Asset Prices, Market Selection and Belief Heterogeneity

Complete Markets with Homogeneous Beliefs

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Literature

- This lecture is based on:
  1. Abel, A. B. (2002), "An exploration of the effects of pessimism and doubt on asset returns," Journal of Economic Dynamic and Control, 26, 1075-1092.
  2. Cogley, T. and T. Sargent (2008): The Market Price of Risk and the Equity Premium: A Legacy of the Great Depression? Journal of Monetary Economics, Vol. 5, No. 3 (April), 454-476.
  3. Lucas, R. E. (1978), "Asset Prices in an Exchange Economy." Econometrica, 66(6), 1429-45.
  4. Mehra, R. and E. C. Prescott (1985), The equity premium: A puzzle. Journal of Monetary Economics, 15, 145161.

Motivation

- The Lucas' Tree Model is a theoretical examination of the stochastic behaviour of equilibrium asset prices in a one-good, pure exchange economy with a single (or many identical) consumers.
- The objective of the paper is to understand the relationship between the exogenously determined productivity changes and market determined movements in asset prices.
- The main contribution of the paper is the derivation and application of a functional equation in the vector of equilibrium asset prices. This equation is a generalisation of the Martingale property.
- The model thus serves as a simple context for examining the conditions under which a pricer series' failure to possess the Martingale property can be viewed as evidence of non-competitive or "irrational" behaviour.

The Stochastic Process of Dividends

- The consumption good is produced on J distinct productive units.
- d_{jt} be the output of unit j in period t.
- $d_t = (d_{1t}, ...d_{Jt})$ be the output vector in t.
- Since output is perishable, feasibility is

$$0 \leq c_t \leq \sum_{j=1}^{J} d_{jt}$$

- Production is entirely "exogenous": no resources are utilised, and there is no possibility of affecting output at any unit at any time.
- d_t is a Markov process with transition function:

$$F(d', d) = P\left\{ d_{t+1} \leq d' \,\middle|\, d_t = d \right\}$$

- In equilibrium $c_t = \sum_j d_{jt}$ and $\theta_t = (1, 1, ..., 1) = \underline{1}$ for all t.
- The main analytical issue is the equilibrium price behavior.

The Euler Equation

- Note that all relevant information on the current and future physical state of the economy is summarised in the current output vector y.
- Since the market "solves" a problem of the same form each period, equilibrium prices should depend only on the state of the economy $q_t = q(d_t)$.
- Consumer behaviour can be described time independent decision rules $c(\cdot)$ and $g(\cdot)$: $c_t = c(\theta_t, d_t, q_t)$ and $\theta_{t+1} = g(\theta_t, d_t, q_t)$.
- In equilibrium, $\theta = \underline{1}$, $c_t = \sum_j d_j$ and $c_{t+1} = \sum_j d_j'$. Thus,

$$U'\left(\sum_i d_i\right) q_i(d) = \beta \int U'\left(\sum_i d_i'\right)(d_j' + q_j(d'))dF(d', d), \text{ j=1,...,J } (EE_j)$$

Solving the Euler Equation

- Let $g_j(d) = \beta \int U'(\sum_j d'_j) d'_j dF(d', d)$, j = 1, ... , J
- Let $(f(d_1), ..., f_J(d))$ solve the J independent functional equations:

$$f(d) = g_j(d) + \beta \int f(d') dF(d', d) \ \ \text{j=1,...,J} \quad (EE'_j)$$

- Then, $q_j(d) = \frac{f_j(d)}{\sum_i U'_j(\sum_j d_j)}$ j = 1,...,J solve the EE.

Solving the Euler Equation

- If f is any continuous bounded, nonnegative function on $\Re^J_+$,

$$(T_j f)(d) = g_j(d) + \beta \int f(d')df(d', d)$$

is well-defined, $T_j f : \Re^J_+ \mapsto \Re_+$ and is continuous in d.

- Since U is concave and bounded (by B, say), we have that for any c:

$$0 \leq U(0) \leq U(c) + U'(c)(-c) \leq B - cU'(c) \Rightarrow cU'(c) \leq B, \forall c.$$

- It follows that the functions $g_j(d)$ are bounded, since they are nonnegative and their sum is bounded by βB.

Solving the Euler Equation

- T_i maps the set of continuous, bounded functions on R_+^J into itself.
- Evidently, solutions to $T_j f = f$ are solutions to $\left(EE_j'\right)$, and conversely.

Proposition

There is exactly one continuous, bounded solution f_j to $T_j f = f$. For any $f_0 \in \mathbb{C}$, $\lim_{n\to\infty} T^n f_0 = f_j$.

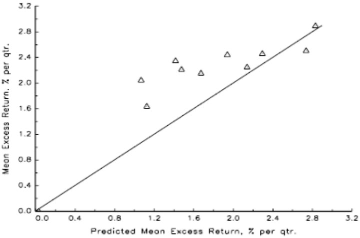
The Equity Premium Puzzle

- Over the period 1889 - 1978, the average annual return to stocks (S&P 500) has been about 7% per year while the average annual return to (3-month) Treasury bills has been only about 1% per year.

- Mehra and Prescott (JME, 1985) show that the difference in the covariance of these returns with consumption growth is only large enough to explain the difference in the average returns if the typical investor is implausible averse to risk.

Mean risk-free rate R_f	1.008
Mean return on equity $E\{R_e\}$	1.0698
Mean growth rate of consumption $E\{x\}$	1.018
Standard deviation of the growth rate of consumption $\sigma\{x\}$	0.036
Mean equity premium $E\{R_e\} - R_f$	0.0618

- $\delta = 0.98$ and $\gamma = 241$,



Mehra and Prescott (JME, 1985)

- Mehra and Prescott KEY assumptions

1. Individuals have preferences

$$U(c) = E_P\left(\sum_{t=0}^{\infty} \delta^t \frac{c_t^{1-\gamma}}{1-\gamma}\right) \quad \text{with } \gamma \geq 0.$$

2. Individual's consumption plan satisfies

$$\delta \cdot E_P\left[\left(\frac{c_{t+1}}{c_{i,t}}\right)^{-\gamma} \cdot R^s_{t+1} \middle| \mathcal{F}_t\right] = 1$$

$$\delta \cdot E_P\left[\left(\frac{c_{t+1}}{c_{i,t}}\right)^{-\gamma} \cdot R^f_t \middle| \mathcal{F}_t\right] = 1$$

3. The above conditions are satisfied for per capita consumption.

Assumptions to Obtain Analytical Formulas

1. Per capita consumption growth follows a two state Markov chain.
   - Constructed in such a way that the mean, variance, and autocorrelation are equivalent to their corresponding sample means in the US data.
2. The individuals know the realization of current and past g_t.
3. The growth rate of total dividends paid by the stocks in the S&P 500 is perfectly correlated with the growth rate of per capita consumption and the real return to the (nominally risk free) Treasury bill is perfectly correlated with the return to a bond that is risk free in real terms.

- Formulas for returns

$$R_t^f = \frac{1}{\delta \cdot \left[\pi_{s_t} \cdot g_H^{-\gamma} + (1-\pi_{s_t}) \cdot g_L^{-\gamma}\right]}$$

$$R_{t+1}^s \equiv \frac{p_{t+1}^s + D_{t+1} - p_t^s}{p_t^s} = g_{t+1} \cdot \frac{\rho(g_{t+1})+1}{\rho(g_t)} - 1$$

where $\rho\left(g_t\right) = \frac{\delta \cdot \left[\pi_{g_t} \cdot g_H^{1-\gamma} + (1-\pi_H) \cdot g_L^{1-\gamma} - \delta \cdot g_H^{1-\gamma} \cdot g_L^{1-\gamma} \cdot (\pi_H - \pi_L)\right]}{1-\delta \cdot \left[\pi_{s_t} \cdot g_H^{1-\gamma} + (1-\pi_{s_t}) \cdot g_L^{1-\gamma} - \delta \cdot g_H^{1-\gamma} \cdot g_L^{1-\gamma} \cdot (\pi_H - \pi_L)\right]}$.

Calibration

- Calibration:
  1. $g_H = 1 + \mu + \delta \quad g_L = 1 + \mu - \delta.$
  2. $\pi_H = 1 - \pi_L = \pi \quad \pi_L = 1 - \pi_H = 1 - \pi.$
  3. Choose (μ, δ, π) to match the sample for the US between 1889-1978.
     - $E(g_t) = 1.018, \sigma(g_t) = 0.036$ and $\rho(g_t, g_{t+1}) = -0.14.$
     - $\mu = 0.018,\ \delta = 0.036,\ \pi = 0.43.$

- Choose "reasonable" (microeconometric data and introspection) γ, δ:
  - $\delta \in (0, 1)$ and $0 \leq \gamma \leq 10$.

- For "reasonable" (γ, δ) such that $1.0 \leq R_t^f \leq 1.04\%$

$$E_P \left[\left(R_{t+1}^s - R_t^f \right) \middle| \mathcal{F}_t \right] \leq 0.35\%$$

while it is about 6% in the data!! $\rightarrow$ The (HIGH) Equity Premium Puzzle.

Hansen-Jagannathan Bounds

$$Cov_{P_i}(m_{t+1}, R_{j,t+1} | \mathcal{F}_t) = E_{P_i}(m_{t+1} R_{j,t+1} | \mathcal{F}_t) - E_{P_i}(m_{t+1} | \mathcal{F}_t) E_{P_i}(R_{j,t+1} | \mathcal{F}_t$$

$$Cov_{P_i}\left(m_{t+1}, R_{j,t+1} - R^f_t \middle| \mathcal{F}_t\right) = -E_{P_i}(m_{t+1} | \mathcal{F}_t) E_{P_i}\left(R_{j,t+1} - R^f_t \middle| \mathcal{F}_t\right)$$

$$E_{P_i}\left(R_{j,t+1} - R^f_t \middle| \mathcal{F}_t\right) = \frac{-Cov_{P_i}(m_{t+1}, R_{j,t+1} | \mathcal{F}_t)}{\sigma_{P_i}(m_{t+1} | \mathcal{F}_t) \cdot \sigma_{P_i}(R_{j,t+1} | \mathcal{F}_t)} \frac{\sigma_{P_i}(m_{t+1} | \mathcal{F}_t) \cdot \sigma_{P_i}(R_{j,t+1} | \mathcal{F}_t)}{E_{P_i}(m_{t+1} | \mathcal{F}_t)}$$

$$\left| E_{P_i}\left(R_{j,t+1} - R^f_t \middle| \mathcal{F}_t\right) \right| = |\rho_i(m_{t+1}, R_{j,t+1})| \cdot \frac{\sigma_{P_i}(m_{t+1} | \mathcal{F}_t) \cdot \sigma_{P_i}(R_{j,t+1} | \mathcal{F}_t)}{E_{P_i}(m_{t+1} | \mathcal{F}_t)}$$

- Since $|\rho_i(m_{t+1}, R_{j,t+1})| \leq 1$, then

$$\frac{\left| E_{P_i}(R_{j,t+1} | \mathcal{F}_t) - R^f_t \right|}{\sigma_{P_i}(R_{j,t+1} | \mathcal{F}_t)} \leq \underbrace{\frac{\sigma_{P_i}(m_{t+1} | \mathcal{F}_t)}{E_{P_i}(m_{t+1} | \mathcal{F}_t)}}_{\textit{Hansen-Jagannathan}\ \text{bound}}$$

Understanding the Problem

- Recall that:

$$\frac{\left|E_{P_i}\left[R_{j,t+1}\middle|\mathcal{F}_t\right]-R_t^f\right|}{\sigma_{P_i}\left(R_{j,t+1}\middle|\mathcal{F}_t\right)}\leq\frac{\sigma_{P_i}\left(m_{t+1}\middle|\mathcal{F}_t\right)}{E_{P_i}\left(m_{t+1}\middle|\mathcal{F}_t\right)}=\frac{\sigma_{P_i}\left(u'\left(c_{t+1}\right)\middle|\mathcal{F}_t\right)}{E_{P_i}\left(u'\left(c_{t+1}\right)\middle|\mathcal{F}_t\right)}$$

- $u'\left(c_{t+1}\right)\approx u'\left(E_t\left(c_{t+1}\right)\right)+\left[c_{t+1}-E_t\left(c_{t+1}\right)\right]u''\left(E_t\left(c_{t+1}\right)\right)$,

$$\frac{\left|E_{P_i}\left[R_{j,t+1}\middle|\mathcal{F}_t\right]-R_t^f\right|}{\sigma_{P_i}\left(R_{j,t+1}\middle|\mathcal{F}_t\right)}\leq\frac{\sigma_{P_i}\left(c_{t+1}\middle|\mathcal{F}_t\right)}{E_{P_i}\left(c_{t+1}\middle|\mathcal{F}_t\right)}\frac{E_{P_i}\left(c_{t+1}\middle|\mathcal{F}_t\right)\cdot u''\left(E_{P_i}\left(c_{t+1}\middle|\mathcal{F}_t\right)\right)}{u'\left(E_{P_i}\left(c_{t+1}\middle|\mathcal{F}_t\right)\right)}$$

- Over the last 50 years in the US, real stock returns have averaged 9%, with a st-dev of 16%, while the real return of treasury bills has been 1%.
  - The US market Sharpe ratio has been $8\%/16\% = 0.5$
  - The volatility of consumption growth is about 1% per year.
  - The mean of consumption growth is about 1.1% per year.
  - If agent i has correct beliefs ($P_i = P$), one need, at least, $RRA = 50$!!!

Abel (JEDC, 2002)

- Abel uses the Mehra-Prescott model with iid growth rates.
- The agent knows the growth rates are iid but has incorrect beliefs.
- The sdf is $m_{t+1} = \beta g_{t+1}^{-\gamma}$.
- $R_t^f = \frac{1}{\beta E_{P_i}\left(g_{t+1}^{-\gamma}\right)}$ $\qquad \hat{R}_t^f = \frac{1}{\beta E_P\left(g_{t+1}^{-\gamma}\right)}$
- $R_{t+1}^s = \frac{g_{t+1}}{\beta E_{P_i}\left(g_{t+1}^{1-\gamma}\right)}$ $\qquad \hat{R}_{t+1}^s = \frac{g_{t+1}}{\beta E_P\left(g_{t+1}^{1-\gamma}\right)}$
- $\frac{R_{t+1}^s}{R_{t+1}^f} = \frac{g_{t+1} E_{P_i}\left(g_{t+1}^{-\gamma}\right)}{E_{P_i}\left(g_{t+1}^{1-\gamma}\right)}$ $\qquad \frac{\hat{R}_{t+1}^s}{\hat{R}_{t+1}^f} = \frac{g_{t+1} E_{P_i}\left(g_{t+1}^{-\gamma}\right)}{E_P\left(g_{t+1}^{1-\gamma}\right)}$
- $\frac{1}{T}\sum_{t=1}^{T} \frac{R_{t+1}^s}{R_{t+1}^f} \rightarrow \frac{E_P(g_{t+1}) E_{P_i}\left(g_{t+1}^{-\gamma}\right)}{E_{P_i}\left(g_{t+1}^{1-\gamma}\right)}$

Dogmatic Pessimism

- Abel assumes the agent is pessimistic in that the consumer's subjective cumulative distribution of the growth rates F_i, is FOSD by the true cumulative distribution F.
- Since $g_{t+1}^{-\gamma}$ is a strictly decreasing function of g_{t+1}, pessimism implies $E_{P_i}\left(g_{t+1}^{-\gamma}\right)$ is larger than $E_P\left(g_{t+1}^{-\gamma}\right)$.
- Pessimism reduces the risk-free rate relative to the correct belief case.

Definition

The cumulative distribution $F_i(g)$ is characterised by uniform pessimism if $F_i(g) = F(e^{-\Delta} g)$ for some $\Delta > 0$.

- $E_{P_i}\left(g_{t+1}^{a}\right) = e^{-a\Delta} E(g^a)$ for any $a \in \Re$.

The Qualitative Effect of Dogmatic Pessimism

- Pessimism reduces the subjective average equity return & the risk-free:
- $E_{P_i}\left(R^s_{t+1}\right) = \frac{E_{P_i}(g_{t+1})}{\beta E_{P_i}\left(g^{1-\gamma}_{t+1}\right)} = \frac{e^{-\Delta}E_P(g_{t+1})}{\beta e^{-(1-\gamma)\Delta}E_P\left(g^{1-\gamma}_{t+1}\right)} = \frac{E_P(g_{t+1})}{\beta e^{\gamma\Delta}E_P\left(g^{1-\gamma}_{t+1}\right)}$
- $R^f_t = \frac{1}{\beta E_{P_i}\left(g^{-\gamma}_{t+1}\right)} = \frac{1}{\beta e^{\gamma\Delta}E_P\left(g^{-\gamma}_{t+1}\right)}$
- The subjective equity premium is identical to the equity premium in an economy with correct beliefs.
- $\frac{E_{P_i}\left(R^s_{t+1}\right)}{R^f_t} = \frac{E_P(g_{t+1})}{\beta E_P\left(g^{1-\gamma}_{t+1}\right)} = \frac{E_P\left(\hat{R}^s_{t+1}\right)}{\hat{R}^f_t}$
- The actual equity premium is:
- $\frac{1}{T}\sum_{t=1}^{T}\frac{R^s_{t+1}}{R^f_{t+1}} \rightarrow \frac{E_P(g_{t+1})E_{P_i}\left(g^{-\gamma}_{t+1}\right)}{E_{P_i}\left(g^{1-\gamma}_{t+1}\right)} = \frac{E_P(g_{t+1})e^{\gamma\Delta}E_P\left(g^{-\gamma}_{t+1}\right)}{e^{-(1-\gamma)\Delta}E_P\left(g^{1-\gamma}_{t+1}\right)} = \frac{E_P(g_{t+1})E_P\left(g^{-\gamma}_{t+1}\right)}{e^{-\Delta}E_P\left(g^{1-\gamma}_{t+1}\right)}$
- The observed equity premium, $\frac{1}{T}\sum_{t=1}^{T}\frac{R^s_{t+1}}{R^f_{t+1}}$, is larger than it would be if agents had correct beliefs as $\frac{1}{T}\sum_{t=1}^{T}\frac{\hat{R}^s_{t+1}}{\hat{R}^f_{t+1}} \rightarrow \frac{E_P(g_{t+1})E_P\left(g^{-\gamma}_{t+1}\right)}{E_P\left(g^{1-\gamma}_{t+1}\right)}$,

The Quantitative Effect of Dogmatic Pessimism

Table 1
Values of α^* and β^* under pessimism (Δ) and doubt (θ)

θ	Δ					
$SD^*\{X\}$ for $\Delta = 0$	0	0.005	0.01	0.02	0.03	0.055
0 0.0357	48.44 0.549	44.37 0.516	40.30 0.495	32.16 0.485	24.02 0.516	3.66 0.858
0.0010 0.0481	26.70 0.706	24.46 0.684	22.21 0.669	17.73 0.663	13.24 0.687	2.02 0.915
0.0020 0.0579	18.43 0.777	16.88 0.761	15.33 0.750	12.24 0.747	9.14 0.766	1.39 0.937
0.0040 0.0737	11.38 0.844	10.42 0.833	9.47 0.827	7.55 0.826	5.64 0.841	0.86 0.957
0.0060 0.0867	8.23 0.875	7.54 0.868	6.85 0.864	5.46 0.864	4.08 0.877	0.62 0.966
0.0100 0.1082	5.30 0.905	4.85 0.901	4.41 0.899	3.52 0.901	2.63 0.911	0.40 0.974
0.0150 0.1302	3.67 0.923	3.36 0.921	3.05 0.920	2.43 0.923	1.82 0.931	0.28 0.979
0.0163 0.1354	3.39 0.926	3.11 0.924	2.82 0.923	2.25 0.926	1.68 0.935	0.26 0.980

Cogley and Sargent, JME 2008

- Cogley and Sargent uses the Mehra-Prescott version of the Lucas' Tree model where the representative agent does not know the true distribution of the growth rates (as in Abel) but it puts positive probability on a set of distributions (unlike Abel) that contains the truth.
- Stock returns, R_t^s, satisfy the Euler equation:

$$E_{P_i}\left(m_{t+1} R_{t+1}^s - R_t^f \middle| \mathcal{F}_t \right) = 0$$

- This can be re-written as:

$$E_P\left(m_{t+1} \frac{P_{i,t+1}}{P_{t+1}} \left(R_{t+1}^s - R_t^f \right) \middle| \mathcal{F}_t \right) = 0$$

$$E_P\left(m_{t+1}^* \left(R_{t+1}^s - R_t^f \right) \middle| \mathcal{F}_t \right) = 0$$

- Using the H-J bounds we see why incorrect beliefs might help:

$$\frac{\left| E_P\left(R_{t+1}^s \middle| \mathcal{F}_t \right) - R_t^f \right|}{\sigma_P\left(R_{t+1}^s \middle| \mathcal{F}_t \right)} \leq \frac{\sigma_P\left(m_{t+1}^* \middle| \mathcal{F}_t \right)}{E_P\left(m_{t+1}^* \middle| \mathcal{F}_t \right)} = \frac{\sigma_P\left(m_{t+1} \frac{P_{i,t+1}}{P_{t+1}} \middle| \mathcal{F}_t \right)}{E_P\left(m_{t+1} \frac{P_{i,t+1}}{P_{t+1}} \middle| \mathcal{F}_t \right)}$$

H-J bounds with incorrect beliefs

- Cogley and Sargent assume the agent is risk neutral, $\gamma = 0$.
- Then, $m_{t+1} = \beta$ and $E_{P_i}\left(R^s_{t+1}\middle|\,\mathcal{F}_t\right) - R^f_t = 0$.
- No equity premium if agent had correct beliefs, $P = P_i$.
- If $P \neq P_i$,

$$\begin{aligned}\sigma_P(m^*_{t+1}) &= \beta\sigma_P\left(\frac{P_i(g_{t+1}|g_t, g^{t-1})}{P\left(g_{t+1}\middle|\, g_t\right)}\right)\\ E_P(m^*_{t+1}) &= \beta\end{aligned}$$

- Hence, $\dfrac{\sigma_P\left(m_{t+1}\frac{P_{i,t+1}}{P_{t+1}}\middle|\mathcal{F}_t\right)}{E_P\left(m_{t+1}\frac{P_{i,t+1}}{P_{t+1}}\middle|\mathcal{F}_t\right)} = \sigma_P\left(\frac{P_i(g_{t+1}|g_t,g^{t-1})}{P(g_{t+1}|g_t)}\right) > 0$

H-J bounds with incorrect beliefs

Table 2
Priors at the beginning of the training sample

$\mu_h \sim N(0.02251, 0.005^2)$
$\mu_l \sim N(-0.06785, 0.015^2)$
$\sigma_\varepsilon^2 \sim IG(0.03^2, 4)$
$F_{hh} \sim beta(1, 1)$
$F_{ll} \sim beta(1, 1)$

Table 3
Posterior means and standard deviations in 1933

	F_{hh}	F_{ll}	μ_h	μ_l	σ_ε
1890–1933	0.977 (0.023)	0.693 (0.269)	2.03 (0.41)	−6.40 (1.45)	4.34 (0.50)
1919–1933	0.915 (0.081)	0.805 (0.191)	2.32 (0.48)	−5.98 (1.25)	3.54 (0.85)

Table 7
Mean excess returns

	Full sample	First half	Second half
Data: 1934–2003	**0.0680**	**0.0990**	**0.0360**
Prior: 1890–1933			
Beta	0.0090	0.0153	0.0027
$2\log B = 5$	0.0317	0.0485	0.0148
$2\log B = 10$	0.0515	0.0747	0.0283
Prior: 1919–1933			
Beta	0.0267	0.0425	0.0109
$2\log B = 5$	0.0421	0.0566	0.0277
$2\log B = 10$	**0.0511**	**0.0654**	**0.0369**