

Economic Growth  
Problem Set 1 - Solow Model

1. Production is given by:

$$Y_t = F(K_t, L_t) = AK_t^\alpha L_t^{1-\alpha}$$

where  $L_{t+1} = (1+n)L_t$ , and  $\alpha \in (0, 1)$ .

- a. Show that  $F$  exhibits a constant return to scale technology.
- b. Express output as a function of the capital labor ratio,  $k_t = K_t/L_t$ .
- c. Find the wage rate per worker and the rental rate per capital.
- d. Find the dynamical system (describing the evolution of  $k_t$  over time) under the assumption that the saving rate is  $s \in (0, 1)$  and the depreciation rate is  $\delta \in (0, 1]$ .
- e. What is the growth rate of  $k_t$ ,  $\gamma_{k_t} \equiv (k_{t+1} - k_t)/k_t$ ?
- f. Find the steady state level of  $k$ ,  $y = Y/L$ , and  $c =$  consumption per capita.
- g. What is the Golden Rule value of  $k$ ? ( $k$  such that the consumption in the steady state is maximized)
- h. What saving rate is needed to yield the Golden Rule?
- i. Find the elasticity of  $y$  with respect to  $s$ . Can observed differences in saving rates explain the observed differences in income per-capita across the world?
- j. Find the dynamical system describing the evolution of  $y_t$  under the assumption that the depreciation rate  $\delta = 1$ .
- k. Suppose you estimate a regression in which  $\ln(y_{t+1}^i/y_t^i)$  is on the left hand side (i.e., you estimate the  $\ln$  of  $\gamma_{y_t}^i + 1$  across countries - the  $i$  is country  $i$ ) and  $\ln s^i$ ,  $\ln(1+n^i)$ , and  $\ln y_t^i$  are on the right hand side. According to the dynamical system you defined in section j, what would be the coefficients on your explanatory variables? How would you interpret these coefficients? Is there “ $\beta$  convergence”? How would you interpret the constant term?

2. Output per worker is an increasing and concave function of capital per worker, given by:  $y_t = f(k_t)$ . Output is divided between labor income and capital income according to their marginal productivity. Namely,

$$f(k_t) \equiv [f(k_t) - f'(k_t)k_t] + f'(k_t)k_t \equiv w_t + r_t k_t.$$

Suppose that the rate of saving from wage income is  $s^w \in [0, 1]$ , and the rate of saving from capital income is  $s^r \in [0, 1]$ . Therefore, total saving are given by

$$s_t = s^w [f(k_t) - f'(k_t)k_t] + s^r f'(k_t)k_t.$$

Population and technology are constant and the rate of capital depreciation,  $\delta \in (0, 1)$ .

a. Derive the dynamical system governing the evolution of capital per capita:  $k_{t+1} = \phi(k_t)$ .

b. Suppose there exist a range of  $k_t$  where  $f'''(k) = 0$  (the third derivative is zero). Find a condition on the saving rates  $s^w$  and  $s^r$  such that the dynamical system,  $k_{t+1} = \phi(k_t)$ , is convex ( $\phi''(k_t) > 0$ ) in the range of  $f'''(k) = 0$ ?

Suppose now that  $s^r = 0$ , and  $f(k_t) = \ln(1 + k_t)$ :

c. Derive the dynamical system  $k_{t+1} = \phi(k_t)$

d. Find  $\phi'(k_t)$ ,  $\lim_{k_t \rightarrow 0} \phi'(k_t)$ ,  $\lim_{k_t \rightarrow \infty} \phi'(k_t)$ , and  $\phi''(k_t)$ .

e. Is the trivial steady state,  $k = 0$ , locally stable? Explain.

f. Find the range of  $k_t$  in which the dynamical system is strictly convex.

g. Show that for  $\delta = 0$  a non-trivial steady state level of  $k$  does not exist (that is, explain why there exists no  $\bar{k} > 0$  such that  $\bar{k} = \phi(\bar{k})$ ). Find the growth rate of  $k_t$  (i.e.,  $k_{t+1}/k_t - 1$ ) as  $k_t \rightarrow \infty$  for  $\delta = 0$ .

3. Production is given by:

$$Y_t = F(K_t, L_t) = K_t^\alpha (B_t L_t)^{1-\alpha}$$

where  $L_{t+1} = (1+n)L_t$ ,  $B_{t+1} = (1+g)B_t$ , and  $\alpha \in (0, 1)$ .

a. Find the dynamical system describing the evolution of  $k_t = K_t/B_t L_t$ . Find the steady state level of  $k_t$ . What is the growth rate of output per worker,  $y_t = Y_t/L_t$ , in the steady state? What is the growth rate of capital per worker,  $K_t/L_t = B_t k_t$  in the steady state?

b. According to Solow's growth accounting, the growth in Total Factor Productivity (TFP) is calculated by

$$\frac{\Delta A_t}{A_t} = \frac{\Delta y_t}{y_t} - \tilde{\alpha} \frac{\Delta \tilde{k}_t}{\tilde{k}_t},$$

where  $y_t \equiv Y_t/L_t$ ,  $\tilde{k}_t \equiv K_t/L_t$ ,  $\tilde{\alpha}$  is the elasticity of  $y_t$  with respect to  $k_t$  (which is equal to the share of capital in a competitive economy), and  $\Delta j$  is the change in the variable  $j = y_t, k_t, A_t$ .

Find TFP growth rate,  $\Delta A_t/A_t$ , in the steady state according to the model in part a?

Problem Set 2 - Overlapping Generations Model

1. Consider the following production function

$$Y_t = F(K_t, L_t) = \begin{cases} 6K_t - K_t^2/L_t & \text{for } K_t/L_t \leq 2 \\ 2K_t + 4L_t & \text{for } K_t/L_t > 2 \end{cases}$$

- a. Show that this function is characterized by constant returns to scale (CRS) and that output per worker is given by:

$$y_t = f(k_t) = \begin{cases} 6k_t - k_t^2 & \text{for } k_t \leq 2 \\ 2k_t + 4 & \text{for } k_t > 2 \end{cases}$$

For the following parts assume that population and technology are constant and that capital fully depreciates at the end of each period.

- b. Calculate the share of labor income as a function of  $k_t$ . Show your results in a figure.
- c. Find the dynamical system for the Solow model under the assumption that the saving rate  $s = 1/3$ . Show that there exists a unique, globally stable, non-trivial steady state.
- d. Find the dynamical system of the OLG model (where individuals work in their first life period and retire in the second). Assume utility is a function of only second period consumption (that is, individuals save their entire first period income). Is the non-trivial steady state unique and globally stable?
- e. Why do the two models generate results that are qualitatively different?

2. Consider the following specific case for the overlapping generations (OLG) model:

Technology and population are constant.

Output per worker is

$$y_t = Ak_t^\alpha \quad k_t = K_t/L_t$$

Utility of individuals who are born in  $t$  is

$$U(c_t, c_{t+1}) = c_t + \frac{1}{\rho}c_{t+1}$$

Capital fully depreciates at the end of each period:  $\delta = 1$ .

- a. Find the dynamical system. (hint: the system is characterized by two qualitatively different regimes, separated by a threshold level of  $k_{t+1}$ , where, for low levels of  $k_{t+1}$  there is no consumption in the first period).
- b. Find sufficient conditions on the parameters that assure that the steady state is characterized by consumption in the first period.

### Problem Set 3 - Ak Model and Endogenous Technology

1. Consider a competitive economy, in which the level of technology, which is external to the firm, is:

$$A_t = A(k_t) = k_t^\beta; k_t = K_t/L_t, \beta \in (0, 1].$$

All firms share the same technology. In particular, production of firm  $j$  is:

$$Y_{j,t} = A_t K_{j,t}^\alpha L_{j,t}^{1-\alpha}; \alpha \in (0, 1)$$

where,  $K_{j,t}$  and  $L_{j,t}$  are the quantities of capital and labor employed by firm  $j$ .

- a. Find the factor shares, show that  $k_{j,t} = k_t$  for all  $j$ , and that aggregate output is

$$Y_t = \sum_j Y_j = A_t K_t^\alpha L_t^{1-\alpha}$$

- b. Find the dynamical system  $k_{t+1} = \phi(k_t)$ , assuming that saving per worker is  $sy_t$ ,  $s \in (0, 1)$ . Calculate  $\lim_{k \rightarrow 0} \phi'(k_t)$  and  $\lim_{k \rightarrow \infty} \phi'(k_t)$  for  $\beta + \alpha < 1$ ,  $\beta + \alpha = 1$  and  $\beta + \alpha > 1$ .
- c. Show that for  $\beta = 1 - \alpha$  the economy grows at a constant rate in the steady state if  $s > \delta + n$ . What is the steady state growth rate if  $s < \delta + n$ ?
- d. Find the economy's steady state for  $\beta < 1 - \alpha$ . Is it unique? Is it stable? How is it affected by the saving rate?
- e. Find the economy's steady state for  $\beta > 1 - \alpha$ . Is it unique? Is it stable? How is it affected by the saving rate? How do you interpret this result?

2. Production is given by:

$$Y_t = F(K_t, h_t) = BK_t^\alpha h_t^{1-\alpha},$$

where  $h_{t+1} = h(e_t) = \rho e_t$ , is human capital and  $e_t$  is investment in human capital. The saving rate is  $s \in (0, 1)$  (i.e.,  $S_t = sY_t$ ), half of the savings is invested in physical capital and the other half in education. physical capital fully depreciates at the end of the period, ( $\delta = 1$ ):

- a. Find the dynamical system governing the evolution of output,  $Y_t$ .
- b. What is the condition on the parameters that assures steady state growth?
- c. Suppose that  $h(e_t) = e_t^\beta$ , where  $\beta \in (0, 1)$ . find the dynamical system governing the evolution of output,  $Y_t$ . Is there growth in the steady state?

Problem Set 4 - Endogenous R&D

1. Consider the quality ladder R&D model. Suppose  $\beta = 1$ .
  - a. Suppose the economy exists only one period, (period 1). Is the competitive allocation of R&D workers,  $H$ , efficient?
  - b. Suppose the economy exists for two periods, (periods 1 and 2) and the interest rate between the two periods is zero. Is the competitive allocation in the first period,  $H_1$ , efficient? Hint: consider the effect of a small change in  $H_1$  on total output in both periods, holding  $H_2$  in its equilibrium level.
  - c. Explain the different results for parts a and b.
  
2. Consider the quality ladder R&D model. Suppose that the number of new inventions (new knowledge) produced in period  $t$  is  $A_{t-1}\sqrt{H_t}$ , where each of the R&D workers develops an equal number of inventions. Therefore,

$$A_t = A_{t-1} + A_{t-1}\sqrt{H_t} = A_{t-1}(1 + \sqrt{H_t}).$$

- a. For  $\beta = 1$  find an implicit function defining  $H_t$  as a function of  $N$ .
- b. For  $\beta = 1$  show (using a figure) that there exists a unique  $H > 0$  for any  $N$ , and that  $H$  is increasing with  $N$ . What is the economic intuition behind these results? (i.e., the uniqueness of  $H$ , that  $H$  is strictly positive, and that it is increasing in  $N$ ).
- c. For  $\beta < 1$  find an implicit function defining  $H$  as a function of  $N$ .

Problem Set 5 - income inequality and growth

1. Consider the Galor-Zeira model. Suppose that technological progress increases the wage rate of skilled and unskilled workers by a factor of  $A > 1$ . Namely, the wage of skilled workers is  $Aw^s$  and the wage of unskilled workers is  $Aw^u$ . The interest rate remains unchanged.
  - a. Find a *sufficiently* high level of  $A$  for which income distribution would have *no* effect on *long-run* output.
  - b. Suppose that the cost of education is indexed to wages. Namely, the cost of education is  $Ah$ . Derive the conditions under which technological change has *no* effect on the qualitative results of the model. (i.e., revise the model's assumptions if and when needed).
2. Consider the Moav 2002 model, with the following human capital production technology:

$$h_{t+1} = h(e_t) = 1 + A \ln[1 + e_t]$$

- a. Find the optimal unconstrained level of investment in human capital.
- b. Find the dynamical system governing the evolution of income within a dynasty
- c. Define *sufficient* conditions on the model's parameters to assure that income distribution will have an effect on long run output.