Bidding Markets with Financial Constraints

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Abstract

We develop a model of bidding markets with financial constraints a la Che and Gale (1998b) in which two firms optimally choose their budgets. We provide four main results. First, we show that firms keep small budgets in the long run even when its opportunity cost is arbitrary low. Second, we provide an alternative explanation for the dispersion of markups and “money left on the table” across procurement auctions. Interestingly, this explanation does not hinge on significant private information but on differences, both endogenous and exogenous, in the availability of financial resources. Third, we explain why the empirical analysis of the size of markups may be biased downwards or upwards, with a bias positively correlated with the availability of financial resources, when the researcher assumes that the data are generated by the standard auction model. Four, we show that large concentration and persistent asymmetries in market shares together with occasional leadership reversals can arise as a consequence of the firms internal financial decisions even in the absence of exogenous shocks.

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1 Introduction

Bidding markets are those in which trade is organized through bidding. The most commonly cited example is public procurement which amounts to between 10% and 20% of GDP in OECD countries. Other examples include procurement in the private sector and auctions both in the private and the public sector.\(^1\)

The standard model of bidding implicitly assumes that the size of the projects is relatively small compared to the financial resources of the firms. Unfortunately, the current financial crisis has made evident that this assumption is not realistic for many bidding markets:

"Offers submitted on Monday by Global Infrastructure Partners and a consortium led by Manchester Airport Group have been depressed [...] by the problems of raising the necessary bank finance."

*Ferrovial receives depressed bids for Gatwick*, Financial Times, 28/Apr/2009

Che and Gale (1998b) show that the predictions of the standard model do not extend to the model where firms are financially constrained. The extent to which a firm is financially constrained in their model depends on its budget, *working capital* hereafter, which they assume to be exogenous. In this paper, as it happens in reality, the firm’s working capital is not exogenous but chosen out of the firm’s internal financial resources, the *cash* hereafter, which in turn depends on the past performance of the firm.

Our first main result challenges the conventional wisdom in economics that “auctions [still] work well if raising cash for bids is easy” (Aghion, Hart, and Moore (1992, p. 527)).\(^2\) Although the standard model arises in our framework when the firms’ working capitals are sufficiently abundant, firms keep surprisingly little working capital in the long run even when its opportunity cost is arbitrary low.

Besides, our model displays sensible features regarding the behaviour of markups, “money left on the table” and market shares that suggests that we should reconsider the

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\(^1\) A detailed description of bidding markets and a wide range of examples can be found in Klemperer (2005), OECD (2006), OFT (2007) and Einav and Levin (2010).

\(^2\) This conjecture has been recently questioned by Rhodes-Kropf and Viswanathan (2005) under the assumption that firms finance their bids by borrowing in a competitive financial market.
empirical analysis of bidding markets. Our second main result provides a new explanation for the dispersion of markups and “money left on the table” across auctions observed in procurement. Interestingly, this explanation does not hinge on significant private information about working capitals and costs, but on differences in the availability of financial resources across auctions in a sense that we formalise later. This casts doubts about the usual interpretation for the dispersion of markups and “money left on the table” observed in procurement as indicative of incomplete information and large heterogeneity in production cost. Our third main result explains why the empirical analysis of the size of markups may be biased downwards or upwards with a bias positively correlated with the availability of financial resources when the researcher assumes that the data are generated by the standard model. Our fourth main result shows that large concentration and persistent asymmetries in market shares together with occasional leadership reversals can arise as a consequence of the firms internal financial decisions even in the absence of exogenous shocks. This effect is greater for larger projects than for smaller projects, a prediction in line with the empirical evidence.

We are interested in markets in which only bids that have secured financing can be submitted, i.e. are acceptable. For instance, this is the case of markets in which surety bonds are required. We also follow Che and Gale’s (1998b) insight that the set of acceptable

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3“Money left on the table,” also known as bid spreads, is defined as the difference between the lowest and the second lowest bids in first price procurement auctions.

4Indeed, as Weber (1981) pointed out: “Some authors have cited the substantial uncertainty concerning the extractable resources present on a tract, as a factor which makes large bid spreads [i.e. ‘money left on the table’] unavoidable.” More recently, Krasnokutskaya (2011) noted that “The magnitude of the ‘money left on the table’ variable [...] indicates that cost uncertainty may be substantial.”

5Exogenous shocks give rise to similar predictions when they introduce randomness in the processes of either capacity accumulation, see Besanko and Doraszelski (2004), or learning by doing, see Besanko, Doraszelski, Kryukov, and Satterthwaite (2010). See also the discussion in page 6.

6Porter and Zona (1993) explain that “the market for large jobs [in procurement of highway maintenance] was highly concentrated. Only 22 firms submitted bids on jobs over $1 million. On the 25 largest jobs, 45 percent of the 76 bids were submitted by the four largest firms.”

7Alternatively, we could have assumed that it was costly for the firm to submit a bid and not complying, e.g. the firm may bear a direct cost in case of default.

8In the U.S., the Miller Act regulates the provision of surety bonds for federal construction projects. State legislatures have enacted “Little Miller Acts” that establish similar requirements for state contracts. A surety bond plays two roles: first, it certifies that the proposed bid is not jeopardized by the technological
bids increases with the working capital. This feature is present in a number of settings in which firms have limited access to external financial resources. One example is an auction in which the price must be paid upfront, and hence the maximum acceptable bid increases with the firm’s working capital. Another example is a procurement contest in which the firm must be able to finance the difference between its working capital and the cost of production. Due to financing constraints, if the external funds that are available to the firm increase with its bid or its profitability, it follows that the firm’s minimum acceptable bid decreases in the firm’s working capital. The latter property arises when the sponsor pays in advance a fraction of the price, a feature of the common practice of progress payments, or when the amount banks are willing to lend depends on the profitability of the project as it is usually the case.

A representative example of the institutional details of the bidding markets we are interested in is highway maintenance procurement. As Hong and Shum (2002) pointed out “many of the contractors in these auctions bid on many contracts over time, and likely derive a large part of their revenues from doing contract work for the state.” Besides, Porter and Zona (1993) explain that “The set of firms submitting bids on large projects was small and fairly stable [...] There may have been significant barriers to entry, and there was little entry in a growing market.”

Motivated by these observations, we build a static model to explain the second and third main results and a dynamic model to give a broader perspective on the second main result and to explain the first and fourth one. In the static model, two firms endowed with some cash choose working capitals to compete in a first price auction for a procurement contract. We assume that the cost of complying is known and identical across firms, the financial conditions of the firm, and second, it insures against the losses in case of non-compliance. Indeed, the Surety Information Office highlights that “the surety [...] may require a financial statement [that] [...] helps the surety company evaluate the working capital and overall financial condition of the company.” See http://www.sio.org/html/Obtain.html#Financial and Calveras, Ganuza, and Hauk (2004).

9A numerical illustration can be found in Beker and Hernando-Veiana (2011).

10We show in Appendix C that this is also the theoretical prediction of a model inspired by the observation of Tirole (2006), page 114, that “The borrower must [...] keep a sufficient stake in the outcome of the project in order to have an incentive not to waste the money.”

11Moreover, it can be shown that in a model with many firms and entry the natural extension of the equilibrium we study has the feature that only the two firms with more cash enter the market.
minimum acceptable bid increases with the firm’s working capital and only cash is publicly observable.\textsuperscript{12} Since using cash as working capital means postponing consumption, it is costly,\textsuperscript{13} though all our results still hold true when that cost is arbitrarily low. Firms choose their working capitals and bids optimally. The dynamic model consists of the infinite repetition of the static model. The cash at the beginning of each period is equal to the last period unspent working capital plus the earnings in previous procurement contract and some exogenous cash-flow.

In our static model, to carry more working capital than strictly necessary to make the bid acceptable is strictly dominated because working capital is costly. Thus, the firm that carries more working capital wins the contract\textsuperscript{14} and both firms incur the cost of their working capital.

The strategic considerations that shape the equilibrium working capitals are the same as in the all pay auction with complete information.\textsuperscript{15} Not surprisingly, in a version of our game with unlimited cash, there is a unique symmetric equilibrium in which firms randomize in a bounded interval with an atomless distribution. This is also the unique equilibrium in our game when the firms’ cash is larger than the upper bound of the support of the equilibrium randomization. We call the scenario symmetric if this is the case, and laggard-leader otherwise. In this latter case, firms also randomize in a bounded interval, though the firm with less cash, the laggard hereafter, puts an atom at zero and the other firm, the leader, at the laggard’s cash.

The second and third main results arise in the laggard-leader scenario. The dispersion

\textsuperscript{12}Our first main result, the part of our second main result regarding markups and the fourth main results also hold in a version of our model with observable working capital, see Beker and Hernando-Veciana (2011).

\textsuperscript{13}Any other motivation for the cost of working capital would deliver similar results.

\textsuperscript{14}This feature seems realistic in many procurement contracts:

It is thought that Siemens’ superior financial firepower was a significant factor in it beating Canada’s Bombardier to preferred bidder status on Thameslink.


\textsuperscript{15}In particular, it resembles Che and Gale’s (1998a) model of an all pay auction with caps in that working capitals are bounded by cash. Our model is more general in that they assume exogenous caps that are common to all agents. Another difference is that in our dynamic model the size of the prize varies with the rival’s action. To the best of our knowledge, the literature on all pay auctions, see Kaplan, Luski, Sela, and Wettstein (2002) and Siegel (2009), has only considered the case of prizes that vary with the agent’s action.
of markups and “money left on the table” is due to heterogeneity across auctions in the availability of financial resources, i.e. either the firms’ cash or the minimum acceptable bids. Either of these two variables affect the equilibrium working capitals which determine the bids, and hence the markups and “money left on the table”. Biases in the structural estimation of the size of markups can also arise if, as it is often the case, the researcher does not observe costs. Imagine bid data from several auctions with identical financial conditions and suppose the data are generated by our model. On the one hand, there are large markups and little “money left on the table” if the laggard has little cash. However, a researcher who assumed the standard model would conclude that there is little cost heterogeneity and, as a consequence, small markups, i.e. the estimation would be biased downwards. On the other hand, if the laggard has relatively large cash, though not too large, there is sizable “money left on the table” and relatively low markups. However, a researcher that assumed the standard model would conclude that there is large cost heterogeneity and, as a consequence, large markups, i.e. the estimation would be biased upwards.

In our dynamic model, we consider the unique equilibrium that is the limit of the sequence of equilibria of models with an increasing number of periods. Remarkably, the marginal continuation value of cash is equal to its marginal consumption value under a mild assumption about the minimum acceptable bid. Consequently, as in the static model, it is suboptimal to increase the working capital while keeping the bid constant. One can also argue that firms do not carry more working capital than strictly necessary to make the bid acceptable and that the strategic interaction each period is, again, similar to an all pay auction.

On the equilibrium path, the frequency of each scenario depends on the severity of the financial constraint, that we define as the ratio between the working capital for which the minimum acceptable bid equals the cost of the procurement contract and the exogenous cash flow. If this ratio is large, the laggard-leader scenario occurs most of the time, as the cost of working capital becomes negligible. This implies our first main result. Another consequence is that a firm tends to win consecutive procurement contracts. In contrast,

\[\text{16}\] The uniqueness result is proved in the supplementary material.

\[\text{17}\] To the extent that joint profits are larger in the laggard-leader scenario than in the symmetric scenario, our result is related to the literature on increasing dominance due to efficiency effects (see Budd, Harris, and
when the ratio is so small that the symmetric scenario occurs each period, the probability that a firm wins a contract is constant across periods. The previous results explain our prediction of greater market concentration and asymmetric market shares, together with occasional leadership reversals, for larger projects to the extent that one can associate the severity of the financial constraint to the project’s size.\textsuperscript{18}

Che and Gale (1998b) and Zheng (2001) already showed that the dispersion of markups can reflect heterogeneity of working capital if this is assumed to be sufficiently scarce.\textsuperscript{19} We show that scarcity is the typical situation once we allow firms to choose their working capital. Note, however, that whereas they assume that the distribution of working capitals is constant across firms, our results show that this distribution is seldom constant across firms. This difference is important because the lack of asymmetries in the distribution of working capitals precludes the possibility of large expected money left on the table when private information is small.

In Galenianos and Kircher’s (2008) model of monetary policy, firms also choose working capital before competing in an auction. In their equilibrium firms also randomize their working capital due to the all pay auction structure. However, since their working capital is not bounded by cash, the laggard-leader scenario does not arise.

Our paper contributes to the dynamic oligopoly literature “an area where much work needs to be done and much work can be done,” as emphasized by Cabral (2012). In his terminology our model is a properly defined dynamic oligopoly model since cash acts as a “physical” link across periods. In particular, it contributes to a recent literature that explains how asymmetries in market shares arise and persist in otherwise symmetric models. In particular, Besanko and Doraszelski (2004), and Besanko, Doraszelski, Kryukov, and Satterthwaite (2010) show that firm-specific shocks can give rise to a dynamic of market shares similar to ours. The difference, though, is that the dynamic in our model arises

\textsuperscript{18}As we do in Appendix C.

\textsuperscript{19}Che and Gale (1996, 2000), DeMarzo, Kremer, and Skrzypacz (2005) and Rhodes-Kropf and Viswanathan (2005) also studied the effect of some given financial constraints in auctions and Pitchik and Schotter (1988), Maskin (2000), Benoit and Krishna (2001) and Pitchik (2009) how bidders distribute a fixed budget in a sequence of auctions. The latter is not a concern in our setup because the profits are realized before the beginning of the next auction.
because firms randomize their working capital due to the all pay auction structure.

Our characterization of the dynamics resembles that of Kandori, Mailath, and Rob (1993) in that we study a Markov process in which two persistent scenarios occur infinitely often and we ask which of the two occurs most of the time as the randomness vanishes. We want to underscore that while the transition function of their stochastic process is exogenous, ours stems from the equilibrium strategies of the infinite horizon game. As in Cabral (2011), a typical time series of market shares displays not only a lot of concentration but also, and more importantly, tipping, i.e. the system is very persistent but moves across extremely asymmetric states.

Other explanations for the persistency of markups are asymmetric information (i.e. the standard model), capacity constraints and collusion. Our model is empirically distinguishable from these models in that it predicts negative correlation between the laggard’s cash and the bids (or the price). An alternative to distinguish our model from the standard model and the model of capacity constraints when the laggard’s cash is not observable by the econometrician is to use as a proxy either the progress payments of the firms uncompleted contracts\(^{20}\) or past bids\(^{21}\). These proxies do not explain the current bids in the standard model once one controls for costs or in the models of capacity constraints once one controls for backlog and costs, see Bajari and Ye (2003) and Jofre-Bonet and Pesendorfer (2003).

Another way in which our model is empirically distinguishable from the models of collusion is that the sample average of the price may decrease with patience\(^{22}\) and winning today increases the probability of winning tomorrow. Collusive models predict that patience increases the sample average of the price, see Green and Porter (1984) and Athey and Bagwell (2001), and that winning today has either no effect on winning tomorrow, see Athey, Bagwell, and Sanchirico (2004), or decreases its probability, see McAfee and McMillan (1992), Athey and Bagwell (2001) and Aoyagi (2003).

Section 2 defines our canonical model of procurement with financial constraints. Section 3 analyzes the static model and Section 4 the dynamic model. Section 5 concludes.

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\(^{20}\) The California Department of Transportation makes this information available in http://www.dot.ca.gov/hq/asc/oap/payments/public/ctnums.htm

\(^{21}\) The latter holds true because the laggard’s past and current cash are positively correlated.

\(^{22}\) As happens when the ratio that defines the severity of the financial constraint is sufficiently small.
also include an appendix with the more technical proofs (Appendix A), the laggard-leader scenario of the model of Section 3 when firms start with the same amount of cash (Appendix B) and an extension of our model that endogenizes financial constraints (Appendix C).

2 A Reduced Form Model of Procurement with Financial Constraints

In this section, we describe a model of procurement that we later embed in the models of Sections 3 and 4. In this model, two firms\(^{23}\) compete for a procurement contract of common and known cost \(c\) in a first price auction: each firm submits a bid, and the firm who submits the lowest bid gets the contract at a price equal to its bid.\(^{24}\) We assume that firms can only submit bids that have secured financing, i.e. acceptable bids. We assume that the minimum acceptable bid of a firm with working capital \(w\geq 0\) is given by\(^{25}\)

\[ b^* (w) \equiv \pi (w) + c \]

where \(\pi\) is strictly decreasing and continuously differentiable.

As we discuss in the Introduction, our assumption that firms can submit only acceptable bids captures a wide range of institutional arrangements whose aim is to preclude firms from submitting unacceptable bids such as bids that cannot be financed.\(^{26}\) Alternatively, the sponsor may provide incentives to guarantee that firms submit only acceptable bids by making them bear some of the cost of default. The monotonicity of the set of acceptable bids arises naturally in markets in which firms have limited access to external

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\(^{23}\)As in all pay auctions, see Baye, Kovenock, and de Vries (1996), if there are more than two firms then there are multiple equilibria. One such equilibrium is that in which two firms choose the equilibrium strategies of the two-firm model and the other firms choose zero working capital.

\(^{24}\)A sale auction of a good with common and known value \(V\) can be easily encompassed in our analysis assuming that \(c = -V < 0\) and bids are negative numbers.

\(^{25}\)Thus, the model of auctions with budget constraints analyzed by Che and Gale (1998b) in Section 3.2 corresponds in our framework with \(b^* (w) = -w\) and \(\pi (w) = V - w\), and the interpretation of our model as a sale auction, see Footnote 24.

\(^{26}\)For instance, Meaney (2012) says that:

As well as considering the financial aspects of bids, the DfT[the sponsor] assesses the deliverability and quality of the bidders’ proposals so as to be confident that the successful bidder is able to deliver on the commitments made in the bidding process.
financial resources, as we discussed in the Introduction and illustrate in Appendix C.

For any given bids \( b_1 \) and \( b_2 \), we use \textit{markup} to denote \( \frac{\min(b_1, b_2) - c}{c} \) and we use “\textit{money left on the table}” to denote \( \frac{|b_1 - b_2|}{c} \). We denote by \( \theta \) the working capital for which the minimum acceptable bid is equal to the cost of the procurement contract \( c \) so that \( \pi(\theta) = 0 \) or, equivalently, \( \theta = \pi^{-1}(0) \). We assume that \( \theta \in (0, \infty) \).

3 The Static Model

Each firm \( i \in \{1, 2\} \) starts with some cash \( m_i \geq 0 \). We assume the firm’s cash to be publicly observable. Each firm \( i \) chooses simultaneously and independently (i) how much of its cash to keep as working capital \( w_i \in [0, m_i] \) and (ii) an acceptable bid \( b_i \geq b^*(w_i) \) for a market as described in Section 2. A pure strategy is thus denoted by the vector \( (b_i, w_i) \in \{(b, w) : b \geq b^*(w), w \in [0, m_i]\} \). Firm \( i \)’s expected profit in the market against another firm with cash \( m_j \) that bids \( b_j \) is equal to:

\[
\Pi(b_i, b_j, m_i, m_j) = \begin{cases} 
  b_i - c & \text{if } b_i < b_j \text{ or } b_i = b_j \text{ and } m_i > m_j, \\
  \frac{1}{2}(b_i - c) & \text{if } b_i = b_j \text{ and } m_i = m_j, \\
  0 & \text{otherwise,}
\end{cases}
\]

where we are applying the usual uniformly random tie breaking rule except in the case in which one firm has strictly more cash than the other. In this case, we assume that the former firm wins.\textsuperscript{27} We assume that the firm maximises

\[
m_i - w_i + \beta(w_i + \Pi(b_i, b_j, m_i, m_j)),
\]

that is, \( m_i - w_i \), its \textit{consumption} hereafter, plus the discounted sum, at rate \( \beta \in (0, 1) \), of the working capital and the expected profit in the market. Note that a unit increase in working capital is costly in the sense that it reduces the current utility in one unit and increases the future utility in \( \beta \).

\textbf{Definition 1.} The cost of working capital becomes negligible when \( \beta \) increases to 1.

\textsuperscript{27}We deviate from the more natural uniformly random tie-breaking rule that is usual in Bertrand games and all pay auctions in order to guarantee the existence of an equilibrium. In our game, a sufficiently fine discretisation of the action space would overcome the existence problem and yield our results at the cost of a more cumbersome notation.
We start by simplifying the strategy space. First, any strategy \((b, w)\) in which \(b > b^*(w)\) is strictly dominated by the strategy \((b, \tilde{w})\) where \(\tilde{w}\) satisfies \(b = b^*(\tilde{w})\) so that it is never optimal to carry more working capital than is strictly necessary.\(^{28}\) As a consequence, we restrict to the set of pure strategies \(\{(b, w) : b = b^*(w), w \in [0, m]\}\) where \(m\) denotes the firm’s cash.

In our second simplification of the strategy space, we use the following definition:

**Definition 2.** \(\nu^\beta \in [0, \theta)\) is the unique solution\(^{29}\) in \(w\) to \(\beta \pi(w) = (1 - \beta)w\).

Thus, \(\nu^\beta\) denotes the working capital for which \(\beta \pi(\nu^\beta)\), the discounted procurement profits associated with the minimum acceptable bid corresponding to working capital \(\nu^\beta\), equals \((1 - \beta)\nu^\beta\), the implicit costs of selecting working capital \(\nu^\beta\) that are associated with postponing consumption. Any pure strategy \((b^*(w), w)\) in which \(w > \nu^\beta\) is strictly dominated by \((b^*(\nu^\beta), \nu^\beta)\). As a consequence, we further restrict the set of pure strategies to \(\{(b, w) : b = b^*(w), w \in [0, \min\{m, \nu^\beta\}]\}\) where \(m\) denotes the firm’s cash.

Once we eliminate the above strictly dominated strategies, the resulting reduced game has a unidimensional strategy space as in an all pay auction. Each firm chooses a working capital and its corresponding minimum acceptable bid. The firm with the higher working capital wins the procurement contract and carrying working capital is costly for each firm. As in all pay auctions, there is no pure strategy equilibrium. This can be easily understood when both firms’ cash is weakly larger than \(\nu^\beta\). If both firms choose different working capitals, the one with more working capital has a strictly profitable deviation: to decrease marginally its working capital.\(^{30}\) If both firms choose the same working capital \(w\), there is also a strictly profitable deviation: to increase marginally its working capital if \(w < \nu^\beta\), and to choose zero working capital if \(w = \nu^\beta\).\(^{31}\)

\(^{28}\)The probability that a firm wins the contract is unaffected but the cost of working capital decreases.

\(^{29}\)Note that this equation is equivalent to \(m - w + \beta w + \beta \pi(w) = m\).

\(^{30}\)It saves on the cost of working capital without affecting to the cases in which the firm wins and increases the profits from the procurement contract because it increases the price.

\(^{31}\)In the former case, the deviation is profitable because winning the procurement contract at \(w < \nu^\beta\) gives strictly positive profits and the deviation breaks the tie in favor of the deviating firm with an arbitrarily small increase in the cost of working capital and an arbitrarily small decrease in the profits from the procurement contract. In the latter case, \(w = \nu^\beta\) implies that one of the firms is winning with a probability strictly less one, and hence the definition of \(\nu^\beta\), see Footnote 29, means that this firm makes strictly lower payoffs than with zero working capital.
A mixed strategy over the set of strictly undominated strategies is described by a distribution function with support in the set \{(b, w) : b = b^*(w), w \in [0, \min\{m, \nu\beta\}]\} where \(m\) denotes the firm’s cash. This randomization can be described by the marginal distribution over working capitals \(F\). We denote the corresponding mixed strategy by \((b^*, F)\), with a slight abuse of notation. If a firm uses \((b^*, F)\) where \(F\) is differentiable in the support \([w, \bar{w}]\), then the payoff to the remaining firm with cash \(m \geq \bar{w}\) from choosing \(w \in [w, \bar{w}]\) is

\[m - w + \beta w + \beta \pi(w)F(w)\]

so that indifference across the support results if and only if \(F\) satisfies the differential equation

\[1 - \beta = \beta F'(w)\pi(w) + F(w)\beta \pi'(w)\]  

(1)

for any \(w \in (w, \bar{w})\). Thus, \((1 - \beta)\), the increase in the cost of working capital \(w(1 - \beta)\), must equal \(\beta F'(w)\pi(w) + F(w)\beta \pi'(w)\), the change in the expected discounted profits \(\beta \pi(w)F(w)\). There is both a positive effect and a negative effect of an increase in \(w\) on the change in expected discounted profits. The positive effect of an increase in \(w\) arises due to the higher probability of winning a contract and the negative effect arises due to the lower profits associated with a win.

We distinguish two scenarios:

**Definition 3.** Let \(m_l \equiv \min\{m_1, m_2\}\). The symmetric scenario denotes the case in which \(m_l \geq \nu \beta\). The laggard-leader scenario denotes the complementary case.

**Proposition 1.** If \(m_l \geq \nu \beta\), then the unique equilibrium is symmetric and denoted by the single (mixed) strategy \((b^*, F^*)\) where

\[F^*(w) \equiv \frac{(1 - \beta)w}{\beta \pi(w)}\]

with support\(^{32}\) \([0, \nu \beta]\) solves the differential equation (1) with initial condition \(F(0) = 0\).

This equilibrium verifies the usual property of all pay auctions that bidders without competitive advantage get their outside opportunity, i.e. the payoff of carrying zero working capital and losing the procurement contract.

\(^{32}\)We use the definition of support of a probability measure in Stokey and Lucas (1999). According to their definition, the support is the smallest closed set with probability one.
Corollary 1. If $m_l \geq \nu^3$, then (i) the equilibrium expected probability of winning the contract is common across firms and (ii) the equilibrium is unaffected by any change in cash that leaves $m_l \geq \nu^3$.

Besides, one can deduce the following corollary by inspection of $F^*$. Let $\chi_y$ denote the degenerate distribution that puts weight 1 on $y \in \mathbb{R}$.

Corollary 2. If $m_l \geq \nu^3$, then as $\beta$ increases to 1, $F^*(w)$ converges to $\chi_\theta(w)$ and, in equilibrium, $|\pi(w_1) - \pi(w_2)|$ and $\pi(\max\{w_1, w_2\})$ converge in distribution to $\chi_0$.

In the standard auction model, cost heterogeneity vanishes as the distribution of costs converges to the degenerate distribution that puts all the weight on one value. As cost heterogeneity vanishes, the markup and "money left on the table" vanish (Krishna (2002), Chapter 2). Corollary 2 says that this limit outcome also arises in the symmetric scenario when the cost of working capital becomes negligible, since the markup $\min\{b_1, b_2\} - c$ is equal to $\pi(\max\{w_1, w_2\})$ and "money left on the table" $(b_1 - b_2)$ is equal to $|\pi(w_1) - \pi(w_2)|$. In this sense, financial constraints become irrelevant as the cost of working capital becomes negligible.

We next consider the laggard-leader scenario. In the main text, we only analyze the case in which both firms start with different cash. The case in which both firms start with identical cash is analyzed in Appendix B. Our interest in the former case is justified in that in our dynamic model the case in which both firms cash is less than $\theta$ does not arise along the game tree. See our discussion after Assumption 1. In what follows, the leader refers to the firm that starts with more cash and the laggard refers to the other firm.

Proposition 2. If $m_l < \nu^3$ and $m_1 \neq m_2$, then the unique equilibrium$^{33}$ is denoted by the laggard and leader strategies $(b^*, F_l)$ and $(b^*, F_L)$, respectively, where

$$F_l(w) \equiv \frac{\beta \pi(m_l) - (1 - \beta)(m_l - w)}{\beta \pi(w)} \quad \text{if } w \in [0, m_l],$$

$$F_L(w) \equiv \begin{cases} 
\frac{(1 - \beta)w}{\beta \pi(w)} & \text{if } w \in [0, m_l), \\
1 & \text{if } w = m_l,
\end{cases}$$

have both support $[0, m_l]$ and solve the differential equation (1) with initial condition $F(m_l) = 1$ and $F(0) = 0$, respectively.

$^{33}$Interestingly, this equilibrium has similar qualitative features as the equilibrium of an all pay auction in which both agents have the same cap but the tie-breaking rule allocates to one of the agents only. The latter model has been studied in an independent and simultaneous work by Szec (2010).
Both firms put their atom of probability at points that do not upset the incentives of the rival to play its equilibrium randomization. There is only one such point for the laggard, whereas the leader’s atom is at the minimum working capital which ensures that it wins the procurement contract. Interestingly, it can be shown that the laggard gets its outside opportunity, as in the symmetric scenario, whereas the leader gets an additive positive premium. The latter is a consequence of the leader’s ability to undercut any acceptable bid of the laggard and the fact that any such bid is strictly profitable.

**Corollary 3.** If \( m_l < \nu^3 \) and \( m_1 \neq m_2 \), then (i) the leader is more likely to win the contract in equilibrium, (ii) an increase in \( m_l \) for which \( m_l < \nu^3 \) increases (in the sense of first order stochastic dominance) both equilibrium distributions of the working capitals and hence, decreases the equilibrium expectation of \( \pi(\max\{w_1, w_2\}) \), (iii) the equilibrium probability that the laggard chooses working capital \( 0 \) and the leader chooses working capital \( m_l \) is:

\[
\frac{\pi(m_l)}{\pi(0)} \left(1 - \frac{(1 - \beta)m_l}{\beta \pi(m_l)}\right)^2,
\]

and (iv) the expected value of \( |\pi(w_1) - \pi(w_2)| \) in equilibrium is at least:

\[
\frac{\pi(m_l)}{\pi(0)} \left(1 - \frac{(1 - \beta)m_l}{\beta \pi(m_l)}\right)^2 (\pi(0) - \pi(m_l)).
\]

Corollary 3 (i) follows from the comparison of the laggard and leader strategies, (ii) is direct from the definition of \( F_l \) and \( F_L \). To understand (iii) note that the statement about the mass points is direct and when both firms play at their mass points, the difference between the bids is equal to \( \pi(0) - \pi(m_l) \).

Corollary 3 (ii)-(iv) captures the second main result. Point (ii) suggests why the dispersion of markups, \( \frac{\min\{b_1, b_2\} - c}{c} = \frac{\pi(\max\{w_1, w_2\})}{c} \), and “money left on the table”, \( \frac{|b_1 - b_2|}{c} = \frac{|\pi(w_1) - \pi(w_2)|}{c} \), observed across auctions can be explained by variations in the laggard’s cash. Note that a similar argument also applies with respect to changes in \( \pi \). Corollary 4 below shows that this result persist even as the cost of working capital becomes negligible. Points (iii)-(iv) cast doubts about the usual interpretation of the dispersion of markups and “money left on the table” as indicative of incomplete information. For instance, in the linear specification of Appendix C an application of (iii) means that the probability that
both firms play at their mass points is at least \(1 - \frac{m_i}{\beta} \) and the expected “money left on the table” at least \((1 - \frac{m_i}{\beta}) \frac{m_i}{c}\) as the cost of working capital vanishes. Thus, a sufficiently large \(\theta\) implies almost no uncertainty together with sizable “money left on the table.”

Here, the laggard’s cash is exogenous but in the model of Section 4 we show, by means of numerical simulations, that the endogenous distribution of the laggard’s cash has sufficient variability to generate significant dispersion of markups and “money left on the table” across otherwise identical auctions. Interestingly, these results are provided for parameter values for which there is little uncertainty and the cost of working capital is small.

**Corollary 4.** If \(m_i < \overline{\beta}\) and \(m_1 \neq m_2\), then as \(\beta\) increases to 1, (i) \(F_\beta(w)\) converges to \(\chi_{m_i}(w)\), (ii) the equilibrium probability that the winner is the leader converges to \(1\), (iii) in equilibrium, \(\pi(\max\{w_1, w_2\})\) converges in distribution to \(\chi_{\pi(m_i)}\), and (iv) the equilibrium expectation of \(|\pi(w_1) - \pi(w_2)|\) converges to \(^{34}\pi(m_i)\left(\ln\left(\frac{\pi(0)}{\pi(m_i)}\right)\right).\)

The corollary follows by inspection of the laggard and leader strategies. Intuitively, as the cost of working capital becomes negligible, the leader finds it optimal to ensure that it wins by choosing the maximum working capital feasible for the laggard. Thus, the laggard has no chance of winning and it is indifferent between any feasible working capital in the limit. Its randomization makes sure that the leader does not have incentives to deviate.

Corollary 4 (iii)-(iv) imply that when \(\beta\) is close to one and \(m_i < \hat{m}\), where\(^{35}\hat{m} \equiv \pi^{-1}(\frac{\pi(0)}{c})\), the markup, \(\frac{\min(b_1, b_2) - c}{c} = \frac{\pi(\max\{w_1, w_2\})}{c}\), decreases and “money left on the table”, \(\frac{|b_1 - b_2|}{c} = \frac{|\pi(w_1) - \pi(w_2)|}{c}\), increases as the laggard’s cash \(m_i\) increases. This is the basis for our third main result. Suppose that \(\beta\) is close to one and that the bid data from several auctions with identical financial constraints are generated by the model with

\(^{34}\)Proving (iv) requires some non-trivial computations. \(F_\beta\) converges to a distribution with an atom of probability \(\frac{\pi(m_i)}{\pi(0)}\) at zero and a density \(-\pi'(w)\frac{\pi(m_i)}{\pi(w)^2}\) for \(w \in (0, m_i]\). This together with (i) implies that the expectation of \(|b_1 - b_2| = \pi(\min\{w_1, w_2\}) - \pi(\max\{w_1, w_2\})\) converges to:

\[
\pi(0)\frac{\pi(m_i)}{\pi(0)} + \int_0^{m_i} \pi(w) \left(-\pi'(w)\frac{\pi(m_i)}{\pi(w)^2}\right) dw - \pi(m_i) = \pi(m_i) \int_0^{m_i} \left(-\pi'(w)\frac{\pi(m_i)}{\pi(w)}\right) dw = \pi(m_i) \left(\ln\left(\frac{\pi(0)}{\pi(m_i)}\right)\right).
\]

\(^{35}\)Since \(\frac{\partial}{\partial m} \left(\pi(m)\left(\ln\left(\frac{\pi(0)}{\pi(m)}\right)\right)\right) > 0 \text{ if } m < \hat{m}.

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constant procurement cost \( c \). If \( m_l < \nu^\beta \), then Corollary 4 states that there will be almost no “money left on the table” and there will be large markups when \( m_l \) is close to zero but that there will be a lot of “money left on the table” and small markups when \( m_l = \hat{m} \). In what follows we assume that \( m_l < \nu^\beta \). The bid data reveals the “money left on the table” but costs and, therefore, markups are not observable. If there is little “money left on the table”, as would happen if \( m_l \) is close to zero, an interpretation of the bid data using the standard model would conclude that there was little cost heterogeneity and small markups even though there were large markups in the generated data. That is, the results would be biased downward. If there is substantial amount of money on the table, as would happen if \( m_l = \hat{m} \), then an interpretation of the bid data using the standard model would conclude that there was large cost heterogeneity and therefore large markups even though there were small markups in the generated data. That is, the results on markups would be biased upwards.

4 The Dynamic Model

In this section, we endogenise the distribution of cash by assuming that it is derived from the past market outcomes. This approach provides a natural framework to analyse the conventional wisdom in economics that “auctions [still] work well if raising cash for bids is easy.” We provide conditions under which the laggard-leader scenario occurs most of the time. This is the basis for our first main result. Besides, we provide formal and numerical results that, on the one hand, complement the previous section analysis of the second main result and, on the other hand, shed some light on the concentration and persistency of market shares, i.e. our fourth main result.

4.1 The Game

We consider the infinite repetition of the time structure of the game in the last section. We assume that both firms have the same amount of cash in the first period. Afterwards each firm’s cash is equal to its working capital in the previous period plus the profits in the procurement contract and some exogenous cash flow \( m > 0 \).\textsuperscript{36} We assume that \( m > 0 \).

\textsuperscript{36}The equilibrium when \( m = 0 \) has the same structure but the dynamics of the laggards cash differs because zero becomes an absorbing stage. Before that, the dynamics of prices and market shares display
is constant across time and firms, and interpret it as derived from other activities of the firm. Hence, in any period \( t \) in which firms start with cash \((m_{1,t}, m_{2,t})\), choose working capitals \((w_{1,t}, w_{2,t})\) and bids \((b_{1,t}, b_{2,t})\), and Firm 1 wins the procurement contract with profits \(b_{1,t} - c\) the next period distribution of cash is equal to:

\[
(m_{1,t+1}, m_{2,t+1}) = (w_{1,t} + b_{1,t} - c + m, w_{2,t} + m).
\]

Firm \( i \) wins in period \( t \) with probability one if \( b_{i,t} < b_{j,t} \) or if \( b_{i,t} = b_{j,t} \) and \( m_{i,t} > m_{j,t} \), with probability 1/2 if \( b_{i,t} = b_{j,t} \) and \( m_{i,t} = m_{j,t} \), and loses otherwise. The payoff in period \( t \) of a firm with cash \( m_t \) that chooses working \( w_t \) is equal to its consumption \( m_t - w_t \). The firm’s lifetime payoff in a subgame beginning at period \( \tau \) if its cash and working capital in the subsequent periods are given by \( \{m_t, w_t\}_{t=\tau}^{\infty} \) is equal to:

\[
\sum_{t=\tau}^{\infty} \beta^{t-\tau} (m_t - w_t).
\]

We assume that the firm maximizes its expected lifetime payoff at any period \( \tau \).

The following assumption is used in the proof of Proposition 3.

**Assumption 1.** \( \pi(w) \geq \theta - m - w \) for any \( w \in \mathbb{R}_+ \). \(^{37}\)

Since \( \pi(w) \) is the minimum profit that a firm with working capital \( w \) can make when it wins the procurement contract, this assumption implies that the firm that wins the procurement contract one period, starts next period with cash at least \( \theta \). As we explain after Proposition 3, this assumption guarantees that firms do not want to carry more working capital than strictly necessary to make the bid acceptable. Besides, it implies that in any information set after the first period in which firms have the same amount of cash, this is greater than \( \theta \). \(^{38}\) We show in Appendix B, for the case of the static model, that a tedious case differentiation is necessary if one allows firms to have identical cash less than \( \theta \). For the same reason, we assume that both firms start in the first period with the same features as in the model with \( m > 0 \). In that sense, our results extend to the case where \( m = 0 \).

\(^{37}\)For instance, the function \( \pi(w) = \theta - w \) derived in Appendix C satisfies this assumption.

\(^{38}\)Beker and Hernando-Veciana (2011) analysis a variation of the model of this section where information sets in which firms may have identical cash and less than \( \theta \) exist because they do not impose Assumption 1. The equilibrium dynamics is similar to ours as there is zero probability that the firms have identical cash along the equilibrium path because of the mixed strategies they play.
We denote by $\Omega$ the set of cash vectors that may arise in the information sets of the game tree. A Markov mixed strategy consists of a randomization over the set of working capitals and acceptable bids for each point $(m, m')$ in $\Omega$, where $m$ denotes the firm’s cash and $m'$ the rival’s. We shall restrict to equilibria in Markov mixed strategies with support in the set $\{(b, w) : b = \tilde{b}(w|m, m'), w \in [0, m]\}$ for some function $\tilde{b}(\cdot|m, m') : [0, m] \to \mathbb{R}$ that satisfies that $\tilde{b}(w|m, m') \geq \pi(w) + c$ for any $w \in [0, m]$. This Markov mixed strategy can be described by its marginal distribution function $\sigma(\cdot|m, m')$ over working capitals and the bid function $\tilde{b}(\cdot|m, m')$.

We let the value function $W(m, m')$ denote the lifetime payoff of a firm that has cash $m$ when its rival has $m'$.

**Definition 4.** A (symmetric) Bidding and Investment (BI) equilibrium\(^{40}\) is a value function $W$, a working capital distribution $\sigma$ and a bid function $b$ such that for every $(m, m') \in \Omega$, $W$ is the value function and $\sigma(\cdot|m, m')$ and $b(\cdot|m, m')$ are the optimisers of the right hand side of the following Bellman equation:

$$W(m, m') = \max_{\tilde{\sigma} \in \Delta(m)} \int \left[ m - w + \beta \int \tilde{W}_{k,b}(w, w', m, m') \sigma(dw') | m', m \right] \tilde{\sigma}(dw),$$

where $\Delta(m)$ denotes the set of distributions with support in $[0, m]$ and $\tilde{W}_{k,b}(w, w', m, m')$ is equal to:

- If either $\tilde{b}(w|m, m') < b(w'|m', m)$, or $\tilde{b}(w|m, m') = b(w'|m', m)$ and $m > m'$:
  $$W(w + m + \tilde{b}(w|m, m') - c, w' + m).$$

- If either $\tilde{b}(w|m, m') > b(w'|m', m)$, or $\tilde{b}(w|m, m') = b(w'|m', m)$ and $m < m'$:
  $$W(w + m, w' + m + b(w'|m', m) - c).$$

- If $\tilde{b}(w|m, m') = b(w'|m', m)$ and $m = m'$:
  $$\frac{1}{2}W(w + m + \tilde{b}(w|m, m') - c, w' + m) + \frac{1}{2}W(w + m, w' + m + b(w'|m', m) - c).$$

\(^{39}\)In this sense, our result that firms carry too little cash in the long term arises even when firms start with sufficiently large amounts of cash.

\(^{40}\)In a version of our model with finitely many periods studied in the supplementary material there is a unique equilibrium that is symmetric. This equilibrium is a natural extension of our equilibrium.
4.2 The Equilibrium Strategies

In what follows, we define a value function, a bid function and a working capital distribution and show that they are a BI equilibrium. The strategies we propose correspond to a generalization of the equilibrium strategies in Section 3. The bid function is, as in the static model, the minimum acceptable bid (with a slight abuse of notation):

\[ b^*(w|m, m') \equiv \pi(w) + c. \]

To define the working capital distribution, we use some auxiliary functions. The set \( \mathcal{P} \) is the set of continuous and weakly decreasing functions \( \Psi : \mathbb{R}_+ \to [0, \frac{\beta}{1-\beta} \pi(0)] \) that satisfy \( \Psi(\theta) = 0 \). Let \( x \) be the unique solution in \[ (\theta - m)^+, \theta \] to:

\[ \frac{1 - \beta}{\beta} \frac{(x - (\theta - m)^+)}{\pi(x)} = 1. \]

Lemma 2 in the Appendix characterizes the properties of the solutions to:

\[ \frac{1 - \beta}{\beta} = F'(w)(\pi(w) + \Psi(w + m)) + F(w)\pi'(w), \tag{2} \]

for any \( \Psi \in \mathcal{P} \). Each point in \([0, \pi] \times [0, 1]\) is associated with the graph of a unique continuous solution to Equation (2) that passes by that point.

**Definition 5.** We denote by \( \hat{\nu}^\Psi \) the unique value of \( x \) such that \((x, 1)\) belongs to the graph of the same continuous solution to Equation (2) that contains \((0, 0)\).

Lemma 2 in the Appendix shows that \( \hat{\nu}^\Psi \in [\nu^\beta, \pi] \). We let \( \hat{F}(\cdot, \Psi, m) \) be a distribution function that agrees in \([0, \min\{m, \hat{\nu}^\Psi\}]\) with that solution to Equation (2) whose graph includes \((\min\{m, \hat{\nu}^\Psi\}, 1)\).

Consider the following functional equation:

\[ \Psi(m) = \beta \hat{F}(0, \Psi, m)(\pi(0) + \Psi(m)) \tag{3} \]

Lemma 3 in the Appendix applies Schauder Fixed-Point Theorem to show that this functional equation has a solution in \( \mathcal{P} \) that we denote by \( \Psi^\beta \).

Let \( \nu^\beta = \hat{\nu}^{\Psi^\beta} \). For \( m \geq \nu^\beta \), let \( F_{l,m}^\beta(w) \) and \( F_{L,m}^\beta(w) \) be both equal to \( \hat{F}(w, \Psi^\beta, \nu^\beta) \), and for \( m \in [0, \nu^\beta) \), let \( F_{l,m}^\beta(w) \) be equal to \( \hat{F}(w, \Psi^\beta, m) \) and,

\[ F_{L,m}^\beta(w) = \begin{cases} \hat{F}(w, \Psi^\beta, \nu^\beta) & \text{if } w \in [0, m) \\ 1 & \text{if } w \geq m. \end{cases} \]

\[^{41}\text{We adopt the convention that } a^+ = a \text{ if } a \geq 0 \text{ and } a^+ = 0 \text{ otherwise.}\]
For any \((m, m') \in \Omega\) let:

\[
\sigma^*(w|m, m') \equiv \begin{cases} 
F_{\beta l,m}(w) & \text{if } m \leq m' \\
F_{\beta L,m'}(w) & \text{if } m > m', 
\end{cases}
\]

and:

\[
W^*(m, m') \equiv \begin{cases} 
m + \frac{\beta}{1-\beta} m & \text{if } m \leq m' \\
\frac{\beta}{1-\beta} m + \Psi^\beta(m') & \text{if } m > m'. 
\end{cases}
\]

Thus, \(\Psi^\beta(m')\) is an additive premium associated to being leader.

Note that \(m = m'\) and \((m, m') \in \Omega\) imply, because of our Assumption 1, that \(m = m' \geq \theta\) and hence \(F_{\beta l,m}(w) = F_{\beta L,m'}(w)\) and \(\Psi^\beta(m') = 0\).

**Proposition 3.** \((W^*, \sigma^*, b^*)\) is a BI equilibrium.

The intuition behind the proposition is based on our results in the static model. There we use the property that the game is equivalent to a reduced game with a one-dimensional strategy space and with a payoff function similar to an all-pay auction. This argument also applies here because this property is inherited from one period to the previous one in the following sense: if the payoffs of the reduced game in period \(t\) satisfy the property, the payoffs of the reduced game in period \(t - 1\) also satisfy it. To see why, note that the usual result of all-pay auctions that bidders without competitive advantage get their outside opportunity implies that the laggard’s continuation payoffs in period \(t - 1\) are equal to the discounted consumption value of its cash in period \(t\). The leader’s continuation payoffs have an additive premium which is a consequence of the leader’s ability to carry sufficient working capital to undercut any acceptable bid of the laggard. This ability is independent of the amount of cash the leader has and so it is the premium. Consequently, the value of a marginal increase in cash in period \(t\) is equal to its discounted consumption value plus the value of switching from laggard to leader. Note that a marginal increase in cash switches the leadership only when the cash of the firms is the same and, by Assumption 1, no less than \(\theta\). In this case, the premium is zero by definition of \(\Psi^\beta\). We can thus conclude that a unit increase in working capital, keeping constant the bid, is costly in the sense that it reduces the current consumption in one unit but only increases the future utility in its discounted value \(\beta\). This means, as in the static model, that it is not profitable to carry
more working capital than strictly necessary to make the bid acceptable, and that in the corresponding unidimensional simplification of the strategy space, the firm that carries more working capital wins but carrying working capital is costly for both firms.\footnote{Note that the property that firms do not want to carry more working capital than strictly necessary to make the bid acceptable is also a property of the unique equilibrium of the finite version of our model. This is because the recursive argument in the previous paragraph can be applied starting from the last period since the last period is the same game as the static model. A formal argument is provided in the supplementary material.}

We can also distinguish here between a symmetric and a laggard-leader scenarios and it may be shown that an analogous version of Corollaries 1-4 holds true as well.

### 4.3 The Equilibrium Dynamics

To study the frequency of the symmetric and the laggard-leader scenarios, we study the stochastic process of the laggard’s cash induced by our equilibrium. Its state space is equal to $[m, \nu^\beta + m]$ because the procurement profits are non negative and none of the firms’ working capitals is larger than $\nu^\beta$. Moreover, the laggard’s cash is determined by the equilibrium working capitals and bids in the previous period. The latter are a function of the former which have a distribution that only depends on the laggard’s cash in the previous period. Thus, the laggard’s cash follows a Markov process. Its transition probabilities $Q^\beta : [m, \nu^\beta + m] \times \mathcal{B} \to [0, 1]$, for $\mathcal{B}$ the Borel sets of $[m, \nu^\beta + m]$, can be easily deduced from the equilibrium. In particular, they are defined by:\footnote{As a convention, we denote by $[m, m]$ the singleton $\{m\}$.}

$$Q^\beta (m, \left[ m, x \right]) = 1 - \left( 1 - F_{L,m}^\beta (x - m) \right) \left( 1 - F_{L,m}^\beta (x - m) \right).$$

(5)

This expression is equal to one minus the probability that both the laggard and the leader’s working capitals are strictly larger than $x - m$.

**Definition 6.** A distribution $\mu : \mathcal{B} \to [0, 1]$ is invariant if it verifies:

$$\mu (\mathcal{M}) = \int Q^\beta (m, \mathcal{M}) \mu (dm) \quad \text{for all } \mathcal{M} \in \mathcal{B}.$$

(6)

Standard arguments\footnote{See Hopenhayn and Prescott (1992).} can be used to show that there exists a unique invariant distribution and it is globally stable. A suitable law of large numbers can be applied to show...
that the fraction of the time that the Markov process spends on any set $M \in B$ converges (almost surely) to $\mu(M)$.

Typically, the frequency of each scenario depends on a non trivial way on the transition probabilities. An exception is when the transition probabilities do not depend on the state which corresponds in our model to an exogenous cash flow sufficiently large to guarantee that only the symmetric scenario occurs. Since $\nu^\beta \leq \theta$ for any $\beta < 1$, a sufficient condition is that $\frac{\theta}{m} < 1$. In this case, the transition probabilities do not dependent on the laggard’s current working capital and

$$
\mu^\beta([m, x)) = 1 - \left(1 - \hat{F}(x - m, \Psi^\beta, \nu^\beta)\right)^2.
$$

A version of Corollaries 1 and 2 characterizes the properties of the equilibrium path.\footnote{In the more difficult case in which the transition probabilities depend on the state, the invariant distribution associated to the limit transition probabilities as the cost of working capital becomes negligible, i.e. $\beta$ increases to one, has an easy characterization. This is because the transition probabilities become degenerate and concentrate its probability in one point only, either $m$ or $\theta + m$, and thus any distribution with support in $\{m, \theta + m\}$ is an invariant distribution. Since there are multiple invariant distributions, we cannot apply a continuity argument to characterize what happens when the cost of working capital is small.}

In general, the frequency of each scenario depends on the ratio $\frac{\theta}{m}$:

**Definition 7.** We call $\frac{\theta}{m}$ the severity of the financial constraint.

We also use the following definition to discuss our results:

**Definition 8.** We call the extreme laggard-leader scenario when the laggard’s cash is $m$.

In Figure 1, we plot the frequency of the symmetric scenario and the frequency of the extreme laggard-leader scenario.

Note that the sum of the probability that $\mu^\beta$ puts on the extreme laggard-leader scenario and on the symmetric scenario is close to one for $\beta$ close to one. This is because, as we show in Lemma 6 in the Appendix, the probability that the stationary distribution puts outside these sets tends to zero as the cost of working capital becomes negligible.

The next theorem is the basis for our first main result.
Figure 1: $\mu^\beta(m)$ and $\mu^\beta([\theta, \theta + m])$ as a function of $\frac{\theta}{m}$ for $\pi(x) = \theta - x$ and $\beta = 0.96$.

**Theorem 1.** If $\frac{\theta}{m} > 4$ and $\lim_{\beta \to 1} \Psi^\beta(m) = \infty$, then $\lim_{\beta \to 1} \mu^\beta(m) = 1$.

This theorem says that when the financial constraint is sufficiently severe, then the extreme laggard-leader scenario occurs most of the time as the cost of working capital becomes negligible.

Next lemma gives a sufficient condition for $\lim_{\beta \to 1} \Psi^\beta(m) = \infty$. Interestingly, the function $\pi$ consistent with our model in Appendix C satisfies this sufficient condition when $\frac{\theta}{m} > 4$.

**Lemma 1.** If $\pi(2m) + \pi(m) > \pi(0)$, then $\lim_{\beta \to 1} \Psi^\beta(\{m\}) = \infty$.

The next corollary can be derived from Theorem 1 and the property of the extreme laggard-leader scenario, proved in Lemma 5 in the Appendix, that the laggard and the leader play with probability arbitrarily close to one at theirs atoms when $\beta$ increases to one.

**Corollary 5.** If $\frac{\theta}{m} > 4$ and $\lim_{\beta \to 1} \Psi^\beta(m) = \infty$, the fraction of the time the following properties hold in the equilibrium converges to one (almost surely) as $\beta$ increases to one: (i) the leader wins the procurement contract; (ii) the laggard chooses zero working capital and the leader’s working capital is arbitrarily close to $m$; (iii) $\pi(\max\{w_{1,t}, w_{2,t}\})$ is arbitrarily close to $\pi(m)$, and (iv) $|\pi( w_{1,t}) - \pi( w_{2,t})|$ is arbitrarily close to $\pi(0) - \pi(m)$. 

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As Corollary 3 (ii)-(iii), Corollary 5 (iii)-(iv) support our second main result: the dispersion of markups $\frac{\min\{b_1, b_2\} - c}{c} = \frac{\pi(w_1, w_2)}{c}$, and “money left on the table”, $\frac{|b_1 - b_2|}{c} = \frac{|\pi(w_1) - \pi(w_2)|}{c}$, can be explained by variations in the exogenous cash flow $m$ and the function $\pi$. Similarly, Corollary 5 (ii) and (iv) cast doubts, as Corollary 3 (iii) and (iv), about the usual interpretation of the dispersion of markups and money left on the table as indicative of incomplete information.

4.4 Numerical Solutions

In this section, we compute numerically the invariant distribution for empirically grounded values of the parameters and $\pi(w) = \theta - w$, motivated by our model in Appendix C. Since $\pi$ and, hence $\mu$, are independent of $c$, any measure of markups provided is arbitrary unless we provide a relation of $c$ with the rest of the variables of the model. We assume that $\frac{\theta}{c}$ is constant which is an implication of our analysis in Appendix C.

The left and right panel of Figure 2 show how the expected and standard deviations, respectively, of the markup and the “money left on the table,” change with the severity of the financial constraint.

Figure 2: Expected and st. deviations of $\frac{\min\{b_1, b_2\} - c}{c}$ and $\frac{|b_1 - b_2|}{c}$ for $\frac{\theta}{c} = 1$.

Figure 2 illustrates our second main result. The left panel quantifies our results in

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46Early work of Hong and Shum (2002) suggests that firms that do highway maintenance typically bid in 4 contracts per year:“a data set of bids submitted in procurement contract auctions conducted by the NJDOT in the years 1989-1997 […] firms which are awarded at least one contract bid in an average of 29-43 auctions.” See also Porter and Zona (1993). Thus, we compute annual market shares for years of four periods. We also choose $\beta = 0.9602$ so that the annual discount rate is 0.85, slightly higher than the 0.80 used in Jofre-Bonet and Pesendorfer (2003), implying an annual expected cost of working capital of 0.15.

47It may be shown that one can obtain the graph corresponding to different values of $\frac{\theta}{c}$ simply by multiplying the values in the vertical axis in Figure 2 by $\frac{\theta}{c}$. 
Corollary 3(ii) in the context of our dynamic model. It shows that exogenous differences in the severity of the financial constraint explain differences in markups and “money left on the table.” The right panel shows that markups and “money left on the table” have significant volatility across auctions, for a given ratio $\frac{\theta}{m}$, due to the endogenous volatility of the firm’s working capital and cash. Note that, by continuity, Corollary 5(ii) implies that the uncertainty each firm has about the others working capital vanishes in equilibrium as the ratio $\frac{\theta}{m}$ approaches 4. Indeed, this is also true for any value of the ratio $\frac{\theta}{m}$ greater than 2. This can deduced from Lemma 5 in Appendix A since it implies that the fraction of the time in which the firms choose a working of either zero or close to the minimum between the laggard’s cash and $\theta$ converges to one (almost surely) as $\beta$ increases to one.

Finally, we illustrate our fourth main result in Figure 3.

![Graph 1](image1.png) ![Graph 2](image2.png)

Figure 3: HHI and invariant distribution of annual market shares (1 year= 4 periods)

The left panel shows how the Herfindahl-Hirschman index (HHI) changes with the severity of the financial constraint. The firms internal financial decisions imply that concentration increases with the severity of the financial constraint. The right panel shows that this effect appears together with persistent asymmetries in market shares. It shows that in the case of financial constraints sufficiently severe the same firm wins all the annual procurement contracts 98.92% of the years. On the contrary, if the severity of the financial constraint is so small that the symmetric scenario occurs all the periods there is little concentration in that each firms wins at least 25% of the annual procurement contracts in 87% of the years.

Similar conclusions can be derived with respect to the persistency of the leadership. Figure 4 shows how the probability of a leadership reversal after 22 years changes with the severity of the financial constraint.\(^{48}\)

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\(^{48}\)This time horizon allows the comparison of the predictions of our model with the stylized fact about unchanging industry leadership highlighted by Besanko and Doraszelski (2004).
Therefore, a typical time series of market shares displays not only a lot of concentration but also, and more importantly, tipping as in Cabral (2011), i.e. the system is very persistent but moves across extremely asymmetric states.

The above results explain our prediction of greater market concentration and asymmetric market shares, together with occasional leadership reversals, for larger projects to the extent that one can associate the severity of the financial constraint to the project’s size, as we show in Appendix C.

We underscore that the concentration effects discussed in this section arise even in the absence of exogenous shocks unlike those obtained by Besanko and Doraszelski (2004), and Besanko, Doraszelski, Kryukov, and Satterthwaite (2010).

5 Conclusions

We have studied a model of bidding markets with financial constraints. A key element of our analysis is that the stage at which firms choose their working capitals resembles an all pay auction with caps.

The above features, and thus our results, seem pertinent for other models of investing under winner-take-all competition, like patent races. A natural extension should consider alternative models of winner-take-all competition with financial constraints. Another interesting extension is to allow for private information about costs. This is the natural framework to test our model versus the standard one as it nests both of them. Existing
results for all pay auctions and general contests\textsuperscript{49} suggest these may be fruitful lines of future research.

Finally, our analysis points out a tractable way to incorporate the dynamics of liquidity in Galenianos and Kircher’s (2008) analysis of monetary policy.

Appendix

A Proofs

Proof of Proposition 1

To see why the proposed strategy is an equilibrium note that the expected payoff of Firm $i$ with cash $m_i$ when it chooses working capital $w$ and the other firm randomizes its working capital according to $F^*$ is equal to:

$$m_i - (1 - \beta)w + \beta \pi(w) F^*(w),$$

which by definition of $F^*$ is equal to $m_i$ for any $w \leq \nu \beta$, and strictly less than $m_i$ otherwise, as required.

To prove that there is no other equilibrium, note that, as we argue in the text, we can restrict to mixed strategies $(b^*, F_i)$ in which $F_i$ has support in $[0, \nu \beta]$. We prove two properties that any equilibrium $((b^*, F_1), (b^*, F_2))$ must satisfy. Later, we show that the proposed strategy is the only one that satisfies them. These two properties also hold true in the more general case in which we do not restrict to $\min\{m_1, m_2\} \geq \nu \beta$. To simplify the notation we denote by $\sigma_i(A)$, $i = 1, 2$ and $A \subset \mathbb{R}_+$ the probability that $F_i$ puts on $A$, i.e. $\sigma_i(A) = \int_A F_i(dx)$.

**Claim 1:** If $w \in (0, \nu \beta]$ belongs to the support of $\sigma_i$, then $\sigma_j([w - \epsilon, w]) > 0$ ($j \neq i$) for any $\epsilon > 0$.

\textsuperscript{49}Amann and Leininger (1996) study the relationship between the equilibrium of the all pay auction with and without private information and Alcalde and Dahm (2010) study the similarities between the equilibrium outcome in an all pay auction and in some other models of contests.
In order to get a contradiction suppose that \( w \in (0, \nu \beta] \) belongs to the support of \( \sigma_i \) and \( \sigma_j([w - \epsilon, w]) = 0 \) for some \( \epsilon > 0 \). We shall argue that Firm \( i \) has a profitable deviation when Firm \( j \) plays \((b^*, F_j)\). The contradiction hypothesis has two implications. (a) \( w - \epsilon \) gives Firm \( i \) strictly greater expected payoffs than \( w \) since the former saves on the cost of working capital and increases the profit when winning without affecting the probability of winning. (b) Firm \( i \)'s expected payoffs are continuous in its working capital at \( w \) since \( \sigma_j \) does not put an atom at \( w \). (a) and (b) mean that there exists an \( \epsilon' \in (0, \epsilon) \) such that any working capital in \((w - \epsilon', w + \epsilon')\) gives strictly less expected payoffs than a working capital \( w - \epsilon \). The fact that \( w \) belongs to the support of \( \sigma_i \) means that \( i \) puts strictly positive probability in \((w - \epsilon', w + \epsilon')\) and thus there is a profitable deviation.

**Claim 2:** If there exists some \( w \in [0, \min\{\nu^3, m_j\}) \) such that \( \sigma_j([w - \epsilon, w]) > 0 \) for any \( \epsilon > 0 \), then \( \sigma_i(\{w\}) = 0 \) \((i \neq j)\).

By contradiction, suppose an \( w \in [0, \min\{\nu^3, m_j\}) \) for which \( \sigma_j([w - \epsilon, w]) > 0 \) for any \( \epsilon > 0 \) and \( \sigma_i(\{w\}) > 0 \). For \( \epsilon' > 0 \) small enough, Firm \( j \) can improve by moving the probability that \( \sigma_j \) puts in \([w - \epsilon', w]\), to a point slightly above \( w \). This deviation affects marginally Firm \( j \)'s cost of working capital and profit conditional on winning but allows the firm to win the procurement contract at a strictly positive profit if Firm \( i \) plays the atom at \( w \).

Claim 1 and 2 imply that (i) the only points where there can be a mass point in the strategies is at zero or at \( \nu^3 \), (ii) at most one of the firms' strategies can have an atom at zero, and (iii) the support of both \( \sigma_1 \) and \( \sigma_2 \) must be the same and equal to an interval \([0, \nu]\) for some \( \nu > 0 \). Conditions (i)-(iii) and the usual indifference condition that must hold in a mixed strategy equilibrium implies: (iv) that the distributions of each of the firms, \( \sigma_1 \) and \( \sigma_2 \), is equal to a continuous solution of Equation (1) in \((0, \nu)\). Suppose, first, that there is no atom at \( \nu^3 \). The uniqueness of the solution of our differential equation, see Theorem 7.1 in Coddington and Levinson (1984), pag. 22, implies that there is only one solution that passes by the point \((\nu, 1)\) for each \( \nu \in [0, \nu^3] \). Thus, both firms must use in equilibrium the same distribution. This together with (iv) means that this distribution
must be the solution to our differential equation with initial condition \((0,0)\). This is our proposed distribution function. To conclude the proof we argue by contradiction that there is no atom at \(\nu^\beta\). Suppose there is an atom of probability \(\gamma > 0\) in one of the distributions. By (iv), this distribution must be a solution to our differential equation that passes by \((\nu^\beta, 1-\gamma)\). Since the solution with initial condition \((0,0)\) passes by \((\nu^\beta, 1)\), the uniqueness of the solutions of our differential equation implies that the solution that passes by \((\nu^\beta, 1-\gamma)\) must cross the horizontal axis strictly to the right of \((0,0)\) which is a contradiction with (iv). \(\blacksquare\)

**Proof of Proposition 2**

We start showing that the proposed strategies are an equilibrium. Let \(m_L\) denote the leader’s cash. The laggard’s expected payoffs of a working capital \(w \in [0, m_L]\) when the leader plays \(F_L\) are equal to:

\[m_L - (1 - \beta)w + \beta\pi(w)F_L(w),\]

hence equal to \(m_L\) by definition of \(F_L\). Recall that our tie breaking rule allocates the contract to the leader when both firms carry working capital \(m_L\) which guarantees that payoffs are continuous at \(w = m_L\), and hence also equal to \(m_L\). Consequently, the laggard has no incentive to deviate. Similarly, the leader’s expected payoffs of a working capital \(w \in [0, m_L]\) when the laggard plays \(F_l\) are equal to:

\[m_L - (1 - \beta)w + \beta\pi(w)F_l(w),\]

hence equal to \(m_L - (1 - \beta)m_L + \beta\pi(m_L)\) if \(w \in [0, m_l]\) and equal to \(m_L - (1 - \beta)w + \beta\pi(w) < m_L - (1 - \beta)m_L + \beta\pi(m_L)\) if \(w \in (m_l, m_L]\). Hence there are no incentives to deviate. Note that we are using that our tie breaking rule allocates the good to the leader when both firms choose zero working capital.

To prove that there is no other equilibrium, we use that Claims 1 and 2 in the proof of Proposition 1 also hold true here. We show that the proposed strategy is the only strategy profile that satisfies both claims. Suppose two distributions that satisfy Claims 1 and 2 in the proof of Proposition 1. They imply here that (i) the laggard’s strategy can have a probability mass only at zero and the leader’s only at either zero or \(m_L\), (ii) at
most one of the firms’ strategies can have an atom at zero, and (iii) the support of both distributions must equal a common interval \([0, \nu]\) for some \(\nu \in [0, m_l]\). (i)-(iii) and the usual indifference condition that must hold in a mixed strategy equilibrium implies: (iv) that the distributions of each of the firms is equal to a continuous solution of Equation (1) in \((0, \nu)\). Since the solutions to Equation (1) do not cross in \([0, m_l]\) and the solution with initial condition \((0, 0)\) reaches one at \(\bar{\nu} > m_l\), both firm’s strategies must have atoms. (i) implies that the laggard’s atom is at zero and (ii) that the leader’s is at \(m_l\). These facts and (iv) imply that the proposed strategy profile is the unique equilibrium. \(\blacksquare\)

**Auxiliary Results of Section 4.2**

**Lemma 2.** For any \(\Psi, \hat{\Psi} \in \mathcal{P}\),

(i) For any \((x_0, y_0) \in [0, \bar{\nu}] \times [0, 1]\), there is a unique continuous solution to the differential equation (2) for each initial condition \((x_0, y_0)\). All these solutions are locally increasing.

(ii) For any solution \(F\) to the differential equation (2) and \(w > w'\):

\[
F(w)\pi(w) - F(w')\pi(w') \leq \frac{1 - \beta}{\beta}(w - w'),
\]

with equality if \(w, w' \in [\theta - m, \bar{\nu}]\).

(iii) \(\hat{\nu}^\Psi \in [\bar{\nu}, \bar{\nu}]\).

(iv) \(\Psi \leq \hat{\Psi}\) implies \(\hat{F}(0, \Psi, m) \leq \hat{F}(0, \hat{\Psi}, m)\).

(v) \(\hat{F}(y, \Psi, m)\) is continuous and weakly decreasing in \(m\), strictly if \(y < m < \hat{\nu}^\Psi\).

**Proof.** Note that Equation (2) can be written as:

\[
F'(w) = \frac{1 - \beta}{\pi(w) + \Psi(w + m)} + (-\pi'(w))F(w).
\]

The application of Theorem 7.1 in Coddington and Levinson (1984), pag. 22 to this expression implies the existence and uniqueness in (i). It also implies the continuity in (v). The monotonicity with respect to \(x\) follows from the fact that the right hand side of the
above expression is strictly positive. The inequality of (ii) can be proved integrating both sides of the following inequality implied by Equation (2):

\[ F'(w)\pi(w) + F(w)\pi'(w) \leq \frac{1 - \beta}{\beta}. \]

The equality of (ii) follows because the above inequality holds with equality when \( w \in [\theta - m, x] \) since our assumption that \( \Psi \) is continuous and weakly decreasing and satisfies \( \Psi(\theta) = 0 \) implies that \( \Psi(w + m) = 0 \). (iii) follows from the consequence of (i) that solutions are increasing and do not cross and two facts: (a) a solution with final condition \( F(\nu^\beta) = 1 \) verifies that \( F(0) \geq 0 \), and (b) a solution with final condition \( F(x) = 1 \) verifies that \( F((\theta - m)^+) = 0 \). (a) can be proved using that (ii) for \( w = \nu^\beta \) and \( w' = 0 \) implies that:

\[ F(0) \geq \frac{\pi(\nu^\beta) - \frac{1 - \beta}{\beta}\nu^\beta}{\pi(0)}, \]

and the definition of \( \nu^\beta \), whereas (b) can be shown using that (ii) for \( w = x \) and \( w' = (\theta - m)^+ \) implies that:

\[ F((\theta - m)^+) = \frac{\pi(x) - \frac{1 - \beta}{\beta}(x - (\theta - m)^+)}{\pi((\theta - m)^+)}, \]

and the definition of \( x \). (iv) is straightforward if \( \nu^{\Psi} \leq m \) because \( \nu^{\Psi} \leq m \) implies that \( \hat{F}(0, \Psi, m) = 0 \) by definition of \( \hat{F} \). To prove (iv) when \( \nu^{\Psi} > m \), we use that any given solution to Equation (7) associated to \( \Psi \) crosses at most once, and from below, any given solution to Equation (7) associated to \( \tilde{\Psi} \). This is because the right hand side of Equation (7) decreases when we increase \( \Psi(w + m) \). The former single crossing condition implies that \( \nu^{\Psi} \leq \nu^{\Psi} \). Hence, \( \nu^{\Psi} > m \) means that \( \hat{F}(m, \Psi, m) = \hat{F}(m, \Psi, m) = 1 \) which by the previous single crossing condition implies (iv). The monotonicity in (v) follows from the implication of (i) that the solutions to Equation (7) are increasing and do not cross.

**Lemma 3.** Equation (3) has a solution in \( \mathcal{P} \) that we denote by \( \Psi^\beta \).

**Proof.** Endow \( \mathcal{P} \) with the sup-norm, that we denote by \( \| \cdot \| \), and let \( T \) be an operator defined by a function that maps to each function \( \Psi \in \mathcal{P} \) a function equal to the right hand side of Equation (3).

We prove the lemma showing that the operator \( T \) meets all the conditions required by Schauder Fixed-Point Theorem, see Stokey and Lucas (1999), Theorem 17.4, page 520, in the subset \( \hat{\mathcal{P}} \subset T \) of the functions \( \Psi \) such that \( \Psi(\pi) = 0 \).
Claim 1: $T(\hat{\mathcal{P}}) \subset \hat{\mathcal{P}}$. Lemma 2(v) implies that $T(\Psi)(m)$ is continuous and decreasing in $m$. By Lemma 2(iii) $\dot{\Psi} \leq \pi$ and hence $T(\Psi)(\pi) = 0$. Finally, $\Psi(m) \leq \frac{\beta \pi(0)}{1-\beta}$ implies that $T(\Psi)(m) \leq \beta T(0, \Psi, m) \frac{\pi(0)}{1-\beta} \leq \frac{\beta}{1-\beta} \pi(0)$.

Claim 2: $T$ is continuous. Take any convergent sequence $\{\Psi_n\} \in \hat{\mathcal{P}}$ with limit $\Psi \in \hat{\mathcal{P}}$. Let $\epsilon_n \equiv \sup_{n \geq m} ||\Psi_n - \Psi||$. Since $\Psi_n \to \Psi$, $\{\epsilon_n\}$ is a decreasing sequence converging to zero. We use:

$$
\Psi_n(m) \equiv (\Psi(m) - \epsilon_n)^+.
$$

and,

$$
\Psi_n(m) \equiv \min \left\{ \Psi(m) + \epsilon_n, \frac{\beta}{1-\beta} \pi(0) \right\} \left( 1 - \frac{(m - \pi)^+}{\theta - \pi} \right).
$$

By construction $\Psi_n, \Psi_n \in \mathcal{P}$, $\Psi_n(m) \in [\Psi_n(m), \Psi_n(m)]$, $\{\Psi_n\}$ is an increasing sequence and $\{\Psi_n\}$ is a decreasing sequence, and $\Psi_n$ and $\Psi_n$ converge point-wise to $\Psi$. Thus, $\{\hat{F}(0, \Psi_n, m)\}$ and $\{\hat{F}(0, \Psi_n, m)\}$ are increasing and decreasing sequences, respectively, of continuous functions (in $m$), by Lemma 2(iv)-(v). Both sequences converge pointwise to $\hat{F}(0, \Psi, m)$, by an adaptation of Theorem 7.1 in Coddington and Levinson (1984), pag. 22.\footnote{Just note that Theorem 7.1 in Coddington and Levinson (1984), pag. 22, implies that the solution to the system of differential equations defined by:

$$
F'(w) = \frac{1-\beta}{\pi(w)} (-\rho'(w)) F(w) \frac{1}{\pi(w) \rho(w) + \rho(w)^+} \frac{\pi(0)}{1-\beta} \left( 1 - \frac{(m - \pi)^+}{\theta - \pi} \right)
$$

$$
\rho'(w) = 0,
$$

with initial conditions $F(x_0) = y_0$ and $\rho(x_0) = \epsilon$, is continuous in $\epsilon$.}

Thus, Theorem 7.13 in Rudin (1976), pag. 150, implies that the sequences of functions $\{\hat{F}(0, \Psi_n, \cdot)\}$ and $\{\hat{F}(0, \Psi_n, \cdot)\}$ convergence in the sup-norm to $F(0, \Psi, \cdot)$. This implies the convergence of $T\Psi_n$ to $T\Psi$ since $\hat{F}(0, \Psi_n, m) \in [\hat{F}(0, \Psi_n, m), \hat{F}(0, \Psi_n, m)]$ by Lemma 2(iv). This means that $T$ is continuous as desired.

Claim 3: the family $T(\hat{\mathcal{P}})$ is equicontinuous. Since $T(\Psi)(\cdot)$ is decreasing, we shall show that there exists $\kappa > 0$ such that for any $m' < m$ and $\Psi \in \mathcal{P}$, $T(\Psi)(m') - T(\Psi)(m) \leq \kappa(m - m')$. We start noting that $T(\Psi)(m') - T(\Psi)(m) = 0$ if $m', m \geq \dot{\Psi}$ (where $\dot{\Psi}$ is defined after Equation (2)), and that $T(\Psi)(m') - T(\Psi)(m) = T(\Psi)(m') - T(\Psi)(\dot{\Psi})$ if...
\( m' \leq \bar{\nu} \leq m \). Take now \( m, m' \leq \bar{\nu} \).

\[
T(\Psi)(m') - T(\Psi)(m) = \beta(\pi(0) + \Psi(m))(\hat{F}(0, \Psi, m') - \hat{F}(0, \Psi, m)) \\
\leq \frac{\beta}{1 - \beta} \pi(0)(\hat{F}(0, \Psi, m') - \hat{F}(0, \Psi, m)) \\
= \frac{\beta}{1 - \beta} \pi(0) \left( 1 - \int_{0}^{m'} \frac{1 - \beta}{\pi} + (-\pi'(y))\hat{F}(y, \Psi, m) \, dy \\
- \left(1 - \int_{0}^{m} \frac{1 - \beta}{\pi} + (-\pi'(y))\hat{F}(y, \Psi, m') \, dy \right) \right) \\
\leq \frac{\beta}{1 - \beta} \pi(0) \int_{0}^{m} \frac{1 - \beta}{\pi} + (-\pi'(y))\hat{F}(y, \Psi, m) \, dy \\
\leq \frac{\beta}{1 - \beta} \pi(0) \frac{1 - \beta + \gamma}{\pi(\bar{x})} (m - m'),
\]

where we have used: in the first step, Equation (3); in the second step, that \( \Psi(m) \leq \frac{\beta}{1 - \beta} \pi(0) \); in the third step, that \( \hat{F}(0, \Psi, m') \) and \( \hat{F}(0, \Psi, m) \) satisfy Equation (11) below and that \( \hat{F}(m', \Psi, m') = \hat{F}(m, \Psi, m) = 1 \); in the fourth step, that \( \hat{F}(y, \Psi, m) \leq \hat{F}(y, \Psi, m') \) by application of Lemma 2(v); and in the last step, that \( F(y, \Psi, m) \leq 1 \), that \( \Psi(y + m) \geq 0 \), and that \( -\pi' \) is continuous and hence bounded above in \([0, \theta]\) by some \( \gamma \geq 0 \) finite.

Hence, our \( \kappa \) is equal to \( \frac{\beta}{1 - \beta} \pi(0) \frac{1 - \beta + \gamma}{\pi(\bar{x})} > 0 \) as desired. \( \blacksquare \)

**Proof of Proposition 3**

To show that our bid function \( b^* \) solves the right hand side of the firm’s Bellman equation in Definition 4, we prove the more general argument that for our continuation value \( W^* \), and for any given bid and working capital of the rival, a working capital \( w \) and a bid \( \tilde{b} > \pi(w) + c \) does strictly worse than the same bid \( \tilde{b} \) and the minimum working capital that makes this bid acceptable, i.e. \( \tilde{w} \) such that \( \pi(\tilde{w}) + c = \tilde{b} \). The argument is the same as in the static case: reducing today’s working capital while keeping constant the bid increases today’s utility in the amount of working capital reduced while it decreases tomorrow’s continuation value in its discounted value. This is easy to deduce from the functional form of \( W^* \) when the reduction in today’s working capital (keeping constant the bid) does not change the identity of tomorrow’s leader. Otherwise, the result follows from the fact that the premium of being leader \( \Psi^\beta(m') \) is equal to zero because the change in leadership can only occur when the other firm’s cash is greater than \( \theta \). This is because
the change does not affect to the cash that the other firm has and Assumption 1 implies that the cash of at least one firm must be larger than $\theta$.

Next, we assume our continuation value $W^*$ and that both firms use $b^*$ and show that a firm cannot do better than using $\sigma^*$ when the other firm uses $\sigma^*$. We start with the symmetric scenario. In this case, the other firm’s randomization is equal to $\hat{F}(\cdot, \Psi^\beta, \nu^\beta)$, which we write $\hat{F}(\cdot)$ to simplify the notation, and one can show after substituting the value of $W^*$ and some algebra that the expected payoffs of choosing a working capital $w \in [0, m]$ are equal to:

$$m - (1 - \beta)w + \frac{\beta}{1 - \beta}m + \beta \int_0^{\min\{w, \nu^\beta\}} (\pi(w) + \Psi^\beta(\tilde{w} + m))\hat{F}'(\tilde{w})d\tilde{w}. \quad (8)$$

The differential of this expression with respect to $w$ is equal to zero for $w \in [0, \nu^\beta]$ because $\hat{F}$ verifies Equation (2), and negative for $w > \nu^\beta$. Thus, $\sigma^*$ is optimal because it randomizes in the support $[0, \nu^\beta]$.

In the laggard-leader scenario, the support of $\sigma^*$ is $[0, m]$ but otherwise, the argument is identical. In this case, we use that the expected payoffs of choosing a working capital $w \in [0, m]$ are equal to:

$$m - (1 - \beta)w + \frac{\beta}{1 - \beta}m + \beta \int_0^w (\pi(w) + \Psi^\beta(\tilde{w} + m))(F^\beta_{\nu^\beta}(\tilde{w}))'d\tilde{w}, \quad (9)$$

if $m < m'$, and to,

$$m - (1 - \beta)w + \frac{\beta}{1 - \beta}m + \beta F^\beta_{\nu^\beta}(0)\pi(w) + \Psi^\beta(m)) + \beta \int_0^{\min\{w, m\}} (\pi(w) + \Psi^\beta(\tilde{w} + m))(F^\beta_{\nu^\beta}(\tilde{w}))'d\tilde{w}, \quad (10)$$

if $m > m'$. Note that although ties could occur with positive probability the tie breaking rule always allocates the contract to the leader. We do not need to consider the case $m = m'$ because it does not belong to $\Omega$ in the laggard-leader scenario by Assumption 1.

Finally, to show that $W^*$ is the value of our Bellman equation, note the indifference condition discussed above and the fact that $W^*$ is equal to each of the Equations (8)-(10) evaluated at $w = 0$. ■
Auxiliary Results Used in the Proof of Theorem 1

In the proof of Theorem 1, we use that the solutions $F$ to Equation (2) satisfy:

$$F(w) - F(w') = \int_{w'}^w \frac{1-\beta}{\beta} + (-\pi'(y)) F(y) \, dy,$$

(11)

for $w \geq w'$. We also use that:

$$Q^\beta(m, [0, x]) = F^\beta_{l,m}(x - m) + F^\beta_{L,m}(x - m) - F^\beta_{l,m}(x - m) F^\beta_{L,m}(x - m),$$

(12)

that:

$$Q^\beta(\theta, [0, x]) = F^\beta_{L,\theta}(x - m)(2 - F^\beta_{L,\theta}(x - m)),$$

(13)

since $F^\beta_{l,\theta} = F^\beta_{L,\theta}$, and the following lemmiae.

Lemma 4.

(i) For any $m \geq \nu^\beta$ and $w \in [0, \pi]$:

$$F^\beta_{l,m}(w) = F^\beta_{L,m}(w) \leq \frac{(1 - \beta)w}{\beta \pi(w)}.$$

(ii) For any $m < \nu^\beta$, and $w \in [0, m)$:

$$F^\beta_{L,m}(w) \leq \frac{(1 - \beta)w}{\beta \pi(w)} \text{ and } F^\beta_{l,m}(w) \geq \frac{\pi(m) - \frac{1-\beta}{\beta}(m - w)}{\pi(w)},$$

with equality if $w \geq \theta - m$.

(iii) $\lim_{\beta \to 1} \nu^\beta = \theta$.

Proof. Lemma 2(ii) together with $F^\beta_{l,m}(0) = F^\beta_{L,m}(0) = 0$ for $m \geq \nu^\beta$, and $F^\beta_{l,m}(m) = 1$ $F^\beta_{L,m}(0) = 0$ for $m < \nu^\beta$ imply, respectively, (i) and (ii). To prove the last item we use Lemma 2(iii) and that $\lim_{\beta \to 1} \nu^\beta = \theta$, and that $\pi \leq \theta$.

Lemma 5. Suppose $\lim_{\beta \to 1} \Psi^\beta(m) = \infty$. 

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• If $\theta > 2m$ then:

$$\lim_{\beta \to 1} F_{l,m}^\beta(w) = \begin{cases} 
1 & \text{if } m < \theta - m \text{ and } w \in [0,m] \\
\frac{\pi(m)}{\pi(\theta - m)} & \text{if } m \in [\theta - m, \theta) \text{ and } w \in [\theta - m, m] \\
\frac{\pi(m)}{\pi(\theta - m)} & \text{if } m \in [\theta - m, \theta) \text{ and } w \in [0, \theta - m), \\
0 & \text{if } m \geq \theta \text{ and } w \in [0, \theta), 
\end{cases}$$

$$\lim_{\beta \to 1} F_{L,m}^\beta(w) = \begin{cases} 
1 & \text{if } w \geq \min\{m, \theta\} \\
0 & \text{if } w < \min\{m, \theta\}. 
\end{cases}$$

• If $\theta \geq 3m$ then:

$$\lim_{\beta \to 1} (1 - \beta) \Psi^\beta(m) = \begin{cases} 
\pi(m) & \text{if } m < \theta - m \\
\pi(m) \frac{\pi(m)}{\pi(\theta - m)} & \text{if } m \in [\theta - m, \theta). 
\end{cases}$$

Proof. Along this proof, we use that $\Psi^\beta(m)$, for $m < \theta$, also diverges to infinity. This is because: $\Psi^\beta(m) = \frac{F_{l,m}^\beta(0)}{\pi(m)} \Psi^\beta(m)$ by Equation (3); $F_{l,m}^\beta(0) \leq 1$; $F_{l,m}^\beta(0) \geq \frac{\pi(m) - \frac{1 - \beta}{2} m}{\pi(0)}$ for $m < \nu^\beta$ by Lemma 4(ii); and Lemma 4(iii).

Now, note that $F_{l,m}^\beta(m) = 1$ for $m \leq \nu^\beta$ which together with Equation (11) means that:

$$F_{l,m}^\beta(w) = 1 - \int_w^m \frac{1 - \beta + (-\pi'(y))F_{l,m}^\beta(y)}{\pi(y) + \Psi^\beta(y + m)} dy, \text{ if } m \leq \nu^\beta \text{ and } w \in [0,m].$$

Taking the limit $\beta \to 1$ we get our result for $m < \theta - m$ and $w \in [0,m]$ since the numerator is bounded and the denominator diverges.

The limit of $F_{l,m}^\beta$ when $m \in [\theta - m, \theta)$ and $w \in [\theta - m, m]$ follows directly from the fact that $F_{l,m}^\beta(m) = 1$ for $m \leq \nu^\beta$ and Lemma 4 (ii) and (iii).

This last result and the implication of Equation (11) that,

$$F_{l,m}^\beta(w) = F_{l,m}^\beta(\theta - m) - \int_w^{\theta - m} \frac{1 - \beta + (-\pi'(y))F_{l,m}^\beta(y)}{\pi(y) + \Psi^\beta(y + m)} dy, \text{ if } m \in [\theta - m, \nu^\beta) \text{ and } w < \theta - m$$

implies our result for $m \in [\theta - m, \theta)$ and $w \in [0, \theta - m)$ taking the limit $\beta \to 1$ and applying Lemma 4 (iii).

\[ \text{Here, and in what follows, we compute the limit of integrals applying, without mentioning it explicitly, the bounded convergence theorem (see Royden (1988), page 81).} \]
The remaining limits in the first item can be easily derived from Lemma 4.

We start the proof of the limit in the second item for the case \( m = m \). In this case, we have to show that \( \lim_{\beta \to 1} (1 - \beta)\Psi^\beta(m) = \pi(m) \). Equation (3) implies:

\[
(1 - \beta)\Psi^\beta(m) = \beta F_{l,m}^\beta(0)(\pi(0) + \Psi^\beta(m)) - \beta \Psi^\beta(m) = \beta F_{l,m}^\beta(0)\pi(0) - \beta \Psi^\beta(m)(1 - F_{l,m}^\beta(0)),
\]

where the first term in the last expression tends to \( \pi(0) \) by application of the result in the first line of this lemma, and the last term is equal to:

\[
\beta \Psi^\beta(m) \int_0^m \frac{1 - \beta}{\pi(y)} \Psi^\beta(y + m) \, dy = \beta \int_0^m \frac{1 - \beta}{\pi(y)} \Psi^\beta(y + m) \, dy,
\]

where we have used in the last equality the implication of Equation (3) that \( \Psi^\beta(y + m) = F_{l,m}^\beta(0) \). The limit of this last term when \( \beta \) tends to one is equal to \( \pi(0) - \pi(m) \), as required, since \( \Psi^\beta(m) \) diverges to infinity and, as we showed above, \( F_{l,m}^\beta(w) \) tends to one for \( w \leq m < \theta - m \) and the lemma assumes that \( 2m < \theta - m \).

The result for a general \( m \) follows from the implication of Equation (3) that \( \Psi^\beta(m) = F_{l,m}^\beta(0) \) and the limit results for \( F_{l,m}^\beta \) in this lemma.  

**Lemma 6.** If \( \lim_{\beta \to 1} \Psi^\beta(m) = \infty \) and \( \theta > 2m \), then \( \lim_{\beta \to 1} \mu^\beta([m, \theta]) = 0. \)

**Proof.** The result can be deduced from the application of Equation (6) noting that,

\[
Q^\beta(m, [m, \theta]) = Q^\beta(m, [m, \theta]) - Q^\beta(m, \{m\})
\]

and that the right hand side converges to zero for any \( m \in [m, \theta + m] \) as can be deduced from Equation (12) and Lemma 5.  

**Proof of Theorem 1**

For \( \epsilon > 0 \) and sufficiently small, we define the following sets \( A \equiv \{m\}, B \equiv (m, \theta - 2m - \epsilon], \)

\( C = (\theta - 2m - \epsilon, \theta - m - \epsilon), D \equiv [\theta - m - \epsilon, \theta - \epsilon], E \equiv [\theta - \epsilon, \theta + m]. \)

Lemma 6 implies that it is sufficient to show that \( \lim_{\beta \to 1} \mu^\beta(E) = 0. \)

Note that since \( Q(m, E) > 0 \) only if \( m \in D \cup E: \)

\[
\mu^\beta(E) = \mu^\beta(D) \int_D Q^\beta(m, E) \mu^\beta(dm) + \mu^\beta(E) \int_E Q^\beta(m, E) \mu^\beta(dm) \mu^\beta(E) \\
\leq \mu^\beta(D) + \left( \mu^\beta(E) \int_E Q^\beta(m, E) \mu^\beta(dm) \mu^\beta(E) \right).
\]

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Besides since $Q(m, D) > 0$ only if $m \in C \cup D \cup E$,

$$\mu^\beta(D) = \mu^\beta(C \cup D) \int_{C \cup D} Q^\beta(m, D) \frac{\mu^\beta(dm)}{\mu^\beta(C \cup D)} + \mu^\beta(E) \int_E Q^\beta(m, D) \frac{\mu^\beta(dm)}{\mu^\beta(E)}.$$

Substituting the latter equality in the former, using that $1 - Q^\beta(m, D) - Q^\beta(m, E) = Q^\beta(m, A \cup B \cup C)$ and solving for $\mu^\beta(E)$, one gets:

$$\mu^\beta(E) \leq \frac{\mu^\beta(C \cup D) \int_{C \cup D} Q^\beta(m, D) \frac{\mu^\beta(dm)}{\mu^\beta(C \cup D)}}{\int_E Q^\beta(m, A \cup B \cup C) \mu^\beta(dm)} \leq \frac{\mu^\beta(C \cup D) \int_{C \cup D} Q^\beta(m, D) \frac{\mu^\beta(dm)}{\mu^\beta(C \cup D)}}{Q^\beta(\theta, A \cup B \cup C)},$$

where the last inequality follows from the fact that $Q^\beta(m, [0, x])$ is increasing in $F_{l,m}^\beta(x)$ and $F_{l,m}^\beta(x)$ and that each of these two functions is greater than $F_{l,\theta}^\beta(x)$ and $F_{l,\theta}^\beta(x)$, respectively for all $x$. This last property follows from the definition of $F_{l,m}^\beta$ and $F_{l,m}^\beta$ and in the case of $F_{l,m}$ the monotonicity in Lemma 2(v).

We shall show that the right hand side of Equation (14) tends to zero as $\beta$ tends to 1. We first argue that:

$$\lim_{\beta \to 1} \frac{Q^\beta(\theta, A \cup B \cup C)}{(1 - \beta)^2} = 2 \lim_{\beta \to 1} \frac{F_{l,\theta}^\beta(\theta - 2m - \epsilon)}{(1 - \beta)^2}$$

$$\leq 2 \lim_{\beta \to 1} \int_0^{\theta - 2m - \epsilon} \frac{1}{(1 - \beta)\pi(y) + (1 - \beta)\Psi^\beta(y + m)} dy$$

$$\leq 2 \int_0^{\theta - 2m - \epsilon} \frac{1}{\rho(y)} dy > 0,$$

where we have used in the first step, Equation (13) and that $\lim_{\beta \to 1} F_{l,\theta}^\beta(\theta - 2m - \epsilon) = 0$ by Lemma 5; we have used in the second step, Equation (11) and $F_{l,\theta}^\beta(0) = 0$; and we have used in the third step Lemma 5 and that for any $y < \min\{\theta - m, m\}$:

$$\lim_{\beta \to 1} \frac{F_{l,m}^\beta(y)}{1 - \beta} = \lim_{\beta \to 1} \int_0^y \frac{1}{(1 - \beta)\pi(z) + (1 - \beta)\Psi^\beta(z + m)} dz = 0,$$

that can be deduced from Lemma 5.

Next, we show that for any $m < \theta$ (and hence any $m \in C \cup D$):

$$\lim_{\beta \to 1} \frac{Q^\beta(m, D)}{1 - \beta} = \lim_{\beta \to 1} \frac{F_{l,m}^\beta(\theta - m - \epsilon) - F_{l,m}^\beta(\theta - 2m - \epsilon)}{(1 - \beta)}$$

$$\leq \int_{\min\{m, \theta - m - \epsilon\}}^{\min\{m, \theta - 2m - \epsilon\}} \frac{1}{\rho(y)} dy < \infty,$$
where we use in the first equality that,
\[
\lim_{\beta \to 1} Q^\beta(m, [m, x]) = \lim_{\beta \to 1} F_{l,m}(x - m)
\]
if \(x < \theta\) because either \(x - m \geq m\) and then \(Q^\beta(m, [m, x]) = 1 = F_{l,m}(x - m)\) or \(x - m < m\) and then we can use equations (12) and (15); we use in the second equality, Equation (11); and we use in the inequality Lemma 5 and \(F^\beta_l(m) \leq 1\).

We conclude the proof by showing that \(\lim_{\beta \to 1} \mu^\beta(C \cup D) = 0\). To prove so, note that since \(Q(m, C \cup D) = 0\) if \(m < m^* = \theta - 3m - \epsilon\) and \(Q^\beta(m, [0, x]) = Q^\beta(\theta, [0, x])\) if \(m \in [\theta, \theta + m]\):
\[
\frac{\mu^\beta(C \cup D)}{1 - \beta} = \int_{m^*}^\theta Q^\beta(m, C \cup D) \frac{\mu^\beta(dm)}{1 - \beta} \mu^\beta([m^*, \theta]) + \frac{Q^\beta(\theta, C \cup D)}{1 - \beta} \mu^\beta([\theta, \theta + m]),
\]
The first term in the sum goes to zero because \(Q^\beta(m, C \cup D) \leq Q^\beta(m, D)\) and \(m^* > m\) and we have already shown that \(\lim_{\beta \to 1} Q^\beta(m, D) \to \infty\) if \(m < \theta\) and \(\lim_{\beta \to 1} \mu^\beta([m, \theta]) = 0\).

That the second term also goes to zero can be deduced from equations (13) and (15) and the fact that \(\mu^\beta([\theta, \theta + m]) \leq 1\).

**Proof of Lemma 1**

**Proof.** The lemma follows from the following sequence of inequalities, that start from a transformation of Equation (3), taking the limit \(\beta \to 1\):
\[
(1 - \beta)\Psi^\beta(m) = \beta F^\beta_{l,m}(0) \left( \pi(0) + \Psi^\beta(m) \right) - \beta \Psi^\beta(m)
= \beta F^\beta_{l,m}(0) \left( \pi(0) - \frac{1 - F^\beta_{l,m}(0)}{F^\beta_{l,m}(0)} \Psi^\beta(m) \right)
\geq \beta F^\beta_{l,m}(0) \left( \pi(0) - \frac{1 - \beta}{\beta} m + \pi(0) - \pi(m) \right)
\geq \beta F^\beta_{l,m}(0) \pi(0) \frac{\pi(2m) + \pi(m) - \pi(0) - \frac{1 - \beta}{\beta} 3m}{\pi(2m) - \frac{1 - \beta}{\beta} 2m},
\]
where we have used in the first inequality that,
\[
1 - F^\beta_{l,m}(0) = \int_0^m \frac{1 - \beta}{\beta} + \left( -\pi'(y) \right) \frac{F^\beta_{l,m}(y)}{\pi(y) + \Psi^\beta(y + m)} dy
\leq \int_0^m \frac{1 - \beta}{\beta} + \left( -\pi'(y) \right) \frac{\Psi^\beta(2m)}{\pi(2m)} dy
= \frac{1 - \beta}{\beta} m + \pi(0) - \pi(m) \Psi^\beta(2m)
\]

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and that by Equation (3) $\frac{\psi^\beta(m)}{\psi^\beta(2m)} = \frac{F^\beta_{l,m}(0)}{F^\beta_{l,2m}(0)}$, and in the second inequality Lemma 4 (ii) and (iii), since $2m < \theta$. This last inequality is implied by $\pi(2m) > \pi(0) - \pi(m) \geq 0$. ■

B The Laggard-Leader Scenario when Firms Start with Identical Cash

In this case, we consider the case in which $m_1 = m_2$ and $m_1 < \nu^\beta$ in the static model of Section 3. In what follows, we use $m$ to denote the common amount of cash and assume $m > 0$ to avoid trivialities.

As in Section 3, the elimination of strictly dominated strategies means that we can restrict to pure strategies $(b^*(w), w)$, for $w \in [0, m]$, or mixed strategies $(b^*, F)$, for $F$ a cumulative distribution function with support in $[0, m]$.

We introduce auxiliary notation. Let $\gamma$ be the unique solution to:

$$\frac{\beta}{2} \pi(\gamma) - (1 - \beta)\gamma = 0.$$  

If $m \in (\gamma, \nu^\beta)$, we let $\lambda(m) \in [0, m]$ be implicitly defined by:

$$\left( F^*(\lambda(m)) + \frac{1 - F^*(\lambda(m))}{2} \right) \beta \pi(m) - (1 - \beta)m = 0,$$

where $F^*$ is as defined in Section 3.

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52This equation is equivalent to:

$$m - \gamma + \beta \left( \gamma + \frac{1}{2} \pi(\gamma) \right) = m,$$

which says that a firm is indifferent between no working capital and a working capital equal to $\gamma$ when the other firm is choosing a working capital $\gamma$.

53Existence and uniqueness of the solution follow from the properties of the function in the left hand side of the equation. This is increasing in $\lambda(m)$, it is negative at $\lambda(m) = 0$ and it is strictly positive at $\lambda(m) = m$. The first one is direct, the second can be deduced from the definition of $\gamma$ using that $m > \gamma$, and the third from the definition of $\nu^\beta$ using that $m < \nu^\beta$.

54This equation is equivalent to:

$$\beta \left( m + \left( F^*(\lambda(m)) + \frac{1 - F^*(\lambda(m))}{2} \right) \pi(m) \right) = m,$$

which says that a firm is indifferent between a working capital of $m$ and no working capital when the other firm’s working capital is with probability $F^*(\lambda(m))$ strictly less than $\lambda(m)$ and with the remaining probability at a mass point in $m$. 

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Proposition 4.

- If \( m_1 = m_2 = m \in (0, \gamma] \), then the unique equilibrium is symmetric and denoted by the single pure strategy \((b^*(m), m)\).

- If \( m_1 = m_2 = m \in (\gamma, \nu) \), then the unique equilibrium is symmetric and denoted by the single mixed strategy \((b^*, F^{**})\), where

\[
F^{**}(w) \equiv \begin{cases} 
F^*(w) & \text{if } w \in [0, \lambda(m)], \\
F^*(\lambda(m)) & \text{if } w \in (\lambda(m), m), \\
1 & \text{if } w \geq m.
\end{cases}
\]

Proof. We start by showing that our proposed strategies are an equilibrium. Consider, first, the case \( m \in (0, \gamma] \). The proposed equilibrium specifies that both firms pick a working capital equal to \( m \) with probability one. Thus, each firm’s expected payoffs are equal to

\[
\beta m + \pi(m) - \frac{(1 - \beta)m}{2} \geq \beta \pi(m) - (1 - \beta)m.
\]

Consider, next, the case \( m \in (\gamma, \nu) \). If a firm randomizes its choices of working capital according to the proposed strategy, the expected payoffs of the other firm when it picks a working capital \( w \) are equal to

\[
\begin{cases} 
m - w + \beta (w + \pi(w)F^*(w)) & \text{if } w \in [0, \lambda(m)], \\
m - w + \beta (w + \pi(w)F^*(\lambda(m))) & \text{if } w \in (\lambda(m), m), \\
m - w + \beta \left( w + \left( F^*(\lambda(m)) + \frac{1-F^*(\lambda(m))}{2} \right) \pi(w) \right) & \text{if } w = m.
\end{cases}
\]

The first and last case are equal to \( m \). The first case by definition of \( F^* \), and the last case by definition of \( \lambda \). The second case is a strictly decreasing function of \( w \) and hence it is strictly less than

\[
m - \lambda(m) + \beta (\lambda(m) + \pi(\lambda(m))F^*(\lambda(m))),
\]

which is equal to \( m \) as we have argued for the first case. Thus, the deviations are not profitable.
To prove uniqueness, we assume an equilibrium \((b^*, F_1), (b^*, F_2)\) and show that \(F_1 = F_2\) and they are equal to a mass point at \(m\) if \(m \leq \gamma\) and to \(F^{**}\) if \(m \in (\gamma, \nu^\beta)\). In our argument, we use the notation \(\sigma_i\) introduced in the proof of Proposition 1. Note that Claim 1 and 2 in the proof of Proposition 1 are also verified when \(m_1 = m_2\). They imply here that (i) the only points where there can be a mass point in the strategies is at zero or at \(m\), (ii) at most one of the firms’ strategies can have an atom at zero, and (iii) the support of both \(\sigma_1\) and \(\sigma_2\) must be the same and equal to either \(\{m\}\) or \([0, \nu] \cup \{m\}\) for some \(\nu \in (0, m]\). We consider the two possibilities for the support separately.

We first note that the support can be equal to \(\{m\}\) only in the case \(m \leq \gamma\). This is because each firm plays the pure strategy \((b^*(m), m)\) and, hence, makes expected payoffs equal to \(\beta \left( m + \frac{\pi(m)}{2} \right)\). The definition of \(\gamma\) means that these payoffs are strictly less than \(m\) when \(m > \gamma\). This implies strict incentives to deviate unilaterally when \(m > \gamma\) since carrying no working capital gives a payoff of \(m\).

To conclude the proof, we note that the support of both \(F_1\) and \(F_2\) can be equal to \([0, \nu] \cup \{m\}\), for some \(\nu \in (0, m]\), only if \(m \in (\gamma, \nu^\beta)\) and that in this case the equilibrium conditions imply that \(F_1 = F_2 = F^{**}\). First, note that (i) implies that both distributions must be atomless in \((0, \nu]\). Now, the support of \(F_i\) can be equal to \([0, \nu] \cup \{m\}\) only if Firm \(i\) is indifferent between an amount of working capital \(\nu\) and an amount of working capital \(m\) when the other firm randomizes working capital according to \(F_j\). In this case, the expected payoffs of both options are equal to the left hand side and right hand side, respectively, of:

\[
m - \nu + \beta(\nu + F_j(\nu)\pi(\nu)) = \beta \left( m + \left( F_j(\nu) + \frac{1 - F_j(\nu)}{2} \right)\pi(m) \right).
\] (16)

There is a single value of \(F_j(\nu)\) that solves Equation (16), thus \(F_1(\nu) = F_2(\nu)\). Equilibrium also requires that Firm \(i\) is indifferent between any working capital in \((0, \nu^\beta]\) when the other firm randomizes working capital according to \(F_j\). This, together with (i), implies that \(F_1\) and \(F_2\) must be solutions to Equation (1) in \((0, \nu^\beta]\). Since \(F_1(\nu) = F_2(\nu)\), and there is a unique solution to Equation (1) passing by each point, \(F_1(w) = F_2(w)\) for any \(w \in (0, \nu]\) which together with (ii) imply that \(F_1(0) = F_2(0) = 0\) and hence \(F_1(w) = F_2(w) = F^*(w)\) for any \(w \in [0, \nu]\). The left hand side of Equation (16) for \(F_j(\nu) = F^*(\nu)\) is equal to \(m\) by definition of \(F^*(\nu)\). This implies that Equation (16) is equivalent to the equation that defines \(\lambda(m)\), and hence \(\nu = \lambda(m)\) if \(m \in (\gamma, \nu^\beta]\), and hence \(F_1 = F_2 = F^{**}\) as desired.
Finally, one can adapt the arguments in Footnote 53 about the existence of a solution \( \lambda(m) \) to show that there is no \( \nu \in (0, m] \) that solves Equation (16) when \( m \leq \gamma \) which means that the support cannot be equal to \( [0, \nu] \cup \{m\} \), for some \( \nu \in (0, m] \), in this case, as desired. ■

C  A Model of Financial Constraints

In this Appendix, we endogenize the function \( \pi \) in a model in which moral hazard and limited liability restrict the set of bids for which the firm can secure financing, i.e. the set of acceptable bids. In this model, the firm can borrow from a competitive banking sector but upon winning the procurement contract can divert a fraction of their total available funds at the cost of jeopardizing the success of the procurement contract. The main implication is that the minimum acceptable bid for a firm with working capital \( w \) is given by a function \( \pi(w) \equiv \theta - w \), for some \( \theta \) endogenously determined.

We assume a variation of the dynamic model of Section 4\(^{55}\) in which instead of assuming that the set of acceptable bids is exogenously given, we assume that a bid \( b \) is acceptable if either the firm’s working capital \( w \) is larger than the cost of the procurement contract \( c \) or the firm can borrow the required funds \( c - w \) to pay the cost of the procurement contract\(^{56}\) \( c \). In the latter case, the firm can borrow from a competitive banking sector. The bank transfers the funds to the firm after the firm has won the contract. At that point, the firm can either: (a) use the total funds \( c \), working capital plus bank lending, to pay the cost of the procurement contract and comply with the procurement contract and with the bank; or (b) divert an exogenously given fraction \( \alpha \in (0,1] \) of its total funds and comply neither with the procurement contract nor with the bank. In case (a), the firm’s cash next period is equal to its working capital plus profits minus any net payment to the bank, denoted by \( R \), and plus the exogenous cash flow \( m \):

\[
w + b - c - R + m.
\]

In case (b), limited liability implies that the firm gets expropriated of all non-diverted

\(^{55}\)It is straightforward how to adapt it to the static model of Section 3.

\(^{56}\)As we explain in the Introduction, this may be a consequence of requiring a surety bond and that sureties only issue bonds if one of the two conditions above are met.
funds and starts the next period with cash equal to its diverted funds plus the exogenous cash flow \( m \):

\[
\alpha c + m.
\]

There is no discounting between the moment in which the bank lends the funds and when it is paid back.

One consequence is that a firm with continuation value \( W^* \), and more generally with any continuation value strictly increasing in the firm’s cash, diverts funds if \( w + b - c - R + m < \alpha c + m \). This means that a bank that lends to a firm that diverts funds makes losses since its profits are equal to at most \((1 - \alpha)c - (c - w) < 0\). Hence, banks do not lend to firms expected to divert funds. Since bank lending is risk free, \( R = 0 \) and banks only lend if \( w + b - c + m \geq \alpha c + m \). Therefore (and no proof is required):

**Proposition 5.** Only bids greater than \( c + \alpha c - w \) are acceptable for a firm with working capital \( w \). Hence, \( \pi(w) = \theta - w \), for \( \theta = \alpha c \).
References


