Problem Set 1

Problem 1

a.

$$F(\lambda K_t, \lambda L_t) = A(\lambda K_t)^\alpha (\lambda L_t)^{1-\alpha} = \lambda A K_t^\alpha L_t^{1-\alpha} = \lambda F(K_t, L_t)$$

b.

$$A K_t^\alpha L_t^{1-\alpha} = L_t A \left(\frac{K_t}{L_t}\right)^\alpha = L_t A k_t^\alpha;$$

c.

$$\begin{aligned} w_t &= \frac{\partial F(K_t, L_t)}{\partial L_t} = (1-\alpha) A K_t^\alpha L_t^{-\alpha} = (1-\alpha) A k_t^\alpha \\ R_t &= \frac{\partial F(K_t, L_t)}{\partial K_t} = \alpha A K_t^{\alpha-1} L_t^{1-\alpha} = \alpha A k_t^{\alpha-1} \end{aligned}$$

d.

$$y_t = f(k_t) = A k_t^\alpha$$

$$K_{t+1} = s y_t + (1-\delta) K_t = s L_t A k_t^\alpha + (1-\delta) K_t$$

$$k_{t+1} = \frac{K_{t+1}}{L_{t+1}} = \frac{s A k_t^\alpha + (1-\delta) k_t}{1+n}$$

e.

$$\gamma_{k_t} = \frac{k_{t+1} - k_t}{k_t} = \frac{s A k_t^{\alpha-1} - (\delta + n)}{1+n}$$

f.

$$s A \overline{k}^{\alpha-1} = \delta + n$$

$\Longrightarrow$

$$\overline{k} = \left(\frac{sA}{\delta+n}\right)^{1/(1-\alpha)}$$

$$\overline{y} = f\left(\overline{k}\right) = A \overline{k}^\alpha = A^{\frac{1}{1-\alpha}} \left(\frac{s}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}$$

$$\overline{c} = (1-s)\overline{y} = (1-s) A^{\frac{1}{1-\alpha}} \left(\frac{s}{\delta+n}\right)^{\frac{\alpha}{1-\alpha}}$$

g.

$$c_t = f(k_t) - s f(k_t)$$

at the steady state:

$$s f(k) = (n+\delta) k$$

$\rightarrow$ at the steady state

$$c = f(k) - (n+\delta) k$$

$\rightarrow$ f.o.c. for maximizing c at the steady state:

$$f'(k_t) = n + \delta$$

$$\rightarrow f'(k^{GR}) = n + \delta$$

for $y = Ak^{\alpha}$:

$$\alpha A k^{\alpha - 1} = n + \delta$$

$\rightarrow$

$$k^{GR} = \left(\frac{\alpha A}{\delta + n}\right)^{1/(1-\alpha)}$$

h.
$\rightarrow$

$$s^{GR} = \alpha$$

alternatively:

$$f'(k^{GR}) = n + \delta$$

$\rightarrow$for $k = k^{GR}$

$$\frac{f'(k)k}{f(k)} = \frac{(n+\delta)\,k}{f(k)}$$

note that:

- $\frac{f'(k)k}{f(k)}$ = share of capital (α in this specific example)
- in the steady state $\frac{(n+\delta)k}{f(k)} = \frac{sf(k)}{f(k)} = s$

$\rightarrow$

$$s^{GR} = share\ of\ capital$$

i.

$$\overline{y} = f\left(\overline{k}\right) = A\overline{k}^{\alpha} = A^{\frac{1}{1-\alpha}} \left(\frac{s}{\delta + n}\right)^{\frac{\alpha}{1-\alpha}}$$

$$e_{y,s} = \frac{\partial \ln y}{\partial \ln s} = \frac{\partial \left(\left(1 + \frac{\alpha}{1-\alpha}\right)\ln A + \frac{\alpha}{1-\alpha}\ln s - \frac{\alpha}{1-\alpha}\ln\left(\delta + n\right)\right)}{\partial \ln s} = \frac{\alpha}{1-\alpha}$$

for $\alpha = \frac{1}{3}$, and a difference in savings rate of 3 times across countries, the explainable gap of GDP per capita is about 150%, according to the Solow Model with Cobb-Douglas production, while the observed difference is nearly 20 times.

however, in a model with a poverty trap, a small rise in savings may release an economy from the poverty trap and imply very large differences in income.

j. for $\delta = 1$:

$$k_{t+1} = \frac{sy_t}{1+n}$$

$\rightarrow$

$$y_{t+1} = Ak_{t+1}^{\alpha} = A\left(\frac{sy_t}{1+n}\right)^{\alpha} = \frac{As^{\alpha}y_t^{\alpha}}{(1+n)^{\alpha}}$$

k.

$$\frac{y_{t+1}}{y_t} = \frac{As^{\alpha}y_t^{\alpha-1}}{(1+n)^{\alpha}}$$

taking logs:

$$\ln\frac{y_{t+1}^i}{y_t^i} = \ln(1+g_t^i) = \ln A + \alpha\ln s^i - (1-\alpha)\ln y_t^i - \alpha\ln\left(1+n^i\right)$$

where $\ln(1+g_t^i) \approx g_t^i$

Problem 2

a.

$$\begin{aligned} k_{t+1} &= s_t + (1-\delta)k_t \\ &= s^w\left[f(k_t) - f'(k_t)k_t\right] + s^r f'(k_t)k_t + (1-\delta)k_t \\ &= (s^r - s^w)f'(k_t)k_t + s^w f(k_t) + (1-\delta)k_t \equiv \phi(k_t) \end{aligned}$$

b.

$$\phi'(k_t) = (s^r - s^w)f'(k_t) + (s^r - s^w)f''(k_t)k_t + s^w f'(k_t) + 1 - \delta$$

$$\phi''(k_t) = 2(s^r - s^w)f^{''}(k_t) + (s^r - s^w)f^{'''}(k_t)k_t + s^w f^{''}(k_t)$$

for $f^{'''}(k_t) = 0$

$$\begin{aligned} \phi''(k_t) &= 2(s^r - s^w)f^{''}(k_t) + s^w f^{''}(k_t) \\ &= (2s^r - s^w)f^{''}(k_t) \end{aligned}$$

$\rightarrow$ for $f^{'''}(k_t) = 0$,

$$\phi''(k_t) > 0 \leftrightarrow s^w > 2s^r$$

c. For $s^r = 0$ and $f(k_t) = \ln(1+k_t)$:

$$w_t = \ln(1+k_t) - \frac{k_t}{1+k_t}$$

$$k_{t+1} = s^w w_t + (1-\delta)k_t = s^w\ln(1+k_t) - s^w\frac{k_t}{1+k_t} + (1-\delta)k_t = \phi(k_t)$$

d.

$$\phi'(k_t) = \frac{s^w}{1+k_t} - \frac{s^w}{1+k_t} + \frac{s^w k_t}{(1+k_t)^2} + 1 - \delta = \frac{s^w k_t}{(1+k_t)^2} + 1 - \delta$$

$$\begin{aligned}
\lim_{k_t \to 0} \phi'(k_t) &= 1-\delta \\
\lim_{k_t \to \infty} \phi'(k_t) &= 1-\delta
\end{aligned}$$

$$\begin{aligned}
\phi''(k_t) &= s^w \frac{(1+k_t)^2 - 2(1+k_t)k_t}{(1+k_t)^4} \\
&= s^w \frac{1+2k_t+k_t^2-2k_t-2k_t^2}{(1+k_t)^4} \\
&= s^w \frac{1-k_t^2}{(1+k_t)^4} = s^w \frac{1-k_t}{(1+k_t)^3}
\end{aligned}$$

e. Since $\phi(0) = 0$ and since $\lim_{k_t \to 0} \phi'(k_t) = 1-\delta$, for $\delta > 0$, $k = 0$ is a localy stable steady state.

f.

$$\phi'' > 0 \Longleftrightarrow s^w \frac{1-k_t}{(1+k_t)^3} > 0 \Longleftrightarrow k_t < 1$$

g. since for $\delta = 0$, $\lim_{k_t \to 0} \phi'(k_t) = \lim_{k_t \to \infty} \phi'(k_t) = 1$. and since the function is convex for $k_t < 1$ and concave for $k_t > 1$ there exists no $\overline{k} \neq 0$.