1 Exogenous Growth Model

(Based on Solow 1956)

1.1 Constant Technology

Discrete time: $t = 0, 1, 2, ..., \infty$
Two factors of production:
L_t - Labor
K_t - Capital
Produce one final good that can be used for consumption or as capital in the production process.

Factor supply
Labor supply at $t + 1$:

$$L_{t+1} = (1 + n)L_t$$

where:
L_0 is given
$n > -1$
capital supply at $t + 1$:

$$K_{t+1} = S_t + (1 - \delta)K_t$$

where:
K_0 is given
S_t - aggregate saving
$\delta \in [0, 1]$.

A1:

$$n + \delta > 0$$

Production
output produced at time t :

$$Y_t = F(K_t, L_t)$$

A2:

$F_k(K_t, L_t), F_L(K_t, L_t) > 0$, $F_{KK}(K_t, L_t), F_{LL}(K_t, L_t) < 0$, for all $K_t, L_t > 0$

$\lim_{k \to 0} F_k(K_t, L_t) = \infty$
$\lim_{k \to \infty} F_k(K_t, L_t) = 0$
$F(0, L_t) = 0$
$\lambda F(K_t, L_t) = F(\lambda K_t, \lambda L_t)$
$\rightarrow$

$$Y_t = F(K_t, L_t) = L_t F(K_t/L_t, 1) \equiv L_t f(k_t)$$

where $k_t \equiv K_t/L_t$

It follows from A2:

$$f(0) = 0 \quad (L_t f(0) = F(0, L_t) = 0)$$

for all $k_t > 0$:

$$f'(k_t) = F_K(K_t, L_t) > 0 \quad (F_K(K_t, L_t) = dL_t f(K_t/L_t)/dK_t = f'(k_t))$$

and

$$f''(k_t) = L_t F_{KK} < 0 \quad (F_{KK} = df'(K_t/L_t)/dK_t = f''(k_t)/L_t)$$

$$\lim_{k_t \to 0} f'(k_t) = \infty \quad \lim_{k_t \to \infty} f'(k_t) = 0$$

Moreover:
since: $\lambda F(K_t, L_t) = F(\lambda K_t, \lambda L_t)$,differentiating with respect to λ :

$$F(K_t, L_t) = F_K K_t + F_L L_t$$

and dividing by L_t :

$$f(k_t) = f'(k_t) k_t + F_L$$

$\rightarrow$

$$f(k_t) - f'(k_t) k_t = F_L > 0$$

Remark:
In a competitive environment:
the rate of return per unit of capital (rental rate):

$$F_K = f'(k_t)$$

the wage rate per unit of labor:

$$F_L = f(k_t) - f'(k_t)k_t$$

Remark:

Since $F(K_t, L_t) = F_K K_t + F_L L_t$, it follows from differentiating with respect to L_t that

$$F_L = F_{KL}K_t + F_{LL}L_t + F_L$$

$\rightarrow$

$$F_{KL}K_t + F_{LL}L_t = 0$$

$\rightarrow$

$$F_{KL} > 0$$

Consumption, Saving and Investment

$$S_t = sY_t$$

where $s \in [0, 1]$

Capital Accumulation:

$$\begin{aligned} K_{t+1} &= S_t + (1-\delta)K_t \\ &= sL_t f(k_t) + (1-\delta)K_t \end{aligned}$$

$\rightarrow$

$$k_{t+1} = \frac{K_{t+1}}{L_{t+1}} = \frac{sL_t f(k_t) + (1-\delta)K_t}{L_t}\frac{L_t}{L_{t+1}}$$

$\rightarrow$

$$k_{t+1} = \frac{sf(k_t) + (1-\delta)k_t}{1+n} \equiv \phi(k_t)$$

The Dynamical System

$\{k_t\}_0^\infty$ such that

$$k_{t+1} = \phi(k_t)\ \forall t$$

where k_0 is given

Let y_t be output per worker

$$y_t = Y_t/L_t = f(k_t)$$

$\rightarrow$
$\{k_t\}_0^\infty$ uniquely determines $\{y_t\}_0^\infty$

Properties of $\phi(k_t)$:

$$\phi(0) = 0$$
$$\phi'(k_t) = \frac{sf'(k_t) + (1-\delta)}{1+n} > 0 \ \forall k_t > 0$$
$$\phi''(k_t) = \frac{sf''(k_t)}{1+n} < 0 \ \forall k_t > 0$$
$$\lim_{k_t \to 0} \phi'(k_t) = \infty$$
$$\lim_{k_t \to \infty} \phi'(k_t) = \frac{1-\delta}{1+n} \in [0,1)$$

Remark:
The strict concavity of $\phi(k_t)$ follows from:
1. the strict concavity of $f(k_t)$
2. saving is a constant fraction of output

Steady states
$\bar{k}$ such that:

$$\bar{k} = \phi(\bar{k}) = \frac{sf(\bar{k}) + (1-\delta)\bar{k}}{1+n}$$

$\rightarrow$

$$(n+\delta)\bar{k} = sf(\bar{k})$$

$\rightarrow$ there exist 2 steady states:
$\bar{k} = 0$ unstable
$\bar{k} > 0$ stable
Remark:

$$n + \delta > sf'(\bar{k})$$

Comparative Statics
Proposition.

$$\frac{d\bar{k}}{dn} < 0$$
$$\frac{d\bar{k}}{ds} > 0$$

$$\frac{d\bar{k}}{dk_0} = 0$$

Proof.
Let

$$G(\bar{k}, n, s) \equiv (n+\delta)\bar{k} - sf(\bar{k}) = 0$$

$\rightarrow$

$$\frac{d\bar{k}}{dn} = -\frac{\frac{\partial G}{\partial n}}{\frac{\partial G}{\partial \bar{k}}} = -\frac{\bar{k}}{n+\delta - sf'(\bar{k})} < 0$$

$$\frac{d\bar{k}}{ds} = -\frac{\frac{\partial G}{\partial s}}{\frac{\partial G}{\partial \bar{k}}} = \frac{f(\bar{k})}{n+\delta - sf'(\bar{k})} > 0$$

initial condition do not matter since there exists a unique globally stable steady state equilibrium

Comparative Dynamics
Let

$$\gamma_{k_t} \equiv \frac{k_{t+1} - k_t}{k_t}$$

Proposition.

$$\frac{d\gamma_{k_t}}{dn} < 0$$

$$\frac{d\gamma_{k_t}}{ds} > 0$$

$$\frac{d\gamma_{k_t}}{dk_t} < 0$$

Proof.

$$\begin{aligned}
\gamma_{k_t} &= \left[\frac{sf(k_t) + (1-\delta)k_t}{1+n} - k_t\right]/k_t \\
&= \frac{sf(k_t) - (n+\delta)k_t}{(1+n)k_t} \\
&= \frac{sf(k_t)}{(1+n)k_t} - \frac{n+\delta}{1+n}
\end{aligned}$$

$\rightarrow$

$$\frac{d\gamma_{k_t}}{dn} = -\frac{sf(k_t)}{(1+n)^2 k_t} - \frac{1-\delta}{(1+n)^2} < 0$$

$$\frac{d\gamma_{k_t}}{ds} = \frac{f(k_t)}{(1+n)k_t} > 0$$

$$\frac{d\gamma_{k_t}}{dk_t} = -\frac{s}{(1+n)k_t^2}[f(k_t) - f'(k_t)k_t] < 0$$

Conclusion: no growth in the long-run without technological progress
Testable Implications and Evidence
conditional convergence
β convergence
σ convergence

1.2 Threshold Externalities

$$Y_t = A_t F(K_t, L_t) = A_t L_t f(k_t)$$

where $k_t \equiv K_t/L_t$
and $A_t = A(k_t)$

$$A(k_t) = \begin{cases} A^H & if \quad k_t > \tilde{k} \\ A^L & if \quad k_t \leq \tilde{k} \end{cases}$$

Dynamics:

$$k_{t+1} = \begin{cases} \phi^H(k_t) & if \quad k_t > \tilde{k} \\ \phi^L(k_t) & if \quad k_t \leq \tilde{k} \end{cases}$$

Testable Implications
club convergence

2 Growth Accounting

(Solow 1956)

Production

$$Y = AF(K, L)$$

$\rightarrow$

$$\Delta Y = \Delta A \frac{\partial Y}{\partial A} + \Delta K \frac{\partial Y}{\partial K} + \Delta L \frac{\partial Y}{\partial L}$$

Δj - the change in the variable between two periods, $j = Y, A, K, L$.

$\rightarrow$

$$\frac{\Delta Y}{Y} = \frac{\Delta A}{A} + \frac{\Delta K}{K} \frac{\partial Y}{\partial K} \frac{K}{Y} + \frac{\Delta L}{L} \frac{\partial Y}{\partial L} \frac{L}{Y}$$

$\rightarrow$

$$\frac{\Delta A}{A} = \frac{\Delta Y}{Y} - S_K \frac{\Delta K}{K} - S_L \frac{\Delta L}{L}$$

S_K - the share of capital = the elasticity of output with respect to capital
S_L - the share of labor = the elasticity of output with respect to labor
If $F(K, L)$ is characterized by CRS:

$$y = Af(k)$$

$y = Y/L$
$k = K/L$
$\rightarrow$

$$\frac{\Delta A}{A} = \frac{\Delta y}{y} - S_K \frac{\Delta k}{k}$$

3 Overlapping Generations Model

(Based on Diamond 1965)
Production

$$Y_t = F(K_t, L_t)$$

F satisfies A2
$\rightarrow$

$$Y_t = F(K_t, L_t) = L_t F(k_t, 1) = L_t f(k_t),$$

where $k_t \equiv K_t/L_t$
Wage per worker

$$w_t = f(k_t) - f'(k_t)k_t$$

Return to capital ($= 1 + \textit{intrest rate}$ between $t-1$ and t)

$$R_t = f'(k_t) + 1 - \delta$$

Individuals
A generation of size L is born every period and lives for two periods
Individuals:
supply labor inelastically, consume and save in their first life period
consume in the second
Utility of the working generation:

$$u_t = u(c_t^y, c_{t+1}^o)$$

Budget constraint

$$c_{t+1}^o = R_{t+1} s_t^y = R_{t+1}[w_t - c_t^y]$$

$\rightarrow$

$$c_t^y + \frac{c_{t+1}^o}{R_{t+1}} = w_t$$

Optimization:

$$s_t^y = s(w_t, R_{t+1})$$

since consumption (in second period) is normal:

$$\frac{ds_t}{dw_t} > 0$$

A3

$$\frac{ds_t}{dR_{t+1}} \geq 0$$

Remark: a sufficient assumption instead of A3 is that the absolute slope of the supply of saving with respect to the return to saving is larger than the slope of the demand for capital. In particular, since consumption in second period is normal, substitution effect and income effect operate in the same direction, implying that second period consumption is increasing with R_{t+1} and therefore the elasticity of saving with respect to R_{t+1} is larger than -1. Therefore, $df'(k_t)k_t/dk_t > 0$ can replace A3.

The evolution of capital

$$K_{t+1} = Ls_t^y$$

$\rightarrow$

$$k_{t+1} = s(w_t, R_{t+1}) = s(f(k_t) - f'(k_t)k_t, f'(k_{t+1}) + 1 - \delta)$$

$\rightarrow$ under A3,

$$k_{t+1} = \phi(k_t)$$

Properties of $\phi(k_t)$:

$$\phi'(k_t) > 0$$

(follows from the normality of consumption)

$$\phi(0) = 0$$

$$\phi(k_t) \leq f(k_t)$$

Comment:

Aggregate saving per capita in the economy

$$s_t = s_t^y + s_t^o$$

where

$$s_t^o = -(1-\delta)k_t$$

$$s_t^y = k_{t+1}$$

$\rightarrow$

$$s_t = s_t^y + s_t^o = k_{t+1} - (1-\delta)k_t$$

in the steady state:

$$s_t = s_t^y + s_t^o = \delta k$$

3.1 Cobb-Douglas production and utility

Production

$$Y_t = F(K_t, L_t) = AK_t^{\alpha} {L_t}^{1-\alpha} = L_t A k_t^{\alpha}$$

$k_t \equiv K_t / L_t$
wage per worker

$$w_t = (1-\alpha) A k_t^{\alpha}$$

return to capital

$$R_t = \alpha A k_t^{\alpha - 1} + 1 - \delta$$

utility

$$u_t = u(c_t, c_{t+1}) = \ln c_t + \frac{1}{1+\rho} \ln c_{t+1}$$

optimization:

$$s_t = \frac{1}{2+\rho} w_t$$

The evolution of capital

$$k_{t+1} = s_t = \frac{1}{(2+\rho)} (1-\alpha) A k_t^{\alpha} \equiv \phi(k_t)$$

Properties of $\phi(k_t)$:

$$\phi(0) = 0$$

$$\phi'(k_t) > 0 \ \forall k_t > 0$$

$$\phi''(k_t) < 0 \ \forall k_t > 0$$

$$\lim_{k_t \to 0} \phi'(k_t) = \infty$$

$$\lim_{k_t \to \infty} \phi'(k_t) = 0$$

steady state

$$\bar{k} = \left(\frac{(1-\alpha) A}{2+\rho} \right)^{1/(1-\alpha)}$$

3.2 The general case

Multiple steady state equilibria are possible

4 Endogenous Growth

4.1 Ak Basic Model

(Based on Rebelo JPE 1991)
Production:

$$Y_t = AK_t \quad \rightarrow \quad y_t = Ak_t$$

where $A > (n+\delta)/s$
The Dynamical System:

$$\begin{aligned} k_{t+1} &= \frac{sf(k_t) + (1-\delta)k_t}{1+n} \\ &= \frac{sAk_t + (1-\delta)k_t}{1+n} \\ &= \left[\frac{sA + 1 - \delta}{1+n}\right] k_t \equiv \phi(k_t) \end{aligned}$$

therefore:

$$\phi(0) = 0$$
$$\phi'(k_t) = \left[\frac{sA + 1 - \delta}{1+n}\right] > 1 \ \forall k_t > 0$$
$$\phi''(k_t) = 0 \ \forall k_t > 0$$

growth rate:

$$\begin{aligned} \gamma_{k_t} &= \left[\frac{sf(k_t) + (1-\delta)k_t}{1+n} - k_t\right] / k_t \\ &= \frac{sA - n - \delta}{1+n} \\ &= \frac{sA}{1+n} - \frac{n+\delta}{1+n} \end{aligned}$$

$\rightarrow$

$$\frac{d\gamma_{k_t}}{dn} < 0$$
$$\frac{d\gamma_{k_t}}{ds} > 0$$
$$\frac{d\gamma_{k_t}}{dk_t} = 0$$

$\rightarrow$
no conditional convergence

4.2 Human Capital Accumulation

(Related to Uzawa IER 1965, Barro JPE 1990)

Production:

$$Y_t = F(K_t, H_t)$$

satisfies A2.

$$H_t = L_t h_t$$

where $L_t = L = 1$, and $h_{t+1} = h(e_t) = \rho e_t$
note that this implies that human capital fully depreciate at the end of each period
output per capita.
(In Uzawa's model labor's productivity growth rate is an increasing function of the fraction of workers in the education sector).

$$Y_t = y_t = h_t f(k_t)$$

where $k_t = K_t/H_t = K_t/h_t$
sum of investment in physical and human capital is

$$S_t = sy_t = sh_t f(k_t)$$

Assumption: physical capital fully depreciate at the end of each period ($\delta = 1$)
efficient allocation of investment:

$$\begin{aligned}
&\max Y_{t+1} = F(K_{t+1}, H_{t+1}) \\
&s.t. \\
&K_{t+1} + e_t = S_t \\
&H_{t+1} = h_{t+1} = h(e_t)
\end{aligned}$$

it follows from the optimization that

$$F_K(K_{t+1}, H_{t+1}) = F_H(K_{t+1}, H_{t+1})h'(e_t)$$

$\rightarrow$ the capital labor ratio that maximizes Y_{t+1}, $\bar{k}$, satisfies:

$$f'(\bar{k}) = \rho[f(\bar{k}) - f'(\bar{k})\bar{k}]$$

$\bar{k}$ is unique (and independent of S_t) since $f'(\bar{k})$ decreases with $\bar{k}$ and $\rho[f(\bar{k}) - f'(\bar{k})\bar{k}]$ increases with $\bar{k}$

$\rightarrow k_t = \bar{k}$ for all t
Remark: $\bar{k}$ is strictly decreasing in ρ (follows from implicit differentiation)
Let θ_t be the fraction of physical capital in investment, and therefore, $1-\theta_t$ is the fraction of human capital investment:

$$K_{t+1} = \theta_t s h_t f(k_t)$$

$$h_{t+1} = \rho(1-\theta_t) s h_t f(k_t)$$

$\rightarrow$

$$k_{t+1} = \frac{K_{t+1}}{h_{t+1}} = \frac{\theta_t}{\rho(1-\theta_t)}$$

where θ_t is efficient
$\rightarrow$
$\theta_t = \theta$ such that

$$k_{t+1} = \bar{k} = \frac{\theta}{\rho(1-\theta)}$$

$\rightarrow$

$$\theta = \frac{\rho\bar{k}}{1+\rho\bar{k}}$$

$\rightarrow$

$$h_{t+1} = \rho(1-\theta_t) s h_t f(k_t) = \frac{\rho s f(\bar{k})}{1+\rho\bar{k}} h_t$$

$\rightarrow$ (multiplying both sides of the equation with $f(\bar{k})$):

$$\begin{aligned} y_{t+1} &= h_{t+1} f(\bar{k}) = \frac{\rho s f(\bar{k})}{1+\rho\bar{k}} h_t f(\bar{k}) \\ &= \frac{\rho s f(\bar{k})}{1+\rho\bar{k}} y_t \end{aligned}$$

$$\gamma_y = \frac{y_{t+1}-y_t}{y_t} = \frac{\rho s f(\bar{k})}{1+\rho\bar{k}} - 1$$

$\rightarrow$ for sufficiently high: productivity of education ρ, saving rate s, and productivity of final output $f(k_t)$, $\frac{\rho s f(\bar{k})}{1+\rho\bar{k}} - 1 > 0$.
The positive effect of ρ on y_{t+1} (despite its negative effect on, θ_t and thus on $\bar{k}$) follows from the envelop theorem noting that y_{t+1} is strictly increasing in ρ for any θ_t where $\theta_t = \arg\max y_{t+1}$.

- no convergence
- no limit to human capital accumulation

4.3 Endogenous Technical Change

(Based on Frankel AER 1962, Romer JPE 1986, Lucas JME 1988)
The level of technology is:

$$A_t = A(k_t)$$

is external to the firm
(in Lucas 1988 $A(h)$ is a function of human capital)
Production:

$$y_t = A(k_t)f(k_t)$$

where $f(k_t)$ is derived from a function $F(K_t, L_t)$ that satisfies A2
Saving per worker is sy_t, $s \in (0, 1)$.

a. $A(k_t)f(k_t)$ is linear in k_t
example:

$$A(k_t) = k_t^{1-\alpha}$$
$$f(k_t) = Bk_t^{\alpha}$$

$\rightarrow$ Ak model with constant factor shares.
inconsistence with conditional convergence.

b. $\lim_{k_t \rightarrow \infty} A(k_t)f(k_t)$ is linear in k_t
$\rightarrow$
share of capital $\nrightarrow 1$
conditional convergence

4.4 Endogenous R&D

4.4.1 Quality Ladder Model

(Related to Lucas JME 1988, Grossman Helpman 1991; Aghion Howitt Econometrica 1992)

Production of the final good:
The final good produced by each worker in the final good sector is

$$y_t = A_t$$

where,

$$A_t = A_{t-1} + i_t$$

A_{t-1} is the non-excludable existing technology and i_t is new knowledge (inventions) purchased by the worker.

Individuals
In each period a population of size N joins the economy
Individuals are active one period in which they work in the final good sector *or* in the R&D sector
The number of workers in the R&D sector is H_t
The number of workers in the final good sector (production) is L_t

$$L_t + H_t = N$$

Production of technology:
The number of non-rival inventions each worker in the R&D sector produces in t is:

$$\mu A_{t-1}$$

inventions are made at the beginning of the period and sold to producers

Equilibrium
In equilibrium all workers purchase all inventions:

$$i_t = \mu A_{t-1} H_t$$

$\rightarrow$

$$A_t = A_{t-1} + \mu A_{t-1} H_t = A_{t-1}(1 + \mu H_t)$$

The surplus generated by each invention used by each worker is 1.
The surplus is divided between production workers and R&D workers: a fraction $\beta \in (0,1]$ is allocated to the R&D worker and $1-\beta$ to the production worker.
income of each R&D worker in t is

$$I_t^H = \mu A_{t-1} \beta L_t$$

income of each production worker in t is

$$I_t^L = A_{t-1} + (1-\beta)\mu A_{t-1} H_t$$

for $\mu\beta L > 1$, equilibrium in the labor market (individuals are indifferent between the two occupations) implies:

$$\begin{aligned} I_t^H &= I_t^R \\ \mu A_{t-1}\beta L_t &= A_{t-1} + (1-\beta)\mu A_{t-1} H_t \\ \beta L_t \mu &= 1 + (1-\beta)\mu H_t \end{aligned}$$

$\rightarrow$

$$\begin{aligned} L_t &= \frac{1+(1-\beta)\mu H_t}{\mu\beta} \\ N - H_t &= \frac{1+\mu H_t - \beta\mu H_t}{\mu\beta} \\ N &= \frac{1+\mu H_t}{\mu\beta} \end{aligned}$$

$\rightarrow$ if $\mu\beta N > 1$, for all t:

$$\begin{aligned} H &= \beta N - 1/\mu \\ L &= (1-\beta)N + 1/\mu \end{aligned}$$

if $\mu\beta N \leq 1$

$$\begin{aligned} H &= 0 \\ L &= N \end{aligned}$$

Since $A_t = A_{t-1} + \mu A_{t-1} H$

$$g_t = g = \frac{A_t - A_{t-1}}{A_{t-1}} = \mu H = \begin{cases} \mu\beta N - 1 & if \quad \mu\beta N > 1 \\ \\ 0 & if \quad \mu\beta N \leq 1 \end{cases}$$

Conclusions

1. Growth is affected by:
   scale
   R&D productivity
   patents property rights

2. Crucial elements:
   technology is non-rival and excludable
   linearity of technological progress with respect to the technological level

Comments

1. monopolistic competition
   → may generate over investment in R&D
2. externality to technology
   → may generate under investment in R&D
3. if investment in technology takes place before benefits from the technology are exhausted
   → the interest rate/time preference have an effect on R&D investment

4.4.2 Criticism

(Jones 1995)

1. Economies of scale
2. Non decreasing productivity in R&D

   inconsistent with empirical evidence from the 20th century

3. New technology is proportional to the stock of old technology

Define γ_t

$$\gamma_t \equiv \frac{\Delta A_t}{A_t} = \frac{A_{t+1} - A_t}{A_t}$$

Suppose

$$A_{t+1} = A_t + g(R\&D)A_t = [1 + g(R\&D)]\,A_t$$

where R&D is constant over time (it can be replaced by your favorite candidate, human capital, population, or anything else)

$\rightarrow$

$$\begin{aligned} \gamma_t &= \gamma = g(R\&D) \\ & \text{and} \\ \Delta A_t &= g(R\&D)A_t \end{aligned}$$

Suppose, in contrast

$$\Delta A_t = g(R\&D)A_t^{\beta} \quad \beta \neq 1$$

$$\gamma_t = \frac{\Delta A_t}{A_t} = g(R\&D)A_t^{\beta-1} \quad \beta \neq 1$$

$\rightarrow$ if $\beta > 1$, γ_t is growing over time converging to infinity
$\rightarrow$ if $\beta < 1$, γ_t is declining over time converging to zero

4.4.3 Scale Effect in a Malthusian Economy

(based on Kremer 1993)
Production

$$Y_t = (A_t X)^\alpha L_t^{1-\alpha} = L_t \left(\frac{A_t X}{L_t}\right)^\alpha$$

where L_t is the adult population in t, X is the constant land size, augmented by a productivity coefficient, A_t.
→ income per adult individual is

$$y_t = Y_t/L_t = \left(\frac{A_t X}{L_t}\right)^\alpha$$

Individuals

live two periods: childhood and adulthood.
adults work, consume and raise children

Preferences:

$$u_t = (1-\beta)\log c_t + \beta \log n_t \quad \beta \in (0,1)$$

c_t - consumption in the household
n_t - number of children

Budget constraint

$$c_t + \lambda n_t = \left(\frac{A_t X}{L_t}\right)^\alpha$$

λ is the cost of raising a child

Optimization

$$n_t = \mu \left(\frac{A_t X}{L_t}\right)^\alpha$$

where $\mu = \beta/\lambda$

The evolution of population

$$L_{t+1} = n_t L_t = \mu y_t L_t = \mu Y_t = \mu \left(A_t X\right)^\alpha L_t^{1-\alpha}$$

→ for any given A there exists a unique globally stable steady state ,

$$L = \mu^{1/\alpha} A X$$

Suppose A_t evolves sufficiently slow → the economy is at the proximity of the Malthusian equilibrium:

$$L_t = \mu^{1/\alpha} X A_t$$

Consider population dynamics under:

1. Technological progress is constant

$$\frac{A_{t+1} - A_t}{A_t} = g$$

$\rightarrow$

$$L_{t+1} = (1 + g) L_t$$

where L_0 is given.
$\rightarrow$

$$\begin{aligned} L_t &= (1+g)^t L_0 \\ \ln(L_t) &= \ln(L_0) + t\ln(1+g) \end{aligned}$$

$\rightarrow$ Prediction: log population evolves linearly over time.

2. Technological progress is increasing with population size

$$\frac{A_{t+1} - A_t}{A_t} = g(L_t); \quad g'(L_t) > 0$$

$\rightarrow$ Prediction: log population is a convex function of time.

Evidence (from million BC until the 20th century)
Consistent with #2
Interpretation:
A larger population generates more non-excludable inventions.
A growing population allows for increasing scope for division of labor.

5 Inequality and Growth

5.1 The credit market imperfection approach

(Galor and Zeira 1993)

Production of the final good:

$$Y_t = F^A(L_t) + F^M(H_t, K_t)$$

where, L_t is the number of unskilled workers producing in the agricultural sector, H_t is the number of skilled workers producing in the manufacturing sector, $L_t + H_t = 1$ is the constant population size of each generation.
Production in the Agricultural sector is

$$F^A(L_t) = w^u L_t,$$

and the production in the Manufacturing sector, $F^M(H_t, K_t)$ is CRS production function characterized by decreasing positive marginal products and boundary conditions that assure an interior solution to the producers maximization problem.

Individuals

The population consists of overlapping generations
A generation of size 1 is born every period and lives for two periods
Each individual has one parent and one child
Individuals:

in their first life period: are endowed with a parental bequest, invest in human capital

in their second life period: supply labor inelastically, consume and bequeath

Preferences of individual i born in t are defined by the utility function:

$$u_t^i = (1-\beta)\log c_{t+1}^i + \beta \log b_{t+1}^i,$$

where $\beta \in (0, 1)$.
Budget constraint

$$c_{t+1} + b_{t+1}^i = I_{t+1}^i,$$

Hence the optimal, non-negative, transfer of individual i born in period t is given by,

$$b_{t+1}^i = b(I_{t+1}^i) = \beta I_{t+1}^i$$

The production of human capital

there is an indivisible cost, h, invested in t to become skilled in $t+1$

Capital markets

unrestricted international capital flows at the world interest rate r.

$\rightarrow k_t = k$ for all t such that:

$$f'(k) + 1 - \delta = 1 + r = R$$

$\rightarrow$

$$w_t^s = w^s = f(k) - f'(k)k$$

as follows from the production function

$$w_t^u = w^u$$

A1: R is sufficiently small such that

$$w^s - w^u > hR$$

the interest rate for borrowers for sake of investment in human capital is

$$\theta R$$

where $\theta > 1$.

A2: θ is sufficiently large such that:

$$w^s - w^u < h\theta R$$

Investment decisions and income

if $b_t^i \geq h$

$$I_{t+1}^i = w^s + (b_t^i - h)R$$

if $b_t^i < h$

$$I_{t+1}^i = \max\{w^s - (h - b_t^i)\theta R, w^u + b_t^i R\}$$

where, as follows A1 and A2 there exists

$$\hat{b} = \frac{w^u - w^s + h\theta R}{R(\theta - 1)} \in (0, h)$$

such that

$$w^s - (h - \hat{b})\theta R = w^u + \hat{b}R$$

and individuals choose to invest in human capital if and only if $b_t^i \geq \hat{b}$.

alternative presentation: the cost of education, which is strictly decreasing in b_t^i for $b_t^i < h$, is equal to the return, $(h - b_t^i)\theta R + b_t^i R = h\theta R - b_t^i(\theta - 1)R = w^s - w^u$.

The dynamical system

$$b_{t+1}^i = \left\{ \begin{array}{ccc} \beta[w^u + b_t^i R] & for & b_t^i < \hat{b} \\ \beta[w^s - (h - b_t^i)\theta R] & for & b_t^i \in [\hat{b}, h) \\ \beta[w^s + (b_t^i - h)R] & for & b_t^i \geq h \end{array} \right\} \equiv \phi(b_t^i)$$

A3: R is sufficiently small and w^s is thereby sufficiently large such that

$$\begin{aligned} \beta R &< 1 \\ \beta w^s &> h \end{aligned}$$

A4: w^u is sufficiently small such that

$$w^u < \hat{b}(1/\beta - R)$$

implying that

$$\beta(w^u + \hat{b}R) < \hat{b}$$

Note that: (1) this assumption can be expressed as an assumption on the parameters: $w^u < (1 - \beta R)(hR\theta - w^s)/[\beta R\theta - 1]$. (2) A4 implies that $\beta R < 1$. Assumptions A1 - A4 assure that the dynamical system is characterized by 2 stable steady states:

$$b^L = \frac{\beta w^u}{1 - \beta R}$$

i.e., $b^L = \beta(b^L R + w^u)$

$$b^H = \frac{\beta(w^s - hR)}{1 - \beta R}$$

i.e., $b^H = \beta((b^H - h)R + w^s)$

and a threshold unstable steady state

$$b^T = \frac{\beta(w^s - h\theta R)}{1 - \beta\theta R}$$

i.e., $b^T = \beta((b^T - h)\theta R + w^s)$

5.1.1 Replacing the non-convexities of the technology

(Moav 2002)
Production

$$Y_t = F(K_t, H_t),$$

where H_t is efficiency units of human capital.
Individuals
as in the Galor-Zeira model, with the following utility function:

$$u_t^i = (1-\beta)\log c_{t+1}^i + \beta \log(\overline{\pi} + b_{t+1}^i), \tag{1}$$

where $\beta \in (0,1)$ and $\overline{\pi} > 0$. from maximization subject to the budget constraint, $I_{t+1}^i = b_{t+1}^i + c_{t+1}^i$ the optimal, non-negative, transfer of individual i born in period t is,

$$b_{t+1}^i = b(I_{t+1}^i) = \begin{cases} 0 & if \quad I_{t+1}^i \leq \pi; \\ \\ \beta(I_{t+1}^i - \pi) & if \quad I_{t+1}^i > \pi; \end{cases}$$

where $\pi \equiv \overline{\pi}(1-\beta)/\beta$.
Capital markets
unrestricted international capital flows at the world capital rate of return, R, uniquely determine the wage per efficiency unit of human capital w.
individuals can not borrow for sake of investment in human capital.
The formation of human capital
the level of human capital of an individual i, h_{t+1}^i, is an increasing concave function of real resources invested in education, e_t^i,

$$h_{t+1}^i = h(e_t^i) = \begin{cases} 1 + \gamma e_t^i & if \quad e_t^i < \overline{e}; \\ \\ 1 + \gamma \overline{e} & if \quad e_t^i \geq \overline{e}. \end{cases}$$

It is assumed that the marginal return to human capital, for $e_t^i < \overline{e}$, is larger than the marginal return to physical capital:

$$w\gamma > R,$$

assuring that individuals invest in human capital. Noting that w is a decreasing function of R, Assumption A1 implies that R is sufficiently low.

The evolution of income

second life period income, I_{t+1}^{i}, is uniquely determined by first life period bequest, b_t^i,

$$I_{t+1}^{i}=I(b_t^i)\equiv\begin{cases} w(1+\gamma b_t^i) & if \quad b_t^i<\overline{e};\\ \\ w(1+\gamma\overline{e})+R(b_t^i-\overline{e}) & if \quad b_t^i\geq\overline{e}.\end{cases}$$

the evolution of income within a dynasty is uniquely determined:

$$I_{t+1}^{i}=\psi(I_t^i)=\begin{cases} w & if \quad \beta(I_t^i-\pi)<0;\\ \\ w(1+\gamma\beta(I_t^i-\pi)) & if \quad \beta(I_t^i-\pi)\in[0,\overline{e}];\\ \\ w(1+\gamma\overline{e})+R(\beta(I_t^i-\pi)-\overline{e}) & if \quad \beta(I_t^i-\pi)>\overline{e},\end{cases}$$

where I_0^i is given. Note that, $I_{t+1}^{i}=\psi(I_t^i)\geq w$ for all I_t^i.
Additional restrictions on the parameter values are required in order for the dynamical system to generate multiple income level steady states:

$$\pi>w$$

$$\begin{aligned}\beta[w(1+\gamma\overline{e})-\pi] &> \overline{e}\\ &\rightarrow\\ \overline{e} &> \frac{\beta(\pi-w)}{\beta w\gamma-1}\end{aligned}$$

$$R\beta<1.$$

$\rightarrow$ there exists an income threshold, $\hat{I}$, such that dynasties with income below the threshold, $I_t^i<\hat{I}$, converge to the poverty trap income level, and dynasties with income above the threshold, $I_0^i>\hat{I}$, converge to the high income steady state. The threshold is,

$$\hat{I}=\frac{(\gamma\beta\pi-1)w}{\gamma\beta w-1},$$

the dynamical system, $I_{t+1}^{i}=\psi(I_t^i)$, generates multiple steady states. A poverty trap, $I^L=w$, a high income steady state, I^H, and a threshold income $\hat{I}\in(I^L,I^H)$, where,

$$I^H=\frac{w(\gamma\overline{e}+1)-R(\beta\pi+\overline{e})}{1-R\beta}.$$

5.1.2 Robustness and endogenous wages

(Related to Banerjee and Newman 1993)
Consider the Galor Zeira model with random shock to income and endogenous wages:
Random shocks
Suppose that:

a fraction ε_1 of individuals who did not invest in human capital become skilled workers

a fraction ε_2 of individuals who did invest in human capital are unskilled workers

ε_1 and ε_2 are small and, for sake of simplicity, are ignored by individuals when making investment decisions.
Assumption A3 assures that the positive shock places individuals in the basin of attraction of the high income steady state and it is assumed that the negative shock places individuals in the basin of attraction of the low income steady state. (this assumption holds for sufficient low values of β, R and w^u).
The fraction of skilled individuals in t (once the range $\hat{b} - b^T$ is empty):

$$H_{t+1} = H_t(1-\varepsilon_2) + \varepsilon_1(1-H_t) = \varepsilon_1 + (1-\varepsilon_1-\varepsilon_2)H_t.$$

Therefore, the steady state value of H,

$$H = \frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2},$$

is independent of the initial wealth distribution.
Endogenous wages
output in the agricultural sector, Y_t^A,

$$Y_t^A = F^A(T, L_t),$$

where, T is the constant land size, F^A is a CRS production function characterized by decreasing positive marginal products and boundary conditions that assure an interior solution to the producers maximization problem.
therefore:

$$w^u = w^u(L_t),$$

$dw^u(L_t)/dL_t < 0$, and $\lim_{L_t \to 0} w^u(L_t) = \infty$.

In addition it is assumed that $w^u(1) < \hat{w}$, where the dynamical system describing the evolution of b_t^i is characterized by two steady states for all $w^u < \hat{w}$ and by one steady state for all $w^u > \hat{w}$.
$\rightarrow$ there exists $\hat{L}$ such that $w^u(\hat{L}) = \hat{w}$.
Assume that

$$\frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2} \equiv \tilde{L} > \hat{L}$$

Therefore:
If $L_0 > \hat{L}$ the economy will converge to the steady state:

$$L = \tilde{L}, \ H = \frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_2}.$$

If $L_0 < \hat{L}$ (and ε_2 is sufficiently small) the dynamical system governing the evolution of b_t^i is characterized by a unique high income steady state. In equilibrium therefore a sufficient fraction of the population will have a bequest $b > h$ and the net return to education declines to zero, uniquely determining the number of uneducated workers in the steady state, $\bar{L}$:

$$w^s - w^u(\bar{L}) = hR.$$

Replacing in the dynamical system:

$$b_{t+1}^i = \beta[w^u(\bar{L}) + b_t R] = \beta[w^s + (b_t - h)R]$$

which is linear in equilibrium and has a unique high income globally stable steady state. In the steady state all individuals converge to the high income steady state, but they are subject to negative income shocks that place them in a lower point along the dynamical path. In the steady state $H = 1 - \bar{L}$.

Hence, endogenous wages in the Galor-Zeira model imply that it is robust to random shocks - steady-state is affected by initial conditions.

5.2 The Political Economy Approach

Alesina-Rodrik 1994, Persson-Tabellini 1994, Benabou 2000
(the model is based on Benabou 2000)

One period endowment economy

Wealth distribution
Cumulative distribution of (pre-tax) wealth:

$$F(x) = \begin{cases} 0 & for \quad x < 0 \\ [\alpha/(1+\alpha)]^{\alpha} x^{\alpha} & for \quad x \in [0, (1+\alpha)/\alpha] \\ 1 & for \quad x > (1+\alpha)/\alpha \end{cases}$$

where $\alpha \in (0, 1]$
The density function:

$$f(x) = \begin{cases} 0 & for \quad x < 0 \\ \alpha \left(\frac{\alpha}{1+\alpha}\right)^{\alpha} x^{\alpha-1} & for \quad x \in [0, (1+\alpha)/\alpha] \\ 0 & for \quad x > (1+\alpha)/\alpha \end{cases}$$

Mean income:

$$\begin{aligned} \bar{x} &= \int_0^{(1+\alpha)/\alpha} x f(x) dx \\ &= \int_0^{(1+\alpha)/\alpha} \alpha \left(\frac{\alpha}{1+\alpha}\right)^{\alpha} x^{\alpha} dx \\ &= \left(\frac{\alpha}{1+\alpha}\right)^{1+\alpha} x^{1+\alpha} \Bigg|_0^{(1+\alpha)/\alpha} = 1 \end{aligned}$$

Median income, m

$$F(m) = 1/2$$

Hence,

$$F(m) = \left[\frac{\alpha}{(1+\alpha)}\right]^{\alpha} m^{\alpha} = \frac{1}{2}$$

and therefore

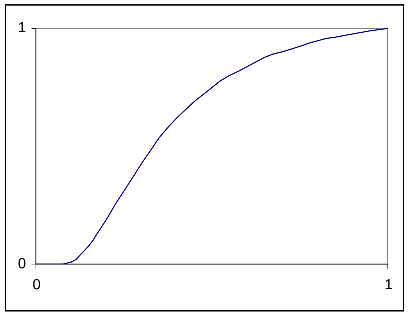
$$m = \frac{1+\alpha}{2^{1/\alpha}\alpha} \equiv m(\alpha).$$

Properties of m :

$$\lim m(\alpha) = 0 \qquad if \quad \alpha \to 0$$

$$\frac{dm(\alpha)}{d\alpha} = \frac{(1+\alpha)\log 2 - \alpha}{2^{1/\alpha}\alpha^3} > 0 \quad if \quad \alpha \in (0,1]$$

$$m = 1 \qquad if \quad \alpha = 1$$



Indices of equality:
1. Median/Mean ratio:

$$\frac{median}{mean} = \frac{m(\alpha)}{\bar{x}} = m(\alpha)$$

$\rightarrow$ the higher is the median, that is the higher is α, the lower is inequality.

2. Income variance:

$$\begin{aligned} var(x) &= E(x^2) - \bar{x}^2 = \int_0^{(1+\alpha)/\alpha} x^2 f(x) dx - 1 \\ &= \int_0^{(1+\alpha)/\alpha} \alpha \left(\frac{\alpha}{1+\alpha}\right)^{\alpha} x^{1+\alpha} dx \\ &= \alpha \left(\frac{\alpha}{1+\alpha}\right)^{\alpha} \frac{1}{2+\alpha} x^{2+\alpha} \Big|_0^{(1+\alpha)/\alpha} \\ &= \frac{1}{(2+\alpha)\alpha} \end{aligned}$$

$\rightarrow$ the variance of x is decreasing in α, that is the higher is α, the lower is inequality
Note that for $\alpha = 1$, $var(x) = 1/3$, equal to the variance of a uniform distribution with a range of 2 (the variance of the uniform distribution is given by $r^2/12$, where r is the range), and for $\alpha \to 0$ $var(x) \to \infty$.

- Equality increases with α

$\rightarrow \alpha$ is a measure of equality.

Redistribution

Post-tax Income of individual i, $\tilde{x}^i$:

$$\tilde{x}^i = (1-\tau)x^i + \tau\theta\bar{x}$$

τ - fraction of wealth taxed and redistributed equally among individuals
$\theta < 1$ - distortionary taxation
$\theta > 1$ - beneficial taxation

Redistribution is preferred by i if

$$\tilde{x}^i > x^i \leftrightarrow \theta\bar{x} > x^i$$

Hence, if $\theta = 1$ (i.e., taxation is neither distortionary nor beneficial) individual i supports redistribution if and only if her income is below the mean. Since $\bar{x} = 1$, redistribution is beneficial for i if

$$\tilde{x}^i > x^i \leftrightarrow \theta > x^i$$

Hence, redistribution is supported by a fraction $F(\theta)$ of the population:

$$F(\theta) = \begin{cases} 0 & for \quad \theta < 0 \\ [\alpha/(1+\alpha)]^\alpha\theta^\alpha & for \quad \theta \in [0,(1+\alpha)/\alpha] \\ 1 & for \quad \theta > (1+\alpha)/\alpha \end{cases}$$

For $\theta \in (0,1)$

$$\begin{cases} F(\theta) = 1 & if \quad \alpha \rightarrow 0 \\ \partial F(\theta)/\partial\alpha < 0 & if \quad \alpha \in (0,1] \\ F(\theta) = \theta/2 & if \quad \alpha = 1 \end{cases}$$

Note that:
For $\alpha \rightarrow 0$:
$\lim_{\alpha\rightarrow 0} m = 0$ and thus $F(\theta) = 1$ since $\theta > 0$.
For $\alpha = 1$:
$m = \bar{x}$ and thus $F(\theta) < 1/2$ since $\theta < 1$.
Hence for distortionary taxations:

- More equality reduces the pressure for redistribution.

For $\theta \in (1,2)$

$$\begin{array}{ccc} F(\theta) > 1/2 & & \forall\alpha \\ \partial F(\theta)/\partial\alpha < 0 & for & low\ \alpha \\ \partial F(\theta)/\partial\alpha > 0 & for & \begin{array}{l} high\ \alpha \\ \theta \in (\exp(\ln 4 - 1)/2, 2] \end{array} \\ F(2) = 1 & for & \alpha = 1 \end{array}$$

The non-monotonic impact of inequality captures two effects:

1. More inequality increases the proportion of less than average income individuals who support redistribution.

2. More inequality increases the cost of redistribution for high income individuals who object redistribution

Inequality and Growth
The theory predicts that inequality:

1. Has a negative effect on growth if taxation is distorting $\theta < 1$.

2. Has a negative effect on growth if $\theta > 1.213$, and inequality is sufficiently low (high α).

6 Two Dimensional Dynamical System

An Example

$$y_t = Ak_t^{\alpha}h_t^{\beta}; \quad \alpha > 0, \beta > 0, \alpha + \beta < 1$$
$$k_{t+1} = sy_t; \quad s > 0$$
$$h_{t+1} = h_t^{\gamma}\lambda y_t; \quad \lambda > 0$$

It is further assumed that:

$$\lambda + s < 1; \quad \gamma + \beta < 1$$

$\rightarrow$ the dynamical system:

$$\begin{aligned} k_{t+1} &= sAk_t^{\alpha}h_t^{\beta} \equiv \psi(k_t, h_t) \\ h_{t+1} &= \lambda Ak_t^{\alpha}h_t^{\beta+\gamma} \equiv \phi(k_t, h_t) \end{aligned}$$

where $\psi_k(k_t, h_t), \psi_h(k_t, h_t) > 0, \psi_{kk}(k_t, h_t), \psi_{hh}(k_t, h_t) < 0$

The kk Locus

Let kk be the locus of all pairs (k_t, h_t) such that k_t, is in a steady-state: $kk \equiv \{(k_t, h_t) : k_{t+1} = k_t\}$. As follows from the dynamical system there exists a function

$$\bar{k}(h_t) = (sA)^{1/(1-\alpha)} h_t^{\beta/(1-\alpha)},$$

such that if $k_t = \bar{k}(h_t)$, then $k_{t+1} = \psi(k_t, h_t) = k_t$. That is, the kk Locus consists of all the pairs $(\bar{k}(h_t), h_t)$.

As follows from the properties of the dynamical system

The hh Locus

Let hh be the locus of all pairs (k_t, h_t) such that h_t, is in a steady-state: $hh \equiv \{(k_t, h_t) : h_{t+1} = h_t\}$. As follows from the dynamical system there exists a function

$$\bar{h}(k_t) = (\lambda A)^{1/(1-\beta-\gamma)} k_t^{\alpha/(1-\beta-\gamma)},$$

such that if $h_t = \bar{h}(k_t)$, then $h_{t+1} = \phi(k_t, h_t) = h_t$. That is, the hh Locus consists of all the pairs $(\bar{h}(k_t), k_t)$.

As follows from the properties of the dynamical system, the non-trivial levels of $\bar{k}(h_t)$, and $\bar{h}(k_t)$ are unique and globally stable.

$\alpha + \beta < 1 \rightarrow \beta/(1 - \alpha) < 1$
$\rightarrow$ the kk locus is increasing and concave with respect to h_t.
If $\alpha + \beta + \gamma < 1 \rightarrow \alpha/(1 - \beta - \gamma) < 1$
$\rightarrow$ the hh locus is increasing and concave with respect to k_t.

7 Institutions

Endogenous Property Rights

(Based on Mayshar, Moav & Neeman 2013)

The principal-agent problem

- The principal designs the contract to maximize its expected income
- Agents are risk neutral and choose their effort level to maximize their expected welfare
- The economy exists for two periods

Output (per agent in each period):

$$Y = \begin{cases} H & \text{if } e = h \text{ and } \theta = G \\ L & \text{otherwise} \end{cases}$$

- $e \in \{h, l\}$ - effort (unobserved by the principal)
- $\theta \in \{G, B\}$ - state of nature (observed by the agent before exerting effort)
- $p \in (0, 1)$ - the probability that $\theta = G$

Information

$\sigma \in \{\tilde{G}, \tilde{B}\}$ - a public signal about the state of nature

Signal accuracy $q \geq 1/2$

$$q = \Pr(\tilde{G}|G) = \Pr(\tilde{B}|B)$$

$$1 - q = \Pr(\tilde{G}|B) = \Pr(\tilde{B}|G)$$

σ - observed after the effort decision

Interpretation of the signal

a. Observation of output in other plots provides information about the state of nature at a specific plot depending on the correlation across plots.

Interpretation of the signal

b. An observable signal, such as the 'Nilometer' that measures the amount of water in the Nile.

The cost of maintaining the agent

0 if effort is low ($e = l$)

$\gamma > 0$ if effort is high ($e = h$)

Assumptions:

$L \geq \gamma$
(low output is larger than the maintenance cost)

$H - L > \gamma$
(effort is efficient)

Agent's Income and Utility

I - agent's expected income

$U = I - \gamma$ - agent's periodic utility when exerting effort

δ - the agent's discount factor

δV - the value of the agent's employment in the next period

$zero$ - agent's value of unemployment

Incentive scheme - the carrot:

The principal pays the agent:

a bonus $b \geq 0$ if output is high ($Y = H$)

a basic wage $\omega \geq \gamma$ regardless of output

Incentive scheme - the stick:

$d \in \{0, 1\}$ - the probability the agent is dismissed after the first period if:
$Y = L$ and $\sigma = \tilde{G}$
(otherwise the agent is retained)

x - the cost of replacing the agent

$\rightarrow$ Two types of contracts are possible in the first period:
$d = 0$ *"Pure Carrot"*
and

$d = 1$ *"Stick and Carrot"*

The optimization implies that $\omega = \gamma$
$\rightarrow$ An employment contract is fully described by b and d
(a carrot and a stick)

Second period optimization
The principal's objective function, OF_2:

$$\min_{b \geq 0} pb + \gamma$$

subject to the agent's incentive compatibility constraint, IC_2, for $\theta = G$

$$b + \omega - \gamma \geq \omega$$

$$\rightarrow b_2 = \gamma$$

The agent's welfare is independent of the state of nature in the second period since the IC is binding
$\rightarrow$ The value of employment at the second period

$$V = \gamma$$

The principal's objective function at the first period, OF_1:

$$\min_{b \geq 0} pb_1 + \gamma + (1 - p)(1 - q)dx$$

subject to the agent's incentive compatibility constraint, IC_1, for $\theta = G$:

$$\begin{aligned} & b_1 + \omega - \gamma + \delta V \\ \geq \; & \omega + (1-d)\delta V + d(1-q)\delta V \end{aligned}$$

Solution

The IC is binding:

$$b_1 = (1 - dq\delta)\,\gamma$$

Stick & Carrot: $(d = 1)$

$$b_S = (1 - q\delta)\,\gamma$$

Pure Carrot: $(d = 0)$

$$b_C = \gamma$$

If:

$$OF_1(d=0) \leq OF_1(d=1)$$

$\leftrightarrow$

$$pb_C \leq pb_S + (1-p)(1-q)x$$

$\leftrightarrow$

$$q \leq \hat{q} = \frac{(1-p)x}{p\gamma\delta + (1-p)x}$$

$\rightarrow$ *'Pure Carrot'*

- $\hat{q} < 1$
- For $x > p\gamma\delta/(1-p)$, $\hat{q} > 1/2$

$\rightarrow$ For some set of parameters:

- 'pure carrot' contract is optimal for low q

- 'stick & carrot' contract is optimal for high q

Intuition: a principal relying on a "stick" to incentivise the agent has to incur the cost of dismissal x with probability:

$$(1-p)(1-q)$$

→ The expected cost of using the "stick":

$$(1-p)(1-q)x$$

is decreasing with the quality of information q

Expected Income in the first period

Pure Carrot

The expected income of the agent

$$I_c = \gamma + p\gamma$$

The expected income of the principal

$$\pi_c = p(H-L) + L - I_c$$

Efficient outcome:

$$I_c + \pi_c = p(H-L) + L$$

Stick & Carrot

The expected income of the Agent

$$I_s = \gamma + p\left(1 - q\delta\right)\gamma$$

is decreasing with q

The intuition for the decline of I with q above $\hat{q}$:

Holding constant the bonus, b, a higher q implies a lower probability of dismissal, increasing the value of employment. Therefore, as q increases b has to decline to hold the incentive constraint binding.

The expected income of the principal

$$\begin{aligned} \pi_s &= p(H-L)+L-I_s \\ &\quad -(1-p)(1-q)x \end{aligned}$$

is increasing with q

inefficient outcome:

$$I_s+\pi_s=p(H-L)+L-(1-p)(1-q)x$$

inefficiency declines with q within the 'stick & carrot'

Conclusions:

- Opacity can lead to de facto property rights and increase the welfare of subjects
- Ownership of land doesn't increase incentives to exert effort - it is granted by the elite because at low transparency it is too costly for the elite to expropriate the subjects
- Transparency can lead to micro management by the elite and reduced efficiency