$ and $p \in C_i \cup C_j$, there already exists $s \in \Delta^m(Z)$ such that $w \succsim s \succsim p$ if $w \succ p$ and $p \succsim s \succsim w$ if $p \succ w$. Since $[w]$ is the cell that contains all mixtures over w and p and since the preference relation satisfies 3 and 5 by assumption, $\succsim_{ext}$ is separable when restricted to $[w]$. Let $S_{[w]}$ be the countable set of $w \in [w]$ such that the separability condition is satisfied on $[w]$. Then $S \cup S_{[w]}$ satisfies the separability condition for $\succsim_{ext}$ proving the result.

$\square$

The last step is to combine the representations of all the connected components and to extend the combined representation to cover all degenerate cells as well. However, degenerate cells in $\mathscr{N}$ are singletons under any representation. Therefore, any extension of f, f^* that fills any gaps in its domain by choosing any completion that preserves that f^* is strictly increasing will give the desired representation. Notably, the construction of $V^*_{C_i}$ ensures continuity so that the correct representation is already achieved for any $[p] \in \mathscr{N}$ such that, for some i, $V([p]) \in \bigcup_{[q]\in C_i} \overline{V([q])}$.

Proof of Theorem 6.1

($\Rightarrow$) If there exist constants $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}$ such that $u'_{[p]} = \alpha u_{[p]} + \beta$ for every $[p] \in \mathscr{P}_{\odot}$, then for any lottery $r \in \Delta^m(Z)$ we have

$$V'(r) \;=\; \int u'(z,[r])\mathrm{d}r \;=\; \alpha \int u(z,[r])\mathrm{d}r + \beta \;=\; \alpha V(r) + \beta. \tag{15}$$

Hence V' represents the same primitive as V.

($\Leftarrow$) Assume $\mathbf{u} = \langle u_{[p]} \rangle$ and $\mathbf{u}' = \langle u'_{[p]} \rangle$ are two PDEU representations of the same nontrivial primitive that satisfies richness and whose cells are full dimensional. Full dimensionality implies that the Bernoulli utility index inside each cell is unique up to a positive affine transformations. Therefore, for every cell $[p] \in \mathscr{P}_{\odot}$ there exist $(\alpha_{[p]}, \beta_{[p]})$ with $\alpha_{[p]} > 0$ such that

$$u'_{[p]} \;=\; \alpha_{[p]} u_{[p]} + \beta_{[p]}. \tag{16}$$

Integrating over any $r \in [p]$ yields

$$V'(r) \;=\; \alpha_{[p]} V(r) + \beta_{[p]} \qquad \text{for every } r \in [p]. \tag{17}$$

Lemma 8. *Let* $V : \Delta^m(Z) \to \mathbb{R}$ *be any utility representing* $\succsim$. *Assume each cell* $[p] \in \mathscr{P}_\odot$ *is convex and full dimensional, and that richness holds. Fix cells* $[q], [p] \in \mathscr{P}_\odot$ *and lotteries* $a \in [q]$ *and* $b \in [p]$. *Write* $I := [\min\{V(a), V(b)\}, \max\{V(a), V(b)\}]$. *There exist cells* $C_0, \dots, C_m$ *with* $C_0 = [q]$ *and* $C_m = [p]$ *such that each* $J_j := V(C_j)$ *contains a nondegenerate interval, the family* $\{J_j\}_{j=0}^m$ *covers* I, *and consecutive images overlap on a nondegenerate interval*

$$I \subseteq \bigcup_{j=0}^{m} J_j, \qquad \textit{and} \qquad \operatorname{int}\big(J_j \cap J_{j+1}\big) \neq \varnothing \textit{ for } j = 0, \dots, m-1.$$

Proof. Inside a fixed cell $[r]$ the preference admits a vNM representation. Hence there is a mixture–linear $U_{[r]}$ and a strictly increasing map $g_{[r]}$ with $V = g_{[r]} \circ U_{[r]}$ on $[r]$. Because $[r]$ is convex and full dimensional, $U_{[r]}$ is nonconstant on $[r]$. Therefore $V([r])$ contains a nondegenerate interval. For each t in I pick any lottery q_t with $V(q_t) = t$ if such a lottery exists. If no lottery attains t, pick a sequence (t_n) in I with $t_n \to t$ and choose q_{t_n} with $V(q_{t_n}) = t_n$. Apply richness at each q_t or q_{t_n}. Richness yields a cell C that contains some point strictly above and some point strictly below the chosen lottery. This implies that $V(C)$ contains an open interval around the value in question. Collect all such cells and denote their images by $\{J\}$. The family of intervals $\{J\}$ forms an open cover of I in the order topology. Compactness of I gives finitely many $J_1, \dots, J_{m-1}$ that cover I.

Set $C_0 := [q]$ and $C_m := [p]$ with $J_0 := V([q])$ and $J_m := V([p])$. Both J_0 and J_m meet I on nondegenerate intervals. Insert C_0 and C_m at the ends of the finite subcover. Refine the list if needed so that $J_j \cap J_{j+1}$ has nonempty interior for each consecutive pair. This yields the desired chain. ☐

Continuing with the proof, by applying Lemma 8 to $[q]$ and $[p]$, we obtain cells $C_0, \dots, C_m$ with $C_0 = [q]$ and $C_m = [p]$ and with overlapping value ranges as stated. For each j write $\alpha_j := \alpha_{C_j}$ and $\beta_j := \beta_{C_j}$. By (17) we have

$$V'(x) \;=\; \alpha_j V(x) + \beta_j \qquad \text{for every } x \in C_j. \tag{18}$$

Fix $j \in \{0, \dots, m-1\}$. Since $J_j \cap J_{j+1}$ contains a nondegenerate interval, pick two distinct values $s \neq s'$ in this intersection. Choose lotteries $x, y \in C_j$ and $x', y' \in C_{j+1}$ such that

$$V(x) = V(x') = s \quad \text{and} \quad V(y) = V(y') = s'.$$

Because V' represents the same primitive as V, the equalities of V at these pairs imply equalities of V':

$$V'(x) = V'(x') \quad \text{and} \quad V'(y) = V'(y'). \tag{19}$$

By applying (18) to (19), we obtain

$$\alpha_j s + \beta_j \;=\; \alpha_{j+1} s + \beta_{j+1} \qquad \text{and} \qquad \alpha_j s' + \beta_j \;=\; \alpha_{j+1} s' + \beta_{j+1}. \tag{20}$$

Subtracting the two identities in (20) yields $(\alpha_j - \alpha_{j+1})(s - s') = 0$. Since $s \neq s'$, we obtain $\alpha_j = \alpha_{j+1}$. By substituting this back into either equality in (20) we also get $\beta_j = \beta_{j+1}$. Next, by repeating the same argument along the finite chain $C_0, \ldots, C_m$, we find constants $\alpha > 0$ and $\beta \in \mathbb{R}$ with

$$\alpha_{C_0} = \cdots = \alpha_{C_m} = \alpha \quad \text{and} \quad \beta_{C_0} = \cdots = \beta_{C_m} = \beta. \tag{21}$$

Because $C_0 = [q]$ and $C_m = [p]$ were arbitrary cells, (21) holds for every cell in $\mathscr{P}_\odot$. Equation (17) together with (21) yields $V'(r) \;=\; \alpha V(r) + \beta$for every $r \in \Delta^m(Z)$. Hence $u'_{[p]} \;=\; \alpha u_{[p]} + \beta$ for all $[p]$ and $V' \;=\; \alpha V + \beta$. This proves the converse direction and completes the proof.

Proof of Proposition 1

Fix a full–dimensional cell $[p]$. On $[p]$ the PDEU representation is linear in probabilities because the Bernoulli index $u(\cdot, [p])$ is fixed across the cell. Hence for any $p_1, p_2 \in [p]$ and any $\lambda \in [0, 1]$,

$$V\big((1-\lambda)p_1 + \lambda p_2\big) = (1-\lambda)V(p_1) \;+\; \lambda V(p_2). \tag{22}$$

Since $[p]$ is convex, the segment $\{(1-\lambda)p_1 + \lambda p_2 : \lambda \in [0, 1]\}$ lies in $[p]$, and (22) gives

$$V(\{(1-\lambda)p_1 + \lambda p_2 : \lambda \in [0,1]\}) \;=\; \big[\,V(p_1),\, V(p_2)\,\big]. \tag{23}$$

Therefore $V([p])$ is connected in $\mathbb{R}$ and in fact an interval:

$$J_{[p]} \;:=\; V([p]) \text{ is an interval in } \mathbb{R}. \tag{24}$$

(Only if.) Assume $\mathfrak{u}' = \langle v_{[p]} \rangle$ represents the same $\succsim$ as $\mathfrak{u} = \langle u_{[p]} \rangle$. Define a map

$$h:\; V(\Delta(Z)) \to \mathbb{R}, \qquad h(v) \;=\; V'(r) \quad \text{for any } r \text{ with } V(r) = v. \tag{25}$$

This is well defined. If $V(r) = V(s)$ then $r \sim s$, because V represents $\succsim$. Since V' also represents $\succsim$, we have $V'(r) = V'(s)$. The function h is strictly increasing on $V(\Delta(Z))$. If $v_1 < v_2$ in $V(\Delta(Z))$, pick r_1, r_2 with $V(r_i) = v_i$. Then $r_2 \succ r_1$, hence $V'(r_2) > V'(r_1)$, and therefore $h(v_2) > h(v_1)$.

On each cell $[p]$ we have by assumption

$$V'(r) \;=\; a_{[p]}V(r) + b_{[p]} \qquad \text{for all } r \in [p]. \tag{26}$$

Hence, for every $v \in J_{[p]}$ and any $r \in [p]$ with $V(r) = v$,

$$h(v) \;=\; V'(r) \;=\; a_{[p]}v + b_{[p]}. \tag{27}$$

We now verify items 1–3.

Item 1. Suppose $J_{[p]} \cap J_{[q]}$ contains two distinct points $v_1 \neq v_2$. Then by (27) and (25),

$$a_{[p]}v_i + b_{[p]} \;=\; h(v_i) \;=\; a_{[q]}v_i + b_{[q]} \qquad \text{for } i = 1, 2. \tag{28}$$

Subtract the two equalities in (28). Since $v_2 \neq v_1$,

$$(a_{[p]} - a_{[q]})(v_2 - v_1) \;=\; 0 \quad\Rightarrow\quad a_{[p]} = a_{[q]}. \tag{29}$$

Plug $a_{[p]} = a_{[q]}$ back into either equation in (28) to obtain $b_{[p]} = b_{[q]}$.

Item 2. Suppose $J_{[p]} \cap J_{[q]} = \{v_0\}$. Then by (27),

$$a_{[p]}v_0 + b_{[p]} \;=\; h(v_0) \;=\; a_{[q]}v_0 + b_{[q]}. \tag{30}$$

Item 3. Suppose $J_{[p]}$ and $J_{[q]}$ are disjoint and ordered with $\sup J_{[p]} < \inf J_{[q]}$. Since h is strictly increasing and continuous from the left and right at every point of $V(\Delta(Z))$ when restricted to each interval piece, we have

$$h(x) \;<\; h(y) \qquad \text{for all } x \in J_{[p]},\; y \in J_{[q]}. \tag{31}$$

Taking the supremum of the left side over $x \in J_{[p]}$ and the infimum of the right side over $y \in J_{[q]}$ yields

$$\sup\{h(x) : x \in J_{[p]}\} \;<\; \inf\{h(y) : y \in J_{[q]}\}. \tag{32}$$

By (27), $h(x) = a_{[p]}x + b_{[p]}$ for $x \in J_{[p]}$, and $h(y) = a_{[q]}y + b_{[q]}$ for $y \in J_{[q]}$. Therefore (32) becomes

$$\sup\{a_{[p]}v + b_{[p]} : v \in J_{[p]}\} \;<\; \inf\{a_{[q]}v + b_{[q]} : v \in J_{[q]}\}, \tag{33}$$

which is exactly item 3.

(If.) Assume items 1–3 hold. We define a function h on $V(\Delta(Z))$ as follows. For $v \in V(\Delta(Z))$ pick any cell $[p]$ with $v \in J_{[p]}$ and set

$$h(v) \;:=\; a_{[p]}v + b_{[p]}. \tag{34}$$

We first show that (34) is well defined. Suppose $v \in J_{[p]} \cap J_{[q]}$. If the intersection contains two distinct points, item 1 gives $a_{[p]} = a_{[q]}$ and $b_{[p]} = b_{[q]}$, hence both formulas coincide on the whole intersection. If the intersection is $\{v_0\}$ then item 2 gives $a_{[p]}v_0 + b_{[p]} = a_{[q]}v_0 + b_{[q]}$, hence both formulas coincide at v_0. Thus h is well defined on $V(\Delta(Z))$.

We next show that h is strictly increasing. Let $v_1 < v_2$ in $V(\Delta(Z))$. There are three cases.

Case A. If v_1 and v_2 belong to the same interval $J_{[p]}$, then by (34)

$$h(v_2) - h(v_1) \;=\; a_{[p]}(v_2 - v_1) \;>\; 0, \tag{35}$$

because $a_{[p]} > 0$.

Case B. If $v_1 \in J_{[p]}$ and $v_2 \in J_{[q]}$ and $J_{[p]} \cap J_{[q]}$ contains at least one point, then by well definedness the two affine pieces agree on the overlap. Pick any $w \in J_{[p]} \cap J_{[q]}$. Then

$$h(v_1) \;\le\; h(w) \;\le\; h(v_2), \tag{36}$$

with at least one strict inequality because $v_1 < v_2$. Hence $h(v_1) < h(v_2)$.

Case C. If $J_{[p]}$ and $J_{[q]}$ are disjoint and ordered with $\sup J_{[p]} < \inf J_{[q]}$, then by item 3

$$h(v_1) \;\le\; \sup\{a_{[p]}v + b_{[p]} : v \in J_{[p]}\} \;<\; \inf\{a_{[q]}v + b_{[q]} : v \in J_{[q]}\} \;\le\; h(v_2). \tag{37}$$

Therefore $h(v_1) < h(v_2)$.

Thus h is strictly increasing on $V(\Delta(Z))$.

Finally we verify that $V' = h \circ V$. Let $r \in \Delta(Z)$ and put $v := V(r)$. Then $v \in J_{[p]}$ for $[p]$ that contains r. By (34) and the definition of $v_{[p]}$,

$$h(V(r)) \;=\; a_{[p]}V(r) + b_{[p]} \;=\; V'(r). \tag{38}$$

Hence $V' = h \circ V$. Therefore $\mathfrak{u}'$ represents the same $\succsim$ as $\mathfrak{u}$.

Proof of Theorem 6.2

From the proof of Proposition 1, we have that $J_{[p]} := V([p])$ is connected and therefore an interval in $\mathbb{R}$.

(⇒) Suppose $\mathfrak{u}$ and $\mathfrak{u}'$ represent the same primitive. We prove (i) and (ii).

Proof of (i). Fix a cell $[p]$. On the mixture space $[p]$, both

$$W(r) := \int u(z,[p])\,dr \quad \text{and} \quad W'(r) := \int u'(z,[p])\,dr$$

represent the same preference $\succsim$ restricted to $[p]$. By standard vNM cardinal uniqueness arguments, there exist $a_{[p]} > 0$ and $b_{[p]} \in \mathbb{R}$ such that

$$W'(r) = a_{[p]} W(r) + b_{[p]} \qquad \text{for all } r \in [p]. \tag{39}$$

Define the pointwise difference on $\mathrm{dom}([p])$ by

$$w_{[p]}(\cdot) \;:=\; u'(\cdot, [p]) - a_{[p]} u(\cdot, [p]) - b_{[p]}. \tag{40}$$

Integrating (40) against any $r \in [p]$ and using (39) gives

$$\int w_{[p]}\, dr = \int u'(\cdot, [p])\, dr - a_{[p]} \int u(\cdot, [p])\, dr - b_{[p]} = W'(r) - a_{[p]} W(r) - b_{[p]} = 0.$$

Thus $w_{[p]} \in \mathcal{N}([p])$. The identity (3) is exactly (40), and (4) is (39) rewritten using $V|_{[p]} = W$ and $V'|_{[p]} = W'$.

Proof of (ii). Define

$$h : \; V(\Delta^m(Z)) \to \mathbb{R}, \qquad h(v) \;:=\; V'(r) \;\text{ for any } r \text{ with } V(r) = v. \tag{41}$$

We first show that h is well defined. If $V(r) = V(s) = v$, then $r \sim s$ since V represents $\succsim$, and because V' also represents $\succsim$, we have $V'(r) = V'(s)$. Hence h does not depend on the chosen preimage.

Next we show that h is strictly increasing on the set $V(\Delta^m(Z))$. Let $v_1 < v_2$ in $V(\Delta^m(Z))$. Choose r_1, r_2 with $V(r_i) = v_i$. Then $r_2 \succ r_1$, hence $V'(r_2) > V'(r_1)$, so $h(v_2) > h(v_1)$.

Finally we identify h on each cell image. Let $v \in J_{[p]}$. Choose $r \in [p]$ with $V(r) = v$. By (4), $V'(r) = a_{[p]} V(r) + b_{[p]} = a_{[p]} v + b_{[p]}$. Together with (41), this yields

$$h(v) = a_{[p]} v + b_{[p]} \qquad \text{for all } v \in J_{[p]}. \tag{42}$$

This gives (ii). The three bullet conditions are immediate consequences of (42) and strict monotonicity of h:

- If $J_{[p]} \cap J_{[q]}$ contains two distinct values $v_1 \neq v_2$, then $a_{[p]} v_i + b_{[p]} = h(v_i) = a_{[q]} v_i + b_{[q]}$ for $i = 1, 2$. Subtract to get $(a_{[p]} - a_{[q]})(v_2 - v_1) = 0$, hence $a_{[p]} = a_{[q]}$, and then $b_{[p]} = b_{[q]}$.
- If $J_{[p]} \cap J_{[q]} = \{v_0\}$, then $a_{[p]} v_0 + b_{[p]} = h(v_0) = a_{[q]} v_0 + b_{[q]}$.
- If $J_{[p]}$ and $J_{[q]}$ are disjoint with $\sup J_{[p]} < \inf J_{[q]}$, then for all $x \in J_{[p]}$ and $y \in J_{[q]}$ we have $x < y$, hence $h(x) < h(y)$ by strict monotonicity. Taking supremum on $J_{[p]}$ and infimum on $J_{[q]}$ yields

$$\sup\{h(x) : x \in J_{[p]}\} \;<\; \inf\{h(y) : y \in J_{[q]}\}.$$

Using (42) on each cell gives the displayed inequality in the theorem.

($\Leftarrow$) Suppose (i) and (ii) hold. We show that $\mathfrak{u}$ and $\mathfrak{u}'$ represent the same primitive. By (5), for any $r, s \in \Delta^m(Z)$ we have

$$V'(r) = h(V(r)) \quad \text{and} \quad V'(s) = h(V(s)),$$

with h strictly increasing on $V(\Delta^m(Z))$. Hence

$$V(r) \geq V(s) \iff h(V(r)) \geq h(V(s)) \iff V'(r) \geq V'(s).$$

Thus V and V' induce the same weak order on $\Delta^m(Z)$, so $\mathfrak{u}$ and $\mathfrak{u}'$ represent the same primitive.

Finally, we prove the "equivalently" clause in (ii) in both directions.

From h to the bullets. This was already shown above. The identities on overlaps follow from (42). The strict inequality for disjoint ordered intervals follows by taking sup and inf after applying strict monotonicity of h pointwise.

From the bullets to h. Assume the three bullet conditions hold for the coefficients $\{a_{[p]}, b_{[p]}\}$. Define

$$h : V(\Delta^m(Z)) \to \mathbb{R}, \qquad h(v) := a_{[p]}v + b_{[p]} \quad \text{for } v \in J_{[p]}. \tag{43}$$

We first verify that h is well defined. If $v \in J_{[p]} \cap J_{[q]}$ contains two distinct values then by the first bullet $a_{[p]} = a_{[q]}$ and $b_{[p]} = b_{[q]}$, so the two formulas coincide on the whole overlap. If $J_{[p]} \cap J_{[q]} = \{v_0\}$ then the second bullet gives $a_{[p]}v_0 + b_{[p]} = a_{[q]}v_0 + b_{[q]}$, so the formulas coincide at v_0. Hence h is well defined on the union $\bigcup_{[p]} J_{[p]} = V(\Delta^m(Z))$.

We next show that h is strictly increasing on $V(\Delta^m(Z))$. Let $v_1 < v_2$ in $V(\Delta^m(Z))$, and pick cells $[p], [q]$ with $v_1 \in J_{[p]}$ and $v_2 \in J_{[q]}$. There are three cases.

Case A (same cell). If $[p] = [q]$ then

$$h(v_2) - h(v_1) = (a_{[p]}v_2 + b_{[p]}) - (a_{[p]}v_1 + b_{[p]}) = a_{[p]}(v_2 - v_1) > 0,$$

since $a_{[p]} > 0$.

Case B (overlapping cells). If $J_{[p]} \cap J_{[q]}$ is nonempty, pick any w in the overlap. Then $v_1 \leq w \leq v_2$ with at least one strict inequality. On $J_{[p]}$ and $J_{[q]}$ the map is affine with positive slope, and by well definedness the two affine pieces agree on the overlap. Therefore

$$h(v_1) \leq h(w) \leq h(v_2),$$

with at least one strict inequality, hence $h(v_1) < h(v_2)$.

Case C (disjoint ordered cells). If $J_{[p]} \cap J_{[q]} = \varnothing$ and $\sup J_{[p]} < \inf J_{[q]}$, then by the third bullet

$$h(v_1) \le \sup\{a_{[p]}v + b_{[p]} : v \in J_{[p]}\} < \inf\{a_{[q]}v + b_{[q]} : v \in J_{[q]}\} \le h(v_2).$$

Hence $h(v_1) < h(v_2)$.

Thus h is strictly increasing on $V(\Delta^m(Z))$ and agrees with the prescribed affine piece on each $J_{[p]}$. For any $r \in \Delta^m(Z)$, let $[p]$ be its cell. Then $V(r) \in J_{[p]}$ and by (4)

$$h(V(r)) = a_{[p]}V(r) + b_{[p]} = V'(r),$$

so $V' = h \circ V$ and (ii) holds.

Combining the directions gives the equivalence stated in the theorem.

Proof of Corollary 2

Only if. Assume $\mathfrak{u}$ and $\mathfrak{u}'$ represent the same primitive. By Theorem 6.2 there exist, for each cell $[p]$, coefficients $a_{[p]} > 0$ and $b_{[p]} \in \mathbb{R}$ and a null function $w_{[p]} \in \mathcal{N}([p])$ such that

$$u'_{[p]} \;=\; a_{[p]}u_{[p]} + b_{[p]} + w_{[p]} \quad \text{on } \mathrm{dom}([p]), \qquad V'|_{[p]} \;=\; a_{[p]}V|_{[p]} + b_{[p]}. \tag{44}$$

By the same theorem there is a strictly increasing function h on $V(\Delta^m(Z))$ with

$$V' \;=\; h \circ V \quad \text{and} \quad h(v) \;=\; a_{[p]}v + b_{[p]} \;\text{ for all } v \in J_{[p]}. \tag{45}$$

By richness and the nondegeneracy of each $J_{[p]}$, Lemma 8 applies *without change* once we record the following observation. In the proof of Lemma 8 the only use of full dimensionality is to ensure that $J_{[c]} := V([c])$ has nonempty interior for each cell $[c]$. Here we assume directly that $V|_{[c]}$ is nonconstant for every cell, which implies that $J_{[c]}$ has nonempty interior. Indeed, if $V|_{[c]}$ is nonconstant then there exist $r, s \in [c]$ with $V(r) \neq V(s)$. Since $[c]$ is convex and V is affine on $[c]$, for every $\lambda \in [0, 1]$,

$$V\big((1-\lambda)r + \lambda s\big) = (1-\lambda)V(r) + \lambda V(s),$$

hence $[V(r), V(s)] \subseteq J_{[c]}$. With this observation in place the proof of Lemma 8 goes through verbatim under richness, so for any two cells $[p]$ and $[q]$ there exists a finite chain $C_0 = [p], C_1, \ldots, C_m = [q]$ with $\mathrm{int}\big(J_{C_j} \cap J_{C_{j+1}}\big) \neq \varnothing$ for all $j < m$.

Fix adjacent cells C_j and C_{j+1}. On the nondegenerate overlap $I := J_{C_j} \cap J_{C_{j+1}}$ the two affine formulas from (45) agree pointwise:

$$a_{C_j}v + b_{C_j} \;=\; h(v) \;=\; a_{C_{j+1}}v + b_{C_{j+1}} \quad \text{for all } v \in I.$$

Two affine maps that coincide on an interval must have the same slope and the same intercept. Hence

$$a_{C_j} = a_{C_{j+1}} \quad \text{and} \quad b_{C_j} = b_{C_{j+1}}.$$

Propagating this equality along the finite chain gives

$$a_{[p]} = a_{[q]} \quad \text{and} \quad b_{[p]} = b_{[q]}.$$

Since $[p]$ and $[q]$ were arbitrary, there exist global coefficients $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $a_{[p]} = \alpha$ and $b_{[p]} = \beta$ for every cell. Plugging these into (44) yields (6). Integrating on each cell gives $V'|_{[p]} = \alpha V|_{[p]} + \beta$ for all $[p]$. Since the cells cover $\Delta^m(Z)$, we have (7).

If. Conversely, suppose (6) holds. Integrating on each cell gives $V'|_{[p]} = \alpha V|_{[p]} + \beta$. Hence $V' = \alpha V + \beta$ on all of $\Delta^m(Z)$ and V' is a positive affine transform of V. Therefore V and V' represent the same primitive.