Asset Prices, Market Selection and Belief Heterogeneity
Arrow-Debreu and Sequential Markets

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This lecture is based on “Implementing Arrow-Debreu equilibria by trading infinitely-lived securities.” by Huang and Werner, Economic Theory, 24, 2004.
The Economy

- There is a single perishable consumption good every period.
- A consumption plan is a sequence \( \{ c_t \}_{t=0}^{\infty} \) such that \( c_0 \in \mathbb{R}_+ \) and \( c_t : S^\infty \rightarrow \mathbb{R}_+ \) is \( \mathcal{F}_t \)-measurable for all \( t \geq 1 \) and \( \sup_{(t,s)} c_t(s) < \infty \).
- Let \( c(s^t) \equiv c_t(s) \) for any \( t \) and \( s^t \in S^t \).
- Given \( s_0 \), \( C(s_0) \) denotes the set of all consumption plans.

- The economy is populated by \( I \) (types of) infinitely-lived agents where \( i \in I = \{ 1, \ldots, I \} \) denotes an agent’s name.
- Agent \( i \) is endowed with initial endowment \( \omega_i \in C(s_0) \)
- The aggregate endowment \( \bar{\omega} \equiv \sum_i \omega_i \).

- An allocation is a collection of plans \( \{ c_i \}_{i \in I} \).
- An allocation is feasible if \( \sum_i c_i(s^t) \leq \bar{\omega}, \forall s^t, \forall t \).
There exists a market at the initial date 0 for consumption at date \( t \) conditional on event \( s^t \), for every date \( t \) and every event \( s^t \).

Prices are described by a *pricing functional*, that is, a linear functional \( P \) which is positive and well-defined (finitely valued) on each consumer’s initial endowment.

It follows that a pricing functional is well-defined on the aggregate endowment \( \overline{\omega} \) and, therefore, on each feasible allocation. It may or may not be well-defined on the entire consumption set \( C(s_0) \).

The price of one unit of consumption in event \( s^t \) under pricing functional \( P \) is \( p(s^t) \equiv P(e(s^t)) \), where \( e(s^t) \) denotes the consumption plan equal to 1 in event \( s^t \) at date \( t \) and zero in all other events and all other dates.

A pricing functional \( P \) is **countably additive** if and only if \( P(c) = \sum_t \sum_{s^t} p(s^t)c(s^t) \) for every \( c \) for which \( P(c) \) is well-defined.
Arrow Debreu Budget Set

- Trades occur only at date zero.
- Agent \( i \) can only choose a consumption plan such that the value of consumption does not exceed the value of agent \( i \)'s endowment.
- The agent chooses a plan on the budget set \( B_{AD}(P, \omega_i) \) where:

\[
B_{AD}(P, \omega_i) \equiv \left\{ c \in C(s_0) : P(c) \leq P(\omega_i) \right\}
\]

\[
= \left\{ c \in C(s_0) : \sum_{t=0}^{\infty} \sum_{s^t \in S^t} p(s^t)c(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t \in S^t} p(s^t)\omega_i(s^t) \right\}
\]

- Consumer \( i \)'s problem is to choose a consumption plan \( c_i \in C(s_0) \) such that

\[
c_i \succsim_i c, \forall c \in B_{AD}(P, \omega_i)
\]
Arrow Debreu Equilibrium

- Agents trade at date zero under a single budget constraint.

**Definition**

An *Arrow-Debreu equilibrium* is a pricing functional $P$ and a consumption allocation $\{c^i\}_{i=1}^I$ such that $c^i$ solves consumer $i$’s problem and markets clear.

- The Arrow-Debreu model of contingent commodity markets is hardly realistic.
- Yet, it serves as an important tool for the analysis of infinite-time security markets.
- This is because one can show that Arrow-Debreu equilibria and equilibria in sequential security markets with debt constraints have the same consumption allocations when markets are dynamically complete and debt bounds are nonbinding.
Sequential Markets

- There are $J \geq S$ infinitely-lived securities traded at every date.
- Each security $j$ is specified by a dividend process $d_j$ which is adapted to $\{\mathcal{F}_t\}_{t=0}^\infty$ and nonnegative.
- The ex-dividend price of security $j$ in event $s^t$ is denoted by $q_j(s^t)$, and $q_j$ is the price process of security $j$.
- Portfolio strategy $\theta$ specifies a portfolio of $J$ securities $\theta(s^t)$ held after trade in each event $s^t$.
- The payoff of portfolio strategy $\theta$ in event $s^t$ at a price process $q$ is

$$z(q, \theta)(s^t) \equiv \left[ q(s^t) + d(s^t) \right] \theta(s^{t-1}) - q(s^t)\theta(s^t)$$

Definition

Security price process $q$ is one-period-arbitrage free in event $s^t$ if there does not exist a portfolio $\theta(s^t)$ such that:

$$[q(s^t, s_{t+1}) + d(s^t, s_{t+1})] \theta(s^t) \geq 0 \text{ for all } s \text{ and } q(s^t)\theta(s^t) \leq 0,$$

with at least one strict inequality.
No arbitrage

If $q$ is arbitrage free in every event, then there exists a sequence of strictly positive state prices $\{\{\pi_q(s^t)\}_{s^t \in S^t}\}_{t=0}^\infty$ with $\pi_q(s^0) = 1$ such that

$$\pi_q(s^t)q_j(s^t) = \sum_{s_{t+1} \in S} \pi_q(s^t, s_{t+1}) [q_j(s^t, s_{t+1}) + d_j(s^t, s_{t+1})] \quad \forall s^t, \forall j$$

Definition

Security markets are one-period complete in event $s^t$ at prices $q$ if the one-period payoff matrix $[q(s^t, s_{t+1}) + d(s^t, s_{t+1})]_{s_{t+1} \in S}$ has rank $S$. Security markets are complete at $q$ if they are one-period complete at every event.

Suppose the security prices $q$ are one-period arbitrage free and that markets are complete at $q$. Then, the fundamental value of security $j$ at $s^t$ is defined using the unique state prices as

$$\frac{1}{\pi_q(s^t)} \sum_{\tau=1}^\infty \sum_{s^\tau \in S^\tau} \pi_q(s^t, s^\tau) d_j(s^t, s^\tau) \quad (1)$$
Sequential Markets

- Each agent $i$ has an initial portfolio $\alpha_i \in \mathbb{R}^J$ at date 0.

- The dividend stream $\alpha_i d$ on initial portfolio constitutes one part of consumer $i$’s endowment. The rest is $y_i \in C(s_0)$ and becomes available to the consumer at each date in every event. Thus,
  \[ \omega_i(s_t) = y_i(s^t) + \alpha_i d(s^t), \quad \forall s^t \in S^t \]

- The supply of securities is $\bar{\alpha} = \sum_i \alpha_i$.

- The adjusted aggregate endowment is $\bar{y} = \sum_i y_i$. Let’s assume $\bar{\alpha} \geq 0$. 
Sequential Budget Set

- \( \theta_i \) supports \( c_i \) at \( (q, y_i) \) if

\[
c_{i,0} + q(s^0)\theta(s^0) \leq y_i(s_0) + q(s_0)\alpha_i
\]

\[
c_i(s^t) + q(s^t)\theta(s^t) \leq y_i(s^t) + [q(s^t) + d(s^t)]\theta(s^{t-1}), \quad \forall s^t \neq s_0
\]

- Consumers must also face constraints in their portfolio strategies for otherwise they would use Ponzi schemes. There is a set \( \Theta_i \) of feasible supporting portfolios.

- The sequential budget set is:

\[
B(q; y_i) \equiv \left\{ c_i \in C(s_0) : \exists \theta_i \in \Theta_i \; \exists c_i(s^t) + q(s^t)\cdot \theta_i(s^t) \leq y_i(s^t) + r(s^t)\cdot \theta_i(s^{t-1}), \; \forall s \in S^\infty, \; \forall t \geq 0. \right\}
\]
The Wealth Constraint

- A frequently used portfolio constraint is the so-called wealth constraint. It prohibits a consumer from borrowing more than the present value of his future endowment. Formally,

\[ q(s^t)\theta(s^t) \geq -\sum_{\tau=1}^{\infty} \sum_{s^\tau \in S^\tau} \frac{\pi q(s^t, s^\tau)}{\pi q(s^t)} y(s^t, s^\tau) \]

- The set of Arrow-Debreu equilibrium allocations is the same as the set of Sequential Markets equilibrium allocations under the wealth constraint with no bubbles.

- There always exist a sequential equilibria with price bubbles under the wealth constraint if some securities are in zero net supply.
Essentially Bounded Portfolios

- A portfolio constraint for which neither price bubbles nor negative security prices arise in equilibrium and AD equilibria can be implemented in sequential markets.

**Definition**
A portfolio \( \theta \) is bounded from below if \( \min_j \inf_{(t,s^t)} \theta_j(s^t) > -\infty \)

**Definition**
A portfolio strategy \( \theta \) is essentially bounded from below at \( q \) if there is a bounded from below portfolio strategy \( b \) s.t. \( q(s^t)\theta(s^t) \geq q(s^t)b(s^t) \) \( \forall s^t \).

**Proposition**
If security price vector \( q(s^t) \) is positive and nonzero for every partial history \( s^t \), then portfolio \( \theta \) is essentially bounded if and only if \( \inf_{s^t} \frac{q(s^t)}{\sum_j q_j(s^t)} \theta(s^t) > -\infty. \)
**Euler Equations**

We say that $c_i$ satisfies the *Euler* equation at the price process $q$ if

$$u'_i(c_i, t(s)) q_j, t(s) = \beta_i \cdot E_{P_i}[r_j, t_{+1} \cdot u'_i(c_i, t_{+1}) | \mathcal{F}_t](s) \quad \forall j \in J, \forall s \in S^\infty, \forall t \geq 0.$$ 

**Assumption U:** $u_i : R^{++} \rightarrow R$ is

(i) strictly increasing, strictly concave, $C^1$ & $u_i(0) \equiv \lim_{c \rightarrow 0^+} u_i(c)$

(ii) $\beta_i \in (0, 1)$.

**Definition**

For $i$, $c_i$ is a maximizer given $q$ if

1. $c_i \in B(q; y_i)$ and

2. there is no $\tilde{c}_i \in B(q; y_i)$ for which

$$\lim_{T \rightarrow +\infty} \sum_{t=0}^{T} \beta_i^t E_{P_i}[u_i(\tilde{c}_i, t)] > \lim_{T \rightarrow +\infty} \sum_{t=0}^{T} \beta_i^t E_{P_i}[u_i(c_i, t)].$$
Necessary Condition

Suppose the investor can freely buy or sell as much of asset \( j \) as she wishes at a price \( q_{j,t} \).

Denote by \( c_i \) the optimal consumption plan.

She can alter her consumption plan as follows:

\[
\tilde{c}_{i,t} = c_t - q_{j,t} \cdot \xi_t \\
\tilde{c}_{i,t+1} = c_{t+1} + r_{j,t+1} \cdot \xi_t
\]

If \( c_i \) maximises the consumer’s utility, then

\[
q_{j,t} \cdot u_i'(c_{i,t}) = E_{P_i} \left[ \beta_i \cdot u_i'(c_{i,t+1}) \cdot r_{j,t+1} \middle| \mathcal{F}_t \right],
\]
Sufficient Conditions

Theorem

Suppose Assumption \( \mathcal{U} \). Given \((q, y_i)\), let \( c_i \in B(q; y_i) \) be such that

1. \( \lim_{T \to +\infty} \sum_{t=0}^{T} \beta_t^i E_P[i u_i(c_i,t)] > -\infty \),
2. satisfies the Euler equation at the price process \( q \),
3. for every \( \tilde{\theta}_i \) that supports a \( \tilde{c}_i \in B(q; y_i) \) the transversality condition at date 0 holds,

\[
\lim_{T \to +\infty} \beta^T_i E_P[i u'_i(c_i, T) \cdot q_T \cdot (\tilde{\theta}_i, T - \theta_i, T)] \geq 0.
\]

where \( \theta_i \) supports \( c_i \) at \((q, y_i)\). Then \( c_i \) is the maximiser on \( B(q; y_i) \).
Bubbles

If the fundamental value $(1)$ is finite, the price bubble $\sigma_{qj}(s^t)$ is

$$
\sigma_{qj}(s^t) \equiv q_j(s^t) - \frac{1}{\pi_q(s^t)} \sum_{\tau=1}^{\infty} \sum_{s^{\tau} \in S^\tau} \pi_q(s^t, s^{\tau}) d_j(s^t, s^{\tau})
$$

Proposition

If the price of security $j$ is nonnegative in every event, then the fundamental value of security $j$ is finite and does not exceed the price of security $j$, i.e.

$$
0 \leq \sigma_{qj}(s^t) \leq q_j(s^t)
$$

for every $s^t$. If the fundamental value of security $j$ is finite and $\sigma_{qj}(s^t) \geq 0$ for every $s^t$, then $q_j(s^t) \geq 0$ for every $s^t$.

Note that $(1)$ and $(2)$ implies that

$$
\sigma_{qj}(s^t) = \frac{1}{\pi_q(s^t)} \sum_{s_{t+1} \in S} \pi_q(s^t, s_{t+1}) \sigma_{qj}(s^t, s_{t+1})
$$

$$
\sigma_{qj}(s^t) = \lim_{T \to \infty} \frac{1}{\pi_q(s^t)} \sum_{s^{T} \in S^T} \pi_q(s^t, s^{T}) q_j(s^t, s^{T})
$$
Bubbles under the Wealth Constraint

- A representative agent economy without uncertainty: \( y_t = y \) for all \( t \geq 0 \).
- A consol pays \( d_t = d < y \), is in zero net-supply and trades at price \( q^c_t \).
- In any equilibrium \( c_t = y \) and \( \theta_t = 0 \) for all \( t \geq 0 \).
- \( q^c_t = \frac{\beta}{1-\beta} d + \varepsilon_t \) where \( \varepsilon_t = \varepsilon_0 \left( \frac{1}{\beta} \right)^t \). Hence, \( \pi_q(s^t) = \beta^t \).
- The wealth constraint: \( q^c(s^t)\theta(s^t) \geq -\sum_{\tau=1}^{\infty} \frac{\pi_q(s^t,s^\tau)}{\pi_q(s^t)} y(s^t,s^\tau) = -y \frac{\beta}{1-\beta} \)
- \( c \) satisfies the Euler equation:
  \[
  q^c_t = \beta \left( \frac{1}{1-\beta} d + \varepsilon_0 \left( \frac{1}{\beta} \right)^{t+1} \right) = \beta \left( \frac{\beta}{1-\beta} d + d + \varepsilon_{t+1} \right) = \beta (q^c_{t+1} + d)
  \]
- \( c \) satisfies the TC:
  \[
  \lim_{T \to \infty} \beta^T q^c_T (\tilde{\theta}_T - \theta_T) = \lim_{T \to \infty} \beta^T q^c_T \tilde{\theta}_T \geq \lim_{T \to \infty} \beta^T \left( -y \frac{\beta}{1-\beta} \right) = 0.
  \]
Bubbles under the Wealth Constraint

- Suppose there is a risk-free bond with price $q_t^b$. Clearly, $q_t^b = \beta$.
- Let $q_t = (q_t^c, q_t^b)$.
- Suppose the agent shorts the consol in one unit and invest $\frac{\beta}{1-\beta}d$ units of the bond at zero to meet the consol payments?
  - $\tilde{\theta}_t = \left(-1, \frac{\beta}{1-\beta}d \frac{d}{\beta} \right)$ for all $t \geq 0$.
  - $q_0\tilde{\theta}_0 = -q_0^c + \frac{\beta}{1-\beta}d = \varepsilon_0 > 0$.
  - $\left[(q_t^c + d)\tilde{\theta}_{t-1}^c + \tilde{\theta}_{t-1}^b\right] - \left[q_t^c\tilde{\theta}_t^c + q_t^b\tilde{\theta}_t^b\right] = -d + \frac{\beta}{1-\beta}d + \beta \frac{\beta}{1-\beta}d = 0$
  - $\tilde{\theta}_t$ supports $\tilde{c}_0 = c_0 + \varepsilon$, $\tilde{c}_t = c_t$.
- How is this compatible with $c$ being optimal?
- The key is that $\tilde{\theta}$ violates the wealth constraint and so $\tilde{c} \notin \mathcal{B}(q, y)$.
  - $q_t\tilde{\theta}_t = -q_t^c + q_t^b \frac{\beta}{1-\beta}d = -\left(\frac{\beta}{1-\beta}d + \varepsilon_t\right) + \frac{\beta}{1-\beta}d = -\varepsilon_0 \left(\frac{1}{\beta}\right)^t \to -\infty$. 

Arrow Debreu and Sequential Markets
No Bubbles with Essentially Bounded Portfolios

Theorem

If \( q \) is an equilibrium price process such that \( \theta \) is essentially bounded and security markets are complete at \( q \), then \( q(s^t) \geq 0 \) and \( \sigma_{q_j}(s^t) = 0 \) for every \( s^t \).
Equivalence I

Theorem

Let allocation \( \{c_i\}_{i=1}^I \) and pricing functional \( P \) be an Arrow-Debreu equilibrium. If \( P \) is countably additive, \( P(d_j) < \infty \) for each \( j \), security markets are complete at prices \( q \) given by

\[
q_j(s^t) = \frac{1}{p(s^t)} \sum_{\tau=t+1}^{\infty} p(s^\tau) d_j(s^\tau) \quad \forall s^t, \forall j
\]

(3)

and there exists an essentially bounded portfolio strategy \( \eta \) such that

\[
-\frac{1}{p(s^t)} \sum_{\tau=t+1}^{\infty} \sum_{s^\tau \in S^\tau} p(s^\tau) y(s^\tau) \geq q(s^t) \eta(s^t) \quad \forall s^t, \forall j
\]

(4)

then there exists a portfolio allocation \( \{\theta_i\}_{i=1}^I \) such that \( q \) and the allocation \( \{c_i, \theta_i\}_{i=1}^I \) are a sequential equilibrium with essentially bounded portfolios.
Equivalence II

Theorem

Let security prices $q$ and $\{c_i, \theta_i\}_{i=1}^I$ be a sequential equilibrium with essentially bounded portfolios. If security markets are complete at $q$ and there exists an essentially bounded portfolio strategy $\eta$ such that

$$-rac{1}{\pi_q(s^t)} \sum_{\tau=t+1}^{\infty} \sum_{s^\tau \in S^\tau} \pi_q(s^\tau) \bar{y}(s^\tau) \geq q(s^t)\eta(s^t) \quad \forall s^t,$$

then $\{c_i, \theta_i\}_{i=1}^I$ and pricing functional $P$ given by

$$P(c) = \sum_t \sum_{s^t \in S^t} \pi_q(s^t)c(s^t)$$

are an Arrow-Debreu equilibrium.