Market Selection and the Evolution of Bargaining Power in Labor Markets

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April 28, 2025

Abstract

How does firms' bargaining power impact outcomes in labor markets? How does bargaining power evolve driven by market forces? The goal of this paper is to understand the factors determining the distribution of bargaining power in an economy and to seek explanation for the observed empirical facts. To answer the first question, we identify conditions on the firm's technology under which profits are non-monotonic in bargaining power and thus, the firms obtaining maximal profits will not be those with a bargaining power of one. In as far as the optimal bargaining power is technology-dependent, this might explain the observed variability of bargaining power across industries. We also examine the dependence of the optimal bargaining power on market conditions, specifically, on the price of the output good. To address the second question, we propose a dynamic model of an industry with endogenous entry and exit, heterogeneity with respect to bargaining power and demand subject to exogenous shocks. The Markov equilibrium characterizes the long-run distribution of bargaining power. The model allows for heterogeneity in bargaining power within a given period. Furthermore, the distribution of bargaining power can vary in time with the state of the economy. These features allow us to explain observed variability of bargaining power across firms and relate the labor wedge to the state of the economy.

1 Introduction

Institutions provide a framework for economic activity and regulate the behavior of economic agents. In most economic models, the institutional framework is taken as given. Yet, research on political and economic institutions has shown

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that suboptimal institutions can emerge and persist over time leading to inefficient outcomes, see Acemoglu (2006a, b), Acemoglu and Robinson (2008). In this paper, we address the question of institution formation, and in particular, the evolution of bargaining power, in the labor market from an evolutionary perspective.

The two central questions we ask are: How does firms' bargaining power impact outcomes in labor markets? How does bargaining power evolve driven by market forces? The goal is to understand the factors determining the distribution of bargaining power in an economy, to model the interplay between firms' bargaining power and market outcomes and to seek explanation for the observed empirical facts, both in a static, as well as in a dynamic setting.

Labor markets often exhibit some degree of imperfect competition: employers or workers get rents from an existing relationship. It is thus necessary to explain how wages are determined. One of the workhorse models of wage determination in the macroeconomic literature is the Nash Bargaining model where the wage is split according to some sharing rule depending on the bargaining powers of employers and workers. In a seminal paper, Stole and Zwiebel (1996) invoke the Nash bargaining solution to provide a framework of intra-firm bargaining when the firm bargains with individual workers (as opposed to unions) and contracts are incomplete and leave room for renegotiation. As noted by Elsby and Michaels (2013), whenever the marginal productivities of the workers are not independent, the Stole and Zwiebel framework is better suited to capture the interaction between the hiring decision of the firm and its effect on the surplus generated by each of the workers. In particular, this model ilustrates that such interdependence leads to inefficiency in production decisions: firm overhires (or underhires) labor and thus paying a wage above (below) the value of marginal product. Bargaining power determines thus not only the division, but also the size of the surplus. This can have far-reaching implications for the efficiency of labor markets, see Cahuc et al. (2008). Understanding the factors that determine bargaining power is thus crucial for predicting labor market outcomes, as well as for making informed policy decisions.

In theoretical models, little attention has been paid to the emergence and sources of bargaining power. Nash's (1952) normative approach to bargaining proposed equal bargaining powers. In contrast, strategic models such as Binmore, Rubinstein and Wollinsky (1986) emphasize the effect of institutional and preference parameters (the probability with which a new proposal can be made, risk aversion, patience, status-quo bias) on the resulting division. In one interpretation of their model, out-of equilibrium (and therefore, non-verifiable) beliefs about the probability of a negotiation break-down pin down the distribution of the surplus, and thus, the effective bargaining powers of the parties. However, relating bargaining power to technological or preference parameters which themselves are exogenous only pushes the assumption back one level, Rogerson, Shimer and Wright (2005). In the labor market literature, Nash bargaining with equal bargaining powers is often used. In contrast, Hossios (1990) demonstrates that matching externalities render an equal split of the surplus inefficient, both from an individual and a social point of view. Bental and Demougin (2010) model the choice of bargaining power by a social planner who trades-off the interests of the workers and the firms in a setting with moral hazard and hold-up problems. In both cases, the socially optimal bargaining power is neither maximal, nor does it result in equal split.

From an empirical point of view, it has been long recognized that labor market institutions impact bargaining power and are a major determinant of equilibrium outcomes, both in the short- and in the long-run, Breda (2015), Botero et al. (2004) and Caballero et al. (2013). Stansbury and Summers (2018) show that the distribution of bargaining power in labor markets is currently undergoing major changes due to: changes in labor law and the degree of unionization; changes in intra-firm institutions such as shareholder power and shareholder activism; and technological changes, see also Acemoglu and Restrepo (2019). Measuring bargaining power has presented a challenge. A large literature has measured and documented a positive rent-sharing elasticity, see Table 1 in Card et al. (2016), without however disentangling the role of bargaining power from other factors. Most works instrumentalize bargaining power by the labor market power of the firms, Berger et al. (2022), Azar et al. (2019), Mertens (2022), or by the worker's outside option, Caldwell and Danieli (2022), Caldwell and Harmon (2019). The theoretical framework of Stole and Zwiebel (1996) however shows that these variables have distinct impact on outcomes from that of the actual bargaining power.

Three studies have measured bargaining power directly by estimating the share of the surplus obtained by the workers, Cahuc et al. (2006), Wong (2021) and Cheremukhin and Restrepo-Echavarria (2014). While firms tend to have substantial bargaining power, it is in general different from 1. Furthermore, bargaining power exhibits large variability across industries, Cahuc et al. (2006), as well as within an industry, Wong (2021). These results are at odds with the idea that markets select for firms with maximal bargaining power.

Even more puzzling is the fact that the fraction of the aggregate surplus obtained by the workers in the economy (the so-called labor wedge) varies over time and exhibits a countercyclical behavior, Shimer (2010). This presents a major theoretical challenge. Examining possible causes, Shimer (2010, p. 17) notes: "A closely related theoretical possibility is that workers have time-varying market power in labor supply. [...] Recessions are periods when different types of labor are poor substitutes, so households are better able to exploit their market power, reducing hours to drive up wages". Indeed, Cheremukhin and Restrepo-Echavarria (2014) document such a countercyclical variation of the workers' bargaining power. Yet, so far a mechanism which would explain the persistent heterogeneity of bargaining power in labor markets, as well as a variation in the distribution of bargaining powers related to the business cycle has not been identified.

In this paper, we wish to understand the evolution of bargaining power in labor markets with the aim to provide an explanation of the observed empirical phenomena discussed above. We study a model, with the following properties. First, in a static framework, the model establishes a link between the firm's bargaining power and its profits. For a given technology and given price of output, it determines the profit-maximizing bargaining power of the firm. Second, the model generates a selection dynamic operating on the firms' profits, which, through the dependence of profits on bargaining power, favors those firms whose bargaining power is closer to the optimal. The dynamic equilibrium determines the long-run distribution of bargaining powers in the economy. We wish to understand whether this distribution can be non-degenerate and evolve with the state of the economy. If so, the model could help explain the empirical stylized facts enumerated above.

In a first step, we are interested in the impact of the bargaining power of the firm on market outcomes. To capture the fact that labor contracts are incomplete and renegotiation can occur at any time, we work with the bargaining setting of Stole and Zwiebel (1996). Using the solution techniques developed by Cahuc et al. (2008), we analyze the hiring decision of a firm employing two types of labor, high-skilled and low-skilled. When firm's bargaining powers are different from 1, the production decision of the firm is in general inefficient and a profit-maximizing production plan may exhibit under- or overemployment as compared to the decision of a neo-classical firm as captured by the overemployment factor defined by Cahuc et al. (2008). A change in the bargaining power of one type of workers impacts profits by affecting the overemployment factors of the two types of labor. Defining the elasticities of overemployment with respect to the bargaining power of the workers, we can characterize the profit-maximizing bargaining powers. A necessary condition for the firm's profit to increase in the bargaining power of low-skilled workers is for the overemployment elasticity (w.r.t. the bargaining power of low-skilled workers) of low-skilled workers to be positive, while the overemployment elasticity of high-skilled workers to be negative. An increase in bargaining power thus leads to a reduction in the overemployment of high-skilled workers and an increase in the overemployment of low-skilled workers. This can bring the firm's labor demand closer to the optimal one and increase profits provided that initially the firm is overemploying skilled workers and underemploying unskilled ones. We show that this is indeed the case when two conditions are met. First, institutional constraints restrict the firm's bargaining power with respect to high-skilled workers, and, second, labor inputs are complements in the production. In contrast, if the bargaining power of both types of workers is unconstrained or if the labor inputs are substitutes, the profit-maximizing bargaining power of the firm is 1.

Assuming that the bargaining power w.r.t. one type of labor is constrained, e.g., due to the nature of the employment relationship and underlying informational deficits, to stronger unionization or stakeholder participation, implies that the profit-maximizing bargaining power depends on the production technology. The possibility that institutional constraints may differ across industries provides a second source of variability worth studying. These findings thus provide insight into explaining the empirically observed distribution of bargaining power across industries in a static context.

The static model also allows us to examine the dependence of the optimal bargaining power on the market price of output. For homothetic production functions, the optimal bargaining power is independent of the price. Yet, for the non-homothetic generalized CES production function (see, e.g., Sato, 1977), the optimal bargaining power depends on the price of output. Hence, shocks in economic fundamentals will affect the profit-maximizing bargaining power within a given industry.

The second part of the paper studies the evolution of bargaining power. We construct a dynamic entry-exit model of an industry subject to demand shocks, similar in spirit to Hopenhayn (1992). The potential pool of firms exhibits bargaining power heterogeneity: while bargaining power w.r.t. one type of workers is constant and identical for all firms, firms differ with respect to their bargaining power w.r.t. the other type of workers. To be active in the market, firms have to pay a fixed cost in each period. An exit decision is irreversible.

In each period, the pool of potential entrants contains the firms who were active in the previous period and newcomers, whose number and distribution are exogenously fixed. Each firm decides whether to pay the fixed cost to stay in / enter the market and if so, makes hiring and production decisions as in the static model studied in the first part. The demand function for the output of the firms is assumed given and can depend on the realization of an exogenous state. The equilibrium price is determined so that the market for output clears. The entry decision is forward-looking and made under perfect foresight, taking into account the equilibrium sequence of state-dependent prices.

We propose to study Markovian equilibria of this economy described, for each observed history, by the entry / exit decisions of the firms, the market composition and the equilibrium price of output.

We first consider the case of certain demand. Even if the initial distribution of bargaining power is non-degenerate, in the long-run steady state, a single type of firms dominates the market: those whose bargaining power maximizes profits. This steady state is also globally stable. In general, the long-run bargaining power is strictly lower than 1 and depends on the production technology and the institutional constraints within the industry. We thus obtain a dynamic foundation for the first two stylized facts noted earlier: evolution does not select for the firms with the highest bargaining power, and bargaining power may vary across industries even in the long-run. This model, however, does not explain the within-industry-heterogeneity, nor the dynamic effects documented by Shimer (2010) and Cheremukhin and Restrepo-Echavarria (2014).

We thus consider the dynamics with demand shocks. We construct Markovian equilibria, in which the set of firms active in the market (in the long-run) and thus, the distribution of bargaining powers, depends on the state of the economy. In particular, for non-homothetic production functions, two types of equilibria may exist[1]. In a regular equilibrium, the optimal bargaining power is constant over the equilibrium price range. This means that if a given state persists fo a large enough number of periods, a single type of firms dominates the market: those with the profit-maximizing (but not necessarily maximal) bargaining power. However, a transition from a low-demand state to a high-

[1]These equilibria occur for distinct parameter values: we have not been able to identify a case in which two equilibria co-exist.

demand state leads to an increase in output price and, when the pool of potential entrants is limited, allows firms with non-profit-maximizing bargaining power to enter temporarily the market. These firms are active during the initial stages of the boom and are eventually driven out of the market by the additional entry of more-competitive firms with profit-maximizing bargaining power. During such periods the distribution of bargaining power is non-degenerate. The Markov process defined by the equilibrium specifies a non-trivial invariant distribution of bargaining powers. Notably, average bargaining power of the workers is maximal during recessions, but initially drops and then increases over the duration of a boom. This type of equilibria provides a first insight into the possibility that the (average) bargaining power of workers can vary across time in a way correlated with the business cycle.

A second and more interesting type of equilibria may exist when the optimal bargaining power depends on the price in the relevant equilibrium price range. In this equilibrium, the two states (recession and boom) will attract different types of firms with respect to their bargaining power: less efficient firms with lower bargaining power of the workers are present in the market during recessions, whereas the more profitable firms, which have higher bargaining power of the workers enter the market during booms. The invariant distribution of bargaining power again exhibits cyclical variation. Furthermore, the long-run distribution of bargaining power within a given state may also be non-degenerate: firms might optimally choose to stay in the market making current losses while expecting a change in the economic fundamentals which would render them more profitable and compensate (in expectation) for the current loss. This suggests that the observed variation of bargaining power within a given industry may be consistent with an entry-exit model with demand shocks and a non-homothetic production function.

Finally we discuss the implications of the model presented here for empirical work. First, the model identifies a novel (and testable) relation between the production function, the elasticity of overemployment with respect to bargaining powers and the profit-maximizing bargaining powers for a given industry. Second, it has implications for the dependence between the extent to which a given sector is exposed to aggregate shocks and the heterogeneity of bargaining powers observed among the firms in this sector. Finally, the dynamics studied in this model endogenizes the exogenous wage markup shocks often used in the literature, see Smets and Wouters (2007), Cheremukhin and Restrepo-Echavarria (2014). An important insight of the model is that the dynamics of the labor-wedge is both firm- and labor-type specific (it may be procyclical for unskilled labor and countercyclical for skilled labor). Thus, to understand the dynamics, both the aggregate effect due to the exogenous demand shock as well as the composition effect due to firm heterogeneity and entry-exit decisions have to be taken into account.

Need to talk about:

- Dependence between the production function and the elasticity of overemployment and the bargaining powers within an industry:

$$\frac{\beta_u}{\bar{\beta}_s} = -\frac{\eta_{u,s}}{\eta_{u,u}} \frac{w_s \ell_s^*}{w_u \ell_u^*}$$

- Relationship between volatility and heterogeneity
  - demand in sectors with higher income elasticity is more sensitive to aggregate shocks
  - such sectors will have heterogeneous bargaining powers within states and across states

2 Intra-Firm Bargaining and the Optimal Demand for Labor

2.1 The Model of the Firm, (Stole and Zwiebel, 1996)

We consider a firm which produces an output according to a technology:

$$f(\ell_u, \ell_s) : \mathbb{R}_+^2 \to \mathbb{R}$$

with ℓ_i – amount of labor of type $i \in \{u, s\}$, where u stands for unskilled and s for skilled labor. We will use the notation $-i$ to denote the type of worker different from i, i.e., $-i = s$ if $i = u$ and $-i = u$ if $i = s$. As usual, we assume that f is twice continuously differentiable, strictly increasing and concave in ℓ_u and ℓ_s and that $f(0, 0) = 0$.

The reservation wage of worker of type i is given by $\bar{w}_i > 0$, $i \in \{u, s\}$. The workers' bargaining power with respect to the firm is given by $\beta_i \in [0, 1]$, $i \in \{u, s\}$. If $\beta_i = 0$, workers of type i have no bargaining power, and the firm can extract the entire surplus generated by the worker. This is tantamount to a competitive market for labor in which workers take the reservation wage to be the market wage, see see Stole and Zwiebel (1996). If $\beta_i = 1$, the firm has no bargaining power and the workers receive the entire surplus of the interaction. Whenever convenient, we will use the notation $\beta = (\beta_u, \beta_s)$.

Contracts are incomplete and renegotiation can occur at any time before production starts. Stole and Zwiebel (1996) consider an extensive-form bargaining game[2] between the firm and the two types of workers with the property that each party can choose to renegotiate the contract at any time before production starts. An equilibrium of this game determines *a wage schedule*, which specifies for any given hiring decision (ℓ_u, ℓ_s), (where ℓ_i is the amount of labor of type i hired), the wages paid to the two types of workers:

$$w(\ell_u, \ell_s) = (w_u(\ell_u, \ell_s), w_s(\ell_u, \ell_s)) : \mathbb{R}_+^2 \to \mathbb{R}_+^2.$$

[2]In fact the equilibrium of the game originally described by Stole and Zwiebel is different from the stable wage schedule they derive. Brügemann et al. (2019) propose a modified bargaining game the equilibrium of which is indeed given by the stable wage schedule.

Stole and Zwiebel are in particular interested in the *stable wage schedule* in the sense that "prior to production, no individual employee can benefit from renegotiating wages with the firm, and the firm cannot benefit from renegotiation with an employee given the further wage negotiations that such a renegotiation would induce", see Stole and Zwiebel (1996, p. 198).

For a given hiring decision (ℓ_u, ℓ_s), a given wage schedule $(w_u(\cdot), w_s(\cdot))$, and price of the output p, the profit of the firm is given by:

$$\pi(\ell_u, \ell_s, p) = pf(\ell_u, \ell_s) - w_u(\ell_u, \ell_s)\,\ell_u - w_s(\ell_u, \ell_s)\,\ell_s$$

The total marginal surplus generated by a worker of type i is given by:

$$\underbrace{p\frac{\partial f(\ell_u, \ell_s)}{\partial \ell_i}}_{\substack{\text{value of marginal}\\ \text{product}}} - \underbrace{\frac{\partial w_i(\ell_u, \ell_s)}{\partial \ell_i}}_{\substack{\text{change in}\\ \text{type } i \text{ wages}}}\ell_u - \underbrace{\frac{\partial w_{-i}(\ell_u, \ell_s)}{\partial \ell_i}}_{\substack{\text{change in}\\ \text{type } -i \text{ wages}}}\ell_s - \underbrace{\bar{w}_i}_{\substack{\text{reservation}\\ \text{wage}}} \tag{1}$$

Notably, the hire of a new worker may affect the marginal productivities of the already hired workers and thus, lead to wage renegotiations. Thus, the surplus generated by a worker also includes the change in total wages as a result of the new hire (the second and the third term on the r.h.s. of (1)). A stable wage schedule takes into account these additional effects for the allocation of the generated surplus.

Definition 1 *A stable wage schedule* $(w_u(\ell_u, \ell_s), w_s(\ell_u, \ell_s)) : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ *satisfies for all* (ℓ_u, ℓ_s)*:*

$$\underbrace{w_i(\ell_u, \ell_s) - \bar{w}_i}_{\substack{\textit{worker's}\\ \textit{surplus}}} = \beta_i \underbrace{\left[p\frac{\partial f(\ell_u, \ell_s)}{\partial \ell_i} - \frac{\partial w_i(\ell_u, \ell_s)}{\partial \ell_i}\ell_u - \frac{\partial w_{-i}(\ell_u, \ell_s)}{\partial \ell_i}\ell_s - \bar{w}_i\right]}_{\textit{total marginal surplus}}. \tag{2}$$

Cahuc et al. (2008) solve the system of differential equations which defines the stable wage schedule obtaining the following condition for wages:

$$w_i(\ell_u, \ell_s) = (1 - \beta_i)\,\bar{w}_i + p\int_0^1 z^{\frac{1-\beta_i}{\beta_i}} \frac{\partial f\left(z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_u}{1-\beta_u}}\ell_u, z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_s}{1-\beta_s}}\ell_s\right)}{\partial \ell_i} dz \tag{3}$$

Given the stable wage schedule, $w[\beta, p]$ as determined by (3), the firm maximizes profits taking the price of output $p > 0$ as given:

$$\pi(\beta, p) \equiv \max_{(\ell_u, \ell_s)\in\mathbb{R}^2_+} pf(\ell_u, \ell_s) - w_u[\beta, p](\ell_u, \ell_s)\,\ell_u - w_s[\beta, p](\ell_u, \ell_s)\,\ell_s \tag{4}$$

Stole and Zwiebel (1996) show that at the optimum labor inputs $(\ell_u(\beta, p), \ell_s(\beta, p))$ are chosen so that marginal surplus for each type of workers $i \in \{u, s\}$ is 0:

$$p\frac{\partial f(\ell_u(\beta,p), \ell_s(\beta,p))}{\partial \ell_i} - \frac{\partial w_i(\ell_u(\beta,p), \ell_s(\beta,p))}{\partial \ell_i}\ell_i(\beta, p) - \frac{\partial w_{-i}(\ell_u(\beta,p), \ell_s(\beta,p))}{\partial \ell_i}\ell_{-i}(\beta, p) - \bar{w}_i = 0 \tag{5}$$

and thus, by (2), wages equal reservation wages:

$$w_i\left[\beta,p\right]\left(\ell_u\left(\beta,p\right),\ell_s\left(\beta,p\right)\right)=\bar{w}_i \tag{6}$$

Whenever the workers' bargaining powers are non-zero, the optimal hiring decisions of the firm is inefficient in that the reservation wage does not equal the marginal product of labor. To capture these effects, Cahuc et al. (2008) define the overemployment factor as:

$$\gamma_i\left(\beta,p\right)=\frac{\bar{w}_i}{p\frac{\partial f\left(\ell_u(\beta,p),\ell_s(\beta,p)\right)}{\partial \ell_i}} \tag{7}$$

The overemployment factor for type i is above 1 if the marginal product of the type i worker is smaller than its wage and below 1 otherwise. To understand the source of this inefficiency, it is useful to consider the case of a single type of workers: individual workers are therefore substitutes in the production. Thus, hiring an extra worker leads to a decrease of the marginal productivity of the workers hired previously and allows the firm to reduce the wage paid. In this case, as shown by Stole and Zwiebel (1996), overemployment occurs at the optimum. As pointed out by Cahuc et al. (2008), underemployment may occur in the case of two labor inputs and complementarities in the production.

2.2 Profit-Maximizing Bargaining Powers

We now study the indirect profit function of the firm $\pi\left(\beta,p\right)$ as a function of the bargaining powers of the workers and identify the profit-maximizing bargaining powers. We do not mean by this that the firm has the ability to choose its bargaining power. Instead, we view the bargaining power of the workers within the firm as a result of the interaction of external factors (such as asymmetry of information, moral hazard, etc.) and (potentially suboptimal) intra-firm institutions which govern the allocation of resources within the firm in view of the existing frictions. Although the firm might be able to take actions to manipulate its bargaining power relative to the workers it employs, such choices might be subject to constraints which prevent the firm bargaining power from being profit-maximizing. Thus, we ask the question from a descriptive point of view: If heterogeneous firms interact in a market, which combination of bargaining powers leads to maximal profits?

We start with the case where the range of bargaining powers is unconstrained between 0 and 1. In this case, profits as a function of β are maximized when the firm mimics the behavior of a neo-classical firm and neither over-, nor underemploys the two types of labor. Our next Lemma shows that this occurs precisely when workers have no bargaining power:

Lemma 2 *The indirect profit is maximized in* β *when* $\beta_u=\beta_s=0$ *and thus,* $\gamma_u\left(\beta,p\right)=\gamma_s\left(\beta,p\right)=1$.

We next explicitly model constraints on the bargaining power of the workers within a given industry by assuming that the bargaining power of one labor type $(-i)$ is fixed at $\bar{\beta}_{-i} \in [0,1]$, and only the bargaining power of the other type of workers (i) may vary across firms. The bargaining power $\beta_i^*\left(p, \bar{\beta}_{-i}\right)$, which leads to maximal profits for the given constraint $\bar{\beta}_{-i}$ and a given market price of the output p is the solution to:

$$\max_{\beta_i \in [0,1]} \left\{\pi\left(\beta, p\right) \mid \beta_{-i} = \bar{\beta}_{-i}\right\} \tag{8}$$

Denote the profit-maximizing bargaining powers by $\beta^*(p) \equiv \left(\beta_i^*\left(p, \bar{\beta}_{-i}\right), \bar{\beta}_{-i}\right)$.

By Lemma 2 $\bar{\beta}_{-i} = 0$ implies $\beta_i^*(p, 0) = 0$. In general, however, the profit-maximizing bargaining power may be distinct from 0. The following result identifies conditions under which this may occur. To state the Proposition, denote by

$$\eta_{i,j} = \beta_i \frac{\frac{\partial \gamma_j(\beta,p)}{\partial \beta_i}}{\gamma_j\left(\beta, p\right)} \tag{9}$$

the elasticity of the overemployment factor of labor type $j \in \{u, s\}$ with respect to the bargaining power of type $i \in \{u, s\}$ workers.

Proposition 3 *An interior solution of (8) satisfies:*

$$\frac{\beta_i}{\beta_{-i}} = -\frac{\eta_{i,-i}\left(\beta, p\right)}{\eta_{ii}\left(\beta, p\right)} \frac{\bar{w}_{-i}\ell_{-i}\left(\beta, p\right)}{\bar{w}_i \ell_i\left(\beta, p\right)} \tag{10}$$

If the two labor inputs are substitutes, then for all $\beta \in [0,1]^2$, $\frac{\partial \pi(\beta,p)}{\partial \beta_i} < 0$ *and the profit-maximizing bargaining power is* $\beta_i^*\left(p, \bar{\beta}_{-i}\right) = 0$.

Necessary conditions for profits to be increasing in β_i *are that* $\eta_{ii}(\beta, p) > 0$ *and inputs are complements:* $\eta_{i,-i}(\beta, p) < 0$. *For* $\ell_{-i}\left(\beta_i, \bar{\beta}_{-i}, p\right) > 0$ *a sufficient condition for profits to be increasing in* β_i *is that*

$$\frac{\beta_i \ell_i\left(\beta_i, \bar{\beta}_{-i}, p\right) \bar{w}_i}{\ell_{-i}\left(\beta_i, \bar{\beta}_{-i}, p\right) \bar{w}_{-i}} < -\bar{\beta}_{-i} \frac{\eta_{i,-i}\left(\beta_i, \bar{\beta}_{-i}, p\right)}{\eta_{i,i}\left(\beta_i, \bar{\beta}_{-i}, p\right)} \tag{11}$$

When $\bar{\beta}_{-i} > 0$ *and* $\ell_{-i}\left(0, \bar{\beta}_{-i}, p\right) > 0$, *necessary and sufficient conditions for an interior optimum* $\beta_i^*\left(p, \bar{\beta}_{-i}\right) > 0$ *for a given* $\bar{\beta}_{-i}$ *are that* $\eta_{i,-i}\left(0, \bar{\beta}_{-i}, p\right) < 0$ *and* $0 < \eta_{ii}\left(0, \bar{\beta}_{-i}, p\right) < \infty$.

Proposition 3 describes the optimal bargaining power β_i for a given $\bar{\beta}_{-i}$ in terms of the bargaining-power elasticities of the overemployment functions γ_i and γ_{-i}. In an interior solution, the ratio of bargaining powers is exactly equal to the (negative of) the inverse of the ratio of the overemployment elasticities multiplied by the inverse of the ratio of the firm's expenditures for the two types of labor.

When labor inputs are substitutes, both factors are overemployed for any values of β, see Cahuc et al. (2008). The indirect profit function is therefore decreasing in β_i (regardless of the fixed bargaining power $\bar{\beta}_{-i}$) and the optimum is obtained at 0. This confirms the common intuition that increasing the bargaining power of the firm with respect to any type of workers results in higher profits.

Complementarity of labor inputs is thus necessary for profits to increase in the bargaining power β_i. In this case, the overemployment of factor $-i$ decreases with the bargaining power of type i, β_i, and $\eta_{i,-i} < 0$. For profits to increase, it must be that the increase in β_i simultaneously "decreases the underemployment" of factor i, ($\eta_{ii} > 0$), thus bringing both overemployment factors closer to 1 and the allocation of inputs closer to the unconstrained optimum as in Lemma (2). Condition (11) is equivalent to the requirement that the ratio of these two effects exceeds the ratio of labor cost for the two factors weighted by their respective bargaining powers.

When the constraint on the bargaining power $\bar{\beta}_{-i} > 0$ is binding and when optimal labor output of this type of labor is strictly positive at $\beta_i = 0$, the two necessary conditions – complementarity and positive (but not infinite) i-overemployment elasticity with respect to β_i are sufficient for an optimal interior bargaining power of i.

The intuition for the result in Proposition 3 is simple. First consider the case of substitutes. If the bargaining power β_i increases, the share of the surplus attributed to i-type workers increases and so does their wage. To countervene this effect, the firm may either hire more workers i, which will drive down their marginal productivity and allow the firm to offer them a lower wage. Alternatively, the firm may higher more workers of type $-i$. By factor substitutability, this also leads to a decrease in the marginal productivity of i-workers and drives down their wage. Thus, an increase in β_i leads to an increase in overemployment of both types of labor: both the own bargaining power elasticity $\eta_{ii}(\beta, p)$ and the cross bargaining power elasticity $\eta_{i,-i}(\beta, p)$ of overemployment have positive signs. An interior solution can therefore not obtain in this case.

In the case of complements, an increase in β_i can again be offset by hiring more type i workers to lower their wage. However, the same effect may also be obtained by hiring fewer workers of type $-i$. Thus, the own overemployment elasticity $\eta_{ii}(\beta, p)$ is positive, whereas the cross-elasticity of overemployment $\eta_{i,-i}(\beta, p)$ is negative, which allows for an interior solution.

We now illustrate these results for the case of the Cobb-Douglas production function, $f(\ell_u, \ell_s) = \ell_u^{\alpha_u} \ell_s^{\alpha_s}$ discussed above. In this case inputs are complements, and we can explicitly compute the profit-maximizing bargaining power $\beta_i^*\left(p, \bar{\beta}_{-i}\right)$ and state the conditions for it to be interior. In this special case, both the overemployment elasticities and the optimal input ratio, $\frac{\ell_i}{\ell_{-i}}$, are independent of p. Thus, the profit-maximizing bargaining power is independent of the ouput price.

Example 4 *Optimal bargaining power with Cobb-Douglas production function*[3]
Let $f(\ell_u, \ell_s) = \ell_u^{\alpha_u} \ell_s^{\alpha_s}$ *with* $\alpha_u > 0$, $\alpha_s > 0$, $\alpha_u + \alpha_s < 1$. *In this case, using the stable wage schedule in (3) derived by Cahuc et al. (2008) and the fact that*

[3]The derivations for this example are contained in Appendix B.

at the firm's optimum wages equal reservation wages, we show in Appendix B that the overemployment (defined in (7)) for $i \in \{u, s\}$ are:

$$\gamma_i(\alpha, \beta) = \frac{1 - \beta_{-i}}{1 + (1 - \alpha_i - \alpha_{-i})\beta_i\beta_{-i} - (1 - \alpha_i)\beta_i - (1 - \alpha_{-i})\beta_{-i}}$$

and using the definition of overemployment elasticities in (9), we obtain:

$$\eta_{ii}(\beta, p) = \beta_i \frac{\frac{\partial \gamma_i(\beta,p)}{\partial \beta_i}}{\gamma_i(\beta, p)} = \beta_i \frac{[(1 - \alpha_i)(1 - \beta_{-i}) + \alpha_{-i}\beta_{-i}]}{[1 + (1 - \alpha_i - \alpha_{-i})\beta_i\beta_{-i} - (1 - \alpha_i)\beta_i - (1 - \alpha_{-i})\beta_{-i}]} > 0$$

$$\eta_{i,-i}(\beta, p) = \beta_i \frac{\frac{\partial \gamma_{-i}(\beta,p)}{\partial \beta_i}}{\gamma_{-i}(\beta, p)} = -\beta_i \frac{\alpha_i(1 - \beta_{-i})}{(1 - \beta_i)[1 + (1 - \alpha_i - \alpha_{-i})\beta_i\beta_{-i} - (1 - \alpha_i)\beta_i - (1 - \alpha_{-i})\beta_{-i}]} < 0$$

Thus, the own overemployment elasticity is positive, whereas the cross-overemployment elasticity is negative, as implied by Proposition 3 for the case of complements.

*Using the optimal demands for labor, (**??**) derived in Appendix B, we conclude that the profit-maximizing bargaining power $\beta_i^*(\bar{\beta}_{-i}, p)$ is determined by:*

$$\beta_i^*(\bar{\beta}_{-i}, p) = \beta_i^*(\bar{\beta}_{-i}) = \frac{\alpha_{-i}\bar{\beta}_{-i}}{(1 - \alpha_i)(1 - \bar{\beta}_{-i}) + \alpha_{-i}\bar{\beta}_{-i}} \qquad (12)$$

Note that for decreasing returns to scale, $\beta_i^*(\bar{\beta}_{-i}) \leq \bar{\beta}_{-i}$ obtains (with strict inequality except at $\bar{\beta}_{-i} = 0$).

We also have:

$$\frac{\partial \pi(\beta, p)}{\partial \beta_i} > 0 \Leftrightarrow \frac{\beta_i}{\beta_{-i}} < -\frac{\eta_{i,-i}(\beta, p)}{\eta_{ii}(\beta, p)} \frac{\bar{w}_{-i}\ell_{-i}(\beta, p)}{\bar{w}_i\ell_i(\beta, p)} \Leftrightarrow \frac{\beta_i}{1 - \beta_i} < \frac{\alpha_{-i}}{1 - \alpha_i} \frac{\beta_{-i}}{1 - \beta_{-i}} \Leftrightarrow \gamma_i(\alpha, \beta) < 1,$$

i.e. profits are increasing in the bargaining power of workers of type i exactly when labor input i is underemployed, and thus, as shown in Appendix B, the labor input $-i$ is overemployed. Thus,

$$\frac{\partial \pi(\beta, p)}{\partial \beta_i} > 0 \Longrightarrow \frac{\partial \pi(\beta, p)}{\partial \beta_{-i}} < 0. \qquad (13)$$

The following table summarizes these results.

$\frac{\beta_u}{1-\beta_u} \in \left[0, \frac{\beta_s}{1-\beta_s}\frac{\alpha_s}{1-\alpha_u}\right)$	$\frac{\partial \pi(\cdot)}{\partial \beta_u} > 0$	$\frac{\partial \pi(\cdot)}{\partial \beta_s} < 0$	*s overemployed, u underemployed*
$\frac{\beta_u}{1-\beta_u} \in \left(\frac{\beta_s}{1-\beta_s}\frac{\alpha_s}{1-\alpha_u}, \frac{\beta_s}{1-\beta_s}\frac{1-\alpha_s}{\alpha_u}\right)$	$\frac{\partial \pi(\cdot)}{\partial \beta_u} < 0$	$\frac{\partial \pi(\cdot)}{\partial \beta_s} < 0$	*s and u overemployed*
$\frac{\beta_u}{1-\beta_u} \in \left(\frac{\beta_s}{1-\beta_s}\frac{1-\alpha_s}{\alpha_u}, \infty\right)$	$\frac{\partial \pi(\cdot)}{\partial \beta_u} < 0$	$\frac{\partial \pi(\cdot)}{\partial \beta_s} > 0$	*s underemployed, u overemployed*

(14)

*Figure **??** ilustrates this non-monotonicity of profits in the bargaining power of the workers. It depicts the isoprofit curves on the space (β_u, β_s) and shows that if $a_s = 0.1$ and $a_u = 0$ (???), then the firm whose workers have bargaining powers $(\beta_u, \beta_s) = (0.07, 0.105)$ makes more profits than the firm whose workers have bargaining powers $(\beta_u', \beta_s') = (0, 0.10)$.*

ftbpF2.84in2.7778in0ptfigure-iso-profit-cbiso-profit-lines.gif

Remark 5 *We note that in the case of a Cobb-Douglas production function, the optimal interior bargaining power* $\beta_u^*\left(\bar{\beta}_s, p\right)$ *as determined in (12) is independent of the price of output. The reason for this is that for a Cobb-Douglas function, both the overemployment factors (and thus, the overemployment elasticities), as well as the optimal input ratios are independent of the output price.*

In the next section, we explore the dependence of the optimal bargaining power on the price of output.

2.3 The Dependence of the Optimal Bargaining Power on the Price of Output

In this section, we generalize the insight in Remark 5. We show that the optimal bargaining power obtained as the solution to problem (8) does not depend on the price of output whenever the production function is homothetic, such as the commonly used CES production function.

Proposition 6 *If* $f\left(\ell_u, \ell_s\right)$ *is homogeneous of degree* α *in* $\left(\ell_u, \ell_s\right)$*, then the optimal bargaining power is independent of* p*.*

The result of Proposition 6 extends the insight of Remark 5 to arbitrary homogeneous production functions. In this case, the optimal bargaining power does not depend on the price of the output. In contrast, non-homogeneous production functions violate this property. We illustrate this using the example of a generalized CES production function, which violates homotheticity.

Example 7 *Generalized CES production function*
Consider a production function $f\left(\ell_u, \ell_s\right)$ *implicitly defined by*

$$f\left(\ell_u, \ell_s, y\right) = y^{\alpha_u \frac{\sigma-1}{\sigma}} \ell_u^{\frac{\sigma-1}{\sigma}} + y^{\alpha_s \frac{\sigma-1}{\sigma}} \ell_s^{\frac{\sigma-1}{\sigma}} = \frac{1}{A} \tag{15}$$

where A *is a constant which determines the level of output (output increases in* A*),* $\alpha_u < \alpha_s < 0$ *and* $\sigma > 1$*. This production function reduces to the standard CES when* $\alpha_u = \alpha_s$*:*

$$y = A\left(\ell_u^{\frac{\sigma-1}{\sigma}} + \ell_s^{\frac{\sigma-1}{\sigma}}\right)^{-\frac{1}{\alpha}\frac{\sigma}{1-\sigma}}$$

Here, σ *is the elasticity of substitution between the two labor inputs.* α_u *and* α_s *are measures of the elasticity of output of the two types of labor. If* $\alpha_i \in \left[-\frac{\sigma}{\sigma-1}, -1\right]$ *for* $i \in \{u, s\}$*, the two labor types are complements.*

Although the generalized CES production function inherits many of the appealing properties of the standard CES-class, see Comin et al. (2015), it violates homogeneity. Thus, the optimal bargaining power of the firm depends on the price of output p*. The following figure provides an illustration by plotting the profit-maximizing bargaining power as a strictly decreasing function of the price.*

ftbpF3.7395in1.9649in0ptfigure-optimal-barg-priceoptimal$_b$*argaining.jpg*

The following proposition identifies necessary conditions on the parameters of the generalized CES function, for which the optimal bargaining power of the unskilled workers is a decreasing function of the price of output. This is the case in particular, whenever skilled workers have higher output elasticity, $\alpha_s > \alpha_u$, labor inputs are complements, $\alpha_i \in \left[-\frac{\sigma}{\sigma-1}, -1\right]$ for $i \in \{u, s\}$ and σ is relatively small, $\frac{\sigma}{\sigma-1} > -\alpha_s - \alpha_u$, e.g., $\sigma < 2$.

Proposition 8 *Suppose that the production function of the firm is given by (15) with* $0 > \alpha_s \geq \alpha_u$, $\alpha_i \in \left[-\frac{\sigma}{\sigma-1}, -1\right]$ *for* $i \in \{u, s\}$, $\sigma > 1$ *and* $\frac{\sigma}{\sigma-1} > -\alpha_s - \alpha_u$. *For any* $p > p_0 > 0$, $\beta_u^*\left(p_0, \bar{\beta}_s\right) \geq \beta_u^*\left(p, \bar{\beta}_s\right)$. *The inequality is strict if* $\beta_u^*\left(p_0, \bar{\beta}_s\right) \neq 0$ *and* $\alpha_s > \alpha_u$.

3 Market Equilibrium in the Static Economy

In this section, we define the demand side of the market and the equilibrium of the economy in a static setting.

Consider a market for the output good defined by a demand function $D(z, p)$, where p is the price of the output and z is an exogenous random variable interpreted as a shock. z can take finitely many values in a set $\mathbb{Z} = \{z_1 ... z_Z\}$. For most of the discussion, we will consider the case of no uncertainty, $Z = 1$ and the case of two possible shocks, $\mathbb{Z} = \{z_1, z_2\}$.

Assumption D We assume that for each $z \in \mathbb{Z}$, $D(z, p) : \mathbb{Z} \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is continuous and strictly decreasing, continuously differentiable on $[0, \bar{p}_z)$ with $\frac{\partial D(z, p=0)}{\partial p} > -\infty$ and $D(z, p) = 0$ for $p \geq \bar{p}_z$. Furthermore, $D(z_1, p) > D(z_2, p)$ for all $p \leq \bar{p}_{z_1}$.

Assumption D defines, for each shock z, a decreasing demand function with a reservation price $\bar{p}_z$. The shock z_1 induces a uniformly higher demand and is interpreted as a boom, while shock z_2 represents a recession.

The output good is produced by a continuum of firms using the same production technology $f(\ell_u, \ell_s)$. All firms engage in intra-firm bargaining, as in Stole and Zwiebel (1996) and make hiring decisions so as to maximize profits given p. Wages satisfy the stable wage schedule condition. Thus, at the optimum, each firm pays each type of workers $i \in \{u, s\}$ the reservation wage $\bar{w}_i$. Workers are thus indifferent in their choice of firm, as well as between working and their outside option. We assume that the amount of labor available is sufficient to ensure market clearing given the bounds imposed on the number of firms in the market.

The bargaining power of all firms with respect to skilled labor is constrained to be $1 - \bar{\beta}_s \in (0, 1)$. The firms however differ relative to their bargaining power with respect to unskilled workers, $1 - \beta_u$. Let $\mathbb{B} \subseteq [0, 1]$ be the set of possible types of firms with respect to β_u: $\beta_u \in \mathbb{B}$. We assume that $\mathbb{B}$ is either finite or a convex and closed subset of $[0, 1]$.

A market composition is given by a function $m : \mathbb{B} \to \mathbb{R}_0^+$ and indicates for each type of firm $\beta_u \in \mathbb{B}$ the mass of firms of this type present in the market. We assume that the set of possible market compositions is bounded from above and from below:

Assumption M The set of admissible market compositions $\mathbb{M}$ is a set of non-normalized distribution functions $m : \mathbb{B} \to [0, \bar{\mu}^c]$ for which $\int_{\beta_1}^{\beta_2} m(\beta_u)\, d\beta_u$ exists for any two $\beta_1,\ \beta_2 \in \mathbb{B}$.

Assumption M restricts the mass of firms of any given type β_u that can be present in the market in any given period to be between $[0, \bar{\mu}^c]$. Thus, the set of market compositions is a Banach space. Below, we relate the upper bound $\bar{\mu}^c$ to the fixed cost parameter.

We assume that each type of firm has a supply function for the output good which is strictly increasing and continuous in the price of output p:

Assumption S For all $\beta_u \in \mathbb{B}$, the profit-maximizing hiring decisions $\ell_i\left(\beta_u, \bar{\beta}_s, p\right)$ satisfy $\ell_i\left(\beta_u, \bar{\beta}_s, 0\right) = 0$ and are strictly increasing, continuously differentiable in p and in β_u. Furthermore, there is a d such that

$$\frac{\partial y\left(\beta_u, \bar{\beta}_s, p\right)}{\partial p} \leq d$$

for all $p \in [0, \bar{p}_{z_1}]$ and all $\beta_u \in \mathbb{B}$.

Thus, the supply function of a firm with bargaining power β_u,

$$y\left(\beta_u, \bar{\beta}_s, p\right) = f\left(\ell_u\left(\beta_u, \bar{\beta}_s, p\right), \ell_s\left(\beta_u, \bar{\beta}_s, p\right)\right)$$

satisfies $y\left(\beta_u, \bar{\beta}_s, 0\right) = 0$ and is also strictly increasing and continuous in p, as well as continuous in β_u.

Given Assumption S, we can define the aggregate supply in the economy for a given market composition m as:

$$S(p, m) = \int_{\beta_u \in \mathbb{B}} y\left(\beta_u, \bar{\beta}_s, p\right) m(\beta_u)\, d\beta_u$$

Clearly, $S(0, m) = 0$.

We now define the market equilibrium[4]. The market is in equilibrium when the price of output p equates the aggregate supply determined by the sum of the optimal production decisions of the firms and the demand of the economy at the given shock realization z:

[4]The notion of equilibrium we use does not require the labor market to be in equilibrium. This is because we assume that an outside option (e.g., home production) is always available to the workers. The condition that the labor market be in equilibrium thus reduces to the requirement that the amount of labor available in the economy is sufficiently large so as to satisfy the (maximal) firm demand for labor.

Definition 9 *Suppose that the market composition at a given period is given by $m \in \mathbb{M}$ and the realization of the shock is $z \in \mathbb{Z}$. The market equilibrium of the economy for a given shock z and a given market composition m is described by*

(i) *firms' labor hiring decisions $\hat{\ell}_u : \mathbb{B} \to \mathbb{R}_0^+$, $\hat{\ell}_s : \mathbb{B} \to \mathbb{R}_0^+$;*

(ii) *firm's output quantities $\hat{y} : \mathbb{B} \to \mathbb{R}_0^+$;*

(iii) *price of output $\rho \in \mathbb{R}_0^+$*

such that for all $\beta_u \in \mathbb{B}$,

$$\begin{aligned} \hat{\ell}_i(\beta_u) &= \ell_i\left(\beta_u, \bar{\beta}_s, p\right), \; i \in \{u, s\} \\ \hat{y}(\beta_u) &= y\left(\beta_u, \bar{\beta}_s, p\right) = f\left(\ell_u\left(\beta_u, \bar{\beta}_s, p\right), \ell_s\left(\beta_u, \bar{\beta}_s, p\right)\right) \end{aligned}$$

and ρ solves

$$D(z, \rho) = \int_{\beta_u \in \mathbb{B}} \hat{y}(\beta_u)\, m(\beta_u)\, d\beta_u \tag{16}$$

The following proposition proves existence and describes the properties of the equilibrium price function and the firms' equilibrium profits:

Proposition 10 *Under the assumptions D, M and S, a market equilibrium of the economy $(\rho, \hat{y}(\cdot))$ exists for any $z \in \mathbb{Z}$ and $m \in \mathbb{M}$ provided that either $\bar{p}_z$ is finite or that $\underline{m} \not\equiv 0$. Furthermore,*

(i) *the equilibrium price function $\rho(z, m)$ is continuously differentiable, strictly decreasing in m, and satisfies for any m, $\rho(z_1, m) > \rho(z_2, m)$;*

(ii) *firms' equilibrium profits $\pi(\beta_u, \rho(z, m))$ are continuously differentiable and strictly decreasing in m and satisfy for any m, $\pi(\beta_u, \rho(z_1, m)) > \pi(\beta_u, \rho(z_2, m))$.*

Assume that in order to be active in the market a firm has to pay a fixed cost $c > 0$. Let $r_c(\beta_u)$ be the static reservation price of firm β_u implicitly defined by:

$$\pi(\beta_u, r_c(\beta_u)) = c$$

Note that the static reservation price is independent of the exogenous demand shock z. We assume that fixed cost are sufficiently low so that equilibria with non-0 production exist for at least some market compositions:

Assumption C For each $z \in \mathbb{Z}$, there exists a $\beta_u \in \mathbb{B}$ such that $r_c(\beta_u) < \bar{p}_z$.

Denote by $r_c\left(\hat{\beta}_u\right)$ the minimal static reservation price[5] on the set $\mathbb{B}$:

$$\min_{\beta_u \in \mathbb{B}} r_c(\beta_u) = r_c\left(\hat{\beta}_u\right)$$

[5] The fact that $r_c(\beta_u)$ achieves a minimum on $\mathbb{B}$ is shown in the proof of Lemma 35 in Appendix 7.2.

Note that since profits are in general non-monotonic in β_u, the minimal $\hat{\beta}_u$ does not in general coincide with $\min\{\beta_u \in \mathbb{B}\}$.

Define the market composition $\bar{m}^c_{\beta_u}(z)$ as:

$$\bar{m}^c_{\beta_u}\left(z, \tilde{\beta}_u\right) = \begin{cases} \mu^c_{\beta_u}(z) & \text{if } \tilde{\beta}_u = \beta_u \\ 0 & \text{else} \end{cases}$$

where $\mu^c_{\beta_u}(z)$ is the maximal mass of firms of type β_u present in the market when the shock is z so that each firm breaks even and is implicitly defined by:

$$\pi\left(\beta_u, \bar{\beta}_s, \rho\left(z, \bar{m}^c_{\beta_u}\right)\right) = c$$

Clearly, $\mu^c_{\beta_u}(z_1) > \mu^c_{\beta_u}(z_2)$, and thus, the maximal number of firms of type β_u in the market at any given time is $\mu^c_{\beta_u} = \mu^c_{\beta_u}(z_1)$ with the corresponding market composition given by: $\bar{m}^c_{\beta_u} = \bar{m}^c_{\beta_u}(z_1)$. The maximal $\mu^c_{\beta_u}$ can thus be taken as an upper limit on the market composition, $\bar{\mu}^c$, imposed in Assumption M, and we set:

$$\bar{\mu}^c = \max_{\beta_u \in \mathbb{B}} \mu^c_{\beta_u} \tag{17}$$

4 Introducing Dynamics

We now introduce dynamics into the model.

Time is discrete, $t = \{0, 1, 2...\}$, and in each period, a publicly observed shock $z_t \in \mathbb{Z}$ is realized according to an i.i.d. probability distribution $q : \mathbb{Z} \to [0,1]$, $\sum_{z\in\mathbb{Z}} q(z) = 1$. $m_t \in \mathbb{M}$ denotes the market composition of firms at the beginning of period t. Thus, the state of the economy at the beginning of time t is described by the vector $(z_t, m_t) \in \mathbb{Z} \times \mathbb{M}$, which is publicly observable. In each period t, the distribution of potential producers in the economy is $m_t + \phi : \mathbb{B} \to \mathbb{R}^+$, where $\phi : \mathbb{B} \to \mathbb{R}^+$ describes the distribution of new producers born at time t. ϕ is exogenously given, constant over time and $\phi(\beta_u) > 0$ for any $\beta_u \in \mathbb{B}$.

Potential producers become active by paying a fixed cost $c > 0$. Firms that are not active exit forever. The common discount factor is given by $\delta \in (0, 1)$.

We will look for a Markovian equilibrium of this economy which determines for each state of the economy, (z, m), the set of firms active in the market, the market price for output and the continuation values of the active firms, given by their expected discounted stream of future profits. Formally, the state space is given by $\Sigma = \mathbb{Z} \times \mathbb{M}$. The equilibrium is described by a market-composition mapping: $M : \Sigma \to \mathbb{M}$ defined as the market composition at the end of period t as a function of the shock z_t and beginning of the period market composition, m_t. Given the market composition mapping M, we can deduce

(i) the price mapping: $P : \Sigma \to \mathbb{R}_+$ defined as the market price as a function of the shock z_t and market composition m_t at the beginning of period t and given by the market equilibrium price at shock z_t and the market composition at the end of period t implied by M:

$$P(z_t, m_t) = \rho(z_t, M(m_t))$$

(ii) the value function of a type β_u firm defined as:

$$v(z,m,\beta_u) = \max\{\pi(\beta_u, P(z,m)) - c + \delta E_q [v(\tilde{z}, M(z,m), \beta_u)], 0\}$$

and

(iii) the entry decision mapping: $X : \Sigma \to \{x : \mathbb{B} \to [0,1]\}$ defined as the decision of each potential producer $\beta_u \in \mathbb{B}$ to become active, $X(z_t, m_t, \beta_u) = 1$, or not, $X(z_t, m_t, \beta_u) = 0$, as a function of the shock z_t and market composition m_t, which has to satisfy:

$$X(z,m,\beta_u) \in \arg\max_{x\in[0,1]} x[\pi(\beta_u, P(z,m)) - c + \delta E_q[v(\tilde{z}, M(z,m), \beta_u)]].$$

Definition 11 *For given $\Sigma = \mathbb{Z} \times \mathbb{M}$, $\mathbb{B}$, q, ϕ, and $\bar{\beta}_s$, a Markovian equilibrium of the economy is defined by a market composition mapping M such that M together with the induced price mapping P, the induced value functions $(v(\cdot, \beta_u))_{\beta_u\in\mathbb{B}}$ and the induced entry decision mapping, X satisfies for every $(z,m) \in \mathbb{Z} \times \mathbb{M}$,*

(i) *the decision of a potential producer of type $\beta_u \in \mathbb{B}$ to become active is optimal given M and the induced P and $v(\cdot, \beta_u)$:*

$$X(z,m,\beta_u) \in \arg\max_{x\in[0,1]} x[\pi(\beta_u, P(z,m)) - c + \delta E_q[v(\tilde{z}, M(z,m), \beta_u)]], \tag{18}$$

where type β_u's value function solves:

$$v(z,m,\beta_u) = \max\{\pi(\beta_u, P(z,m)) - c + \delta E_q[v(\tilde{z}, M(z,m), \beta_u)], 0\} \tag{19}$$

(ii) *the economy is in equilibrium, the induced price mapping is:*

$$P(z,m) = \rho(z, M(z,m))$$

(iii) *expectations are consistent: the set of active firms is determined by the entry / exit decisions of potential producers:*

$$M(z,m,\beta_u) = X(z,m,\beta_u)[m+\phi](\beta_u) \text{ for } \beta_u \in \mathbb{B}. \tag{20}$$

The definition of a Markovian equilibrium ensures that (i) potential producers decide to become active, $X(z,m,\beta_u) > 0$, iff their expected (w.r.t. the distribution of shocks q and future market compositions determined by M) discounted profits are non-negative; (ii) in each state, the price is given by the static equilibrium price, $\rho(z, M(z,m))$, given the actually active set of firms, $M(z,m)$ in this state; (iii) the producers' expectations about the future market composition M are consistent with the entry / exit decisions X.

As usual, we define the t-step forward operator $M^t(m)$ as:

$$\begin{aligned} M^0(m) &= m \\ M^1(m) &= M(m) \\ M^t(m) &= \underbrace{M(M...(M(m)))}_{t\text{-times}} \end{aligned}$$

4.1 The Deterministic Case

Let $q(z_1) = 1$. This corresponds to the case of a deterministic economy, in which demand is constant over time. We obtain:

Proposition 12 *Suppose that $r^c_{\beta_u}$ has a single global minimum on $\mathbb{B}$, $r^c_{\hat{\beta}_u}$. Suppose that ϕ is a continuous density on $\mathbb{B}$ which assigns a strictly positive measure to every $\beta_u \in \mathbb{B}$ when $\mathbb{B}$ is finite and corresponds to a distribution absolutely continuous w.r.t. the Lebesgue measure when $\mathbb{B}$ is an interval. Then, there exist two functions $\underline{\beta}(m) : \mathbb{M} \to \mathbb{B}$ and $\bar{\beta}(m) : \mathbb{M} \to \mathbb{B}$, a respectively increasing and decreasing function of m that satisfy for every m:*

$$r^c_{\underline{\beta}(m)} = r^c_{\bar{\beta}(m)},\ or\ \underline{\beta}(m) = \min\{\beta_u \in \mathbb{B}\},\ or\ \bar{\beta}(m) = \max\{\beta_u \in \mathbb{B}\}$$

and

$$\int_{\underline{\beta}(m)}^{\bar{\beta}(m)} y\left(\beta_u, \bar{\beta}_s, \max\left\{r^c_{\underline{\beta}(m)}, r^c_{\bar{\beta}(m)}\right\}\right) X^*(m, \beta_u)[m+\phi](\beta_u)\, d\beta_u = D\left(z_1, \max\left\{r^c_{\underline{\beta}(m)}, r^c_{\bar{\beta}(m)}\right\}\right) \tag{21}$$

such that the Markovian equilibrium of the economy is given by:

(i) the decision of a potential producer of type $\beta_u \in \mathbb{B}$ to become active is given by:

$$X^*(m, \beta_u) = \begin{cases} 1 & if\ r^c_{\beta_u} < \max\left\{r^c_{\underline{\beta}(m)}, r^c_{\bar{\beta}(m)}\right\} \\ 0 & if\ r^c_{\beta_u} > \max\left\{r^c_{\underline{\beta}(m)}, r^c_{\bar{\beta}(m)}\right\} \end{cases}$$

and $X^*\left(m, \arg\max\left\{r^c_{\underline{\beta}(m)}, r^c_{\bar{\beta}(m)}\right\}\right)$ is determined so that (21) is satisfied;

the value function of firm β_u is given as:

$$v^*(m, \beta_u) = \sum_{t=0}^{\min\left\{\tilde{t} | P^*\left(M^{*\tilde{t}}(m)\right) < r_{\beta^c_u}\right\}} \delta^t \left[\pi\left(\beta_u, P^*\left(M^{*t}(m)\right)\right) - c\right]$$

(ii) the price mapping satisfies:

$$P^*(m) = \rho(z_1, M(m)) = \max\left\{r^c_{\underline{\beta}(m)}, r^c_{\bar{\beta}(m)}\right\}$$

(iii) the set of active firms is given by:

$$\begin{aligned} M^*(m, \beta_u) &= X^*(m, \beta_u)[m+\phi](\beta_u) = \\ &= \begin{cases} [m+\phi](\beta_u) & if\ r^c_{\beta_u} < \max\left\{r^c_{\underline{\beta}(m)}, r^c_{\bar{\beta}(m)}\right\} \\ X^*(m, \beta_u)[m+\phi](\beta_u) & if\ r^c_{\beta_u} = \max\left\{r^c_{\underline{\beta}(m)}, r^c_{\bar{\beta}(m)}\right\} \\ 0 & if\ r^c_{\beta_u} > \max\left\{r^c_{\underline{\beta}(m)}, r^c_{\bar{\beta}(m)}\right\} \end{cases} \end{aligned} \tag{22}$$

Furthermore, $\hat{\beta}_u \in \left(\underline{\beta}(m), \bar{\beta}(m)\right)$ *for each* $m \in \mathbb{M}$*, and for every* $m_0 \in \mathbb{M}$*,*

$$\lim_{t\to\infty} M^{*t}(m_0) = \bar{m}^c_{\hat{\beta}_u}$$

and

$$\lim_{t\to\infty} P^*\left(M^{*t-1}(m_0)\right) = r^c_{\hat{\beta}_u}$$

The proposition considers an economy, in which the profit function has a single peak on the set of bargaining powers $\mathbb{B}$. However, the set of potential producers contains a strictly positive mass of firms with bargaining powers different from the optimal $\hat{\beta}_u$. Even though the firms with bargaining power $\hat{\beta}_u$ have the lowest static reservation price $r^c_{\hat{\beta}_u}$, their mass $[m+\phi]\left(\hat{\beta}_u\right)$ in a given period may not be sufficient to reduce the price to $r^c_{\hat{\beta}_u}$. Thus, firms with non-profit-maximizing bargaining powers β_u (between $\underline{\beta}(m)$ and $\bar{\beta}(m)$) find it optimal to be active. The resulting equilibrium price is given by the maximum of the two cut-off reservation prices, $\max\left\{r^c_{\underline{\beta}(m)}, r^c_{\bar{\beta}(m)}\right\}$. The cut-off bargaining powers $\underline{\beta}(m)$ and $\bar{\beta}(m)$ are determined endogenously by the market-clearing condition (21): the set of firms remaining in the market are those who make non-negative (net of fixed cost) current profits at the market price. All other firms leave the market.

For this equilibrium, the dynamics along a path starting from any initial market composition m_0 is monotonic. With time, the set of bargaining powers of active firms shrinks around the profit-maximizing bargaining power $\hat{\beta}_u$ and eventually converges to $\hat{\beta}_u$. The equilibrium price sequence is decreasing: as more firms with bargaining power closer to the optimal enter the market, the equilibrium price converges to the lowest reservation price, $r^c_{\hat{\beta}_u}$.

This monotonicity of prices has an interesting implication. If a firm cannot break even at the current price $P(m)$, it will make losses in all subsequent periods. Thus, even though firms optimize over an infinite horizon, the decision to become active only depends on the current market price: a firm enters the market iff the current price exceeds its reservation price.

Proposition 12 illustrates how markets may select for the firms' bargaining power. When the economy is deterministic, the Markovian equilibrium "selects" for a single type of firms: those with the minimal static reservation price, $r^c_{\hat{\beta}_u}$, and the corresponding bargaining power, $\hat{\beta}_u$. This dynamics is independent of the initial conditions.

We recall that for $\bar{\beta}_s > 0$, $\hat{\beta}_u > 0$ might obtain. The surviving firms are therefore not necessarily those with maximal bargaining power relative to the unskilled workers. This finding sheds some light on the empirical results cited in the introduction, namely that firms do not have maximal bargaining power with respect to their workers. Furthermore, in as far as $\hat{\beta}_u$ depends on the technology of production, f, as well as on the institutional constraints captured by the restrictions on β_s, $\bar{\beta}_s$, the limit bargaining power may vary across industries within a given country, or across countries reflecting institutional and cultural differences.

Note also that when the production function is not homogeneous, the surviving firm type need not achieve maximal profits over the entire relevant price range. In particular, $\hat{\beta}_u$ might result in lower profits than other bargaining powers in $\mathbb{B}$ for relatively high prices. In this case, the equilibrium dynamics may exhibit "reversal of fortunes": the long-run survivors will be initially underperforming relative to the rest of the population.

The analysis of the deterministic model shows the potential of our approach to capture the effect of market forces on the evolution of bargaining power. However, the deterministic model does not explain the observed heterogeneity of bargaining powers within an industry, or over time. In the limit, the equilibrium of the economy is characterized by a representative firm with a "constrained-optimal" bargaining power. In the next section, we will show that this result is special to the case of deterministic economies by studying the effect of exogenous stochastic shocks on the properties of the Markovian equilibrium.

5 Stochastic Model

In this section we consider an economy with exogenous shocks and heterogeneity in bargaining powers.

We first show that in the special case, in which the mass of newly born producers $\hat{\beta}_u$ with minimal static reservation price $r^c_{\hat{\beta}_u}$ is sufficiently high, only these most efficient firms become active. In the Markovian equilibrium, the price is constant at $r^c_{\hat{\beta}_u}$ independently of the state and initial market composition. All firms make 0 profits and have 0 continuation values. A transition from one state to another simply leads to an adjustment in the entry rate of the most efficient firms $\hat{\beta}_u$:

Proposition 13 *If $\phi_{\hat{\beta}_u} \geq \bar{\mu}^c$, the Markovian equilibrium is given by $M^*(z_i, m) = m^c_{\hat{\beta}_u}(z)$, $P^*(z, m) = r^c_{\hat{\beta}_u}$ and*

$$X^*(z, m, \beta_u) = \begin{cases} \bar{m}^c_{\hat{\beta}_u}(z) & \text{if } \beta_u = \hat{\beta}_u \\ 0 & \beta_u \neq \hat{\beta}_u \end{cases}$$

$v^*(z, m, \beta_u) = 0$ for any z and m and β_u.

We next consider the case in which the mass of new-born consumers is insufficient to allow markets to instantly adjust to exogenous demand shocks. To simplify the analysis, we restrict attention to the case of two shocks, $\mathbb{Z} = \{z_1, z_2\}$ and two types of firms with bargaining powers $\mathbb{B} = \{\beta^1_u, \beta^2_u\}$. We refer to the state z_1 as a boom and z_2 as recession. We set $q(z_1) = q$. We assume that $r^c_{\beta^1_u} < r^c_{\beta^2_u}$: firms of type β^1_u has a lower static reservation price. As we know from Proposition 12, these would be the only survivors in an economy with no uncertainty.

To illustrate how uncertainty affects equilibrium, we describe two types of equilibria which exhibit distinct characteristics: a "regular" equilibrium and a "predatory" equilibrium. In both types of equilibria, after a transition from a recession to a boom, the price is initially strictly higher than the highest reservation price $r^c_{\beta^2_u}$. This implies that both types of firms make strictly positive profits and enter the market. If the boom persists, continuous entry causes the price to drop and (provided the boom lasts sufficiently long) to reach $r^c_{\beta^2_u}$. At $r^c_{\beta^2_u}$, firms of type β^1_u are still making strictly positive profits so all of them enter. Firms β^2_u, however, are just breaking even from a static point of view. Thus, β^2_u-firms start decreasing their market participation. In a regular equilibrium, β^2_u-firms exit the market once the price falls below $r^c_{\beta^2_u}$. In the predatory equilibrium, their market participation may be strictly positive even for prices below their static reservation price.

5.1 Regular Markovian Equilibrium

A regular Markovian equilibrium is defined by a critical mass of firms of type β^1_u, $\tilde{\mu}_{\beta^1_u}(z_2)$: this is the maximal number of such firms active during a recession. This mass is chosen in such a way that the following market composition M^* together with P^*, X^* and v^* constitutes a Markovian equilibrium:

$$\tilde{v}\left(\beta^i_u, z_1, m\right) = \pi\left(\beta^i_u, \rho\left(z_1, m+\phi\right)\right) - c + \delta q \max\left\{\tilde{v}\left(\beta^i_u, z_1, m+\phi, \beta^i_u\right), 0\right\} \quad (23)$$

$$(24)$$

Definition 14 *A regular market composition mapping M^* is:*

$$M^*_{\beta^1_u}(z,m) = \min\left\{m_{\beta^1_u} + \phi_{\beta^1_u}, \tilde{\mu}_{\beta^1_u}(z)\right\}$$

$$M^*_{\beta^2_u}(z,m) = \min\left\{m_{\beta^2_u} + \phi_{\beta^1_u}, \max\left\{k\left(z, m_{\beta^1_u}\right), 0\right\}\right\}$$

where $\tilde{\mu}_{\beta^1_u} : \mathbb{Z} \mapsto \mathbb{R}_+$ satisfy:

$$\tilde{\mu}_{\beta^1_u}(z_1) = \bar{\mu}^c \quad (25)$$

$$\pi\left(\beta^1_u, \rho\left(z_2, \tilde{\mu}_{\beta^1_u}(z_2), 0\right)\right) - c + \tilde{v}\left(\beta^1_u, z_1, \tilde{\mu}_{\beta^1_u}(z_2), 0\right) = 0 \quad (26)$$

and $k : \mathbb{Z} \times [0, \bar{\mu}^c] \mapsto \mathbb{R}$ satisfies:

$$\rho\left(z_1, \min\left\{m_{\beta^1_u} + \phi_{\beta^1_u}, \bar{\mu}^c\right\}, k\left(z_1, m_{\beta^1_u}\right)\right) = r^c_{\beta^2_u} \quad (27)$$

$$\pi\left(\beta^2_u, \rho\left(z_2, M^*_{\beta^1_u}(z_2,m), k\left(z_2, m_{\beta^1_u}\right)\right)\right) - c + \tilde{v}\left(\beta^1_u, z_1, M^*_{\beta^1_u}(z_2,m), k\left(z_2, m_{\beta^1_u}\right)\right) = 0 \quad (28)$$

The regular market compositions mapping specifies the active firms for the two states: recession, z_2 and boom, z_1. During a recession with a sufficiently low m, both firms will be active in the market and making positive current profits. As however m increases, the induced equilibrium price (provided all potential producers are active) $\rho(z_2, m)$ decreases. For a sufficiently high mass of β_u^1-firms, the firm with the higher static reservation price β_u^2 would make current losses which would not be compensated by expected future profits: expression (28) is negative for all non-negative values of k. Thus, β_u^2-firms thus exit the market and only the more efficient β_u^1-firms produce. The maximal number in a recession $\tilde{\mu}_{\beta_u^1}(z_2)$ of β_u^1-firms is defined by (26). The first line corresponds to current losses that β_u^1-firms make during recession. Firms accept these losses due to the prospect of a boom: $\tilde{\mu}_{\beta_u^1}(z_2)$ is chosen so that the number of potential producers is not sufficient to saturate the market in case of a boom. Thus, prices during recession drop below the lowest reservation price $r^c_{\beta_u^1}$ and rise above it during a boom. Furthermore, if the price increase during a boom is sufficient (above $r^c_{\beta_u^2}$), both types of firms make strictly positive profits and are active in the market. As the boom persists and more firms enter, prices eventually drop to $r^c_{\beta_u^2}$: at this point β_u^2-firms start exiting the market, while β_u^1-firms still make strictly positive profits. Eventually, the number of β_u^1-firms reaches its maximum $\bar{\mu}^c$ driving the price to their reservation price $r^c_{\beta_u^1}$ and both profits and continuation value to 0.

We now provide conditions for the regular Markovian equilibrium to exist.

Proposition 15 *Assume that*

$$\phi_{\beta_u^1} y\left(\beta_u^1, r^c_{\beta_u^1}\right) + \phi_{\beta_u^2} y\left(\beta_u^2, r^c_{\beta_u^1}\right) < D\left(z_1, r^c_{\beta_u^1}\right) - D\left(z_2, r^c_{\beta_u^1}\right) \tag{29}$$

then there is a unique $\tilde{\mu}_{\beta_u^1}(z_2) \in \left(0, \bar{\mu}^c_{\beta_u^1}(z_1)\right)$ *which solves (26). If, furthermore,*

$$\pi\left(\beta_u^1, p\right) > \pi\left(\beta_u^2, p\right) \tag{30}$$

for all $p \in [0, \bar{p}_z]$*, then* M^* *as defined in Definition 14 is a Markovian equilibrium of the economy.*

To state the next proposition, we define recursively the t-step ahead market composition mapping for a given sequence of shocks $z^1...z^t$ as:

$$\begin{aligned} M^{*1} &= M^* \\ M^{*t}\left(z^1...z^t, m\right) &= M^*\left(z^t, M^{*t-1}\left(z^1...z^{t-1}, m\right)\right) \end{aligned}$$

Proposition 16 *The Markov process on* $\mathbb{Z} \times \mathbb{M}$ *defined by* q *and* M^* *converges a.s. to an invariant distribution with finite support* $\bar{\Sigma} \subseteq \mathbb{Z} \times \mathbb{M}$*: for*

$$n = \min\left\{\tau \mid \rho\left(z_1, M^{*t}\left(z_1...z_1, \left(\tilde{\mu}_{\beta_u^1}(z_2), 0\right)\right)\right) \leq r^c_{\beta_u^1}\right\}$$

the support of the invariant distribution has a cardinality $n+1$ *and is given by*

$$\bar{\Sigma} = \left\{\sigma_0 = z_2, \left(\tilde{\mu}_{\beta_u^1}(z_2), 0\right), \left(\sigma_\tau = \left(z_1, M^{*\tau}\left(z_1...z_1, \left(\tilde{\mu}_{\beta_u^1}(z_2), 0\right)\right)\right)\right)_{\tau=1}^n\right\}$$

The invariant distribution is $Q = \left((1-q), q(1-q)...q^{n-2}(1-q), q^{n-1}\right)$.

Proposition 15 determines the Markovian equilibrium as well as the corresponding invariant distribution. Note that each of the recurrent states $\sigma \in \bar{\Sigma}$ can be associated with a distribution of bargaining powers given by:

$$\tilde{\beta}_u(\sigma) = \left(\frac{m_{\beta_u^1}(\sigma)}{m_{\beta_u^1}(\sigma) + m_{\beta_u^2}(\sigma)}, \frac{m_{\beta_u^2}(\sigma)}{m_{\beta_u^1}(\sigma) + m_{\beta_u^2}(\sigma)} \right)$$

The average bargaining power in state σ is:

$$\beta_u^{av}(\sigma) = \frac{m_{\beta_u^1}(\sigma)}{m_{\beta_u^1}(\sigma) + m_{\beta_u^2}(\sigma)} \beta_u^1 + \frac{m_{\beta_u^2}(\sigma)}{m_{\beta_u^1}(\sigma) + m_{\beta_u^2}(\sigma)} \beta_u^2 \tag{31}$$

Notably, in states σ_0 and σ_n, the bargaining power of the workers in all active firms is β_u^1. In a recession, or when a boom lasts sufficiently long, only the firms with the lowest static reservation price $r^c_{\beta_u^1}$ are active and β_u^1 is the average bargaining power of the unskilled workers in such periods. During the initial periods of a boom, however, β_u^2-firms also enter the market. The average bargaining power is thus the weighted average of β_u^1 and β_u^2.

As shown in Proposition 3, the firm with the minimal static reservation price can have a lower bargaining power w.r.t. the workers, which corresponds to $\beta_u^1 > \beta_u^2$. If this is the case, then the average bargaining power of the workers will be at its maximum, β_u^1, during a recession (z_2). It will then initially decrease at the start of a boom when β_u^2-firms enter the market. It will continue increasing as long as

$$\rho\left(z_1, \left(m + \tau\phi_{\beta_u^1}, \tau\phi_{\beta_u^2}\right)\right) > r^c_{\beta_u^2}$$

Once the reservation price $r^c_{\beta_u^2}$ is reached, β_u^2-firms start exiting the market, whereas the mass of β_u^1-firms continues increasing. The average bargaining power will start increasing again and reach its maximum β_u^1 once $\rho\left(z_1, \left(m + \tau\phi_{\beta_u^1}, 0\right)\right) < r^c_{\beta_u^2}$ and all β_u^2-firms have exited the market.

We conclude that the average bargaining power of the unskilled workers during a recession will indeed be larger than the average bargaining power of the workers conditional on the economy being in a boom:

$$\beta_u^{av}(\sigma_0) = \beta_u^1 > \sum_{\sigma \in \{\sigma_1 \ldots \sigma_n\}} Q(\sigma)\, \beta_u^{av}(\sigma)$$

for as conjectured by Shimer (2010). This effect is consistent with the countercyclical behavior of the labor wedge.

5.2 Predatory Markovian Equilibrium

In the proof of existence of a regular equilibrium, we made the assumption that the profit functions of the two types of firms are totally ordered on the range of relevant prices. We now turn to the case in which the profit functions $\pi\left(\beta_u^1, p\right)$

and $\pi\left(\beta_u^2, p\right)$ cross at least once. Specifically, we will assume that there is a price $\tilde{p} \in \left(r^c_{\beta_u^1}, \bar{p}_{z_1}\right)$ such that:

$$\begin{array}{rcl} \pi\left(\beta_u^1, p\right) & > & \pi\left(\beta_u^2, p\right) \text{ for all } p \in (0, \tilde{p}) \\ \pi\left(\beta_u^1, p\right) & < & \pi\left(\beta_u^2, p\right) \text{ for all } p \in (\tilde{p}, \bar{p}_{z_1}) \end{array}$$

This means that on the relevant price range, β_1^u-firms are more profitable when prices are below $\tilde{p}$ and β_u^2-firms are more profitable for prices above $\tilde{p}$. We will also assume that $\bar{p}_{z_2} > r^c_{\beta_u^2}$ and

$$\phi_{\beta_u^1} y\left(\beta_u^1, r^c_{\beta_u^2}\right) + \phi_{\beta_u^2} y\left(\beta_u^2, r^c_{\beta_u^2}\right) < D\left(z_1, r^c_{\beta_u^2}\right) - D\left(z_2, r^c_{\beta_u^2}\right) \tag{32}$$

We define the predatory Markovian equilibrium as follows:

Definition 17 *A predatory Markovian Equilibrium is defined by thresholds* $\hat{\mu}_{\beta_u^2}(z_2)$, $\check{\mu}_{\beta_u^2}(z_1)$ *with* $\hat{\mu}_{\beta_u^2}(z_2) > \phi_{\beta_u^2} > \check{\mu}_{\beta_u^2}(z_1)$, *and* $\hat{\mu}_{\beta_u^1}(z_1) \in [0, \bar{\mu}^c]$ *a function:*

$$k_2(z_1, m) : [0, \bar{\mu}^c] \to \left[0, m_{\beta_u^2} + \phi_{\beta_u^2}\right]$$

such that the market composition mapping M^* *is given by*

$$M^*(z_2, m) = \left(0, \min\left\{m_{\beta_u^2} + \phi_{\beta_u^2}, \hat{\mu}_{\beta_u^2}(z_2)\right\}\right)$$

$$M^*(z_1, m) = \left(\min\left\{m_{\beta_u^1} + \phi_{\beta_u^1}, \hat{\mu}_{\beta_u^1}(z_1)\right\}, k_2(z_1, m)\right)$$

and the value functions derived from M^*, $v^*\left(z, m, \beta_u^1\right)$ *and* $v^*\left(z, m, \beta_u^2\right)$ *together with the function* $k_2(z_1, m)$ *satisfy:*

(*i*) *non-positive discounted expected profits for* β_u^1*-firms in state* z_2*:*

$$\begin{array}{rcl} 0 & \geq & \pi\left(\beta_u^1, \rho\left(z_2, \left(0, \phi_{\beta_u^2}\right)\right)\right) - c + \\ & & +\delta q v^*\left(z_1, 0, \min\left\{m_{\beta_u^2} + \phi_{\beta_u^2}, \hat{\mu}_{\beta_u^2}(z_2)\right\}, \beta_u^1\right) \end{array} \tag{33}$$

(*ii*) β_u^2*-firms' entry in state* z_1 *is determined by* $k_2(z_1, m)$*, such that*

- $k_2(z_1, m)$ *is the solution* k *to*

$$\begin{array}{rcl} 0 & = & \pi\left(\beta_u^2, \rho\left(z_1, \left(\min\left\{m_{\beta_u^1} + \phi_{\beta_u^1}, \hat{\mu}_{\beta_u^1}(z_1)\right\}, k\right)\right)\right) - c + \\ & & +\delta(1-q) v^*\left(z_2, \min\left\{m_{\beta_u^1} + \phi_{\beta_u^1}, \hat{\mu}_{\beta_u^1}(z_1)\right\}, k, \beta_u^2\right) \end{array} \tag{34}$$

  provided a solution $k \in \left[0, m_{\beta_u^2} + \phi_{\beta_u^2}\right]$ *exists*

- $k_2(z_1, m) = 0$ *if the r.h.s. of (34) is negative for all* $k \in \left[0, m_{\beta_u^2} + \phi_{\beta_u^2}\right]$*;*

- $k_2(z_1, m) = m_{\beta_u^2} + \phi_{\beta_u^2}$ *if the r.h.s. of (34) is positive for all* $k \in \left[0, m_{\beta_u^2} + \phi_{\beta_u^2}\right]$.

(*iii*) the maximal number of β_u^2-firms in state z_2, $\hat{\mu}_{\beta_u^2}(z_2)$, is the solution μ_2 of:

$$0 = \pi\left(\beta_u^2, \rho\left(z_2, (0, \mu_2)\right)\right) - c + \delta q v^*\left(z_1, 0, \mu_2, \beta_u^2\right) \tag{35}$$

and $\hat{\mu}_{\beta_u^2}(z_2) > \phi_{\beta_u^2}$

(*iv*) the minimal number of β_u^2-firms in state z_1, $\check{\mu}_{\beta_u^2}(z_1)$, is the solution to:

$$0 = \pi\left(\beta_u^2, r^c_{\beta_u^1}\right) - c + \delta\left(1 - q\right) v^*\left(z_2, 0, \check{\mu}_{\beta_u^2}(z_1), \beta_u^2\right) \tag{36}$$

provided that a positive solution exists and 0, else, whereas the maximal number of β_u^1-firms in state z_1, $\hat{\mu}_{\beta_u^1}(z_1)$, is given by the solution of:

$$\rho\left(z_1, \hat{\mu}_{\beta_u^1}(z_1), \check{\mu}_{\beta_u^2}(z_1)\right) = r^c_{\beta_u^1}$$

In the predatory equilibrium, the market during a recession is dominated by β_u^2-type firms, while β_u^1-type firms opt out of the market. Furthermore, β_u^2-type firms are making current losses expecting a transition to a boom, and thus, an increase in the price above their reservation price $r^c_{\beta_u^2}$. In particular, prices during the initial periods of a boom are in the range in which β_u^2-type firms are more profitable than β_u^1. The associated positive profits during a boom compensate β_u^2-firms for the losses they make during a recession. The maximal number of β_u^2-firms during a recession $\hat{\mu}_{\beta_u^2}(z_2)$ is determined by condition (35) which equates the discounted expected profits during a boom to the losses incurred during a recession. The definition assumes that this number is larger than $\phi_{\beta_u^2}$, the entry rate of β_u^2-firms. For β_u^1-firms, a similar trade-off (condition (33)) implies that being active is not profitable, even if the initial number of β_u^2-firms is 0 and thus, only $\phi_{\beta_u^2}$ enter the market. Notably, this condition may hold, as we show in our simulations, even though β_u^2-firms make larger losses during a recession than β_u^1.

A boom induces an increase in demand and thus, of equilibrium prices above the reservation prices of both types of firms. Consequently, both types of firms enter the market. This process continues, until the market price reaches the higher reservation price, that of β_u^2, $r^c_{\beta_u^2}$. We now argue that, differently from the regular equilibrium, β_u^2-firms will not exit the market once the price falls below $r^c_{\beta_u^2}$. Indeed, suppose this were the case. Since the maximal number of β_u^2-firms in a recession is larger than the entry rate, and since β_u^1-firms do not enter the market in a recession, it follows that there is an $\epsilon > 0$ such that a β_u^2-firm who stays in the market during a boom when the price is given by $r^c_{\beta_u^2} - \epsilon$ will have a continuation value which is strictly positive. Intuitively, this firm would hope for a recession, followed by a new boom before the number of β_u^2-firms in the market reaches $\hat{\mu}_{\beta_u^2}(z_2)$. This is the reason for a (strictly) positive mass of β_u^2-firms to remain in the market during a boom when prices are between the two reservation prices $r^c_{\beta_u^1}$ and $r^c_{\beta_u^2}$. This mass is determined by condition (34) which ensures that the continuation value of β_u^2-firms is 0, making them exactly indifferent between staying active or exiting the market.

Since β_u^1-firms make strictly positive profits until the price falls to their reservation price $r^c_{\beta_u^1}$, they continue to enter the market, while the share of β_u^2-firms decreases. The maximal share of β_u^1-firms, $\hat{\mu}_{\beta_u^1}(z_1)$ obtains simultaneously with the minimal share of β_u^2-firms in a boom, $\check{\mu}_{\beta_u^2}(z_1)$. Whether $\check{\mu}_{\beta_u^2}(z_1)$ is strictly positive or 0 depends on the parameters of the model. For the set of parameters chosen for the simulations, we obtain $\check{\mu}_{\beta_u^2}(z_1) = 0$, and thus, $\hat{\mu}_{\beta_u^1}(z_1) = \bar{\mu}^c$, similarly to the regular equilibrium.

The following proposition demonstrates that Definition 17 is indeed consistent with a Markovian equilibrium. Furthermore, the Markov chain defined by such an equilibrium converges to a unique invariant probability distribution.

Proposition 18 *If there exist M^* and v^* as in Definition 17, then they constitute a Markovian equilibrium of the economy. Furthermore, the Markov process on $\mathbb{Z} \times \mathbb{M}$ defined by q and M^* has a unique invariant distribution Q and converges strongly to Q starting from any initial state (z, m).*

The invariant distribution Q derived in Proposition 18 naturally induces an invariant probability Q_{β_u} over the average bargaining power β_u^{av} in the economy, as defined in (31):

$$Q_{\beta_u}\left(\beta_u^{av}\right) = Q\left\{(z, m) \mid \frac{m_{\beta_u^1}}{m_{\beta_u^1} + m_{\beta_u^2}}\beta_u^1 + \frac{m_{\beta_u^2}}{m_{\beta_u^1} + m_{\beta_u^2}}\beta_u^2 \leq \beta^{av}\right\}$$

Since in a recession, z_2, the economy is dominated by β_u^2-firms, we have that the average bargaining power in a recession is β_u^2. In contrast, during a boom, some β_u^1-firms are present, implying that the average bargaining power β_u^{av} is between β_u^2 and β_u^1. The condition that worker's average bargaining power is larger during a boom is thus tantamount to workers having a higher bargaining power w.r.t. β_u^1-firms.

Lemma 19 *A Markovian equilibrium given by Definition 17 specifies a unique invariant distribution over the average bargaining power $\beta_u^{av}(z, m)$ (as defined by) in the economy, Q_{β_u}. In particular,*

$$E_{Q_{\beta_u}}\left[\beta_u^{av}(z, m) \mid z = z_1\right] >(<) E_{Q_{\beta_u}}\left[\beta_u^{av}(z, m) \mid z = z_2\right]$$

holds iff $\beta_u^1 >(<) \beta_u^2$.

Lemma 19 shows that our model does not in general constrain the way in which average bargaining power varies with the business cycle. This means that the labor wedge can in principle be both procyclical or countercyclical, depending on the specific relation between prices and the profit-maximizing bargaining power. In particular, if the profit-maximizing bargaining power of the workers is decreasing in the price, as illustrated in Figure **??**, the average bargaining power of the workers will be higher during a boom than during a recession, leading to a procyclical labor wedge, contrary to the empirical evidence.

5.3 The Dynamics of the Labor Wedge in a Markovian Equilibrium

The labor wedge is defined by Shimer (2010) as

$$\tau = 1 - \frac{MRS}{MP}$$

Since we do not explicitly model the decisions of the workers, we will assume that the marginal rate of substitution of type i workers at each period equals the reservation wage $\bar{w}_i$, which corresponds to the equilibrium wage: $MRS_i = \frac{\bar{w}}{p}$. Thus, the labor wedge for workers of type i at state (z, m) becomes:

$$\tau^i \left(\beta_u, \bar{\beta}_s, (z, m)\right) = 1 - \frac{\bar{w}_i}{P(z, m)\, MP_i(z, m)}$$

In a model with heterogeneous firms, the marginal productivity of labor and with it, the labor wedge varies across firms and, for a given β_u-type of firm, can be related to the overemployment factor of labor type i, $\gamma_i \left(\beta_u, \bar{\beta}_s, P(z, m)\right)$ as follows:

$$\tau^i \left(\beta_u, P(z, m)\right) = 1 - \gamma_i \left(\beta_u, \bar{\beta}_s, P(z, m)\right) \tag{37}$$

The first insight from (37) is that since in the optimum of the firm at least one of the labor inputs is overemployed, $\gamma_i \left(\beta_u, \bar{\beta}_s, P(z, m)\right) > 1$, the labor wedge can be positive for at most one, but not for both labor inputs simultaneously. Furthermore, if for a given p, a profit-maximizing interior bargaining power $\beta_u^* \left(\bar{\beta}_s, p\right) \in (0, 1)$ exists, then for two firms with $\beta_u^2 < \beta_u^1 \leq \beta_u^* \left(\bar{\beta}_s, p\right)$, Proposition 3 implies:

$$\begin{aligned} \gamma_u \left(\beta_u^2, \bar{\beta}_s, p\right) &< \gamma_u \left(\beta_u^1, \bar{\beta}_s, p\right) \\ \gamma_s \left(\beta_u^2, \bar{\beta}_s, p\right) &> \gamma_s \left(\beta_u^1, \bar{\beta}_s, p\right) \end{aligned} \tag{38}$$

and thus,

$$\begin{aligned} \tau_t^u \left(\beta_u^1, p\right) &< \tau_t^u \left(\beta_u^2, p\right) \\ \tau_t^s \left(\beta_u^1, p\right) &> \tau_t^s \left(\beta_u^2, p\right) \end{aligned} \tag{39}$$

It follows that, in general, the way in which aggregation is done both across labor types and firms will play a crucial role for both the sign and the size of the labor wedge. It will also impact the co-movement of the labor wedge with the cycle of the economy. Nevertheless, we can show that for the case of a Cobb-Douglas production function, the regular Markovian equilibrium determined in Section 5.1 has implications for the labor wedge, which are independent of the specific aggregation rule used. In particular, let $\tau^{av,i}$ specify the aggregation rule for the labor wedge of type i labor as a function of the i-type labor wedges of the two types of firms, $\tau^i \left(\beta_u^1\right)$, $\tau^i \left(\beta_u^2\right)$, and of the market composition, m. We look at a general class of aggregation rules such that the "average" labor wedge is always between the two firm-specific ones and strictly between the two, whenever both firms have a strictly positive mass, $m >> 0$. If one of the firms has a mass of 0, the "average" labor wedge is that of the firm with a strictly positive mass[6]:

$$\begin{aligned} \tau^{av,i}\left(\tau^i\left(\beta_u^1\right),\tau^i\left(\beta_u^2\right),m\right) &\in conv\left(\tau^i\left(\beta_u^1\right),\tau^i\left(\beta_u^2\right)\right) \\ \tau^{av,i}\left(\tau^i\left(\beta_u^1\right),\tau^i\left(\beta_u^2\right),m\right) &\notin \left\{\tau^i\left(\beta_u^1\right),\tau^i\left(\beta_u^2\right)\right\} \text{ if } m >> 0 \\ \tau^{av,i}\left(\tau^i\left(\beta_u^1\right),\tau^i\left(\beta_u^2\right),m\right) &= \tau^i\left(\beta_u^j\right) \text{ if } m_{\beta_u}^{-j} = 0 \end{aligned} \tag{40}$$

Proposition 20 *Assume that the production function is Cobb-Douglas and that*

$$\frac{\beta_u^2}{1-\beta_u^2} < \frac{\beta_u^1}{1-\beta_u^1} \leq \frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{\alpha_s}{1-\alpha_u} \tag{41}$$

Consider a regular Markovian equilibrium as in Definition 14 and described by an invariant probability distribution Q on $\bar{\Sigma}$ as in Proposition 15. Then, the average labor wedge for skilled labor is counter-cyclical:

$$\tau^{av,s}\left(\sigma_0\right) > \sum_{\sigma\in\{\sigma_1\ldots\sigma_n\}} Q\left(\sigma\right)\tau^{av,s}\left(\sigma\right)$$

whereas the labor wedge for unskilled labor is procyclical:

$$\tau^{av,u}\left(\sigma_0\right) < \sum_{\sigma\in\{\sigma_1\ldots\sigma_n\}} Q\left(\sigma\right)\tau^{av,u}\left(\sigma\right)$$

We conclude that the counter-cyclicity of the unskilled workers' bargaining power derived in Proposition implies a countercyclical labor wedge for skilled labor, and a procyclical labor wedge for unskilled labor. Recall in particular, that as shown by Proposition 3, a higher bargaining power of unskilled workers will induce a decrease in the overemployment, and thus, lower labor wedge, of unskilled labor, while increasing overemployment and increasing the labor wedge for skilled workers. This result holds regardless of how aggregation is done across firms. However, the specific aggregation both across firms and across labor types will determine whether the countercyclical or the procyclical part of the dominate.

In the predatory equilibrium, the profit maximizing bargaining power is, by assumption, price-dependent. For a given p, conditions (38) and (39) still hold provided that both β_u^1 and β_u^2 are relatively close to the optimum at p. Thus, β_u^1-type firms will have a lower skilled- and a higher unskilled labor overemployment compared to β_u^2. When the production function is of the generalized CES-type as in (15), exhibits complementarities, and skilled labor is more productive than unskilled labor, we obtain:

Proposition 21 *Suppose that the production function of a firm is given by (15) with $0 > \alpha_s \geq \alpha_u$, $\alpha_i \in \left[-\frac{\sigma}{\sigma-1},-1\right]$ for $i \in \{u,s\}$, $\sigma > 1$ and $\alpha_s + \alpha_u >$*

[6]Note that on an equilibrium path, the case of both firms having a mass of 0 is excluded.

$-\frac{\sigma}{\sigma-1}$. *Then the overemployment factor for unskilled labor* $\gamma_u(\beta,p)$ *is decreasing in* p *and thus, the labor wedge* $\tau^u(\beta,p)$ *is increasing in* p *(procyclical). The overemployment factor for skilled labor* $\gamma_s(\beta,p)$ *is increasing in* p *and thus,* $\tau^s(\beta,p)$ *is decreasing in* p *(countercyclical).*

Combined with condition (39), Proposition 21 demonstrates the impact of the economic cycle on the "aggregate labor wedge[7]" is highly non-trivial and composed of several effects. First, for each individual firm, the type of labor with higher input elasticity will exhibit a countercyclical labor wedge, whereas the wedge will be procyclical for the less productive input. Second, in as far as the market composition changes with the economic cycle, the "aggregate" labor wedge for unskilled labor decreases when β_u^1-type firms dominate the market, whereas the "aggregate" wedge for skilled labor increases in this case. The behavior of the "aggregate labor wedge" will therefore crucially depend on the way in which these effects are aggregated.

Standard estimations of the labor wedge, see, e.g., Shimer (2010), do not consider individual firm characteristics, or distinguish between labor types. Instead, they assume that the production function of the economy is Cobb-Douglas

$$f(\ell_u,\ell_s)=\ell_u^{\alpha_u}\ell_s^{\alpha_s}$$

and workers have no bargaining power, $\beta_u=\beta_s=0$. We now show the implications of such a (misspecified) model for the estimation of the labor wedge in the predatory equilibrium constructed above. Indeed, for this specification, the labor wedge of labor $i\in\{u,s\}$ at state z is given by:

$$\tau^i(P(z,m))=1-\frac{\bar{w}_i}{P(z,m)\,MP_i(P(z,m))}=1-\frac{\bar{w}_i}{P(z,m)\,\alpha_i\frac{y(P(z,m))}{\ell_i(P(z,m))}} \tag{42}$$

The labor wedge of type i labor is countercyclical if

$$\begin{aligned}\tau^i(P(z_1,m)) &< \tau^i(P(z_2,m))\\ \frac{\ell_i P(z_1,m)}{\ell_i P(z_2,m)} &> \frac{P(z_1,m)\,y(P(z_1,m))}{P(z_2,m)\,y(P(z_2,m))}\end{aligned}$$

The following graphs illustrate the "misspecified" labor wedges of the two labor inputs for the case of a predatory equilibrium for two parameter constellations. In Figure **??**, skilled labor has higher output elasticity, $\sigma=1.5$, $\alpha_u=-2.1$, $\alpha_s=-1.1$. In this case, the misspecified labor wedge for skilled labor is procyclical, whereas the labor wedge for unskilled labor is countercyclical. Note that this parameter constellation does not satisfy the conditions of Proposition 21 and thus, we cannot say whether the model misspecification can indeed reverse

[7]We take no stand here as to how such aggregation should be done. Standard estimations do not disaggregate the labor wedge across different types of labor or different firms. Instead, they assume a representative Cobb-Douglas technology for all firms and a representative worker, whose marginal rate of substitution is estimated. We discuss the implications of these assumptions below.

the cyclical behavior of the wedge. ftbpFU4.4962in0.9072in0ptMisspecified labor wedges in the predatory equilibrium when skilled labor has higher ouput elasiticity.figure-missp-wedge-skilled-more-prodwedges$_e l_1 1_e s_2 1_b s_0 80.jpg$

Figure **??** illustrates the results for the case where unskilled labor is more productive, $\sigma = 1.5$, $\alpha_u = -1.1$, $\alpha_s = -2.1$ and shows that this reverses the cyclical behavior of the wedges.

ftbpFU4.4962in0.8795in0ptMisspecified labor wedges in the predatory equilibrium when unskilled labor has higher outputr elasticityfigure-wedge-missp-unskilled-more-prodwedges$_e l_2 1_e s_1 1_b s_0 80.jpg$

6 Conclusion

In this paper, we present a model, which provides a direct link between the firms' bargaining power with respect to the workers and their profits. When production employs different types of labor which are complements and when labor contracts are incomplete, profits need not increase with the firm's bargaining power with respect to unskilled labor, in particular when the bargaining power with respect to skilled labor is fixed. The mechanism behind this result is simple: increasing the bargaining power of the unskilled workers is profitable for the firm, whenever the resulting increase in the overemployment of unskilled labor is (more than) compensated by the decrease in the overemployment of skilled labor. We show that this can occur even for the most commonly used Cobb-Douglas production function. Moreover, for non-homogeneous production functions, bargaining power might depend on the price of the output good as illustrated by the case of the generalized CES function.

We use this static result in a model with endogenous entry and exit to show that in a deterministic setting, firms with lower bargaining power can earn higher profits and dominate the market in the long-run. In as far as the profit-maximizing bargaining power depends on the production technology and industry-specific institutional constraints, the bargaining power of the dominant firm can vary across industries, consistent with the findings of Cahuc et al. (2006).

In a stochastic model with exogenous demand shocks and limited entry, market composition within an industry depends on the shock realization. The average bargaining power of firms w.r.t. the workers is thus non-degenerate and depends on the state of the economy, at least in the short-run. In the regular Markovian equilibrium, this leads to a non-degenerate distribution of bargaining power in the initial periods of a boom, which is consistent with the observed heterogeneity of firms' bargaining powers with a given industry, as documented by Wong (2021).

When the profit-maximizing bargaining power is price-dependent, different shock realizations favor firms with different bargaining powers. This leads to a non-degenerate distribution of bargaining powers, both within a given time period, as well as across the two states of the economy, even when each of the states is persistent. Notably, the market is dominated by a different type of firms during a lasting recession than during a long lasting boom, as illustrated by the predatory Markovian equilibrium.

The invariant distribution over states generated by a Markovian equilibrium implies an invariant probability over the average bargaining power in the economy. This allows us to study how the distribution of bargaining power varies with the cycle of the economy. If firms with lower bargaining power are more profitable, as in the regular equilibrium, the average bargaining power of the unskilled workers is higher in a recession than in a boom, thus providing an explanation for the intertemporal pattern documented by Cheremukhin and Restrepo-Echavarria (2014) and for the observed behavior of the labor wedge as in Shimer (2010). The model, however, can accommodate the reverse dependence, as well, as illustrated by the case of the predatory equilibrium.

We thus obtain a rich framework, which allows us to explicitly model and study the evolution of bargaining power as driven by market forces and explore the role of institutional constraints on the long-run properties of market composition, prices and equilibrium allocations. One possible extension of the model would incorporate the effects of investment in capital and technological innovation, which would have a non-trivial effect on entry- and exit-dynamics.

The methods used here could also potentially be extended to provide a general framework for studying the evolution of intra-firm institutions, such as distribution mechanisms within the firm. The main difference between such mechanisms and the allocation mechanism based on bargaining power studied here would consist in the fact that the share of the marginal product allocated to the firm / the worker would no longer be constant, but depend on prices.

7 Appendix A:

7.1 Proofs for the Static Model, Section 2

Proof of Lemma 2:

The indirect profit function is:

$$\pi(\beta,p) \equiv pf(\ell_u(\beta,p),\ell_s(\beta,p)) - w_u(\ell_u(\beta,p),\ell_s(\beta,p))\ell_u(\beta,p) - w_s(\ell_u(\beta,p),\ell_s(\beta,p))\ell_s(\beta,p)$$

By (2) and (5), at the optimal employment plan,

$$w_i[\beta,p](\ell_u(\beta,p),\ell_s(\beta,p)) = \bar{w}_i$$

we conclude that the indirect profit function can be re-written as:

$$\pi(\beta,p) = pf(\ell_u(\beta,p),\ell_s(\beta,p)) - \bar{w}_u\ell_u(\beta,p) - \bar{w}_s\ell_s(\beta,p)$$

Let

$$(\ell_u^*,\ell_s^*) \equiv \arg\max_{\ell_u,\ell_s} pf(\ell_u,\ell_s) - \bar{w}_u\ell_u - \bar{w}_s\ell_s$$

For the case $\beta_i = 0$, equation (2) implies that $w_i(\ell_u, \ell_s) = \bar{w}_i$ for all (ℓ_u, ℓ_s) and by equation (5) we must have

$$p\frac{\partial f(\ell_u(0,p), \ell_s(0,p))}{\partial \ell_i} - \bar{w}_i = 0 \Rightarrow \gamma_i(0,p) = 1 \tag{43}$$

which implies:

$$\ell_i(0,p) = \ell_i^* \quad \text{for } i = u, s$$

Thus,

$$\begin{aligned}\pi(0,p) &= pf(\ell_u^*, \ell_s^*) - \bar{w}_u\ell_u^* - \bar{w}_s\ell_s^* \\ &= \max_{\ell_u,\ell_s} pf(\ell_u, \ell_s) - \bar{w}_u\ell_u - \bar{w}_s\ell_s \\ &\geq \pi(\beta, p)\end{aligned}$$

and we conclude the indirect profit function is maximised at $\beta_u = \beta_s = 0$ and thus $\gamma_u(0,p) = \gamma_s(0,p) = 1$. ■

Lemma 22 *The effect of bargaining powers on wages are given as follows*[8]*:*

$$\begin{aligned}\frac{\partial w_i[\beta,p](\ell_u,\ell_s)}{\partial \beta_i} &= \bar{w}_i\ell_i + \ell_i p\int_0^1 \frac{\ln z}{\beta_i^2} z^{\frac{1-\beta_i}{\beta_i}} \frac{\partial f\left(z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_u}{1-\beta_u}}\ell_u, z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_s}{1-\beta_s}}\ell_s\right)}{\partial \ell_i} dz \\ &+ \ell_i p\int_0^1 \frac{\ln z}{\beta_i^2} z^{\frac{1-\beta_i}{\beta_i}} \frac{\partial^2 f\left(z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_u}{1-\beta_u}}\ell_u, z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_s}{1-\beta_s}}\ell_s\right)}{\partial \ell_i \partial \ell_{-i}} \frac{\beta_{-i}}{1-\beta_{-i}} z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_{-i}}{1-\beta_{-i}}}\ell_{-i} dz \\ &- p\ell_i\ell_{-i}\int_0^1 (-\ln z)\frac{1-\beta_{-i}}{\beta_{-i}(1-\beta_i)^2} z^{\frac{1-\beta_{-i}}{\beta_{-i}}\left(1+\frac{\beta_i}{1-\beta_i}\right)} \frac{\partial^2 f\left(z^{\frac{1-\beta_{-i}}{\beta_{-i}}\frac{\beta_u}{1-\beta_u}}\ell_u, z^{\frac{1-\beta_{-i}}{\beta_{-i}}\frac{\beta_s}{1-\beta_s}}\ell_s\right)}{\partial \ell_u \partial \ell_s} dz\end{aligned}$$

$$\frac{\partial w_i[\beta,p](\ell_u,\ell_s)}{\partial \beta_{-i}} = -p\int_0^1 \left[(-\ln z)\frac{1-\beta_{-i}}{\beta_{-i}(1-\beta_i)^2} z^{\frac{1-\beta_{-i}}{\beta_{-i}}\left(1+\frac{\beta_i}{1-\beta_i}\right)} \cdot \cdot \ell_{-i} \frac{\partial^2 f\left(z^{\frac{1-\beta_{-i}}{\beta_{-i}}\frac{\beta_u}{1-\beta_u}}\ell_u, z^{\frac{1-\beta_{-i}}{\beta_{-i}}\frac{\beta_s}{1-\beta_s}}\ell_s\right)}{\partial \ell_i \partial \ell_{-i}}\right] dz$$

If production exhibits substitutes,

$$\frac{\partial w_i[\beta,p](\ell_u,\ell_s)}{\partial \beta_{-i}} > 0$$

and

$$\frac{\partial w_i[\beta,p](\ell_u,\ell_s)}{\partial \beta_i} < 0$$

[8] The results on the sign of $\frac{\partial w_i[\beta,p](\ell_u,\ell_s)}{\partial \beta_{-i}}$ can be found in Cahuc et al. (2008, p. ??).

provided that

$$p\int_0^1 \frac{-\log z}{\beta_i^2} z^{\frac{1-\beta_i}{\beta_i}} \frac{\partial f\left(z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_u}{1-\beta_u}}\ell_u, z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_s}{1-\beta_s}}\ell_s\right)}{\partial \ell_i} dz - \bar{w}_i < 0 \tag{44}$$

If production exhibits complementarities,

$$\frac{\partial w_i\left[\beta, p\right](\ell_u, \ell_s)}{\partial \beta_{-i}} < 0 \tag{45}$$

Proof of Lemma 22:

Differentiating (3) with respect to β_i, we obtain:

$$\frac{\partial w_i(\ell_u, \ell_s)}{\partial \beta_{-i}} = -p\int_0^1 (-\ln z)\frac{1-\beta_{-i}}{\beta_{-i}(1-\beta_i)^2} z^{\frac{1-\beta_{-i}}{\beta_{-i}}\left(1+\frac{\beta_i}{1-\beta_i}\right)}\ell_{-i}\frac{\partial^2 f\left(z^{\frac{1-\beta_{-i}}{\beta_{-i}}\frac{\beta_u}{1-\beta_u}}\ell_u, z^{\frac{1-\beta_{-i}}{\beta_{-i}}\frac{\beta_s}{1-\beta_s}}\ell_s\right)}{\partial \ell_u \partial \ell_s} dz$$

Note that $\ln z \leq 0$ for $z \in [0,1]$ and that

$$\frac{\partial^2 f\left(z^{\frac{1-\beta_{-i}}{\beta_{-i}}\frac{\beta_u}{1-\beta_u}}\ell_u, z^{\frac{1-\beta_{-i}}{\beta_{-i}}\frac{\beta_s}{1-\beta_s}}\ell_s\right)}{\partial \ell_u \partial \ell_s} > (<) 0$$

whenever the two types of labor are complements (substitutes). Thus, $\frac{\partial w_i(\ell_u,\ell_s)}{\partial \beta_{-i}} < (>) 0$ whenever ℓ_u and ℓ_s are complements (substitutes).

Note also that

$$\begin{aligned}
\frac{\partial w_i}{\partial \beta_i} &= -\bar{w}_i + \int_0^1 \frac{(-\log(z))}{\beta_i^2} z^{\frac{1-\beta_i}{\beta_i}} p \left(\begin{array}{c} \frac{\partial f\left(z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_u}{1-\beta_u}}\ell_u, z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_s}{1-\beta_s}}\ell_s\right)}{\partial \ell_i} + \\ + \frac{\partial^2 f\left(z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_u}{1-\beta_u}}\ell_u, z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_s}{1-\beta_s}}\ell_s\right)}{\partial \ell_i \partial \ell_{-i}} \frac{\beta_{-i}}{1-\beta_{-i}} z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_{-i}}{1-\beta_{-i}}}\ell_{-i} \end{array} \right) dz \\
&= \int_0^1 \frac{(-\log(z))}{\beta_i^2} z^{\frac{1-\beta_i}{\beta_i}} \left(\begin{array}{c} p\frac{\partial f\left(z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_u}{1-\beta_u}}\ell_u, z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_s}{1-\beta_s}}\ell_s\right)}{\partial \ell_i} - \bar{w}^i \\ + p\frac{\partial^2 f\left(z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_u}{1-\beta_u}}\ell_u, z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_s}{1-\beta_s}}\ell_s\right)}{\partial \ell_i \partial \ell_{-i}} \frac{\beta_{-i}}{1-\beta_{-i}} z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_{-i}}{1-\beta_{-i}}}\ell_{-i} \end{array} \right) dz
\end{aligned}$$

where the second line uses $\int_0^1 \frac{(-\log(z))}{\beta_i^2} z^{\frac{1-\beta_i}{\beta_i}} dz = 1$.

Finally, note that when inputs are substitutes,

$$\frac{\partial^2 f\left(z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_u}{1-\beta_u}}\ell_u, z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_s}{1-\beta_s}}\ell_s\right)}{\partial \ell_i \partial \ell_{-i}} < 0$$

and if

$$\int_0^1 \frac{(-\log(z))}{\beta_i^2} z^{\frac{1-\beta_i}{\beta_i}} p\frac{\partial f\left(z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_u}{1-\beta_u}}\ell_u, z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_s}{1-\beta_s}}\ell_s\right)}{\partial \ell_i} dz - \bar{w}^i < 0$$

$\frac{\partial w_i}{\partial \beta_i} < 0$ obtains. ■

The intuition for this result is easy to understand for the case of complements. Suppose indeed that workers of type u and of type s are complements in the production. An additional hire of a skilled worker s increases the marginal productivity of workers of type u, and thus, as explained above, the total wages paid to type u workers. By (2), this increase depends positively on the bargaining power of u workers, β_u. The marginal total surplus of worker s (1) is thus decreasing in β_u and so is the wage $w_s[\beta, p](\ell_u, \ell_s)$ the firm is willing to pay.

While the effect of β_{-i} on w_i in the case of complementarities is unambiguously negative, the effect of β_i in general depends on the hiring decision. It is composed of two terms: the first, (44) relates the "average" inframarginal products of type s workers to their reservation wage (see condition (**??**) in the following example for an illustration) and can be positive or negative depending on whether type s workers are under- or overemployed. The second is the marginal effect of a worker of type s on the productivity of type u labor and is always positive in the case of complements.

The following Lemma will be used in the proofs of some of the results.

Lemma 23 *The effect of increasing the bargaining power of workers of type i, β_i, on profits $\pi(\beta, p)$ can be represented equivalently as follows:*

(i)

$$\frac{\partial \pi(\beta, p)}{\partial \beta_i} = \ell_i(\beta, p)\left[\bar{w}_i - p\int_0^1 \frac{(-\ln z)}{\beta_i^2} z^{\frac{1-\beta_i}{\beta_i}} \frac{\partial f\left(z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_u}{1-\beta_u}}\ell_u(\beta, p), z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_s}{1-\beta_s}}\ell_s(\beta, p)\right)}{\partial \ell_i} dz\right] \tag{46}$$

(ii)

$$\frac{\partial \pi(\beta, p)}{\partial \beta_i} = -p\left[\eta_{ii}(\beta, p)\bar{w}_i\ell_i(\beta, p) + \frac{\beta_{-i}}{\beta_i}\eta_{i,-i}(\beta, p)\bar{w}_{-i}\ell_{-i}(\beta, p)\right] \tag{47}$$

where $\eta_{ii}(\beta, p)$ and $\eta_{i,-i}(\beta, p)$ are the bargaining-power overemployment elasticities defined in (9).

Proof of Lemma 23:
(i) Differentiating (3) with respect to β_i, we obtain the expressions[9] for $\frac{\partial w_i}{\partial \beta_i}$

[9] To shorten notation, in the following proofs, we write (ℓ_u, ℓ_s) as a short-hand for the optimal factor inputs $(\ell_u(\beta, p), \ell_s(\beta, p))$.

and $\frac{\partial w_{-i}}{\partial \beta_i}$:

$$\begin{aligned}
\frac{\partial \pi(\beta,p)}{\partial \beta_i} &= -\left(\frac{\partial w_i[\beta,p](\ell_u,\ell_s)}{\partial \beta_i}\ell_i + \frac{\partial w_{-i}[\beta,p](\ell_u,\ell_s)}{\partial \beta_i}\ell_{-i}\right) \\
&= \bar{w}_i \ell_i + \ell_i p \int_0^1 \frac{\ln z}{\beta_i^2} z^{\frac{1-\beta_i}{\beta_i}} \frac{\partial f\left(z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_u}{1-\beta_u}}\ell_u, z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_s}{1-\beta_s}}\ell_s\right)}{\partial \ell_i} dz \\
&+ \ell_i p \int_0^1 \frac{\ln z}{\beta_i^2} z^{\frac{1-\beta_i}{\beta_i}} \frac{\partial^2 f\left(z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_u}{1-\beta_u}}\ell_u, z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_s}{1-\beta_s}}\ell_s\right)}{\partial \ell_i \partial \ell_{-i}} \frac{\beta_{-i}}{1-\beta_{-i}} z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_{-i}}{1-\beta_{-i}}} \ell_{-i} dz \\
&- p\ell_i\ell_{-i}\int_0^1 (-\ln z)\, \frac{1-\beta_{-i}}{\beta_{-i}(1-\beta_i)^2} z^{\frac{1-\beta_{-i}}{\beta_{-i}}\left(1+\frac{\beta_i}{1-\beta_i}\right)} \frac{\partial^2 f\left(z^{\frac{1-\beta_{-i}}{\beta_{-i}}\frac{\beta_u}{1-\beta_u}}\ell_u, z^{\frac{1-\beta_{-i}}{\beta_{-i}}\frac{\beta_s}{1-\beta_s}}\ell_s\right)}{\partial \ell_u \partial \ell_s} dz
\end{aligned}$$

To simplify the expression, we make a change of variables. Define $\bar{z}$ so that for each $z \in [0,1]$,

$$\bar{z} = \psi(z) = z^{\frac{\beta_i}{1-\beta_i}\frac{1-\beta_{-i}}{\beta_{-i}}}$$

or equivalently,

$$\bar{z}^{\frac{\beta_{-i}}{1-\beta_{-i}}\frac{1-\beta_i}{\beta_i}} = z.$$

Then,

$$z^{\frac{1-\beta_{-i}}{\beta_{-i}}} = \bar{z}^{\frac{\beta_{-i}}{1-\beta_{-i}}\frac{1-\beta_i}{\beta_i}\frac{1-\beta_{-i}}{\beta_{-i}}} = \bar{z}^{\frac{1-\beta_i}{\beta_i}}.$$

Then,

$$\begin{aligned}
d\bar{z} &= \psi'(z)\, dz = \frac{\beta_i}{1-\beta_i}\frac{1-\beta_{-i}}{\beta_{-i}} z^{\frac{\beta_i}{1-\beta_i}\frac{1-\beta_{-i}}{\beta_{-i}}-1} dz \\
&= \frac{\beta_i}{1-\beta_i}\frac{1-\beta_{-i}}{\beta_{-i}}\frac{\bar{z}}{z} dz = \frac{\beta_i}{1-\beta_i}\frac{1-\beta_{-i}}{\beta_{-i}} \bar{z}^{1-\frac{\beta_{-i}}{1-\beta_{-i}}\frac{1-\beta_i}{\beta_i}} dz
\end{aligned}$$

and, for any function g, we have:

$$\int_0^1 g(\psi(z))\, \psi'(z)\, dz = \int_0^1 g(\bar{z})\, d\bar{z}$$

We now make the change of variables in the last term of $\frac{\partial \pi(\beta,p)}{\partial \beta_i}$ (and change the name of the variable of integration to $\bar{z}$ in the other terms):

$$\begin{aligned}\frac{\partial\pi\left(\beta,p\right)}{\partial\beta_i} &= \bar{w}_i\ell_i+p\ell_i\int_0^1\frac{\ln\bar{z}}{\beta_i^2}\bar{z}^{\frac{1-\beta_i}{\beta_i}}\frac{\partial f\left(\bar{z}^{\frac{1-\beta_i}{\beta_i}\frac{\beta_u}{1-\beta_u}}\ell_u,\bar{z}^{\frac{1-\beta_i}{\beta_i}\frac{\beta_s}{1-\beta_s}}\ell_s\right)}{\partial\ell_i}d\bar{z}+\\ &+p\ell_i\ell_{-i}\int_0^1\frac{\ln\bar{z}}{\beta_i^2}\frac{\beta_{-i}}{1-\beta_{-i}}\bar{z}^{\frac{1-\beta_i}{\beta_i}\left(1+\frac{\beta_{-i}}{1-\beta_{-i}}\right)}\frac{\partial^2 f\left(\bar{z}^{\frac{1-\beta_i}{\beta_i}\frac{\beta_u}{1-\beta_u}}\ell_u,\bar{z}^{\frac{1-\beta_i}{\beta_i}\frac{\beta_s}{1-\beta_s}}\ell_s\right)}{\partial\ell_i\partial\ell_{-i}}d\bar{z}-\\ &-p\ell_i\ell_{-i}\int_0^1\frac{\bar{z}^{\frac{1-\beta_i}{\beta_i}}\bar{z}\frac{1-\beta_{-i}}{\beta_{-i}}\frac{\ln\bar{z}^{\frac{\beta_{-i}}{1-\beta_{-i}}\frac{1-\beta_i}{\beta_i}}}{(1-\beta_i)^2}}{\frac{\beta_i}{1-\beta_i}\frac{1-\beta_{-i}}{\beta_{-i}}\bar{z}^{1-\frac{\beta_{-i}}{1-\beta_{-i}}\frac{1-\beta_i}{\beta_i}}}\frac{\partial^2 f\left(\bar{z}^{\frac{1-\beta_i}{\beta_i}\frac{\beta_u}{1-\beta_u}}\ell_u,\bar{z}^{\frac{1-\beta_i}{\beta_i}\frac{\beta_s}{1-\beta_s}}\ell_s\right)}{\partial\ell_u\partial\ell_s}d\bar{z}\end{aligned}$$

Standard algebra shows that this is equivalent to:

$$\begin{aligned}\frac{\partial\pi\left(\beta,p\right)}{\partial\beta_i} &= \bar{w}_i\ell_i+p\ell_i\int_0^1\frac{\ln\bar{z}}{\beta_i^2}\bar{z}^{\frac{1-\beta_i}{\beta_i}}\frac{\partial f\left(\bar{z}^{\frac{1-\beta_i}{\beta_i}\frac{\beta_u}{1-\beta_u}}\ell_u,\bar{z}^{\frac{1-\beta_i}{\beta_i}\frac{\beta_s}{1-\beta_s}}\ell_s\right)}{\partial\ell_i}d\bar{z}+\\ &+p\ell_i\ell_{-i}\int_0^1\frac{\ln\bar{z}}{\beta_i^2}\frac{\beta_{-i}}{1-\beta_{-i}}\bar{z}^{\frac{1-\beta_i}{\beta_i}\left(1+\frac{\beta_{-i}}{1-\beta_{-i}}\right)}\frac{\partial^2 f\left(\bar{z}^{\frac{1-\beta_i}{\beta_i}\frac{\beta_u}{1-\beta_u}}\ell_u,\bar{z}^{\frac{1-\beta_i}{\beta_i}\frac{\beta_s}{1-\beta_s}}\ell_s\right)}{\partial\ell_i\partial\ell_{-i}}d\bar{z}-\\ &-p\ell_i\ell_{-i}\int_0^1\frac{\ln\bar{z}}{\beta_i^2}\frac{\beta_{-i}}{1-\beta_{-i}}\bar{z}^{\frac{1-\beta_i}{\beta_i}\left(1+\frac{\beta_{-i}}{1-\beta_{-i}}\right)}\frac{\partial^2 f\left(\bar{z}^{\frac{1-\beta_i}{\beta_i}\frac{\beta_u}{1-\beta_u}}\ell_u,\bar{z}^{\frac{1-\beta_i}{\beta_i}\frac{\beta_s}{1-\beta_s}}\ell_s\right)}{\partial\ell_i\partial\ell_{-i}}d\bar{z}\end{aligned}$$

or

$$\begin{aligned}\frac{\partial\pi\left(\beta,p\right)}{\partial\beta_i} &= \bar{w}_i\ell_i+p\ell_i\int_0^1\frac{\ln\bar{z}}{\beta_i^2}\bar{z}^{\frac{1-\beta_i}{\beta_i}}\frac{\partial f\left(\bar{z}^{\frac{1-\beta_i}{\beta_i}\frac{\beta_u}{1-\beta_u}}\ell_u,\bar{z}^{\frac{1-\beta_i}{\beta_i}\frac{\beta_s}{1-\beta_s}}\ell_s\right)}{\partial\ell_i}d\bar{z}\\ &= \ell_i\left[\bar{w}_i-p\int_0^1\frac{\left(-\ln\bar{z}\right)}{\beta_i^2}\bar{z}^{\frac{1-\beta_i}{\beta_i}}\frac{\partial f\left(\bar{z}^{\frac{1-\beta_i}{\beta_i}\frac{\beta_u}{1-\beta_u}}\ell_u,\bar{z}^{\frac{1-\beta_i}{\beta_i}\frac{\beta_s}{1-\beta_s}}\ell_s\right)}{\partial\ell_i}d\bar{z}\right]\end{aligned}$$

(ii) At the stable wage schedule,

$$\begin{aligned}\pi\left(\beta,p\right) &= pf\left(\ell_u\left(\beta,p\right),\ell_s\left(\beta,p\right)\right)-p\beta_u\frac{\partial f\left(\ell_u\left(\beta,p\right),\ell_s\left(\beta,p\right)\right)}{\partial\ell_u}\gamma_u\left(\beta,p\right)\ell_u\left(\beta,p\right)-\left(1-\beta_u\right)\bar{w}_u-\\ &\quad -p\beta_s\frac{\partial f\left(\ell_u\left(\beta,p\right),\ell_s\left(\beta,p\right)\right)}{\partial\ell_s}\gamma_s\left(\beta,p\right)\ell_s\left(\beta,p\right)-\left(1-\beta_s\right)\bar{w}_s\end{aligned}$$

Differentiating with respect to β_i, we thus have

$$\begin{aligned}
\frac{\partial \pi(\beta,p)}{\partial \beta_i} &= -p\frac{\partial f(\ell_u(\beta,p),\ell_s(\beta,p))}{\partial \ell_i}\frac{\partial \gamma_i(\beta,p)}{\partial \beta_i}\beta_i \ell_i(\beta,p) - \\
&\quad \underbrace{-p\frac{\partial f(\ell_u(\beta,p),\ell_s(\beta,p))}{\partial \ell_i}\gamma_i(\beta,p)\ell_i(\beta,p) + \bar{w}_i -}_{=0} \\
&\quad -p\frac{\partial f(\ell_u(\beta,p),\ell_s(\beta,p))}{\partial \ell_{-i}}\frac{\partial \gamma_{-i}(\beta,p)}{\partial \beta_i}\beta_{-i}\ell_{-i}(\beta,p) \\
&= -p\frac{\partial f(\ell_u(\beta,p),\ell_s(\beta,p))}{\partial \ell_i}\frac{\partial \gamma_i(\beta,p)}{\partial \beta_i}\beta_i \ell_i(\beta,p) - \\
&\quad -p\frac{\partial f(\ell_u(\beta,p),\ell_s(\beta,p))}{\partial \ell_{-i}}\frac{\partial \gamma_{-i}(\beta,p)}{\partial \beta_i}\beta_{-i}\ell_{-i}(\beta,p) \\
&= -p\bar{w}_i\frac{\frac{\partial \gamma_i(\beta,p)}{\partial \beta_i}\beta_i}{\gamma_i(\beta,p)}\ell_i(\beta,p) - p\bar{w}_{-i}\frac{\frac{\partial \gamma_{-i}(\beta,p)}{\partial \beta_i}\beta_{-i}}{\gamma_{-i}(\beta,p)}\ell_{-i}(\beta,p) \\
&= -p\left[\eta_{ii}(\beta,p)\bar{w}_i\ell_i(\beta,p) + \frac{\beta_{-i}}{\beta_i}\eta_{i,-i}(\beta,p)\bar{w}_{-i}\ell_{-i}(\beta,p)\right]
\end{aligned}$$

where the second and the third equality follow from the definition of the overemployment factors $\gamma_i(\beta,p)$ and $\gamma_{-i}(\beta,p)$ in (7) given the optimality of the labor inputs at (β,p). The last equality uses the definitions of η_{ii} and $\eta_{i,-i}$ in (9).

Proof of Proposition 3:
Consider first the case in which the two labor inputs are substitutes. In equilibrium (3) and (6) imply that

$$\bar{w}_i = p\frac{\partial f(\ell_u,\ell_s)}{\partial \ell_i}\left[\int \frac{1}{\beta_i}\bar{z}^{\frac{1-\beta_i}{\beta_i}}\frac{\partial f\left(\bar{z}^{\frac{1-\beta_i}{\beta_i}\frac{\beta_u}{1-\beta_u}}\ell_u, \bar{z}^{\frac{1-\beta_i}{\beta_i}\frac{\beta_s}{1-\beta_s}}\ell_s\right)}{\partial \ell_i}d\bar{z}\right]$$

From (46), we thus obtain:

$$\frac{\partial \pi(\beta,p)}{\partial \beta_i} = -p\ell_i\left[\int_0^1 \frac{\bar{z}^{\frac{1-\beta_i}{\beta_i}}}{\beta_i}\frac{\partial f\left(\bar{z}^{\frac{1-\beta_i}{\beta_i}\frac{\beta_u}{1-\beta_u}}\ell_u, \bar{z}^{\frac{1-\beta_i}{\beta_i}\frac{\beta_s}{1-\beta_s}}\ell_s\right)}{\partial \ell_i}\left[\frac{(-\ln \bar{z})}{\beta_i} - 1\right]d\bar{z}\right] \tag{48}$$

Now note that in the case of substitutes $\frac{\partial f\left(\bar{z}^{\frac{1-\beta_i}{\beta_i}\frac{\beta_u}{1-\beta_u}}\ell_u, \bar{z}^{\frac{1-\beta_i}{\beta_i}\frac{\beta_s}{1-\beta_s}}\ell_s\right)}{\partial \ell_i}$ is strictly decreasing in $\bar{z}$. Furthermore, we have

$$\int_0^1 \frac{\bar{z}^{\frac{1-\beta_i}{\beta_i}}}{\beta_i} d\bar{z} = \left. \frac{\bar{z}^{\frac{1-\beta_i}{\beta_i}+1}}{\left(\frac{1-\beta_i}{\beta_i}+1\right)\beta_i} \right|_0^1 = \bar{z}^{\frac{1-\beta_i}{\beta_i}+1}\Big|_0^1 = 1$$

$$\int_0^1 \frac{(-\ln \bar{z})}{\beta_i^2} \bar{z}^{\frac{1-\beta_i}{\beta_i}} d\bar{z} = -\left. \frac{\bar{z}^{\frac{1}{\beta_i}} \ln \bar{z}}{\beta_i} \right|_0^1 + \int_0^1 \frac{\bar{z}^{\frac{1}{\beta_i}-1}}{\beta_i} d\bar{z} = -\lim_{\bar{z}\to 0} \frac{\bar{z}^{\frac{1}{\beta_i}} \ln \bar{z}}{\beta_i} + \bar{z}^{\frac{1}{\beta_i}}\Big|_0^1$$

$$= -\lim_{\bar{z}\to 0} \left(-\frac{1}{\beta_i} \frac{\frac{1}{z}}{\frac{1}{\bar{z}^{\frac{1}{\beta_i}+1}}} \right) + 1 = \lim_{\bar{z}\to 0} \left(\frac{1}{\beta_i} \frac{\bar{z}^{\frac{1}{\beta_i}+1}}{z} \right) + 1 = 1$$

Thus, the expression in the big square brackets in (48) is a difference of two expectations of $\frac{\partial f\left(\bar{z}^{\frac{1-\beta_i}{\beta_i}\frac{\beta_u}{1-\beta_u}}\ell_u, \bar{z}^{\frac{1-\beta_i}{\beta_i}\frac{\beta_s}{1-\beta_s}}\ell_s\right)}{\partial \ell_i}$, the first with a density function $\frac{(-\ln \bar{z})}{\beta_i^2}\bar{z}^{\frac{1-\beta_i}{\beta_i}}$ and the second with a density $\frac{\bar{z}^{\frac{1-\beta_i}{\beta_i}}}{\beta_i}$.

We will now show that the probability distribution defined by the density function $\frac{(-\ln \bar{z})}{\beta_i^2}\bar{z}^{\frac{1-\beta_i}{\beta_i}}$ first-order stochastically dominates that with density $\frac{\bar{z}^{\frac{1-\beta_i}{\beta_i}}}{\beta_i}$. Since $\frac{\partial f\left(\bar{z}^{\frac{1-\beta_i}{\beta_i}\frac{\beta_u}{1-\beta_u}}\ell_u, \bar{z}^{\frac{1-\beta_i}{\beta_i}\frac{\beta_s}{1-\beta_s}}\ell_s\right)}{\partial \ell_i}$ is strictly decreasing in $\bar{z}$, the condition to check is that

$$\int_{\tilde{z}}^1 \frac{(-\ln \bar{z})}{\beta_i^2} \bar{z}^{\frac{1-\beta_i}{\beta_i}} d\bar{z} > \int_{\tilde{z}}^1 \frac{\bar{z}^{\frac{1-\beta_i}{\beta_i}}}{\beta_i} d\bar{z}$$

for each $\tilde{z}$. This is equivalent to

$$\left. -\frac{\bar{z}^{\frac{1}{\beta_i}} \ln \bar{z}}{\beta_i} + \bar{z}^{\frac{1}{\beta_i}} - \bar{z}^{\frac{1}{\beta_i}} \right|_{\tilde{z}}^1 > 0$$

$$-\frac{\tilde{z}^{\frac{1}{\beta_i}} \ln \tilde{z}}{\beta_i} > 0$$

which is indeed satisfied for any $\tilde{z} \in (0,1)$. It follows that $\frac{\partial \pi(\beta,p)}{\partial \beta_i} < 0$ whenever the labor inputs are substitutes.

Thus, if $\frac{\partial^2 f(\ell_u,\ell_s)}{\partial \ell_u \partial \ell_s} < 0$, $\beta_i^*\left(p, \bar{\beta}_{-i}\right) = 0$ for any $\bar{\beta}_{-i} \in [0,1]$.

It follows that complementarity, $\frac{\partial^2 f(\ell_u,\ell_s)}{\partial \ell_u \partial \ell_s} > 0$ is necessary for $\frac{\partial \pi\left(\beta_i,\bar{\beta}_{-i},p\right)}{\partial \beta_i} > 0$. Complementarity, $\frac{\partial^2 f(\ell_u,\ell_s)}{\partial \ell_u \partial \ell_s} > 0$, further implies that

$$sign\left(\eta_{i,-i}\right) = sign\left(\frac{\partial \gamma_{-i}\left(\beta_i, \bar{\beta}_{-i}, p\right)}{\partial \beta_i}\right) = sign\left(\frac{\partial w_{-i}\left(\beta_i, \bar{\beta}_{-i}, p\right)}{\partial \beta_i}\right) < 0.$$

Furthermore, by (47),

$$\frac{\partial \pi\left(\beta_i, \bar{\beta}_{-i}, p\right)}{\partial \beta_i} > 0$$

is equivalent to:

$$\frac{\beta_i \ell_i\left(\beta_i, \bar{\beta}_{-i}, p\right) \bar{w}_i}{\ell_{-i}\left(\beta_i, \bar{\beta}_{-i}, p\right) \bar{w}_{-i}} < -\bar{\beta}_{-i} \frac{\eta_{i,-i}\left(\beta_i, \bar{\beta}_{-i}, p\right)}{\eta_{i,i}\left(\beta_i, \bar{\beta}_{-i}, p\right)} \tag{49}$$

which can only be satisfied for $\eta_{ii} > 0$ (and thus, $\frac{\partial \gamma_i\left(\beta_i, \bar{\beta}_{-i}, p\right)}{\partial \beta_i} > 0$) and provides a sufficient condition for profits to be increasing in β_i.

Since for $\ell_{-i}\left(0, \bar{\beta}_{-i}, p\right) > 0$ the l.h.s. of (49) is 0 for $\beta_i = 0$ and $\bar{\beta}_{-i} > 0$, (49) is trivially satisfied whenever

$$-\frac{\eta_{i,-i}\left(0, \bar{\beta}_{-i}, p\right)}{\eta_{i,i}\left(0, \bar{\beta}_{-i}, p\right)} > 0$$

and thus, whenever $0 < \eta_{ii}\left(0, \bar{\beta}_{-i}, p\right) < \infty$.

We prove Proposition 6 using a series of Lemmata:

The first two results provide necessary and sufficient conditions for the optimal bargaining power to be independent of the price:

Lemma 24 *Suppose the optimal bargaining power $\beta_i^*\left(\bar{\beta}_{-i}, p\right)$ is interior, differentiable and $\frac{\partial^2 \pi\left(\beta_i^*\left(\bar{\beta}_{-i}, p\right), \bar{\beta}_{-i}, p\right)}{\partial \beta_i^2} \neq 0$ for all $p > 0$. Then, $\beta_i^*\left(\bar{\beta}_{-i}, p\right)$ is independent of p if and only if $\frac{\partial^2 \pi\left(\beta_i^*\left(\bar{\beta}_{-i}, p\right), p\right)}{\partial \beta_i \partial p} = 0$ for all $p > 0$.*

<u>Proof of Lemma 24:</u>

Since $\frac{\partial \pi\left(\beta_i^*\left(\bar{\beta}_{-i}, p\right), p\right)}{\partial \beta_i} = 0$ for all $p > 0$ and $\beta_i^*\left(\bar{\beta}_{-i}, p\right)$ is differentiable for all $p > 0$, it follows that

$$\frac{\partial^2 \pi\left(\beta_i^*\left(\bar{\beta}_{-i}, p\right), p\right)}{\partial \beta_i^2} \frac{\partial \beta_i^*\left(\bar{\beta}_{-i}, p\right)}{\partial p} + \frac{\partial^2 \pi\left(\beta_i^*\left(\bar{\beta}_{-i}, p\right), p\right)}{\partial \beta_i \partial p} = 0$$

Since $\frac{\partial^2 \pi\left(\beta_i^*\left(\bar{\beta}_{-i}, p\right), p\right)}{\partial \beta_i^2} \neq 0$ for all p by assumption, then $\frac{\partial \beta_i^*\left(\bar{\beta}_{-i}, p\right)}{\partial p} = 0$ if and only if $\frac{\partial^2 \pi\left(\beta_i^*\left(\bar{\beta}_{-i}, p\right), p\right)}{\partial \beta_i \partial p} = 0$.

Lemma 25 *Suppose the optimal bargaining power $\beta_i^*\left(\bar{\beta}_{-i}, p\right)$ is interior and differentiable for all $p > 0$. If there exist differentiable functions $H : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$, $\pi_1 : \mathbb{R} \mapsto \mathbb{R}_+$ and $\pi_2 : \mathbb{R}_+ \mapsto \mathbb{R}_{++}$ such that $\pi(\beta, p) = H\left(\pi_1(\beta), \pi_2(p)\right)$, $H_1' > 0$ and $\frac{\partial^2 \pi_1\left(\beta_i^*\left(\bar{\beta}_{-i}, p\right)\right)}{\partial \beta_i^2} \neq 0$ for all $p \in \mathbb{R}_{++}$, then $\beta_i^*\left(\bar{\beta}_{-i}, p\right)$ is independent of p.*

Proof of Lemma 25:

Note that
$$\frac{\partial \pi(\beta,p)}{\partial \beta_i} = H_1'(\cdot)\frac{\partial \pi_1(\beta)}{\partial \beta_i}$$

Because any interior optimal bargaining power satisfies $\frac{\partial \pi\left(\beta_i^*\left(\bar{\beta}_{-i},p\right),\bar{\beta}_{-i},p\right)}{\partial \beta_i} = 0$, we conclude that $H_1'(\cdot)\frac{\partial \pi_1\left(\beta_i^*\left(\bar{\beta}_{-i},p\right),\bar{\beta}_{-i},p\right)}{\partial \beta_i} = 0$. Since $H_1'(\cdot) > 0$ for all $p > 0$, it must be the case that $\frac{\partial \pi_1\left(\beta_i^*\left(\bar{\beta}_{-i},p\right),\bar{\beta}_{-i},p\right)}{\partial \beta_i} = 0$. Note also that

$$\frac{\partial^2 \pi\left(\beta_i^*\left(\bar{\beta}_{-i},p\right),p\right)}{\partial \beta_i^2} = H_{11}''(\cdot)\frac{\partial \pi_1\left(\beta_i^*\left(\bar{\beta}_{-i},p\right),\bar{\beta}_{-i},p\right)}{\partial \beta_i} + H_1'(\cdot)\frac{\partial^2 \pi_1\left(\beta_i^*\left(\bar{\beta}_{-i},p\right),\bar{\beta}_{-i},p\right)}{\partial \beta_i^2}$$

Since $\frac{\partial \pi_1\left(\beta_i^*\left(\bar{\beta}_{-i},p\right),\bar{\beta}_{-i},p\right)}{\partial \beta_i} = 0$, $H_1'(\cdot) > 0$ and $\frac{\partial^2 \pi_1\left(\beta_i^*\left(\bar{\beta}_{-i},p\right),\bar{\beta}_{-i},p\right)}{\partial \beta_i^2} \neq 0$ for all $p > 0$, we conclude that $\frac{\partial^2 \pi\left(\beta_i^*\left(\bar{\beta}_{-i},p\right),\bar{\beta}_{-i},p\right)}{\partial \beta_i^2} \neq 0$ for all $p > 0$. By Proposition 24, therefore, it suffices to show that $\frac{\partial^2 \pi_1\left(\beta_i^*\left(\bar{\beta}_{-i},p\right),\bar{\beta}_{-i},p\right)}{\partial \beta_i \partial p} = 0$. Note that

$$\frac{\partial^2 \pi_1\left(\beta_i^*\left(\bar{\beta}_{-i},p\right),\bar{\beta}_{-i},p\right)}{\partial \beta_i \partial p} = \left(H_{12}''(\cdot) + H_1'(\cdot)\frac{\partial \pi_2(p)}{\partial p}\right)\frac{\partial \pi_1\left(\beta_i^*\left(\bar{\beta}_{-i},p\right),\bar{\beta}_{-i},p\right)}{\partial \beta_i} = 0$$

since $\frac{\partial \pi_1\left(\beta_i^*\left(\bar{\beta}_{-i},p\right),\bar{\beta}_{-i},p\right)}{\partial \beta_i} = 0$. ■

We now show that the conditions of Lemma 25 are always satisfied when the production function f is homogeneous.

We start showing the implication of homogeneity of f on the stable wage schedule.

Lemma 26 *If $f(\ell_u,\ell_s)$ is homogeneous of degree α in (ℓ_u,ℓ_s), then for $i \in \{u,s\}$,*
$$w_i[\beta,tp]\left(t^{\frac{1}{1-\alpha}}\ell_u, t^{\frac{1}{1-\alpha}}\ell_s\right) = w_i[\beta,p](\ell_u,\ell_s)$$

Proof of Lemma 26:

First note that:
$$\int_0^1 z^{\frac{1-\beta_i}{\beta_i}}\frac{\partial f\left(z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_u}{1-\beta_u}}\ell_u, z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_s}{1-\beta_s}}\ell_s\right)}{\partial \ell_i}dz \tag{50}$$

is homogeneous of degree $(\alpha-1)$ in (ℓ_u,ℓ_s). This is because first derivatives of functions homogeneous of degree α are homogenous of degree $(\alpha-1)$ and the linearity of the integral. Next, note that

$$\begin{aligned}
& w_i[\beta, tp]\left(t^{\frac{1}{1-\alpha}}\ell_u, t^{\frac{1}{1-\alpha}}\ell_s\right) \\
= \quad & \bar{w}_i + tp\int_0^1 z^{\frac{1-\beta_i}{\beta_i}} \frac{\partial f\left(z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_u}{1-\beta_u}} t^{\frac{1}{1-\alpha}}\ell_u, z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_s}{1-\beta_s}} t^{\frac{1}{1-\alpha}}\ell_s\right)}{\partial \ell_i} dz \\
= \quad & \bar{w}_i + tp\int_0^1 z^{\frac{1-\beta_i}{\beta_i}} \frac{1}{t} \frac{\partial f\left(z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_u}{1-\beta_u}}\ell_u, z^{\frac{1-\beta_i}{\beta_i}\frac{\beta_s}{1-\beta_s}}\ell_s\right)}{\partial \ell_i} dz \\
= \quad & w_i[\beta, p](\ell_u, \ell_s)
\end{aligned}$$

where the second equality follows because the term (50) is homogeneous of degree $(\alpha - 1)$ in (ℓ_u, ℓ_s).■

Let

$$L(\beta, p) = \left\{(\ell_u, \ell_s) \in \mathbb{R}_+^2 : (\ell_u, \ell_s) \text{ solves } (4)\right\}$$

denote the optimal input demand correspondence.

Definition 27 *$L(\beta, p)$ is homogeneous of degree α in p if $(\ell_u(\beta, p), \ell_s(\beta, p)) \in L(\beta, p)$ implies $(t^\alpha \ell_u(\beta, p), t^\alpha \ell_s(\beta, p)) \in L(\beta, tp)$ for any $p > 0$ and $t > 0$.*

Next we show that for a homogeneous production function, the optimal demand for labor is homogeneous and the indirect profit functions are homogeneous of degree $\frac{1}{1-\alpha}$.

Lemma 28 *If $f(\ell_u, \ell_s)$ is homogeneous of degree α in (ℓ_u, ℓ_s), then $L(\beta, p)$ and $\pi(\beta, p)$ are homogeneous of degree $\frac{1}{1-\alpha}$ in p.*

Proof of Lemma 28:

Let $(\ell_u, \ell_s) \in L(\beta, p)$ and $(\ell_u(t), \ell_s(t)) \equiv t^{\frac{1}{1-\alpha}}(\ell_u, \ell_s)$. We need to show that $(\ell_u(t), \ell_s(t)) \in L(\beta, tp)$. Suppose not, then since $(\ell_u(t), \ell_s(t)) \in \mathbb{R}_+^2$ there must exists $\left(\tilde{\ell}_u, \tilde{\ell}_s\right) \in L(\beta, tp)$ such that

$$\begin{aligned}
& tpf(\ell_u(t), \ell_s(t)) - w_u[\beta, tp](\ell_u(t), \ell_s(t))\ell_u(t) - w_s[\beta, tp](\ell_u(t), \ell_s(t))\ell_s(t) \\
< \quad & tpf\left(\tilde{\ell}_u, \tilde{\ell}_s\right) - w_u[\beta, tp]\left(\tilde{\ell}_u, \tilde{\ell}_s\right)\tilde{\ell}_u - w_s[\beta, tp]\left(\tilde{\ell}_u, \tilde{\ell}_s\right)\tilde{\ell}_s
\end{aligned}$$

Note that

$$\begin{aligned}
tpf(\ell_u(t), \ell_s(t)) &= t^{1+\frac{\alpha}{1-\alpha}} pf(\ell_u, \ell_s) = t^{\frac{1}{1-\alpha}} pf(\ell_u, \ell_s) \\
w_i[\beta, tp](\ell_u(t), \ell_s(t))\ell_i(t) &= w_i[\beta, tp]\left(t^{\frac{1}{1-\alpha}}\ell_u, t^{\frac{1}{1-\alpha}}\ell_s\right) t^{\frac{1}{1-\alpha}}\ell_i \\
&= t^{\frac{1}{1-\alpha}} w_i[\beta, p](\ell_u, \ell_s)
\end{aligned}$$

for $i \in \{u, s\}$, where the last equality follows by Lemma 26. It follows that

$$\begin{aligned}
& t^{\frac{1}{1-\alpha}}(pf(\ell_u, \ell_s) - w_u[\beta, p](\ell_u, \ell_s)\ell_u - w_s[\beta, p](\ell_u, \ell_s)\ell_s) \\
< \quad & tpf\left(\tilde{\ell}_u, \tilde{\ell}_s\right) - w_u[\beta, tp]\left(\tilde{\ell}_u, \tilde{\ell}_s\right)\tilde{\ell}_u - w_s[\beta, tp]\left(\tilde{\ell}_u, \tilde{\ell}_s\right)\tilde{\ell}_s
\end{aligned}$$

and, therefore,

$$\begin{aligned} & pf\left(\ell_u,\ell_s\right)-w_u\left[\beta,p\right]\left(\ell_u,\ell_s\right)\ell_u-w_s\left[\beta,p\right]\left(\ell_u,\ell_s\right)\ell_s \\ < \quad & t^{-\frac{\alpha}{1-\alpha}}pf\left(\tilde{\ell}_u,\tilde{\ell}_s\right)-w_u\left[\beta,tp\right]\left(\tilde{\ell}_u,\tilde{\ell}_s\right)t^{-\frac{1}{1-\alpha}}\tilde{\ell}_u-w_s\left[\beta,tp\right]\left(\tilde{\ell}_u,\tilde{\ell}_s\right)t^{-\frac{1}{1-\alpha}}\tilde{\ell}_s \\ = \quad & pf\left(t^{-\frac{1}{1-\alpha}}\tilde{\ell}_u,t^{-\frac{1}{1-\alpha}}\tilde{\ell}_s\right)-w_u\left[\beta,p\right]\left(t^{-\frac{1}{1-\alpha}}\tilde{\ell}_u,t^{-\frac{1}{1-\alpha}}\tilde{\ell}_s\right)t^{-\frac{1}{1-\alpha}}\tilde{\ell}_u-w_s\left[\beta,p\right]\left(t^{-\frac{1}{1-\alpha}}\tilde{\ell}_u,t^{-\frac{1}{1-\alpha}}\tilde{\ell}_s\right)t^{-\frac{1}{1-\alpha}}\tilde{\ell}_s \end{aligned}$$

where the second line uses Lemma 26. We conclude that $t^{-\frac{1}{1-\alpha}}\left(\tilde{\ell}_u,\tilde{\ell}_s\right)$ yields larger profits than (ℓ_u,ℓ_s) at p, a contradiction since we assumed $(\ell_u,\ell_s) \in L(\beta,p)$. We conclude that

$$t^{\frac{1}{1-\alpha}}\ell_i(\beta,p)\in L(\beta,tp) \tag{51}$$

Let $(\ell_u,\ell_s)\in L(\beta,p)$ and $(\ell_u(t),\ell_s(t))\equiv t^{\frac{1}{1-\alpha}}(\ell_u,\ell_s)$. It follows that

$$\begin{aligned} \pi(\beta,tp) &= pf\left(\ell_u(\beta,tp),\ell_s(\beta,tp)\right)-w_u(\ell_u(\beta,tp),\ell_s(\beta,tp))\ell_u(\beta,tp)-w_s(\ell_u(\beta,tp),\ell_s(\beta,tp))\ell_s(\beta,tp) \\ &= tpf\left(\ell_u(t),\ell_s(t)\right)-w_u\left[\beta,tp\right]\left(\ell_u(t),\ell_s(t)\right)\ell_u(t)-w_s\left[\beta,tp\right]\left(\ell_u(t),\ell_s(t)\right)\ell_s(t) \\ &= t^{\frac{1}{1-\alpha}}\left(pf\left(\ell_u,\ell_s\right)-w_u\left[\beta,p\right]\left(\ell_u,\ell_s\right)\ell_u-w_s\left[\beta,p\right]\left(\ell_u,\ell_s\right)\ell_s\right) \\ &= t^{\frac{1}{1-\alpha}}\pi(\beta,p) \end{aligned}$$

where the second equality follows by (51) and the third equality by Lemma 26 and so $\pi(\beta,p)$. We conclude that homogeneous of degree $\frac{1}{1-\alpha}$. Thus, one can write $\pi(\beta,p)=p^{\frac{1}{1-\alpha}}\pi(\beta,1)$. ■

Proof of Proposition 6:

The result follows by Lemmata 25 and 28.■

Proposition 8 is proven using a sequence of intermediary results.

Proposition 29 *Suppose that for every $z\in[0,1]$ and $p>p_0>0$,*

$$\frac{\partial f\left(z\ell_u^*\left(p,\beta_u,\bar{\beta}_s\right),z^{\frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{1-\beta_u}{\beta_u}}\ell_s^*\left(p,\beta_u,\bar{\beta}_s\right)\right)}{\partial\ell_u}\geq\frac{p_0}{p}\frac{\partial f\left(z\ell_u^*\left(p_0,\beta_u,\bar{\beta}_s\right),z^{\frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{1-\beta_u}{\beta_u}}\ell_s^*\left(p_0,\beta_u,\bar{\beta}_s\right)\right)}{\partial\ell_u} \tag{52}$$

Then the optimal interior bargaining power is decreasing in the price of output, $\beta_u^*\left(p_0,\bar{\beta}_s\right)\geq\beta_u^*\left(p,\bar{\beta}_s\right)$. The inequality is strict if the inequality in (52) is strict.

Proof of Proposition 29:
According to (46), the condition for the optimal interior bargaining power $\beta_u^*\left(p,\bar{\beta}_s\right)$ can be alternatively written as:

$$\frac{\partial\pi\left(\beta_u,\bar{\beta}_s,p\right)}{\partial\beta_u}=\left[\bar{w}_u+p\int_0^1\frac{\ln z}{\beta_u^2}z^{\frac{1-\beta_u}{\beta_u}}\frac{\partial f\left(z\ell_u^*\left(p,\beta_u,\bar{\beta}_s\right),z^{\frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{1-\beta_u}{\beta_u}}\ell_s^*\left(p,\beta_u,\bar{\beta}_s\right)\right)}{\partial\ell_u}dz\right]=0$$

Since $\ln z < 0$ for $z \in [0,1]$, the marginal productivities $\frac{\partial f(\cdot,\cdot)}{\partial \ell_u}$ appear in the first derivative $\frac{\partial \pi}{\partial \beta_u}$ with a negative sign. At the optimum, $\beta_u^*(\bar{\beta}_s, p_0)$, the first derivative of the profit function satisfies:

$$\frac{\partial \pi\left(\beta_u^*\left(p_0, \bar{\beta}_s\right), \bar{\beta}_s, p_0\right)}{\partial \beta_u} = 0$$

Thus, (52) implies for $p > p_0$,

$$\frac{\partial \pi\left(\beta_u^*\left(p_0, \bar{\beta}_s\right), \bar{\beta}_s, p\right)}{\partial \beta_u} \leq 0$$

Thus, the optimal $\beta_u^*(p, \bar{\beta}_s)$ is to the left of $\beta_u^*(p_0, \bar{\beta}_s)$, or $\beta_u^*(p, \bar{\beta}_s) \leq \beta_u^*(p_0, \bar{\beta}_s)$. The corresponding inequalities are strict, whenever $\beta_u^*(p_0, \bar{\beta}_s) \neq 0$ and the inequality in (52) is strict.

In the following, we fix, w.l.o.g. $p_0 = 1$ and state a condition which will imply (52), and thus, the result of Proposition 29.

Condition B: (Decreasing optimal bargaining power) Suppose that marginal productivity of unskilled labor u satisfies for some $\psi < 0$ for any $p \in \mathbb{R}^+$

$$\frac{\partial f\left(p\ell_u, p\ell_s\right)}{\partial \ell_u} \geq p^{\psi} \frac{\partial f\left(\ell_u, \ell_s\right)}{\partial \ell_u}$$

Furthermore, let optimal demand for the two types of labor satisfy for any $p \in \mathbb{R}^+$

$$\begin{aligned} \ell_u^*\left(p, \beta_u\right) &= p^{\chi_u(p)} \ell_u^*\left(1, \beta_u\right) \\ \ell_s^*\left(p, \beta_u\right) &= p^{\chi_s(p)} \ell_s^*\left(1, \beta_u\right) \end{aligned}$$

for some $\chi_s(p) \geq \chi_u(p) > 0$ such that $\min_{p \in \mathbb{R}^+} \chi_u(p)\, \psi > -1$.

Proposition 30 *If Condition B Decreasing optimal bargaining power is satisfied then $\beta_u\left(p, \bar{\beta}_s\right)$ is decreasing in p.*

Proof of Proposition 30:

If Condition Decreasing optimal bargaining power is satisfied, then there are ψ and for any $p \in \mathbb{R}^+$, $\chi_s(p) \geq \chi_u(p) > 0$ such that:

$$\begin{aligned} \ell_u^*\left(p, \beta_u\right) &= p^{\chi_u(p)} \ell_u^*\left(1, \beta_u\right) \\ \ell_s^*\left(p, \beta_u\right) &= p^{\chi_s(p)} \ell_s^*\left(1, \beta_u\right) \end{aligned}$$

and such that $\min \chi_u(p)\psi > -1$. Thus, for any $z \in [0,1]$ and any $p > 1$,

$$\begin{aligned}
\frac{\partial f\left(z\ell_u^*(p,\beta_u), z^{\frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{1-\beta_u}{\beta_u}}\ell_s^*(p,\beta_u)\right)}{\partial \ell_u} &= \frac{\partial f\left(zp^{\chi_u(p)}\ell_u^*(1,\beta_u), z^{\frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{1-\beta_u}{\beta_u}}p^{\chi_s(p)}\ell_s^*(1,\beta_u)\right)}{\partial \ell_u} \\
&\geq \frac{\partial f\left(zp^{\chi_u(p)}\ell_u^*(1,\beta_u), z^{\frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{1-\beta_u}{\beta_u}}p^{\chi_u(p)}\ell_s^*(1,\beta_u)\right)}{\partial \ell_u} \\
&\geq p^{\chi_u(p)\psi}\frac{\partial f\left(z\ell_u^*(1,\beta_u), z^{\frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{1-\beta_u}{\beta_u}}\ell_s^*(1,\beta_u)\right)}{\partial \ell_u} \\
&\geq p^{\min \chi_u(p)\psi}\frac{\partial f\left(z\ell_u^*(1,\beta_u), z^{\frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{1-\beta_u}{\beta_u}}\ell_s^*(1,\beta_u)\right)}{\partial \ell_u} \\
&\geq \frac{1}{p}\frac{\partial f\left(z\ell_u^*(1,\beta_u), z^{\frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{1-\beta_u}{\beta_u}}\ell_s^*(1,\beta_u)\right)}{\partial \ell_u}
\end{aligned}$$

as required.

Our next results concern the behavior of marginal productivity in a response to a proportional increase of the two factors of production, ℓ_u and ℓ_s. To do so we start by examining the response of output y to such a proportional change.

Proposition 31 *Let $k > 1$ and let $0 > \alpha_s \geq \alpha_u$. Then,*

$$k^{-\frac{1}{\alpha_s}} f(\ell_u, \ell_s) \geq f(k\ell_u, k\ell_s) \geq k^{-\frac{1}{\alpha_u}} f(\ell_u, \ell_s). \tag{53}$$

Proof of Proposition 31:

We have, for $k > 0$,

$$f^{\frac{\alpha_u(\sigma-1)}{\sigma}}(k\ell_u, k\ell_s)\, k^{\frac{\sigma-1}{\sigma}}\ell_u^{\frac{\sigma-1}{\sigma}} + f^{\frac{\alpha_s(\sigma-1)}{\sigma}}(k\ell_u, k\ell_s)\, k^{\frac{\sigma-1}{\sigma}}\ell_s^{\frac{\sigma-1}{\sigma}} = \frac{1}{A}$$

For the given k, ℓ_u, and ℓ_s, let χ be such that $f(k\ell_u, k\ell_s) = k^{\chi} f(\ell_u, \ell_s)$. Then,

$$f^{\frac{\alpha_u(\sigma-1)}{\sigma}}(\ell_u, \ell_s)\, k^{\frac{\alpha_u(\sigma-1)}{\sigma}\chi} k^{\frac{\sigma-1}{\sigma}}\ell_u^{\frac{\sigma-1}{\sigma}} + f^{\frac{\alpha_s(\sigma-1)}{\sigma}}(\ell_u, \ell_s)\, k^{\chi\frac{\alpha_s(\sigma-1)}{\sigma}} k^{\frac{\sigma-1}{\sigma}}\ell_s^{\frac{\sigma-1}{\sigma}} = \frac{1}{A}$$

$$f^{\frac{\alpha_u(\sigma-1)}{\sigma}}(\ell_u, \ell_s)\, k^{\frac{\sigma-1}{\sigma}[1+\alpha_u\chi]}\ell_u^{\frac{\sigma-1}{\sigma}} + f^{\frac{\alpha_s(\sigma-1)}{\sigma}}(\ell_u, \ell_s)\, k^{\frac{\sigma-1}{\sigma}[1+\alpha_s\chi]}\ell_s^{\frac{\sigma-1}{\sigma}} = \frac{1}{A}$$

But because

$$f^{\frac{\alpha_u(\sigma-1)}{\sigma}}(\ell_u, \ell_s)\, \ell_u^{\frac{\sigma-1}{\sigma}} + f^{\frac{\alpha_s(\sigma-1)}{\sigma}}(\ell_u, \ell_s)\, \ell_s^{\frac{\sigma-1}{\sigma}} = \frac{1}{A}Y$$

we must have that $1 + \alpha_u\chi > 0$ holds iff $1 + \alpha_s\chi < 0$. In particular, if $0 > \alpha_s \geq \alpha_u$, then, for $k > 1$:

$$\begin{aligned}
1 + \alpha_u\chi &\leq 0 \leq 1 + \alpha_s\chi, \text{ or} \\
-\frac{1}{\alpha_u} &\leq \chi \leq -\frac{1}{\alpha_s}
\end{aligned}$$

$$\chi \in \left[-\frac{1}{\alpha_u}, -\frac{1}{\alpha_s}\right]. \square$$

Thus, increasing both inputs proportionally by a factor of k leads to an increase of the output y by a factor χ which is between $-\frac{1}{\alpha_u}$ and $-\frac{1}{\alpha_s}$, and these bounds are independent of the values of ℓ_u and ℓ_s. We refer to this as "semi-homogeneity".

We next apply this insight to examine how the marginal productivity with respect to ℓ_u reacts to a proportional increase in the two inputs.

Proposition 32 *Suppose that* $0 > \alpha_s \geq \alpha_u$. *For* $f(\ell_u, \ell_s)$ *implicitly defined as in (15) and* $k > 1$,

$$\frac{\partial f(k\ell_u, k\ell_s)}{\partial \ell_u} \in \left(k^{-1-\frac{1}{\alpha_u}} \frac{\partial f(\ell_u, \ell_s)}{\partial \ell_u}, k^{-\frac{1+\alpha_u}{\alpha_s}+\frac{\alpha_u-\alpha_s}{\alpha_s\sigma}} \frac{\partial f(\ell_u, \ell_s)}{\partial \ell_u}\right). \tag{54}$$

Symmetrically,

$$\frac{\partial f(k\ell_u, k\ell_s)}{\partial \ell_s} \in \left(k^{-\frac{1+\alpha_s}{\alpha_u}+\frac{\alpha_s-\alpha_u}{\alpha_u\sigma}} \frac{\partial f(\ell_u, \ell_s)}{\partial \ell_s}, k^{-1-\frac{1}{\alpha_s}} \frac{\partial f(\ell_u, \ell_s)}{\partial \ell_s}\right). \tag{55}$$

Furthermore, if $-\frac{\sigma}{\sigma-1} < \alpha_i < -1$,

$$\begin{aligned} -1-\frac{1}{\alpha_i} &< 0,\ i \in \{u, s\} \\ -\frac{1+\alpha_i}{\alpha_{-i}} + \frac{\alpha_i - \alpha_{-i}}{\alpha_{-i}\sigma} &< 0,\ i,\ -i \in \{u, s\},\ i \neq -i \end{aligned}$$

Proof of Proposition 32:

We have:

$$\frac{\partial f(k\ell_u, k\ell_s)}{\partial \ell_u} = \frac{f^{\frac{\alpha_u(\sigma-1)}{\sigma}+1}(k\ell_u, k\ell_s)}{k^{\frac{1}{\sigma}} \ell_u^{\frac{1}{\sigma}} \left[-\alpha_u f^{\frac{\alpha_u(\sigma-1)}{\sigma}}(k\ell_u, k\ell_s) k^{\frac{\sigma-1}{\sigma}} \ell_u^{\frac{\sigma-1}{\sigma}} - \alpha_s f^{\frac{\alpha_s(\sigma-1)}{\sigma}}(k\ell_u, k\ell_s) k^{\frac{\sigma-1}{\sigma}} \ell_s^{\frac{\sigma-1}{\sigma}}\right]}$$

We first determine the sign of the direct effect of $y = f(\ell_u, \ell_s)$ on $\frac{\partial f(\ell_u, \ell_s)}{\partial \ell_u}$ (disregarding the indirect effect of ℓ_u and ℓ_s on y). Provided that $\frac{\partial f(\ell_u, \ell_s)}{\partial \ell_u}$ is monotonic in y, we can use (53) to establish boundaries on the change in $\frac{\partial f(k\ell_u, k\ell_s)}{\partial \ell_u}$.

Note that

$$
\begin{aligned}
&\left(\frac{y^{\frac{\alpha_u(\sigma-1)}{\sigma}+1}}{\left[-\alpha_u y^{\frac{\alpha_u(\sigma-1)}{\sigma}}\ell_u^{\frac{\sigma-1}{\sigma}}-\alpha_s y^{\frac{\alpha_s(\sigma-1)}{\sigma}}\ell_s^{\frac{\sigma-1}{\sigma}}\right]}\right)'_y \\
= &\ \frac{\begin{array}{c}\left[\frac{\alpha_u(\sigma-1)}{\sigma}+1\right]y^{\frac{\alpha_u(\sigma-1)}{\sigma}}\left[-\alpha_u y^{\frac{\alpha_u(\sigma-1)}{\sigma}}\ell_u^{\frac{\sigma-1}{\sigma}}-\alpha_s y^{\frac{\alpha_s(\sigma-1)}{\sigma}}\ell_s^{\frac{\sigma-1}{\sigma}}\right] \\ -y^{\frac{\alpha_u(\sigma-1)}{\sigma}+1}\left[-\alpha_u\frac{\alpha_u(\sigma-1)}{\sigma}y^{\frac{\alpha_u(\sigma-1)}{\sigma}-1}\ell_u^{\frac{\sigma-1}{\sigma}}-\alpha_s\frac{\alpha_s(\sigma-1)}{\sigma}y^{\frac{\alpha_s(\sigma-1)}{\sigma}-1}\ell_s^{\frac{\sigma-1}{\sigma}}\right]\end{array}}{\left[-\alpha_u y^{\frac{\alpha_u(\sigma-1)}{\sigma}}\ell_u^{\frac{\sigma-1}{\sigma}}-\alpha_s y^{\frac{\alpha_s(\sigma-1)}{\sigma}}\ell_s^{\frac{\sigma-1}{\sigma}}\right]^2} \\
= &\ \frac{\begin{array}{c}\frac{\alpha_u(\sigma-1)}{\sigma}\left[-\alpha_u y^{2\frac{\alpha_u(\sigma-1)}{\sigma}}\ell_u^{\frac{\sigma-1}{\sigma}}-\alpha_s y^{\frac{(\alpha_u+\alpha_s)(\sigma-1)}{\sigma}}\ell_s^{\frac{\sigma-1}{\sigma}}\right]-\alpha_u y^{2\frac{\alpha_u(\sigma-1)}{\sigma}}\ell_u^{\frac{\sigma-1}{\sigma}}-\alpha_s y^{\frac{(\alpha_u+\alpha_s)(\sigma-1)}{\sigma}}\ell_s^{\frac{\sigma-1}{\sigma}} \\ +\alpha_u\frac{\alpha_u(\sigma-1)}{\sigma}y^{2\frac{\alpha_u(\sigma-1)}{\sigma}}\ell_u^{\frac{\sigma-1}{\sigma}}+\alpha_s\frac{\alpha_s(\sigma-1)}{\sigma}y^{\frac{(\alpha_u+\alpha_s)(\sigma-1)}{\sigma}}\ell_s^{\frac{\sigma-1}{\sigma}}\end{array}}{\left[-\alpha_u y^{\frac{\alpha_u(\sigma-1)}{\sigma}}\ell_u^{\frac{\sigma-1}{\sigma}}-\alpha_s y^{\frac{\alpha_s(\sigma-1)}{\sigma}}\ell_s^{\frac{\sigma-1}{\sigma}}\right]^2} \\
= &\ \frac{-\alpha_u y^{2\frac{\alpha_u(\sigma-1)}{\sigma}}\ell_u^{\frac{\sigma-1}{\sigma}}+\alpha_s\left[\frac{(\alpha_s-\alpha_u)(\sigma-1)}{\sigma}-1\right]y^{\frac{(\alpha_u+\alpha_s)(\sigma-1)}{\sigma}}\ell_s^{\frac{\sigma-1}{\sigma}}}{\left[-\alpha_u y^{\frac{\alpha_u(\sigma-1)}{\sigma}}\ell_u^{\frac{\sigma-1}{\sigma}}-\alpha_s y^{\frac{\alpha_s(\sigma-1)}{\sigma}}\ell_s^{\frac{\sigma-1}{\sigma}}\right]^2} \\
> &\ 0
\end{aligned}
$$

because

$$
\begin{aligned}
\left[\frac{(\alpha_s-\alpha_u)(\sigma-1)}{\sigma}-1\right] &< 0 \\
(\alpha_s-\alpha_u)(\sigma-1) &< \sigma \\
\alpha_s-\alpha_u-\frac{\sigma}{\sigma-1} &< 0
\end{aligned}
$$

is implied by

$$
\alpha_s-\alpha_u-\frac{\sigma}{\sigma-1}<-\alpha_u-\frac{\sigma}{\sigma-1}<0
$$

which is satisfied by assumption. Thus, $\frac{\partial f(\ell_u,\ell_s)}{\partial \ell_u}$ is strictly increasing in y whenever the two labor types are complements.

Using (53), we can thus state that:

$$
\frac{\partial f(k\ell_u,k\ell_s)}{\partial \ell_u}\in\left(\begin{array}{c}\frac{k^{-\frac{1}{\alpha_u}\left(\frac{\alpha_u(\sigma-1)}{\sigma}+1\right)}y^{\frac{\alpha_u(\sigma-1)}{\sigma}+1}(\ell_u,\ell_s)}{k^{\frac{1}{\sigma}}\ell_u^{\frac{1}{\sigma}}\left[-\alpha_u k^{-\frac{(\sigma-1)}{\sigma}}y^{\frac{\alpha_u(\sigma-1)}{\sigma}}(\ell_u,\ell_s)k^{\frac{\sigma-1}{\sigma}}\ell_u^{\frac{\sigma-1}{\sigma}}-\alpha_s k^{-\frac{1}{\alpha_u}\frac{\alpha_s(\sigma-1)}{\sigma}}y^{\frac{\alpha_s(\sigma-1)}{\sigma}}(\ell_u,\ell_s)k^{\frac{\sigma-1}{\sigma}}\ell_s^{\frac{\sigma-1}{\sigma}}\right]}, \\ \frac{k^{-\frac{1}{\alpha_s}\left(\frac{\alpha_u(\sigma-1)}{\sigma}+1\right)}y^{\frac{\alpha_u(\sigma-1)}{\sigma}+1}(\ell_u,\ell_s)}{k^{\frac{1}{\sigma}}\ell_u^{\frac{1}{\sigma}}\left[-\alpha_u k^{-\frac{1}{\alpha_s}\frac{\alpha_u(\sigma-1)}{\sigma}}y^{\frac{\alpha_u(\sigma-1)}{\sigma}}(\ell_u,\ell_s)k^{\frac{\sigma-1}{\sigma}}\ell_u^{\frac{\sigma-1}{\sigma}}-\alpha_s k^{-\frac{(\sigma-1)}{\sigma}}y^{\frac{\alpha_s(\sigma-1)}{\sigma}}(\ell_u,\ell_s)k^{\frac{\sigma-1}{\sigma}}\ell_s^{\frac{\sigma-1}{\sigma}}\right]}\end{array}\right)
$$

$$
\frac{\partial f(k\ell_u,k\ell_s)}{\partial \ell_u}=\left(\begin{array}{c}\frac{k^{-\frac{1}{\alpha_u}\left(\frac{\alpha_u(\sigma-1)}{\sigma}+1\right)}y^{\frac{\alpha_u(\sigma-1)}{\sigma}+1}(\ell_u,\ell_s)}{k^{\frac{1}{\sigma}}\ell_u^{\frac{1}{\sigma}}\left[-\alpha_u y^{\frac{\alpha_u(\sigma-1)}{\sigma}}(\ell_u,\ell_s)\ell_u^{\frac{\sigma-1}{\sigma}}-\alpha_s k^{\left(1-\frac{\alpha_s}{\alpha_u}\right)\frac{\sigma-1}{\sigma}}y^{\frac{\alpha_s(\sigma-1)}{\sigma}}(\ell_u,\ell_s)\ell_s^{\frac{\sigma-1}{\sigma}}\right]}, \\ \frac{k^{-\frac{1}{\alpha_s}\left(\frac{\alpha_u(\sigma-1)}{\sigma}+1\right)}y^{\frac{\alpha_u(\sigma-1)}{\sigma}+1}(\ell_u,\ell_s)}{k^{\frac{1}{\sigma}}\ell_u^{\frac{1}{\sigma}}\left[-\alpha_u k^{\left(1-\frac{\alpha_u}{\alpha_s}\right)\frac{(\sigma-1)}{\sigma}}y^{\frac{\alpha_u(\sigma-1)}{\sigma}}(\ell_u,\ell_s)\ell_u^{\frac{\sigma-1}{\sigma}}-\alpha_s y^{\frac{\alpha_s(\sigma-1)}{\sigma}}(\ell_u,\ell_s)\ell_s^{\frac{\sigma-1}{\sigma}}\right]}\end{array}\right)
$$

We now examine the two boundaries to check whether we can derive a condition similar to (53) relative to marginal productivity. Note that since $0 > \alpha_s > \alpha_u$, we have $\frac{\alpha_u}{\alpha_s} < 1$ and thus,

$$\begin{aligned}
&\frac{k^{-\frac{1}{\alpha_s}\left(\frac{\alpha_u(\sigma-1)}{\sigma}+1\right)}y^{\frac{\alpha_u(\sigma-1)}{\sigma}+1}(\ell_u,\ell_s)}{k^{\frac{1}{\sigma}}\ell_u^{\frac{1}{\sigma}}\left[-\alpha_u k^{\left(1-\frac{\alpha_u}{\alpha_s}\right)\frac{(\sigma-1)}{\sigma}}y^{\frac{\alpha_u(\sigma-1)}{\sigma}}(\ell_u,\ell_s)\ell_u^{\frac{\sigma-1}{\sigma}}-\alpha_s y^{\frac{\alpha_s(\sigma-1)}{\sigma}}(\ell_u,\ell_s)\ell_s^{\frac{\sigma-1}{\sigma}}\right]}\\
\leq\quad&\frac{k^{-\frac{1}{\alpha_s}\left(\frac{\alpha_u(\sigma-1)}{\sigma}+1\right)}y^{\frac{\alpha_u(\sigma-1)}{\sigma}+1}(\ell_u,\ell_s)}{k^{\frac{1}{\sigma}}\ell_u^{\frac{1}{\sigma}}\left[-\alpha_u y^{\frac{\alpha_u(\sigma-1)}{\sigma}}(\ell_u,\ell_s)\ell_u^{\frac{\sigma-1}{\sigma}}-\alpha_s y^{\frac{\alpha_s(\sigma-1)}{\sigma}}(\ell_u,\ell_s)\ell_s^{\frac{\sigma-1}{\sigma}}\right]}
\end{aligned}$$

whereas

$$\begin{aligned}
&\frac{k^{-\frac{1}{\alpha_u}\left(\frac{\alpha_u(\sigma-1)}{\sigma}+1\right)}y^{\frac{\alpha_u(\sigma-1)}{\sigma}+1}(\ell_u,\ell_s)}{k^{\frac{1}{\sigma}}\ell_u^{\frac{1}{\sigma}}\left[-\alpha_u y^{\frac{\alpha_u(\sigma-1)}{\sigma}}(\ell_u,\ell_s)\ell_u^{\frac{\sigma-1}{\sigma}}-\alpha_s k^{\left(1-\frac{\alpha_s}{\alpha_u}\right)\frac{\sigma-1}{\sigma}}y^{\frac{\alpha_s(\sigma-1)}{\sigma}}(\ell_u,\ell_s)\ell_s^{\frac{\sigma-1}{\sigma}}\right]}\\
\geq\quad&\frac{k^{-\frac{1}{\alpha_u}\left(\frac{\alpha_u(\sigma-1)}{\sigma}+1\right)}y^{\frac{\alpha_u(\sigma-1)}{\sigma}+1}(\ell_u,\ell_s)}{k^{\frac{1}{\sigma}}\ell_u^{\frac{1}{\sigma}}\left[-\alpha_u y^{\frac{\alpha_u(\sigma-1)}{\sigma}}(\ell_u,\ell_s)\ell_u^{\frac{\sigma-1}{\sigma}}-\alpha_s y^{\frac{\alpha_s(\sigma-1)}{\sigma}}(\ell_u,\ell_s)\ell_s^{\frac{\sigma-1}{\sigma}}\right]}
\end{aligned}$$

It follows that:

$$\frac{\partial f(k\ell_u,k\ell_s)}{\partial\ell_u}\in\left(\begin{array}{c}k^{-\frac{1}{\alpha_u}\left(\frac{\alpha_u(\sigma-1)}{\sigma}+1\right)-\frac{1}{\sigma}}\frac{y^{\frac{\alpha_u(\sigma-1)}{\sigma}+1}(\ell_u,\ell_s)}{\ell_u^{\frac{1}{\sigma}}\left[-\alpha_u y^{\frac{\alpha_u(\sigma-1)}{\sigma}}(\ell_u,\ell_s)\ell_u^{\frac{\sigma-1}{\sigma}}-\alpha_s y^{\frac{\alpha_s(\sigma-1)}{\sigma}}(\ell_u,\ell_s)\ell_s^{\frac{\sigma-1}{\sigma}}\right]},\\ k^{-\frac{1}{\alpha_s}\left(\frac{\alpha_u(\sigma-1)}{\sigma}+1\right)-\frac{1}{\sigma}}\frac{y^{\frac{\alpha_u(\sigma-1)}{\sigma}+1}(\ell_u,\ell_s)}{\ell_u^{\frac{1}{\sigma}}\left[-\alpha_u k^{\left(1-\frac{\alpha_u}{\alpha_s}\right)\frac{(\sigma-1)}{\sigma}}y^{\frac{\alpha_u(\sigma-1)}{\sigma}}(\ell_u,\ell_s)\ell_u^{\frac{\sigma-1}{\sigma}}-\alpha_s y^{\frac{\alpha_s(\sigma-1)}{\sigma}}(\ell_u,\ell_s)\ell_s^{\frac{\sigma-1}{\sigma}}\right]}\end{array}\right)$$

$$\frac{\partial f(k\ell_u,k\ell_s)}{\partial\ell_u}\in\left(k^{-\frac{1}{\alpha_u}\left(\frac{\alpha_u(\sigma-1)}{\sigma}+1\right)-\frac{1}{\sigma}}\frac{\partial f(\ell_u,\ell_s)}{\partial\ell_u},k^{-\frac{1}{\alpha_s}\left(\frac{\alpha_u(\sigma-1)}{\sigma}+1\right)-\frac{1}{\sigma}}\frac{\partial f(\ell_u,\ell_s)}{\partial\ell_u}\right)$$

Symmetrically,

$$\frac{\partial f(k\ell_u,k\ell_s)}{\partial\ell_s}\in\left(k^{-\frac{1}{\alpha_u}\left(\frac{\alpha_s(\sigma-1)}{\sigma}+1\right)-\frac{1}{\sigma}}\frac{\partial f(\ell_u,\ell_s)}{\partial\ell_s},k^{-\frac{1}{\alpha_s}\left(\frac{\alpha_s(\sigma-1)}{\sigma}+1\right)-\frac{1}{\sigma}}\frac{\partial f(\ell_u,\ell_s)}{\partial\ell_s}\right)$$

Furthermore, we have

$$\begin{aligned}
-\frac{1}{\alpha_u}\left(\frac{\alpha_u(\sigma-1)}{\sigma}+1\right)-\frac{1}{\sigma}&=-1-\frac{1}{\alpha_u}<0\\
-\frac{1}{\alpha_s}\left(\frac{\alpha_s(\sigma-1)}{\sigma}+1\right)-\frac{1}{\sigma}&=-1-\frac{1}{\alpha_s}<0
\end{aligned}$$

$$-\frac{1}{\alpha_s}\left(\frac{\alpha_u(\sigma-1)}{\sigma}+1\right)-\frac{1}{\sigma}=-\frac{1+\alpha_u}{\alpha_s}+\frac{\alpha_u-\alpha_s}{\alpha_s\sigma}<0$$

because

$$\begin{aligned}
-\frac{1+\alpha_u}{\alpha_s}+\frac{\alpha_u-\alpha_s}{\alpha_s\sigma}&<0\\
\sigma(1+\alpha_u)-\alpha_u&<-\alpha_s
\end{aligned}$$

which, since $\alpha_s < -1$, would be satisfied if we could show that:

$$\begin{aligned} \sigma(1+\alpha_u) - \alpha_u &< 1 \\ (\sigma - 1)(1+\alpha_u) &< 0 \end{aligned}$$

which is always satisfied because $\alpha_u < -1$.

Finally,

$$-\frac{1}{\alpha_u}\left(\frac{\alpha_s(\sigma-1)}{\sigma}+1\right)-\frac{1}{\sigma} = -\frac{1+\alpha_s}{\alpha_u}+\frac{\alpha_s-\alpha_u}{\alpha_u\sigma} < 0$$

We conclude that

$$\frac{\partial f(k\ell_u, k\ell_s)}{\partial \ell_u} \in \left(k^{-1-\frac{1}{\alpha_u}}\frac{\partial f(\ell_u,\ell_s)}{\partial \ell_u}, k^{-\frac{1+\alpha_u}{\alpha_s}+\frac{\alpha_u-\alpha_s}{\alpha_s\sigma}}\frac{\partial f(\ell_u,\ell_s)}{\partial \ell_u}\right)$$

$$\frac{\partial f(k\ell_u, k\ell_s)}{\partial \ell_s} \in \left(k^{-\frac{1+\alpha_s}{\alpha_u}+\frac{\alpha_s-\alpha_u}{\alpha_u\sigma}}\frac{\partial f(\ell_u,\ell_s)}{\partial \ell_s}, k^{-1-\frac{1}{\alpha_s}}\frac{\partial f(\ell_u,\ell_s)}{\partial \ell_s}\right)$$

Our next proposition shows that for each p, $\chi_u(p)$ and $\chi_s(p)$ as defined in Condition Decreasing optimal bargaining power exist:

Proposition 33 *For $0 > \alpha_s \geq \alpha_u$ and a given $p > 1$, and $\chi_u > 0$ and $\chi_s > 0$ such that:*

$$\begin{aligned} \ell_u^*(p,\beta_u) &= p^{\chi_u}\ell_u^* \\ \ell_s^*(p,\beta_u) &= p^{\chi_s}\ell_s^* \end{aligned}$$

- *if $\chi_s \geq \chi_u$, then $\chi_i \in \left[\frac{\alpha_s}{1+\alpha_s}, \frac{\alpha_u}{1+\alpha_u}\right]$, $i \in \{u,s\}$;*
- *if $\chi_u > \chi_s$, then $\chi_i \in \left[\frac{\alpha_s\sigma}{(1+\alpha_u)\sigma-\alpha_u+\alpha_s}, \frac{\alpha_u\sigma}{(1+\alpha_s)\sigma-\alpha_s+\alpha_u}\right]$, $i \in \{u,s\}$ provided that*

$$\frac{\alpha_u\sigma}{(1+\alpha_s)\sigma-\alpha_s+\alpha_u} \geq \frac{\alpha_s\sigma}{(1+\alpha_u)\sigma-\alpha_u+\alpha_s}. \tag{56}$$

If $\alpha_i \in \left[-\frac{\sigma}{\sigma-1}, -1\right]$, $\sigma > 1$ and $\alpha_s + \alpha_u > -\frac{\sigma}{\sigma-1}$, then (56) is violated and $\chi_s \geq \chi_u$.

Proof of Proposition 33:

We know from Cahuc et al. (2008) that at the solution to the profit maximization problem, optimal demand for inputs and wages satisfy:

$$\begin{aligned} w_u(p,\beta_u) &= \frac{p}{\beta_u}\int z^{\frac{1-\beta_u}{\beta_u}}\frac{\partial f}{\partial \ell_u}\left(z\ell_u^*(p,\beta_u), z^{\frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{1-\beta_u}{\beta_u}}\ell_s^*(p,\beta_u)\right)dz \\ w_s(p,\beta_u) &= \frac{p}{\bar{\beta}_s}\int z^{\frac{1-\bar{\beta}_s}{\bar{\beta}_s}}\frac{\partial f}{\partial \ell_s}\left(z^{\frac{\beta_u}{1-\beta_u}\frac{1-\bar{\beta}_s}{\bar{\beta}_s}}\ell_u^*(p,\beta_u), z\ell_s^*(p,\beta_u)\right)dz \end{aligned}$$

$$\begin{aligned} w_u(p,\beta_u) &= (1-\beta_u)\bar{w}_u + p\int z^{\frac{1-\beta_u}{\beta_u}}\frac{\partial f}{\partial \ell_u}\left(z\ell_u^*(p,\beta_u), z^{\frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{1-\beta_u}{\beta_u}}\ell_s^*(p,\beta_u)\right)dz \\ w_s(p,\beta_u) &= (1-\bar{\beta}_s)\bar{w}_s + p\int z^{\frac{1-\bar{\beta}_s}{\bar{\beta}_s}}\frac{\partial f}{\partial \ell_s}\left(z^{\frac{\beta_u}{1-\beta_u}\frac{1-\bar{\beta}_s}{\bar{\beta}_s}}\ell_u^*(p,\beta_u), z\ell_s^*(p,\beta_u)\right)dz \end{aligned}$$

Combining these, we obtain that optimal demand is characterized by the equations:

$$\begin{aligned} p\int z^{\frac{1-\beta_u}{\beta_u}}\frac{\partial f}{\partial \ell_u}\left(z\ell_u^*\left(p,\beta_u\right), z^{\frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{1-\beta_u}{\beta_u}}\ell_s^*\left(p,\beta_u\right)\right)dz &= \beta_u\bar{w}_u \\ p\int z^{\frac{1-\bar{\beta}_s}{\bar{\beta}_s}}\frac{\partial f}{\partial \ell_s}\left(z^{\frac{\beta_u}{1-\beta_u}\frac{1-\bar{\beta}_s}{\bar{\beta}_s}}\ell_u^*\left(p,\beta_u\right), z\ell_s^*\left(p,\beta_u\right)\right)dz &= \bar{\beta}_s\bar{w}_s \end{aligned}$$

and in particular, writing $\ell_u^* = \ell_u^*\left(1,\beta_u\right)$ and $\ell_s^* = \ell_s^*\left(1,\beta_u\right)$, we have:

$$\begin{aligned} \int z^{\frac{1-\beta_u}{\beta_u}}\frac{\partial f}{\partial \ell_u}\left(z\ell_u^*, z^{\frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{1-\beta_u}{\beta_u}}\ell_s^*\right)dz &= \beta_u\bar{w}_u \\ \int z^{\frac{1-\bar{\beta}_s}{\bar{\beta}_s}}\frac{\partial f}{\partial \ell_s}\left(z^{\frac{\beta_u}{1-\beta_u}\frac{1-\bar{\beta}_s}{\bar{\beta}_s}}\ell_u^*, z\ell_s^*\right)dz &= \bar{\beta}_s\bar{w}_s \end{aligned}$$

This in turn implies that for any p,

$$p\int z^{\frac{1-\beta_u}{\beta_u}}\frac{\partial f}{\partial \ell_u}\left(z\ell_u^*\left(p,\beta_u\right), z^{\frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{1-\beta_u}{\beta_u}}\ell_s^*\left(p,\beta_u\right)\right)dz = \int z^{\frac{1-\beta_u}{\beta_u}}\frac{\partial f}{\partial \ell_u}\left(z\ell_u^*, z^{\frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{1-\beta_u}{\beta_u}}\ell_s^*\right)dz \qquad (57)$$

$$p\int z^{\frac{1-\bar{\beta}_s}{\bar{\beta}_s}}\frac{\partial f}{\partial \ell_s}\left(z^{\frac{\beta_u}{1-\beta_u}\frac{1-\bar{\beta}_s}{\bar{\beta}_s}}\ell_u^*\left(p,\beta_u\right), z\ell_s^*\left(p,\beta_u\right)\right)dz = \int z^{\frac{1-\bar{\beta}_s}{\bar{\beta}_s}}\frac{\partial f}{\partial \ell_s}\left(z^{\frac{\beta_u}{1-\beta_u}\frac{1-\bar{\beta}_s}{\bar{\beta}_s}}\ell_u^*, z\ell_s^*\right)dz \qquad (58)$$

For a given $p > 1$, let $\chi_u > 0$ and $\chi_s > 0$ be such that[10]:

$$\begin{aligned} \ell_u^*\left(p,\beta_u\right) &= p^{\chi_u}\ell_u^* > \ell_u^* \\ \ell_s^*\left(p,\beta_u\right) &= p^{\chi_s}\ell_s^* > \ell_s^* \end{aligned}$$

Suppose first that $\chi_s \geq \chi_u$. Since the two inputs are complements, we know that $\frac{\partial f}{\partial \ell_u}$ is decreasing in ℓ_u and increasing in ℓ_s. Thus, using (57) and (54),

$$\begin{aligned} & \int z^{\frac{1-\beta_u}{\beta_u}}\frac{\partial f}{\partial \ell_u}\left(z\ell_u^*, z^{\frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{1-\beta_u}{\beta_u}}\ell_s^*\right)dz \\ = \ & p\int z^{\frac{1-\beta_u}{\beta_u}}\frac{\partial f}{\partial \ell_u}\left(zp^{\chi_u}\ell_u^*, z^{\frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{1-\beta_u}{\beta_u}}p^{\chi_s}\ell_s^*\right)dz \\ \geq \ & p\int z^{\frac{1-\beta_u}{\beta_u}}\frac{\partial f}{\partial \ell_u}\left(zp^{\chi_s}\ell_u^*, z^{\frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{1-\beta_u}{\beta_u}}p^{\chi_s}\ell_s^*\right)dz \\ \geq \ & p^{1+\chi_s\left(-1-\frac{1}{\alpha_u}\right)}\int z^{\frac{1-\beta_u}{\beta_u}}\frac{\partial f}{\partial \ell_u}\left(z\ell_u^*, z^{\frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{1-\beta_u}{\beta_u}}\ell_s^*\right)dz \end{aligned}$$

leading to the condition:

$$\begin{aligned} 1+\chi_s\left(-1-\frac{1}{\alpha_u}\right) &\leq 0 \\ \chi_s &\leq \frac{\alpha_u}{1+\alpha_u} \end{aligned}$$

and thus,

$$\ell_s^*\left(p,\beta_u\right) = p^{\chi_S}\ell_s^* \leq p^{\frac{\alpha_u}{1+\alpha_u}}\ell_s^*$$

[10]While χ_u and χ_s in general depend on p, we omit this dependence in the derivations for convenience and reintroduce it when stating the result. We will show that the boundaries on χ_u and χ_s will be independent of p.

We also have from (57) and (54)

$$\begin{aligned}
&\int z^{\frac{1-\beta_u}{\beta_u}} \frac{\partial f}{\partial \ell_u}\left(z\ell_u^*, z^{\frac{\bar{\beta}_s}{1-\beta_s}\frac{1-\beta_u}{\beta_u}}\ell_s^*\right)dz \\
= \;& p\int z^{\frac{1-\beta_u}{\beta_u}} \frac{\partial f}{\partial \ell_u}\left(zp^{\chi_u}\ell_u^*, z^{\frac{\bar{\beta}_s}{1-\beta_s}\frac{1-\beta_u}{\beta_u}}p^{\chi_s}\ell_s^*\right)dz \\
\geq \;& p\int z^{\frac{1-\beta_u}{\beta_u}} \frac{\partial f}{\partial \ell_u}\left(zp^{\chi_u}\ell_u^*, z^{\frac{\bar{\beta}_s}{1-\beta_s}\frac{1-\beta_u}{\beta_u}}p^{\chi_u}\ell_s^*\right)dz \\
\geq \;& p^{1+\chi_u\left(-1-\frac{1}{\alpha_u}\right)}\int z^{\frac{1-\beta_u}{\beta_u}} \frac{\partial f}{\partial \ell_u}\left(z\ell_u^*, z^{\frac{\bar{\beta}_s}{1-\beta_s}\frac{1-\beta_u}{\beta_u}}\ell_s^*\right)dz
\end{aligned}$$

leading to the condition

$$\begin{aligned}
1+\chi_u\left(-1-\frac{1}{\alpha_u}\right) &\leq 0 \\
\chi_u &\leq \frac{\alpha_u}{1+\alpha_u}
\end{aligned}$$

and thus,

$$\ell_u^*(p,\beta_u) = p^{\chi_u}\ell_u^* \leq p^{\frac{\alpha_u}{1+\alpha_u}}\ell_u^*$$

Next, using (58) and (55),

$$\begin{aligned}
&\int z^{\frac{1-\beta_u}{\beta_u}} \frac{\partial f}{\partial \ell_s}\left(z\ell_u^*, z^{\frac{\bar{\beta}_s}{1-\beta_s}\frac{1-\beta_u}{\beta_u}}\ell_s^*\right)dz \\
= \;& p\int z^{\frac{1-\bar{\beta}_s}{\bar{\beta}_s}} \frac{\partial f}{\partial \ell_s}\left(z^{\frac{\beta_u}{1-\beta_u}\frac{1-\bar{\beta}_s}{\bar{\beta}_s}}p^{\chi_u}\ell_u^*, zp^{\chi_s}\ell_s^*\right)dz \\
\leq \;& p\int z^{\frac{1-\bar{\beta}_s}{\bar{\beta}_s}} \frac{\partial f}{\partial \ell_s}\left(z^{\frac{\beta_u}{1-\beta_u}\frac{1-\bar{\beta}_s}{\bar{\beta}_s}}p^{\chi_s}\ell_u^*, zp^{\chi_s}\ell_s^*\right)dz \\
\leq \;& p^{1+\chi_s\left(-1-\frac{1}{\alpha_s}\right)}\int z^{\frac{1-\bar{\beta}_s}{\bar{\beta}_s}} \frac{\partial f}{\partial \ell_s}\left(z^{\frac{\beta_u}{1-\beta_u}\frac{1-\bar{\beta}_s}{\bar{\beta}_s}}\ell_u^*, z\ell_s^*\right)dz
\end{aligned}$$

which implies

$$\begin{aligned}
1+\chi_s\left(-1-\frac{1}{\alpha_s}\right) &\geq 0 \\
\chi_s &\geq \frac{\alpha_s}{1+\alpha_s}
\end{aligned}$$

$$\frac{\alpha_u}{1+\alpha_u} \geq \chi_s \geq \frac{\alpha_s}{1+\alpha_s}$$

$$\begin{aligned}
\alpha_u(1+\alpha_s) &\leq \alpha_s(1+\alpha_u) \\
\alpha_u &\leq \alpha_s
\end{aligned}$$

Similarly, using (58) and (55),

$$\begin{aligned}
&\int z^{\frac{1-\beta_u}{\beta_u}} \frac{\partial f}{\partial \ell_s}\left(z\ell_u^*, z^{\frac{\bar{\beta}_s}{1-\beta_s}\frac{1-\beta_u}{\beta_u}}\ell_s^*\right)dz \\
= \;& p\int z^{\frac{1-\bar{\beta}_s}{\bar{\beta}_s}} \frac{\partial f}{\partial \ell_s}\left(z^{\frac{\beta_u}{1-\beta_u}\frac{1-\bar{\beta}_s}{\bar{\beta}_s}}p^{\chi_u}\ell_u^*, zp^{\chi_s}\ell_s^*\right)dz \\
\leq \;& p\int z^{\frac{1-\bar{\beta}_s}{\bar{\beta}_s}} \frac{\partial f}{\partial \ell_s}\left(z^{\frac{\beta_u}{1-\beta_u}\frac{1-\bar{\beta}_s}{\bar{\beta}_s}}p^{\chi_u}\ell_u^*, zp^{\chi_u}\ell_s^*\right)dz \\
\leq \;& p^{1+\chi_u\left(-1-\frac{1}{\alpha_s}\right)}\int z^{\frac{1-\bar{\beta}_s}{\bar{\beta}_s}} \frac{\partial f}{\partial \ell_s}\left(z^{\frac{\beta_u}{1-\beta_u}\frac{1-\bar{\beta}_s}{\bar{\beta}_s}}\ell_u^*, z\ell_s^*\right)dz
\end{aligned}$$

which implies

$$\begin{aligned} 1+\chi_u\left(-1-\frac{1}{\alpha_s}\right) &\geq 0 \\ \chi_u &\geq \frac{\alpha_s}{1+\alpha_s} \end{aligned}$$

and thus,

$$\frac{\alpha_u}{1+\alpha_u} \geq \chi_u \geq \frac{\alpha_s}{1+\alpha_s}$$

Thus the assumption $\chi_s \geq \chi_u$ leads to well-defined boundaries.

Now, we check whether $\chi_s < \chi_u$ can be satisfied. using (57) and (54),

$$\begin{aligned} & \int z^{\frac{1-\beta_u}{\beta_u}} \frac{\partial f}{\partial \ell_u}\left(z\ell_u^*, z^{\frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{1-\beta_u}{\beta_u}} \ell_s^*\right) dz \\ = \quad & p\int z^{\frac{1-\beta_u}{\beta_u}} \frac{\partial f}{\partial \ell_u}\left(zp^{\chi_u}\ell_u^*, z^{\frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{1-\beta_u}{\beta_u}} p^{\chi_s}\ell_s^*\right) dz \\ \leq \quad & p\int z^{\frac{1-\beta_u}{\beta_u}} \frac{\partial f}{\partial \ell_u}\left(zp^{\chi_s}\ell_u^*, z^{\frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{1-\beta_u}{\beta_u}} p^{\chi_s}\ell_s^*\right) dz \\ \leq \quad & p^{1+\chi_s\left(-\frac{1+\alpha_u}{\alpha_s}+\frac{\alpha_u-\alpha_s}{\alpha_s\sigma}\right)} \int z^{\frac{1-\beta_u}{\beta_u}} \frac{\partial f}{\partial \ell_u}\left(z\ell_u^*, z^{\frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{1-\beta_u}{\beta_u}} \ell_s^*\right) dz \end{aligned}$$

leading to the condition:

$$1+\chi_s\left(-\frac{1+\alpha_u}{\alpha_s}+\frac{\alpha_u-\alpha_s}{\alpha_s\sigma}\right) \geq 0$$

$$\chi_s \geq \frac{1}{\frac{1+\alpha_u}{\alpha_s}-\frac{\alpha_u-\alpha_s}{\alpha_s\sigma}} = \frac{\alpha_s\sigma}{(1+\alpha_u)\sigma-\alpha_u+\alpha_s}$$

and thus,

$$\ell_s^*(p,\beta_u) = p^{\chi_S}\ell_s^* \geq p^{\frac{\alpha_s\sigma}{(1+\alpha_u)\sigma-\alpha_u+\alpha_s}}\ell_s^*$$

We also have from (57) and (54)

$$\begin{aligned} & \int z^{\frac{1-\beta_u}{\beta_u}} \frac{\partial f}{\partial \ell_u}\left(z\ell_u^*, z^{\frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{1-\beta_u}{\beta_u}} \ell_s^*\right) dz \\ = \quad & p\int z^{\frac{1-\beta_u}{\beta_u}} \frac{\partial f}{\partial \ell_u}\left(zp^{\chi_u}\ell_u^*, z^{\frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{1-\beta_u}{\beta_u}} p^{\chi_s}\ell_s^*\right) dz \\ \leq \quad & p\int z^{\frac{1-\beta_u}{\beta_u}} \frac{\partial f}{\partial \ell_u}\left(zp^{\chi_u}\ell_u^*, z^{\frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{1-\beta_u}{\beta_u}} p^{\chi_u}\ell_s^*\right) dz \\ \leq \quad & p^{1+\chi_u\left(-\frac{1+\alpha_u}{\alpha_s}+\frac{\alpha_u-\alpha_s}{\alpha_s\sigma}\right)} \int z^{\frac{1-\beta_u}{\beta_u}} \frac{\partial f}{\partial \ell_u}\left(z\ell_u^*, z^{\frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{1-\beta_u}{\beta_u}} \ell_s^*\right) dz \end{aligned}$$

leading to the condition

$$1+\chi_u\left(-\frac{1+\alpha_u}{\alpha_s}+\frac{\alpha_u-\alpha_s}{\alpha_s\sigma}\right) \leq 0$$

and thus,

$$\ell_u^*(p,\beta_u) = p^{\chi_u}\ell_u^* \geq p^{\frac{\alpha_s\sigma}{(1+\alpha_u)\sigma-\alpha_u+\alpha_s}}\ell_u^*$$

Next, using (58) and (55),

$$\begin{aligned}
& \int z^{\frac{1-\beta_u}{\beta_u}} \frac{\partial f}{\partial \ell_s}\left(z\ell_u^*, z^{\frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{1-\beta_u}{\beta_u}}\ell_s^*\right) dz \\
= \quad & p\int z^{\frac{1-\bar{\beta}_s}{\bar{\beta}_s}} \frac{\partial f}{\partial \ell_s}\left(z^{\frac{\beta_u}{1-\beta_u}\frac{1-\bar{\beta}_s}{\bar{\beta}_s}} p^{\chi_u}\ell_u^*, zp^{\chi_s}\ell_s^*\right) dz \\
\geq \quad & p\int z^{\frac{1-\bar{\beta}_s}{\bar{\beta}_s}} \frac{\partial f}{\partial \ell_s}\left(z^{\frac{\beta_u}{1-\beta_u}\frac{1-\bar{\beta}_s}{\bar{\beta}_s}} p^{\chi_s}\ell_u^*, zp^{\chi_s}\ell_s^*\right) dz \\
\geq \quad & p^{1+\chi_s\left(-\frac{1+\alpha_s}{\alpha_u}+\frac{\alpha_s-\alpha_u}{\alpha_u\sigma}\right)}\int z^{\frac{1-\bar{\beta}_s}{\bar{\beta}_s}} \frac{\partial f}{\partial \ell_s}\left(z^{\frac{\beta_u}{1-\beta_u}\frac{1-\bar{\beta}_s}{\bar{\beta}_s}} \ell_u^*, z\ell_s^*\right) dz
\end{aligned}$$

which implies

$$1+\chi_s\left(-\frac{1+\alpha_s}{\alpha_u}+\frac{\alpha_s-\alpha_u}{\alpha_u\sigma}\right)\leq 0$$

We conclude that

$$1+\chi_s\left(-\frac{1+\alpha_s}{\alpha_u}+\frac{\alpha_s-\alpha_u}{\alpha_u\sigma}\right)\leq 0\leq 1+\chi_s\left(-\frac{1+\alpha_u}{\alpha_s}+\frac{\alpha_u-\alpha_s}{\alpha_s\sigma}\right)$$

or,

$$\begin{aligned}
-\frac{1+\alpha_s}{\alpha_u}+\frac{\alpha_s-\alpha_u}{\alpha_u\sigma} &\leq -\frac{1+\alpha_u}{\alpha_s}+\frac{\alpha_u-\alpha_s}{\alpha_s\sigma} \\
-\alpha_s-\alpha_s^2+\frac{\alpha_s^2-\alpha_s\alpha_u}{\sigma} &\leq -\alpha_u-\alpha_u^2+\frac{\alpha_u^2-\alpha_s\alpha_u}{\sigma} \\
\frac{\alpha_s^2-\alpha_u^2}{\sigma} &\leq (\alpha_s-\alpha_u)(1+\alpha_s+\alpha_u) \\
\frac{\alpha_s+\alpha_u}{\sigma} &\leq (1+\alpha_s+\alpha_u)
\end{aligned}$$

$$(\alpha_s+\alpha_u)\frac{(1-\sigma)}{\sigma}\leq 1$$

$$\alpha_s+\alpha_u\geq -\frac{\sigma}{\sigma-1}$$

Thus, if

$$\alpha_s+\alpha_u > -\frac{\sigma}{\sigma-1}$$

we obtain a contradiction.

Similarly, using (58) and (55),

$$\begin{aligned}
& \int z^{\frac{1-\beta_u}{\beta_u}} \frac{\partial f}{\partial \ell_s}\left(z\ell_u^*, z^{\frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{1-\beta_u}{\beta_u}}\ell_s^*\right) dz \\
= \quad & p\int z^{\frac{1-\bar{\beta}_s}{\bar{\beta}_s}} \frac{\partial f}{\partial \ell_s}\left(z^{\frac{\beta_u}{1-\beta_u}\frac{1-\bar{\beta}_s}{\bar{\beta}_s}} p^{\chi_u}\ell_u^*, zp^{\chi_s}\ell_s^*\right) dz \\
\geq \quad & p\int z^{\frac{1-\bar{\beta}_s}{\bar{\beta}_s}} \frac{\partial f}{\partial \ell_s}\left(z^{\frac{\beta_u}{1-\beta_u}\frac{1-\bar{\beta}_s}{\bar{\beta}_s}} p^{\chi_u}\ell_u^*, zp^{\chi_u}\ell_s^*\right) dz \\
\geq \quad & p^{1+\chi_u\left(-\frac{1+\alpha_s}{\alpha_u}+\frac{\alpha_s-\alpha_u}{\alpha_u\sigma}\right)}\int z^{\frac{1-\bar{\beta}_s}{\bar{\beta}_s}} \frac{\partial f}{\partial \ell_s}\left(z^{\frac{\beta_u}{1-\beta_u}\frac{1-\bar{\beta}_s}{\bar{\beta}_s}} \ell_u^*, z\ell_s^*\right) dz
\end{aligned}$$

which implies

$$1+\chi_u\left(-\frac{1+\alpha_s}{\alpha_u}+\frac{\alpha_s-\alpha_u}{\alpha_u\sigma}\right)\leq 0$$

$$\chi_u \leq \frac{\alpha_u \sigma}{(1+\alpha_s)\,\sigma - \alpha_s + \alpha_u}$$

and thus,

$$\frac{\alpha_s \sigma}{(1+\alpha_u)\,\sigma - \alpha_u + \alpha_s} \leq \chi_u \leq \frac{\alpha_u \sigma}{(1+\alpha_s)\,\sigma - \alpha_s + \alpha_u}$$

which is inconsistent if

$$\alpha_s + \alpha_u > -\frac{\sigma}{\sigma - 1}$$

Now note that $\alpha_s + \alpha_u \leq -2$, whereas

$$\begin{aligned} -\frac{\sigma}{\sigma-1} &< -2 \\ \sigma &> 2(\sigma-1) \\ \sigma &< 2 \end{aligned}$$

Thus, for $\alpha_s + \alpha_u > -\frac{\sigma}{\sigma-1}$, $\chi_s < \chi_u$ cannot occur and we have $\chi_s \geq \chi_u$.

Our next result shows that under the conditions of Proposition 8, Condition B Decreasing optimal bargaining power is satisfied and thus, condition (52) holds.

Proposition 34 *Suppose that* $0 > \alpha_s \geq \alpha_u$, $\alpha_i \in \left[-\frac{\sigma}{\sigma-1}, -1\right]$ *for* $i \in \{u, s\}$, $\sigma > 1$ *and* $\alpha_s + \alpha_u > -\frac{\sigma}{\sigma-1}$. *Then Condition B Decreasing optimal bargaining power is satisfied with* $\psi = -1 - \frac{1}{\alpha_u}$ *and* $\chi_s(p) \geq \chi_u(p) > 0$ *such that*

$$\chi_i(p) \in \left[\frac{\alpha_s}{1+\alpha_s}, \frac{\alpha_u}{1+\alpha_u}\right],\ i \in \{u, s\}\ \text{and}\ p \in \mathbb{R}^+.$$

Furthermore,

$$\min_{p \in \mathbb{R}^+} \chi_u(p)\,\psi = -\frac{\alpha_s}{1+\alpha_s}\frac{\alpha_u + 1}{\alpha_u} \geq -1$$

with strict inequality whenever $\alpha_s > \alpha_u$. *Thus, for any* $p \in \mathbb{R}^+$ *and* $z \in [0,1]$,

$$\frac{\partial f\left(z\ell_u^*(p,\beta_u), z^{\frac{\bar{\beta}_s}{1-\beta_s}\frac{1-\beta_u}{\beta_u}}\ell_s^*(p,\beta_u)\right)}{\partial \ell_u} \geq \frac{1}{p}\frac{\partial f\left(z\ell_u^*(1,\beta_u), z^{\frac{\bar{\beta}_s}{1-\beta_s}\frac{1-\beta_u}{\beta_u}}\ell_s^*(1,\beta_u)\right)}{\partial \ell_u}$$

with strict inequality whenever $\alpha_s > \alpha_u$. *Furthermore, for any* $p > p_0 > 0$, *and* $z \in [0,1]$

$$\frac{\partial f\left(z\ell_u^*(p,\beta_u), z^{\frac{\bar{\beta}_s}{1-\beta_s}\frac{1-\beta_u}{\beta_u}}\ell_s^*(p,\beta_u)\right)}{\partial \ell_u} \geq \frac{p_0}{p'}\frac{\partial f\left(z\ell_u^*(p_0,\beta_u), z^{\frac{\bar{\beta}_s}{1-\beta_s}\frac{1-\beta_u}{\beta_u}}\ell_s^*(p_0,\beta_u)\right)}{\partial \ell_u}$$

Proof of Proposition 34:

By Proposition 32, we can set $\psi = -1 - \frac{1}{\alpha_u}$ to obtain

$$\frac{\partial f(p\ell_u, p\ell_s)}{\partial \ell_u} \geq p^{\psi}\frac{\partial f(\ell_u, \ell_s)}{\partial \ell_u} = p^{-1-\frac{1}{\alpha_u}}\frac{\partial f(\ell_u, \ell_s)}{\partial \ell_u}$$

By Proposition 33, whenever the parameters satisfy the conditions of Proposition 34, optimal demand for the two types of labor satisfies for any $p \in \mathbb{R}^+$

$$\begin{aligned} \ell_u^*(p,\beta_u) &= p^{\chi_u(p)}\ell_u^*(1,\beta_u) \\ \ell_s^*(p,\beta_u) &= p^{\chi_s(p)}\ell_s^*(1,\beta_u) \end{aligned}$$

for some $\chi_s(p) \geq \chi_u(p) > 0$ such that $\chi_i(p) \in \left[\frac{\alpha_s}{1+\alpha_s}, \frac{\alpha_u}{1+\alpha_u}\right]$, $i \in \{u, s\}$ and $p \in \mathbb{R}^+$. Thus,

$$\min_{p\in\mathbb{R}^+} \chi_u(p)\psi = \min_{\chi_u \in \left[\frac{\alpha_s}{1+\alpha_s}, \frac{\alpha_u}{1+\alpha_u}\right]} \chi_u \psi = -\frac{\alpha_s}{1+\alpha_s}\frac{\alpha_u+1}{\alpha_u}$$

We know that

$$\begin{aligned}\frac{\alpha_u}{1+\alpha_u} &\geq \frac{\alpha_s}{1+\alpha_s}, \text{ or} \\ 1 &\geq \frac{\alpha_s}{1+\alpha_s}\frac{1+\alpha_u}{\alpha_u}\end{aligned}$$

and thus,

$$\min_{p\in\mathbb{R}^+} \chi_u(p)\psi = \min_{\chi_u \in \left[\frac{\alpha_s}{1+\alpha_s}, \frac{\alpha_u}{1+\alpha_u}\right]} \chi_u \psi = -\frac{\alpha_s}{1+\alpha_s}\frac{1+\alpha_u}{\alpha_u} \geq -1$$

with strict inequality, whenever $\alpha_s > \alpha_u$. We therefore obtain

$$\begin{aligned}\frac{\partial f\left(z\ell_u^*(p,\beta_u), z^{\frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{1-\beta_u}{\beta_u}}\ell_s^*(p,\beta_u)\right)}{\partial \ell_u} &= \frac{\partial f\left(zp^{\chi_u(p)}\ell_u^*(1,\beta_u), z^{\frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{1-\beta_u}{\beta_u}}p^{\chi_s(p)}\ell_s^*(1,\beta_u)\right)}{\partial \ell_u} \\ &\geq \frac{\partial f\left(zp^{\chi_u(p)}\ell_u^*(1,\beta_u), z^{\frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{1-\beta_u}{\beta_u}}p^{\chi_u(p)}\ell_s^*(1,\beta_u)\right)}{\partial \ell_u} \\ &\geq p^{\chi_u(p)\left(-1-\frac{1}{\alpha_u}\right)}\frac{\partial f\left(z\ell_u^*(1,\beta_u), z^{\frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{1-\beta_u}{\beta_u}}\ell_s^*(1,\beta_u)\right)}{\partial \ell_u} \\ &\geq \frac{1}{p}\frac{\partial f\left(z\ell_u^*(1,\beta_u), z^{\frac{\bar{\beta}_s}{1-\bar{\beta}_s}\frac{1-\beta_u}{\beta_u}}\ell_s^*(1,\beta_u)\right)}{\partial \ell_u}\end{aligned}$$

with strict inequality whenever $\alpha_s > \alpha_u$.

Finally note that the results were derived by setting $p_0 = 1$ and that only $\ell_u^*(1,\beta_u)$ and $\ell_s^*(1,\beta_u)$ were used in the derivations. Thus, replacing $\ell_u^*(1,\beta_u)$ and $\ell_s^*(1,\beta_u)$ by $\ell_u^*(p_0,\beta_u), \ell_s^*(p_0,\beta_u)$ and p by $\frac{p}{p_0}$ in all results derived above gives the last part of the Proposition.

Proof of Proposition 8:

Follows by combining Proposition 29 with Proposition 34.

7.2 Proofs for the Static Equilibrium of the Economy, Section 3

Proof of Proposition 10:

For given z and m, consider first $p = 0$. By Assumption D and the properties of the supply function, we have that

$$\begin{aligned}\int_{\beta_u\in\mathbb{B}} f\left(\ell_u\left(\beta_u,\bar{\beta}_s,0\right), \ell_s\left(\beta_u,\bar{\beta}_s,0\right)\right) m(\beta_u)\, d\beta_u &= 0 \\ D(z,0) &> 0\end{aligned}$$

and thus, aggregate demand is strictly positive:

$$D(z,0) - S(0) > 0 \tag{59}$$

If $p \geq \bar{p}_z$ and $m \gg 0$ then

$$\begin{aligned}\int_{\beta_u \in \mathbb{B}} f\left(\ell_u\left(\beta_u, \bar{\beta}_s, p\right), \ell_s\left(\beta_u, \bar{\beta}_s, p\right)\right) m(\beta_u)\, d\beta_u &> 0 \\ D(z,p) &= 0\end{aligned}$$

and thus, aggregate supply is strictly negative:

$$D(z,p) - S(p) < 0$$

Since $D(z,p)$ and $S(p)$ are continuous, an equilibrium price ρ exists by the Intermediate value theorem.

Finally, suppose that $m \equiv 0$. Then $S(p) = 0$ for all p. For the market to be in equilibrium, we need

$$D(p) = 0$$

Thus, $\rho(z,m) = \bar{p}_z$.

Statement (i): That $\rho(z_1,m) > \rho(z_2,m)$ follows directly from Assumption D.

To prove that ρ is continuously differentiable and decreasing in m for a given z, consider $\rho(z,m)$ as an implicit function defined by:

$$D(z,\rho(z,m)) - S(\rho(z,m), m) = 0$$

Note that by Assumptions D and S,

$$\begin{aligned}\frac{\partial\left[D(z,p) - S(p,m)\right]}{\partial p} &= \frac{\partial D(z,p)}{\partial p} - \frac{\partial S(p,m)}{\partial p} = \\ &= \frac{\partial D(z,p)}{\partial p} - \int_{\beta_u} \frac{\partial y\left(\beta_u, \bar{\beta}_s, p\right)}{\partial p} m(\beta_u)\, d\beta_u < 0\end{aligned}$$

for all values of p and m. Furthermore, on $[0, \bar{p}_z)$, $\frac{\partial D(z,p)}{\partial p}$ and all of $\frac{\partial y\left(\beta_u, \bar{\beta}_s, p\right)}{\partial p}$ are bounded and thus, the Frechet derivative with respect to p is bounded

$$0 > \frac{\partial\left[D(z,p) - S(p,m)\right]}{\partial p} \geq \frac{\partial D(z,p=0)}{\partial p} - d\int_{\beta_u} \bar{\mu}^c d\beta_u > -\infty,$$

The Frechet derivative w.r.t. p thus defines an invertible (isomorphic) linear functional from $\mathbb{R}_0^+$ to $\mathbb{R}_0^+$.

The set $\mathbb{M}$ endowed with the L^1-norm is a Banach space. Furthermore, the Frechet derivative of $D(z,\rho(z,m)) - S(\rho(z,m), m)$ w.r.t. m is given by the linear functional:

$$\left[D(z,\rho(z,m)) - S(\rho(z,m), m)\right]'_m : h \to \int_{\beta_u} y\left(\beta_u, \bar{\beta}_s, \rho(z,m)\right) h(\beta_u)\, d\beta_u$$

which, by assumptions S and M is bounded and thus, continuous.

The conditions of the Implicit Function theorem are therefore satisfied and we conclude that for a given z, the price function $\rho(z,m)$ is continuously differentiable in m and furthermore, the Frechet derivative of $\rho(z,m)$ with respect to m, $\rho'_m(z,m)$, is given by the linear operator:

$$\rho'_m(z,m) : h \to \frac{\int_{\beta_u} y\left(\beta_u, \bar{\beta}_s, \rho(z,m)\right) h(\beta_u)\, d\beta_u}{\frac{\partial D(z,\rho(z,m))}{\partial p} - \int_{\beta_u} \frac{\partial y\left(\beta_u, \bar{\beta}_s, \rho(z,m)\right)}{\partial p} m(\beta_u)\, d\beta_u}$$

for every $m \in M$. Since the numerator is positive, while the denominator is negative, we have

$$\rho'_m(z,m) < 0.$$

We conclude that $\rho(z,m)$ is decreasing in m. In particular,

$$\rho(z, m \equiv \bar{\mu}^c) \leq \rho(z,m) \leq \rho(z, m \equiv 0),$$

with $[\rho(z, m \equiv \bar{\mu}^c), \rho(z, m \equiv 0)] \subseteq [0, \bar{p}_z]$.

Statement (ii) follows directly from the fact that profit is increasing in p, continuous and differentiable in ℓ_u and ℓ_s, the properties imposed on hiring decisions in Assumption S and the results obtained in Statement (i) of the proposition.

The following Lemma shows that a minimal, strictly positive reservation price $r_c\left(\hat{\beta}_u\right)$ exists and that $\mu^c_{\beta_u}$ is bounded above on $\mathbb{B}$, i.e., there is a maximum number of firms (of any given type β_u) that would be willing to be active in the market in presence of fixed cost $c > 0$. Thus, $\bar{\mu}^c$ as specified in (17) is well-defined.

Lemma 35 *A minimal static reservation value $r_c\left(\hat{\beta}_u\right)$ exists and satisfies $r_c\left(\hat{\beta}_u\right) > 0$. Furthermore, there exists a $\bar{\mu}^c > 0$ such that for a given c, $\mu^c_{\beta_u} \leq \bar{\mu}^c$ for all β_u and all $\bar{\beta}_s$.*

Proof of Lemma 35:

Consider first the case in which $\mathbb{B}$ is a convex and closed subset of $[0,1]$. Note that by Assumption S, π is continuously differentiable in β_u and in p, as well as strictly increasing in p. Thus, there exists a continuously differentiable in β_u, $r_c(\beta_u)$ for every $c > 0$. Note furthermore, that for $c > 0$, $r_c(\beta_u)$ is bounded below by $\underline{r}_c$ defined as:

$$\pi\left(\beta_u = 0, \bar{\beta}_s = 0, \underline{r}_c\right) = c$$

where obviously, $\underline{r}_c > 0$. This is because for any β_u, $\bar{\beta}_s$, by Lemma 2,

$$\pi\left(\beta_u = 0, \bar{\beta}_s = 0, \underline{r}_c\right) = c \geq \pi\left(\beta_u, \bar{\beta}_s, r_c(\beta_u)\right)$$

and since profits are strictly increasing in p, we have

$$r_c(\beta_u) \geq \underline{r}_c.$$

It follows that $r_c(\beta_u)$ is a continuous function on a closed and compact set $\mathbb{B}$ and is bounded below by $\underline{r}_c > 0$. We conclude that it has a minimum on this set. Denote this minimum by $r_c\left(\hat{\beta}_u\right)$ where $\hat{\beta}_u \in \mathbb{B}$.

By the definition of $\bar{m}^c_{\beta_u}$, we have:

$$\rho\left(z_1, \bar{m}^c_{\beta_u}\right) = r_c(\beta_u)$$

and thus,

$$\begin{aligned} \mu^c_{\beta_u} y\left(\beta_u, \bar{\beta}_s, r_c(\beta_u)\right) &= D\left(z_1, r_c(\beta_u)\right) \\ \mu^c_{\beta_u} &= \frac{D\left(z_1, r_c(\beta_u)\right)}{y\left(\beta_u, \bar{\beta}_s, r_c(\beta_u)\right)} \end{aligned}$$

The r.h.s. is continuous and strictly decreasing in $r_c(\beta_u)$ on the relevant price range, $r_c(\beta_u) < \bar{p}_{z_1}$ (by Assumption C), and thus,

$$\max_{\beta_u \in \mathbb{B}} \mu^c_{\beta_u} = \frac{D\left(z_1, r_c\left(\hat{\beta}_u\right)\right)}{y\left(\beta_u, \bar{\beta}_s, r_c\left(\hat{\beta}_u\right)\right)} > 0$$

provided that $r_c\left(\hat{\beta}_u\right) < \bar{p}_{z_1}$. Setting $\bar{\mu}^c = \max_{\beta_u \in \mathbb{B}} \mu^c_{\beta_u}$ gives the desired result.

For the case in which $\mathbb{B}$ is a finite set, taking $r_c\left(\hat{\beta}_u\right) = \min_{\beta_u \in \mathbb{B}} r_c(\beta_u)$ and setting

$$\bar{\mu}_c = \max_{\beta_u \in \mathbb{B}} \mu^c_{\beta_u} = \frac{D\left(z_1, r_c\left(\hat{\beta}_u\right)\right)}{y\left(\beta_u, \bar{\beta}_s, r_c\left(\hat{\beta}_u\right)\right)} > 0$$

where the strict inequality is implied by Assumption C, $r_c\left(\hat{\beta}_u\right) < \bar{p}_{z_1}$. This gives the desired result.

7.3 Proofs for the Dynamic Model, Section 4

Proof of Proposition 12:

We first show that the functions $\underline{\beta}(m) : \mathbb{M} \to \mathbb{B}$ and $\bar{\beta}(m) : \mathbb{M} \to \mathbb{B}$ are well-defined. Since, by assumption, $\pi\left(\beta_u, \bar{\beta}_s, p\right)$ is single-peaked, for any β_u, there is at most one other β'_u such that $r^c_{\beta_u} = r^c_{\beta'_u}$ and if (w.l.o.g.) $\beta_u < \beta'_u$, we have $\hat{\beta}_u \in (\beta_u, \beta'_u)$ with $r^c_{\hat{\beta}_u} < r^c_{\beta_u} = r^c_{\beta'_u}$. However, if $\pi\left(\beta_u, \bar{\beta}_s, p\right)$ is not symmetric on $\mathbb{B}$, there may be values of $\beta_u \neq \hat{\beta}_u$ for which no such β'_u exists. In particular, let $\check{\beta}_u \in \arg\max_{\beta_u \in \mathbb{B}} \left\{r^c_{\beta_u}\right\}$ be a type of firm with maximal reservation price on $\mathbb{B}$. If a $\check{\beta}'_u \in \mathbb{B}$ exists such that $r^c_{\check{\beta}_u} = r^c_{\check{\beta}'_u}$, we will call $\pi\left(\beta_u, \bar{\beta}_s, p\right)$ symmetric w.r.t. β_u and we will refer to it as asymmetric, otherwise. For a symmetric π, and for a value $r^c \in \left[r^c_{\hat{\beta}_u}, r^c_{\check{\beta}_u}\right]$, we will denote by $\underline{\beta}_u(r^c)$ and $\bar{\beta}_u(r^c)$ respectively the lower and the upper value of $\beta_u \in \mathbb{B}$ such that the firms of this type have a reservation price r^c:

$$\begin{aligned} \underline{\beta}_u(r^c) &= \min\left\{\beta_u \in \mathbb{B} \,|\, r^c_{\beta_u} = r^c\right\} \\ \bar{\beta}_u(r^c) &= \max\left\{\beta_u \in \mathbb{B} \,|\, r^c_{\beta_u} = r^c\right\} \end{aligned}$$

In the case of an asymmetric π, let $\bar{r}^c$ be the maximal reservation price for which $\underline{\beta}_u(\bar{r}^c)$ and $\bar{\beta}_u(\bar{r}^c)$ exist. For $r^c \leq \bar{r}^c$, we define $\underline{\beta}_u(r^c)$ and $\bar{\beta}_u(r^c)$ just as before. For $r \in \left(\bar{r}^c, r^c_{\hat{\beta}_u}\right]$, one of the values $\underline{\beta}_u(r^c)$ and $\bar{\beta}_u(r^c)$ is constrained by the set $\mathbb{B}$. For the purposes of the proof, we will assume that this is the case for the larger value, $\bar{\beta}_u(r^c)$. A symmetric argument can be used if this is the case for the smaller value $\underline{\beta}_u(r^c)$. Assume thus that $\bar{\beta}_u(\bar{r}^c) = \max\{\beta_u \in \mathbb{B}\}$, whereas $\underline{\beta}_u(\bar{r}^c) > \min\{\beta_u \in \mathbb{B}\}$. We then define for $r^c > \bar{r}^c$

$$\begin{aligned} \underline{\beta}_u(r^c) &= \min\left\{\beta_u \in \mathbb{B} \mid r^c_{\beta_u} = r^c\right\} \\ \bar{\beta}_u(r^c) &= \max\{\beta_u \in \mathbb{B}\} \end{aligned}$$

Note that according to the definition of $\underline{\beta}_u(r^c)$ and $\bar{\beta}_u(r^c)$, $\underline{\beta}_u(r^c)$ is a decreasing function of r^c, whereas $\bar{\beta}_u(r^c)$ is increasing in r^c. Since the interval $\left[\underline{\beta}_u(r^c), \bar{\beta}_u(r^c)\right]$ always contains $\hat{\beta}_u$, we thus have that for $r^c > r^{c\prime}$,

$$\left[\underline{\beta}_u(r^c), \bar{\beta}_u(r^c)\right] \supset \left[\underline{\beta}_u\left(r^{c\prime}\right), \bar{\beta}_u\left(r^{c\prime}\right)\right] \tag{60}$$

Define the correspondence $x(r^c, \beta_u)$ as

$$x(r^c, \beta_u) = \begin{cases} \{1\} & \text{if } r^c_{\beta_u} < r^c \\ [0,1] & \text{if } r^c_{\beta_u} = r^c \\ \{0\} & \text{if } r^c_{\beta_u} > r^c \end{cases}$$

Note that $x(r^c, \beta_u)$ is increasing in r_c in the sense that for $r^{c\prime} > r^c$, r^c, $r^{c\prime} \in \left[\min_{\beta_u \in \mathbb{B}} r^c_{\beta_u}, r^c_{\hat{\beta}_u}\right]$, $\min x(r^{c\prime}, \beta_u) \geq \min x(r^c, \beta_u)$ and $\max x(r^{c\prime}, \beta_u) \geq \max x(r^c, \beta_u)$ with strict inequality for a a subset of $\mathbb{B}$ with strictly positive Lebesgue measure. Define the correspondence $x(r^c, \beta_u)[m + \phi](\beta_u)$ and observe that it is also increasing in r^c in the same sense because ϕ is a.c. w.r.t. the Lebesgue measure on $\mathbb{B}$.

Next, for a given $m \in \mathbb{M}$ consider the market-clearing condition

$$D(z_1, r^c) \in \int_{\underline{\beta}_u(r^c)}^{\bar{\beta}_u(r^c)} y\left(\beta_u, \bar{\beta}_s, r^c\right) x(r^c, \beta_u)[m + \phi](\beta_u)\, d\beta_u \tag{61}$$

Note that the r.h.s. of the equation is an upper-hemicontinuous correspondence in r^c, which is strictly increasing in r^c in the sense that $r^{c\prime} > r^c$ implies

$$\begin{aligned} x\left(r^{c'}, \beta_u\right) &= \{1\} = x(r^c, \beta_u) \text{ for } r^c_{\beta_u} < r^c \\ x\left(r^{c'}, \beta_u\right) &= \{1\},\ x(r^c, \beta_u) = [0,1] \text{ for } r^c_{\beta_u} = r^c \\ x\left(r^{c'}, \beta_u\right) &= \{1\},\ x(r^c, \beta_u) = \{0\} \text{ for } r^c_{\beta_u} \in \left(r^c, r^{c'}\right) \\ x\left(r^{c'}, \beta_u\right) &= [0,1],\ x(r^c, \beta_u) = \{0\} \text{ for } r^c_{\beta_u} = r^{c'} \\ x\left(r^{c'}, \beta_u\right) &= \{0\} = x(r^c, \beta_u) \text{ for } r^c_{\beta_u} > r^{c'} \end{aligned}$$

Thus, $\tilde{x}(r^{c\prime}, \beta_u) \geq \tilde{x}(r^c, \beta_u)$ holds for any selections $\tilde{x}(r^{c\prime}, \beta_u)$ and $\tilde{x}(r^c, \beta_u)$ from the correspondences $x\left(r^{c'}, \beta_u\right)$ and $x(r^c, \beta_u)$ and, furthermore, $\tilde{x}(r^{c\prime}, \beta_u) > \tilde{x}(r^c, \beta_u)$ for all β_u s.t. $r^c_{\beta_u} \in \left(r^c, r^{c'}\right)$, which, by assumption have a strictly positive mass under ϕ, and thus, $m+\phi$. Since also, by Assumption S, supply y is a strictly increasing in the price r^c for each β_u and by (60), we obtain:

$$\begin{aligned} & \int_{\underline{\beta}_u(r^{c\prime})}^{\bar{\beta}_u(r^{c\prime})} y\left(\beta_u, \bar{\beta}_s, r^{c\prime}\right) \tilde{x}\left(r^{c\prime}, \beta_u\right) [m+\phi](\beta_u) \\ > & \int_{\underline{\beta}_u(r^{c})}^{\bar{\beta}_u(r^{c})} y\left(\beta_u, \bar{\beta}_s, r^{c}\right) \tilde{x}\left(r^{c}, \beta_u\right) [m+\phi](\beta_u) \end{aligned}$$

By Assumption D, $D(z_1, r^c)$ is strictly decreasing in r^c for $r^c \leq \bar{p}_{z_1}$. By Assumptions D and S, we further have:

$$D(z_1, r^c = 0) > \int_{\underline{\beta}_u(r^c)}^{\bar{\beta}_u(r^c)} y\left(\beta_u, \bar{\beta}_s, r^c = 0\right) [m+\phi](\beta_u)\, d\beta_u = 0$$

and by Assumptions D, S and C,

$$\begin{aligned} D\left(z_1, r^c = \bar{p}_{z_1}\right) &= 0 \\ &< \int_{\underline{\beta}_u(\bar{p}_{z_1})}^{\bar{\beta}_u(\bar{p}_{z_1})} y\left(\beta_u, \bar{\beta}_s, r^c = \bar{p}_{z_1}\right) [m+\phi](\beta_u)\, d\beta_u \\ &\leq \int_{\underline{\beta}_u(\bar{p}_{z_1})}^{\bar{\beta}_u(\bar{p}_{z_1})} y\left(\beta_u, \bar{\beta}_s, r^c = \bar{p}_{z_1}\right) [\phi](\beta_u)\, d\beta_u \end{aligned}$$

Since y, D, $\bar{\beta}_u$ and $\underline{\beta}_u$ are continuous in r^c, and the correspondence inside the integral on the r.h.s. of (61) is upper-hemicontinuous in r^c, it follows that for any $m \in \mathbb{M}$, there is a unique reservation price $r^c(m)$ such that

$$D\left(z_1, r^c(m)\right) \in \int_{\underline{\beta}_u(r^c(m))}^{\bar{\beta}_u(r^c(m))} y\left(\beta_u, \bar{\beta}_s, r^c(m)\right) x\left(r^c, \beta_u\right) [m+\phi](\beta_u)\, d\beta_u \tag{62}$$

as well as a corresponding $\tilde{x}(r^c, \beta_u) \in x(r^c, \beta_u)$ such that:

$$D\left(z_1, r^c(m)\right) = \int_{\underline{\beta}_u(r^c(m))}^{\bar{\beta}_u(r^c(m))} y\left(\beta_u, \bar{\beta}_s, r^c(m)\right) \tilde{x}\left(r^c, m, \beta_u\right) [m+\phi](\beta_u)\, d\beta_u$$

Note that $\tilde{x}(r^c, m, \beta_u)$ is unique for all $\beta_u \notin \left\{\underline{\beta}_u(r^c(m)), \bar{\beta}_u(r^c(m))\right\}$. When $\bar{\beta}_u(r^c(m)) = \max\{\beta_u \in \mathbb{B}\}$, $\tilde{x}\left(r^c(m), m, \bar{\beta}_u(r^c(m))\right) = 1$ and the corresponding $\tilde{x}\left(r^c(m), m, \underline{\beta}_u(r^c(m))\right)$ is therefore also unique. The only case of non-uniqueness arises when

$$\underline{\beta}_u(r^c(m)), \bar{\beta}_u(r^c(m)) \in int(\mathbb{B})$$

For this case, we will assume (w.l.o.g.) that

$$\frac{\tilde{x}\left(r^c(m), m, \underline{\beta}_u(r^c(m))\right)}{\tilde{x}\left(r^c(m), m, \bar{\beta}_u(r^c(m))\right)} = \frac{y\left(\bar{\beta}_u, \bar{\beta}_c, r^c(m)\right)}{y\left(\underline{\beta}_u, \bar{\beta}_c, r^c(m)\right)} \tag{63}$$

and given this condition, uniquely defined.

We can therefore define the functions $\underline{\beta}(m) : \mathbb{M} \to \mathbb{B}$ and $\bar{\beta}(m) : \mathbb{M} \to \mathbb{B}$ as:

$$\begin{aligned} \underline{\beta}(m) &= \underline{\beta}_u(r^c(m)) \\ \bar{\beta}(m) &= \bar{\beta}_u(r^c(m)) \end{aligned}$$

For a market composition $m \in \mathbb{M}$, and the corresponding $r^c(m)$, define the truncated market composition $m \mid_{\underline{\beta}(m)}^{\bar{\beta}(m)}$ as:

$$m \mid_{\underline{\beta}(m)}^{\bar{\beta}(m)} (\beta_u) = \begin{cases} m(\beta_u) & \text{if } \beta_u \in \left(\underline{\beta}(m), \bar{\beta}(m)\right) \\ \tilde{x}(r^c(m), m, \beta_u) & \text{if } \beta_u \in \left\{\underline{\beta}(m), \bar{\beta}(m)\right\} \\ 0 & \text{if } \beta_u \notin \left(\underline{\beta}(m), \bar{\beta}(m)\right) \end{cases}$$

Note that $\left[\phi + m \mid_{\underline{\beta}(m)}^{\bar{\beta}(m)}\right](\beta_u) > m(\beta_u)$ for all $\beta_u \in \left(\underline{\beta}(m), \bar{\beta}(m)\right)$. Thus,

$$\begin{aligned} & \int_{\underline{\beta}_u(r^c(m))}^{\bar{\beta}_u(r^c(m))} y\left(\beta_u, \bar{\beta}_s, r^c(m)\right) x(r^c, \beta_u) \left[m + \phi \mid_{\underline{\beta}(m)}^{\bar{\beta}(m)}\right](\beta_u)\, d\beta_u \\ > & \int_{\underline{\beta}_u(r^c(m))}^{\bar{\beta}_u(r^c(m))} y\left(\beta_u, \bar{\beta}_s, r^c(m)\right) x(r^c, \beta_u) \left[m \mid_{\underline{\beta}(m)}^{\bar{\beta}(m)}\right](\beta_u)\, d\beta_u \end{aligned}$$

We now show that there exists for each $m \in \mathbb{M}$ and each $\beta_u \in \mathbb{B} \backslash \left\{\hat{\beta}_u\right\}$ a finite $\tilde{t}$ such that

$$\tilde{x}\left(r^c, \tilde{t}\phi + m \mid_{\underline{\beta}(m)}^{\bar{\beta}(m)}, \beta_u\right) = 0$$

Indeed, suppose to the contrary, that for some $\beta_u \neq \hat{\beta}_u$ and each $\tilde{t}$,

$$\tilde{x}\left(r^c, \tilde{t}\phi + m \mid_{\underline{\beta}(m)}^{\bar{\beta}(m)}, \beta_u\right) > 0$$

This in turn implies, by the definition of the correspondence $x(r^c, \beta^u)$ that $\tilde{x}(r^c, \beta^u) = 1$ for all $\tilde{\beta}_u \in (\beta^u, \beta^{u\prime})$. Furthermore, also by the definition of $x(r^c, \beta^u)$, we have: $r^c\left(\tilde{t}\phi + m \mid_{\underline{\beta}(m)}^{\bar{\beta}(m)}\right) \geq r^c(\beta^u)$. However,

$$\begin{aligned} & \int_{\underline{\beta}_u\left(r^c\left(\tilde{t}\phi + m\mid_{\underline{\beta}(m)}^{\bar{\beta}(m)}\right)\right)}^{\bar{\beta}_u\left(r^c\left(\tilde{t}\phi + m\mid_{\underline{\beta}(m)}^{\bar{\beta}(m)}\right)\right)} y\left(\beta_u, \bar{\beta}_s, r^c(m)\right) \tilde{x}\left(r^c\left(\tilde{t}\phi + m \mid_{\underline{\beta}(m)}^{\bar{\beta}(m)}\right), m, \beta_u\right) [m + \phi](\beta_u)\, d\beta_u \\ \geq \;& y\left(\hat{\beta}_u, \bar{\beta}_s, r^c\left(\hat{\beta}^u\right)\right) [m + \tilde{t}\phi]\left(\hat{\beta}_u\right) > D\left(z_1, r^c\left(\hat{\beta}^u\right)\right) \end{aligned}$$

for a sufficiently large $\tilde{t}$, a contradiction.

The fact that $r^c(m)$ is decreasing in m, $\underline{\beta}_u(r^c)$ is decreasing in r^c and $\bar{\beta}_u(r^c)$ is increasing in r^c implies that $\underline{\beta}(m)$ is indeed increasing in m and $\bar{\beta}(m)$ is decreasing in m as required in the statement of the proposition. Note, furthermore, that since in our case,

$$r^c_{\bar{\beta}(m)} \leq r^c_{\underline{\beta}(m)} = r^c(m)$$

we have

$$r^c(m) = \max\left\{r^c_{\underline{\beta}(m)}, r^c_{\bar{\beta}(m)}\right\}$$

Define

$$X(m,\beta_u) = \tilde{x}(r^c(m), m, \beta_u).$$

Thus, (62) can be rewritten as:

$$\int_{\underline{\beta}(m)}^{\bar{\beta}(m)} y\left(\beta_u, \bar{\beta}_s, \max\left\{r^c_{\underline{\beta}(m)}, r^c_{\bar{\beta}(m)}\right\}\right) X(m,\beta_u)[m+\phi](\beta_u)\, d\beta_u = D\left(z_1, \max\left\{r^c_{\underline{\beta}(m)}, r^c_{\bar{\beta}(m)}\right\}\right) \tag{64}$$

as in the statement of the proposition.

We now check that the so-defined $\underline{\beta}(m)$ and $\bar{\beta}(m)$ together with $X(m,\beta_u)$ indeed characterize a Markovian equilibrium of the economy. Suppose first that the set of active firms in the market is given by:

$$M^*(m,\beta_u) = X(m,\beta_u)[m+\phi](\beta_u)$$

and $P^*(m) = \max\left\{\left\{r^c_{\underline{\beta}(m)}, r^c_{\bar{\beta}(m)}\right\}\right\}$.

Consider an initial state m and the subsequent state $M^*(m)$ as defined above. Note that by the definition of X,

$$[M^*(m)+\phi](\beta_u) > X(m,\beta_u)[m+\phi](\beta_u)$$

whenever $\beta_u \in \left[\underline{\beta}(m), \bar{\beta}(m)\right]$. It follows that if (64) is satisfied at $m+\phi$ with equality for given $\underline{\beta}(m)$ and $\bar{\beta}(m)$ such that $\underline{\beta}(m) \neq \bar{\beta}(m)$, then at $M^*(m)+\phi$,

$$\begin{aligned} & \int_{\underline{\beta}(m)}^{\bar{\beta}(m)} y\left(\beta_u, \bar{\beta}_s, \max\left\{r^c_{\underline{\beta}(m)}, r^c_{\bar{\beta}(m)}\right\}\right)[M^*(m)+\phi](\beta_u)\, d\beta_u \\ > & \int_{\underline{\beta}(m)}^{\bar{\beta}(m)} y\left(\beta_u, \bar{\beta}_s, \max\left\{r^c_{\underline{\beta}(m)}, r^c_{\bar{\beta}(m)}\right\}\right) X(m,\beta_u)[m+\phi](\beta_u)\, d\beta_u = D\left(z_1, \max\left\{r^c_{\underline{\beta}(m)}, r^c_{\bar{\beta}(m)}\right\}\right) \end{aligned}$$

Thus, to satisfy (64) at $M^*(m)$, it has to be that $\underline{\beta}(M^*(m)) \geq \underline{\beta}(m)$ and $\bar{\beta}(M^*(m)) \leq \bar{\beta}(m)$. We conclude that

$$\max\left\{r^c_{\underline{\beta}(M^*(m))}, r^c_{\bar{\beta}(M^*(m))}\right\} \leq \max\left\{r^c_{\underline{\beta}(m)}, r^c_{\bar{\beta}(m)}\right\}$$

has to hold. Furthermore, if

$$\max\left\{r^c_{\underline{\beta}(M^*(m))}, r^c_{\bar{\beta}(M^*(m))}\right\} = \max\left\{r^c_{\underline{\beta}(m)}, r^c_{\bar{\beta}(m)}\right\}$$

we have

$$\begin{aligned} X\left(M^*(m), \underline{\beta}(m)\right) &< X\left(m, \underline{\beta}(m)\right) \\ X\left(M^*(m), \bar{\beta}(m)\right) &< X\left(m, \bar{\beta}(m)\right) \end{aligned}$$

As explained above, as long as $\left(r^c_{\underline{\beta}(m)}, r^c_{\bar{\beta}(m)}\right) \neq \varnothing$, (i.e., $r^c_{\underline{\beta}(m)} \neq r^c_{\bar{\beta}(m)} \neq r^c_{\hat{\beta}_u}$), there exists a finite $\tilde{t}(m)$ such that

$$\max\left\{r^c_{\underline{\beta}\left(M^{*\tilde{t}(m)}(m)\right)}, r^c_{\bar{\beta}\left(M^{*\tilde{t}(m)}(m)\right)}\right\} < \max\left\{r^c_{\underline{\beta}(m)}, r^c_{\bar{\beta}(m)}\right\}$$

and

$$X\left(M^{*\bar{t}(m)}(m), \underline{\beta}(m)\right) = X\left(M^{*\bar{t}(m)}(m), \bar{\beta}(m)\right) = 0$$

It follows that for any firm $\beta_u \neq \hat{\beta}_u$, and $m \in \mathbb{M}$, there is a finite $\bar{t}(m, \beta_u)$ (potentially equal to 0) such that

$$\bar{t}(m, \beta_u) = \min\left\{t \mid \beta_u \notin \left[\underline{\beta}\left(M^{*t}(m)\right), \bar{\beta}\left(M^{*t}(m)\right)\right]\right\}$$

Because $P^*(m) = \max\left\{r^c_{\underline{\beta}(m)}, r^c_{\bar{\beta}(m)}\right\}$, it follows that the equilibrium price can only decrease on any given path of the Markovian equilibrium

$$P^*\left(M^{*t}(m)\right) \leq P^*\left(M^{*t-1}(m)\right)$$

for any m and t and

$$P^*\left(M^{*\bar{t}(m)}(m)\right) < P^*(m)$$

It follows that the profits of a firm of type β_u who decides to be active in the market will also be decreasing on any given path. Thus, if at m, a firm of type β_u cannot cover the fixed cost c,

$$\pi\left(\beta_u, \bar{\beta}_s, P^*(m)\right) - c < 0$$

then the same will be true at $M^{*t}(m)$ for any t, because $P^*(m) \geq P^*\left(M^{*t}(m)\right)$:

$$\pi\left(\beta_u, \bar{\beta}_s, P^*\left(M^{*t}(m)\right)\right) - c < 0$$

We conclude that the value function of a firm of type β_u can be written as:

$$v^*(z, m, \beta_u) = \sum_{t=0}^{\bar{t}(m,\beta_u)} \delta^t\left[\pi\left(\beta_u, P^*\left(M^{*t}(m)\right)\right) - c\right]$$

and that

$$v^*(z, m, \beta_u) > 0 \text{ iff } \pi\left(\beta_u, \bar{\beta}_s, P^*(m)\right) - c > 0 \text{ iff } P^*(m) > r^c_{\beta_u}$$

It follows that decision of a firm to stay in the market given the mappings P^* and M^* given by

$$X^*(m, \beta_u) = X(m, \beta_u) = \begin{cases} 1 & \text{if } r^c_{\beta_u} < P^*(m) = \max\left\{r^c_{\underline{\beta}(m)}, r^c_{\bar{\beta}(m)}\right\} \\ X(m, \beta_u) & \text{if } r^c_{\beta_u} = P^*(m) \\ 0 & \text{if } r^c_{\beta_u} > P(m) = \max\left\{r^c_{\underline{\beta}(m)}, r^c_{\bar{\beta}(m)}\right\} \end{cases}$$

is indeed optimal as assumed.

Next note that if X^* and P^* are given as in the statement of the proposition, M^*, as defined in (22) indeed gives the corresponding market composition.

Finally, suppose that M^* and X^* are given as in the statement of the proposition. Then, by the definition of $\underline{\beta}(m)$, $\bar{\beta}(m)$, $r^c_{\underline{\beta}(m)}$ and $r^c_{\bar{\beta}(m)}$, condition (21) implies that

$$\rho(z_1, M^*(m)) = \max\left\{r^c_{\underline{\beta}(m)}, r^c_{\bar{\beta}(m)}\right\}$$

Therefore, setting

$$P^*(m) = \max\left\{r^c_{\underline{\beta}(m)}, r^c_{\bar{\beta}(m)}\right\}$$

indeed results in the equilibrium price mapping. We conclude that (X^*, P^*, M^*) is a Markovian equilibrium of the economy.

The fact that $\hat{\beta}_u \in \left[\underline{\beta}(m), \bar{\beta}(m)\right]$ follows from the definitions of $\underline{\beta}(m)$ and $\bar{\beta}(m)$. As already shown, at each m, we have that

$$\left[\underline{\beta}(M^*(m)), \bar{\beta}(M^*(m))\right] \subseteq \left[\underline{\beta}(m), \bar{\beta}(m)\right]$$

and, furthermore, for every m_0, $\epsilon > 0$ and $\beta_u \notin \left[\hat{\beta}_u - \epsilon, \hat{\beta}_u + \epsilon\right]$, there is a finite time $\bar{t}(m_0, \beta_u)$ such that $\beta_u \notin \left[\underline{\beta}\left(M^t(m_0)\right), \bar{\beta}\left(M^t(m_0)\right)\right]$ for $t \geq \bar{t}(m_0, \beta_u)$. It follows that for any m_0, each $\epsilon > 0$, there is a $\bar{t}(m, \epsilon)$ such that

$$\begin{aligned} \underline{\beta}\left(M^t(m_0)\right) &\geq \hat{\beta}_u - \epsilon \\ \bar{\beta}\left(M^t(m_0)\right) &\leq \hat{\beta}_u + \epsilon \end{aligned}$$

for any $t \geq \bar{t}(m, \epsilon)$. We conclude that

$$\lim_{t\to\infty} \underline{\beta}\left(M^{*t}(m_0)\right) = \lim_{t\to\infty} \bar{\beta}\left(M^{*t}(m_0)\right) = \hat{\beta}_u$$

and thus,

$$\lim_{t\to\infty} M^{*t}(m_0) = \bar{m}^c_{\hat{\beta}_u}$$

This in turn implies that

$$\lim_{t\to\infty} P^*\left(M^{*t}(m_0)\right) = r^c_{\hat{\beta}_u}$$

Proof of Proposition 13:

Checking that the described mappings constitute a Markovian equilibrium is straightforward. Suppose that there existed an equilibrium with a distinct price mapping, $P^{**}(z, m)$. Clearly, if $P^{**}(z, m) \ll P^*(z, m)$, then $v^{**}(z, m, \beta_u) = 0$ and no firms would wish to be active in the market, $X^*(z, m, \beta_u) \equiv 0$, contradicting market clearing. Thus, there must be a (z, m) at which $P^{**}(z, m) > P^*(z, m) = m^c_{\hat{\beta}_u}(z)$. Thus,

$$X^{**}(z, m, \beta_u) = m_{\hat{\beta}_u} + \phi_{\hat{\beta}_u} \geq m_{\hat{\beta}_u} + \bar{\mu}^c \geq \bar{\mu}^c_{\hat{\beta}_u}(z)$$

We know, however that

$$P^*(z, m) = \rho\left(z, m^c_{\hat{\beta}_u}(z)\right) > \rho\left(z, \bar{\mu}^c_{\hat{\beta}_u}(z), M^{**}_{-\hat{\beta}_u}(m, z)\right) = P^{**}(z, m),$$

a contradiction to the assumption that $P^{**}(z, m) > P^*(z, m)$.

Proof of Proposition 15:

Under condition (29), we have that

$$\pi\left(\beta^1_u, \rho\left(z_1, \phi_{\beta^1_u}, \phi_{\beta^2_u}\right)\right) - c > 0$$

By Assumption C, if no β_u^1-firms enter the market in state z_2,

$$\pi\left(\beta_u^1, \rho\left(z_2, 0, 0\right)\right) - c = \pi\left(\beta_u^1, \bar{p}_{z_2}\right) - c > 0$$

This implies that for $\tilde{\mu}_{\beta_u^1}\left(z_2\right) = 0$, the l.h.s. of (26) is strictly positive. Note further, that, by Proposition 10, profits are strictly decreasing in m, and thus, the l.h.s.of (26) is strictly decreasing in m and is strictly negative at $\tilde{\mu}_{\beta_u^1}\left(z_2\right) = \bar{\mu}^c$, because

$$\pi\left(\beta_u^1, \rho\left(z_1, \bar{\mu}^c, \phi_{\beta_u^2}\right)\right) - c < \pi\left(\beta_u^1, \rho\left(z_1, \bar{\mu}^c, 0\right)\right) - c < \pi\left(\beta_u^1, r^c_{\beta_u^1}\right) - c = 0$$

and, by Proposition 10,

$$\pi\left(\beta_u^1, \rho\left(z_2, \bar{\mu}^c, 0\right)\right) - c < \pi\left(\beta_u^1, \rho\left(z_1, \bar{\mu}^c, 0\right)\right) - c = 0$$

Since ρ, and therefore, π is continuous in m_1, the Intermediate value theorem implies the existence of a unique $\tilde{\mu}_{\beta_u^1}\left(z_2\right) \in \left(0, \bar{\mu}^c_{\beta_u^1}\left(z_1\right)\right)$ which solves (26).

At $m = \left(\tilde{\mu}_{\beta_u^1}\left(z_2\right), 0\right)$, (30) implies:

$$\begin{aligned} \pi\left(\beta_u^2, \rho\left(z_1, \tilde{\mu}^c_{\beta_u^1}\left(z_2\right), 0\right)\right) &< \pi\left(\beta_u^1, \rho\left(z_1, \tilde{\mu}^c_{\beta_u^1}\left(z_2\right), 0\right)\right) \\ \pi\left(\beta_u^2, \rho\left(z_1, \tilde{\mu}^c_{\beta_u^1}\left(z_2\right) + \tau\phi_{\beta_u^1}, \tau\phi_{\beta_u^2}\right)\right) &< \pi\left(\beta_u^1, \rho\left(z_1, \tilde{\mu}^c_{\beta_u^1}\left(z_2\right) + \tau\phi_{\beta_u^1}, \tau\phi_{\beta_u^2}\right)\right) \end{aligned}$$

and

$$\begin{aligned} & \min\left\{\tau \mid \rho\left(z_1, M^*_{\beta_u^1}\left(z_2, m\right) + (\tau+1)\,\phi_{\beta_u^1}, k + (\tau+1)\,\phi_{\beta_u^2}\right) \leq r^c_{\beta_u^2}\right\} \\ \leq \quad & \min\left\{\tau \mid \rho\left(z_1, M^*_{\beta_u^1}\left(z_2, m\right) + (\tau+1)\,\phi_{\beta_u^1}, 0\right) \leq r^c_{\beta_u^1}\right\} \end{aligned}$$

Because profits are decreasing in m_2, we have that $\tilde{v}\left(z_2, \tilde{\mu}_{\beta_u^1}\left(z_2\right), k, \beta_u^2\right)$ is decreasing in k, and thus, the definition of $\tilde{\mu}_{\beta_u^1}\left(z_2\right)$ implies that $\tilde{v}\left(z_2, \tilde{\mu}_{\beta_u^1}\left(z_2\right), k, \beta_u^2\right) < 0$ for all $k \in \left[0, m_{\beta_u^2} + \phi_{\beta_u^2}\right]$. Therefore, there exists an $\tilde{\mu}_1 < \tilde{\mu}_{\beta_u^1}\left(z_2\right)$ such that

$$M^*_{\beta_u^2}\left(z_2, m\right) = 0 \text{ for all } m_1 \geq \tilde{\mu}_1$$

We have thus shown that M^* is well-defined. It is non-decreasing, continuous and bounded between $[0, \bar{\mu}^c]$.

The corresponding price sequence is obtained as:

$$P^*\left(z, m\right) = \rho\left(z, M^*\left(m\right)\right)$$

Thus, by a standard contraction-mapping argument, the corresponding value functions $v^*\left(z, m, \beta_u\right)$, $\beta_u \in \left\{\beta_u^1, \beta_u^2\right\}$ exist. Note that since

$$M^*\left(z_2, m = \left(\tilde{\mu}_{\beta_u^1}\left(z_2\right), 0\right)\right) = \left(\tilde{\mu}_{\beta_u^1}\left(z_2\right), 0\right),$$

v^* satisfy:

$$v^*\left(z_2, m=\left(\tilde{\mu}_{\beta_u^1}(z_2), 0\right), \beta_u^i\right) = \pi\left(\beta_u^i, \rho\left(z_1, \tilde{\mu}^c_{\beta_u^1}(z_2), 0\right)\right) - c$$
$$+\delta\left[qv^*\left(z_1, m, \beta_u^i\right) + (1-q)\, v^*\left(z_2, m=\left(\tilde{\mu}_{\beta_u^1}(z_2), 0\right), \beta_u^i\right)\right]$$

for $i \in \{1,2\}$. Thus,

$$v^*\left(z_2, m=\left(\tilde{\mu}_{\beta_u^1}(z_2), 0\right), \beta_u^i\right) = \pi\left(\beta_u^i, \rho\left(z_1, \tilde{\mu}^c_{\beta_u^1}(z_2), 0\right)\right) - c + \delta v^*\left(z_1, m, \beta_u^i\right)$$

Note, furthermore, that since

$$M^*\left(z_1, m=\left(\bar{\mu}^c_{\beta_u^1}(z_1), 0\right)\right) = \left(\bar{\mu}^c_{\beta_u^1}(z_1), 0\right),$$

$$v^*\left(z_1, \left(\bar{\mu}^c_{\beta_u^1}(z_1), 0\right), \beta_u^1\right) = (1-q)\,\delta v^*\left(z_2, m=\left(\tilde{\mu}_{\beta_u^1}(z_2), 0\right), \beta_u^1\right)$$

Recursively, we obtain for m such that $m_{\beta_u^1} \geq \tilde{\mu}_{\beta_u^1}(z_2)$,

$$\begin{aligned} v^*\left(z_1, m, \beta_u^1\right) \quad = \quad & \sum_{\tau=1}^{\min\left\{\tau \mid \rho\left(z_1, M^{*\tau}(z_1 \ldots z_1, m)\right) \leq r^c_{\beta_u^1}\right\}} q^{\tau-1}\delta^{\tau-1}\left(\pi\left(\beta_u^1, \rho\left(z_1, M^{*\tau-1}(z_1 \ldots z_1, m)\right)\right) - c\right) + \\ & + (1-q)\frac{(q\delta)^{\min\left\{\tau \mid \rho\left(z_1, M^{*\tau}(z_1 \ldots z_1, m)\right) \leq r^c_{\beta_u^1}\right\}}}{1-q\delta} v^*\left(z_2, m=\left(\tilde{\mu}_{\beta_u^1}(z_2), 0\right), \beta_u^1\right) \end{aligned}$$

This in turn gives us:

$$\begin{aligned} & v^*\left(z_2, m=\left(\tilde{\mu}_{\beta_u^1}(z_2), 0\right), \beta_u^i\right) = \pi\left(\beta_u^1, \rho\left(z_1, \tilde{\mu}^c_{\beta_u^1}(z_2), 0\right)\right) - c \\ & +\delta \sum_{\tau=1}^{\min\left\{\tau \mid \rho\left(z_1, M^{*\tau}(z_1 \ldots z_1, m)\right) \leq r^c_{\beta_u^1}\right\}} q^{\tau-1}\delta^{\tau-1}\left(\pi\left(\beta_u^i, \rho\left(z_1, M^{*\tau-1}(z_1 \ldots z_1, m)\right)\right) - c\right) \\ & + (1-q)\,\delta\frac{(q\delta)^{\min\left\{\tau \mid \rho\left(z_1, M^{*\tau}(z_1 \ldots z_1, m)\right) \leq r^c_{\beta_u^1}\right\}}}{1-q\delta} v^*\left(z_2, m=\left(\tilde{\mu}_{\beta_u^1}(z_2), 0\right), \beta_u^1\right) \end{aligned}$$

or

$$\begin{aligned} & \left[1-(1-q)\,\delta\frac{(q\delta)^{\min\left\{\tau \mid \rho\left(z_1, M^{*\tau}(z_1 \ldots z_1, m)\right) \leq r^c_{\beta_u^1}\right\}}}{1-q\delta}\right] v^*\left(z_2, m=\left(\tilde{\mu}_{\beta_u^1}(z_2), 0\right), \beta_u^1\right) \\ = \quad & \pi\left(\beta_u^i, \rho\left(z_1, \tilde{\mu}^c_{\beta_u^1}(z_2), 0\right)\right) - c + \\ & +\delta \sum_{\tau=1}^{\min \tau \mid \left\{\rho\left(z_1, M^{*\tau}(z_1 \ldots z_1, m)\right) \leq r^c_{\beta_u^1}\right\}} q^{\tau-1}\delta^{\tau-1}\left(\pi\left(\beta_u^1, \rho\left(z_1, M^{*\tau-1}(z_1 \ldots z_1, m)\right)\right) - c\right) \\ = \quad & 0 \end{aligned}$$

by the definition of $\tilde{\mu}^c_{\beta_u^1}(z_2)$. Therefore,

$$v^*\left(z_1, m, \beta_u^1\right) = \sum_{\tau=1}^{\min\left\{\tau \mid \rho\left(z_1, M^{*\tau}(z_1 \ldots z_1, m)\right) \leq r^c_{\beta_u^1}\right\}} q^{\tau-1}\delta^{\tau-1}\left(\pi\left(\beta_u^1, \rho\left(z_1, M^{*\tau-1}\left((z_1 \ldots z_1), m\right)\right)\right) - c\right)$$

We conclude that $v^*(z_1, m, \beta_u^1) > 0$ exactly when

$$\rho\left(z_1, \left(\min\left\{m_{\beta_u^1} + \phi_{\beta_u^1}, \bar{\mu}^c\right\}, 0\right)\right) > r^c_{\beta_u^1}$$

and the decision given by:

$$X^*(z_1, m, \beta_u^1) = \begin{cases} 1 & \text{if } v^*(z_1, m, \beta_u^1) > 0 \text{ iff } m_{\beta_u^1} + \phi_{\beta_u^1} < \bar{\mu}^c \\ \frac{\bar{\mu}^c}{m_{\beta_u^1} + \phi_{\beta_u^1}} & \text{if } v^*(z_1, m, \beta_u^1) = 0 \text{ iff } m_{\beta_u^1} + \phi_{\beta_u^1} \geq \bar{\mu}^c \end{cases}$$

is indeed both consistent with M^* and optimal.

Next note that at $m = (\bar{\mu}^c, 0)$

$$v^*(z_1, (\bar{\mu}^c, 0), \beta_u^2) = \max\left\{ \begin{array}{c} \pi(\beta_u^2, \rho(z_1, \bar{\mu}^c, 0)) - c + \delta q v^*(z_1, (\bar{\mu}^c, 0), \beta_u^2) \\ +\delta(1-q) v^*(z_2, (\bar{\mu}^c, 0), \beta_u^2) \end{array} \right\}$$

Assume first that $v^*(z_1, (\bar{\mu}^c, 0), \beta_u^2) > 0$, then

$$(1 - \delta q) v^*(z_1, (\bar{\mu}^c, 0), \beta_u^2) = \pi(\beta_u^2, \rho(z_1, \bar{\mu}^c, 0)) - c + \delta(1-q) v^*(z_2, (\bar{\mu}^c, 0), \beta_u^2) \quad (65)$$

Now observe that by (30), we have

$$\begin{aligned} v^*\left(z_2, m = \left(\tilde{\mu}_{\beta_u^1}(z_2), 0\right), \beta_u^2\right) &\leq v^*\left(z_2, m = \left(\tilde{\mu}_{\beta_u^1}(z_2), 0\right), \beta_u^1\right) = 0 \\ v^*\left(z_2, m = \left(\tilde{\mu}_{\beta_u^1}(z_2), 0\right), \beta_u^2\right) &= 0 \end{aligned}$$

But then (65) implies that

$$(1 - \delta q) v^*\left(z_1, \left(\bar{\mu}^c_{\beta_u^1}, 0\right), \beta_u^2\right) = \pi\left(\beta_u^2, \rho\left(z_1, \bar{\mu}^c_{\beta_u^1}, 0\right)\right) - c < 0$$

which contradicts our assumption that $v^*\left(z_2, m = \left(\tilde{\mu}_{\beta_u^1}(z_2), 0\right), \beta_u^2\right) > 0$.

Thus, $v^*\left(z_1, \left(\bar{\mu}^c_{\beta_u^1}, 0\right), \beta_u^2\right) = 0$. Using the fact that $v^*\left(z_2, m = \left(\tilde{\mu}_{\beta_u^1}(z_2), 0\right), \beta_u^2\right) = 0$, we can then recursively write $v^*(z_1, m, \beta_u^2)$ as:

$$v^*(z_1, m, \beta_u^2) = \sum_{\tau=1}^{\min\left\{\tau \mid \rho\left(z_1, M^{*\tau}((z_1 \ldots z_1), m)\right) \leq r^c_{\beta_u^2}\right\}} q^{\tau-1}\delta^{\tau-1}\left(\pi\left(\beta_u^2, \rho\left(z_1, M^{*\tau-1}((z_1 \ldots z_1), m)\right)\right) - c\right)$$

We conclude that $v^*(z_1, m, \beta_u^2) > 0$ exactly when $\rho\left(m_{\beta_u^1} + \phi_{\beta_u^1}, m_{\beta_u^2} + \phi_{\beta_u^2}\right) > r^c_{\beta_u^2}$ and the decision given by:

$$X^*(z_1, m, \beta_u^2) = \begin{cases} 1 & \text{if } v^*(z_1, m, \beta_u^2) > 0 \text{ iff } \rho\left(z_1, \left(\min\left\{m_{\beta_u^1} + \phi_{\beta_u^1}, \bar{\mu}^c\right\}, m_{\beta_u^2} + \phi_{\beta_u^2}\right)\right) > r^c_{\beta_u^2} \\ \frac{k}{m_{\beta_u^2} + \phi_{\beta_u^2}} & \text{if } v^*(z_1, m, \beta_u^2) = 0 \text{ and } \rho\left(z_1, \left(\min\left\{m_{\beta_u^1} + \phi_{\beta_u^1}, \bar{\mu}^c\right\}, k\right)\right) = r^c_{\beta_u^2} \\ & \text{for } k \in \left[0, m_{\beta_u^2} + \phi_{\beta_u^2}\right] \\ 0 & \text{if } v^*(z_1, m, \beta_u^2) = 0 \text{ and } \rho\left(z_1, \left(\min\left\{m_{\beta_u^1} + \phi_{\beta_u^1}, \bar{\mu}^c\right\}, 0\right)\right) < r^c_{\beta_u^2} \end{cases}$$

is both consistent with M^* and optimal.

Next consider z_2. By the definition of $M^*_{\beta_u^2}$ and $\tilde{v}\left(z_2, m_{\beta_u^1}, k, \beta_u^2\right)$, we have that

$$v^*\left(z_2, m, \beta_u^2\right) \geq \tilde{v}\left(z_2, m_{\beta_u^1}, k, \beta_u^2\right)$$

when $\tilde{v}\left(z_2, m_{\beta_u^1}, M^*_{\beta_u^2}(m), \beta_u^2\right) > 0$ and

$$v^*\left(z_2, m, \beta_u^2\right) = 0$$

when $\tilde{v}\left(z_2, m_{\beta_u^1}, M^*_{\beta_u^2}(m), \beta_u^2\right) \leq 0$. This in turn implies that the decision rule

$$X^*\left(z_2, m, \beta_u^2\right) = \begin{cases} 0 & \text{if } \tilde{v}\left(z_2, \left(m_{\beta_u^1}, 0\right), \beta_u^2\right) < 0 \\ & \text{and thus, } v^*\left(z_2, m, \beta_u^2\right) = 0 \\ \frac{k}{m_{\beta_u^2} + \phi_{\beta_u^2}} \text{ s.t.} & \text{if } k \in \left[0, m_{\beta_u^2} + \phi_{\beta_u^2}\right] \\ \tilde{v}\left(z_2, \left(m_{\beta_u^1}, k\right), \beta_u^2\right) = 0 & \text{and thus, } v^*\left(z_2, m, \beta_u^2\right) = 0 \\ 1 & \text{if } \tilde{v}\left(z_2, \left(m_{\beta_u^1}, m_{\beta_u^2} + \phi_{\beta_u^2}\right), \beta_u^2\right) > 0 \\ & \text{and thus, } v^*\left(z_2, m, \beta_u^2\right) > 0 \end{cases}$$

is both consistent with M^* and optimal.

Let $\hat{\mu}_1$ denote the minimal $m_{\beta_u^1}$ such that $\tilde{v}\left(z_2, \left(m_{\beta_u^1}, k\right), \beta_u^2\right) = 0$ for $k = 0$. Note that since $v^*\left(z_2, m, \beta_u^1\right)$ is decreasing in m_1,

$$v^*\left(z_2, m_{\beta_u^1}, 0, \beta_u^1\right) \geq v^*\left(z_2, m = \left(\tilde{\mu}_{\beta_u^1}(z_2), 0\right), \beta_u^1\right) = 0$$

for $m_{\beta_u^1} \in \left[\hat{\mu}_1, \tilde{\mu}_{\beta_u^1}(z_2)\right]$ and by (30), for any $m_{\beta_u^1} < \hat{\mu}_1$,

$$v^*\left(z_2, m, \beta_u^1\right) > v^*\left(z_2, m, \beta_u^2\right).$$

Therefore, the decision

$$X^*\left(z_1, m, \beta_u^1\right) = \min\left\{1, \frac{\bar{\mu}^c}{m_{\beta_u^1} + \phi_{\beta_u^1}}\right\}$$

is both consistent with M^* and optimal. We conclude that M^* is indeed a Markovian equilibrium. ■

Proof of Proposition 16: Suppose that the exogenous shock realization is given by z_2 for $\left\lceil \frac{\tilde{\mu}_{\beta_u^1}(z_2)}{\phi_{\beta_u^1}} \right\rceil$ consecutive periods. The economy then reaches a state (z, m) with $m_{\beta_u^1} \geq \tilde{\mu}_{\beta_u^1}(z_2)$. Since z_2 occurs $\left\lceil \frac{\tilde{\mu}_{\beta_u^1}(z_2)}{\phi_{\beta_u^1}} \right\rceil$ times in a row a.s. in finite time on any path, the economy a.s. reaches a state (z, m) with $m_{\beta_u^1} \geq \tilde{\mu}_{\beta_u^1}(z_2)$ in finite time. Once such a state m with $m_{\beta_u^1} \geq \tilde{\mu}_{\beta_u^1}(z_2)$ has been reached, M^* implies that $M^{*t}_{\beta_u^1}\left(\left(z^1 ... z^t\right), m\right) \geq \tilde{\mu}_{\beta_u^1}(z_2)$ for any t and any sequence of shocks $z^1 ... z^t$. It follows that the only recurrent state with $z = z_2$ is given by $\sigma_0 = \left(z_2, \left(\tilde{\mu}_{\beta_u^1}(z_2), 0\right)\right)$, whereas the recurrent states with $z = z_1$ are those reached from $\left(z_2, \left(\tilde{\mu}_{\beta_u^1}(z_2), 0\right)\right)$ given by

$$(\sigma_1 ... \sigma_n) = \left(z_1, M^{*\tau}\left(\left(z_1 ... z_1\right), \tilde{\mu}_{\beta_u^1}(z_2), 0\right)\right)_{\tau=1}^{n}$$

The set of recurrent states is thus $\bar{\Sigma} = \{\sigma_0, \sigma_1 ... \sigma_n\}$. We note that the Markov process restricted to $\bar{\Sigma}$ is aperiodic. Indeed, the probability of reaching σ_0 from σ_0 is $1 - q > 0$. Suppose that we are at state σ_τ, $\tau \in \{1...n\}$. This state can be reached again with strictly positive probability $(1-q)^l q^\tau > 0$ in $(l+\tau)$-periods, where l is any natural number: in particular, the state is reached if the following l state realizations are z_2, followed by τ realizations of z_1. Clearly, the largest common divisor of the set $\{l + \tau \mid l \in \mathbb{N}\}$ is 1. The Markov process restricted to $\bar{\Sigma}$ is thus recurrent and aperiodic. Its transition matrix is given by

$$\begin{aligned} \Pr\{\sigma_0 \mid \sigma_\tau\} &= 1 - q \text{ for any } \tau \in \{1...n\} \\ \Pr\{\sigma_{\tau+1} \mid \sigma_t\} &= q \text{ for any } \tau \in \{0...n-1\} \\ \Pr\{\sigma_n \mid \sigma_n\} &= q \end{aligned}$$

The unique invariant distribution Q thus satisfies:

$$\begin{aligned} (1-q)\sum_{\tau=0}^{n} Q_\tau &= Q_1 \\ qQ_1 &= Q_2 \\ &... \\ qQ_{n-2} &= Q_{n-1} \\ qQ_{n-1} + qQ_n &= Q_n \end{aligned}$$

Solving the system gives the invariant distribution:

$$Q = \left((1-q), q(1-q) ... q^{n-2}(1-q), q^{n-1}\right).$$

Proof of Proposition 18:

We first show that the continuation values $v^*\left(z, M^*(z,m), \beta_u^i\right)$ are non-increasing in m.

Lemma 36 *The continuation values $v^*\left(z, M^*(z,m), \beta_u^i\right)$ are non-increasing in m.*

Proof of Lemma 36:

Consider first state z_2. Since $M^*_{\beta_u^1}(z_2, m) = M^*_{\beta_u^1}(z_2, m') = 0$ for any m, m', we have that $m_{\beta_u^2} = m'_{\beta_u^2}$,

$$\begin{aligned} & \rho\left(z^1 = z_2 ... z^t, M^{*t-1}\left(z^1 = z_2 ... z^t, \left(m_{\beta_u^1}, m_{\beta_u^2}\right)\right)\right) \\ = \; & \rho\left(z^1 = z_2 ... z^t, M^{*t-1}\left(z^1 = z_2 ... z^t, \left(m'_{\beta_u^1}, m_{\beta_u^2}\right)\right)\right) \end{aligned}$$

and

$$\begin{aligned} & v^*\left(z^1 = z_2 ... z^t, M^{*t-1}\left(z^1 = z_2 ... z^t, \left(m_{\beta_u^1}, m_{\beta_u^2}\right)\right), \beta_u^i\right) \\ = \; & v^*\left(z^1 = z_2 ... z^t, M^{*t-1}\left(z^1 = z_2 ... z^t, \left(m'_{\beta_u^1}, m_{\beta_u^2}\right)\right)\right) \end{aligned}$$

Thus for z_2, we only need to consider changes in $m_{\beta_u^2}$. Let thus, (as shown above, w.l.o.g.), $m' = \left(0, m'_{\beta_u^2}\right) > m = \left(0, m_{\beta_u^2}\right)$ and consider the resulting price sequence. Since $M^*_{\beta^2_u}\left(0,m'_{\beta^2_u}\right) \geq M^*_{\beta^2_u}\left(0,m_{\beta^2_u}\right)$, we have by Proposition 10,

$$\rho\left(z_2, M^*_{\beta^2_u}\left(0,m'_{\beta^2_u}\right)\right) \leq \rho\left(z_2, M^*_{\beta^2_u}\left(0,m'_{\beta^2_u}\right)\right)$$

and, by induction,

$$\begin{aligned} & \rho\left(z^1 = z_2...z^t = z_2, M^{*t-1}\left(z^1 = z_2...z^t = z_2, \left(0,m'_{\beta^2_u}\right)\right)\right) \\ \leq \quad & \rho\left(z^1 = z_2...z^t = z_2, M^{*t-1}\left(z^1 = z_2...z^t = z_2, \left(0,m_{\beta^2_u}\right)\right)\right) \end{aligned}$$

for all t (i.e., for any number of repetitions of state z_2). Suppose now that state z_1 occurs after τ repetitions of state z_2 (where $\tau = 1$ is of course possible), i.e.,

$$z^1 = z_2...z^\tau = z_2, z^{\tau+1} = z_1$$

By the previous argument, we have

$$\begin{aligned} m'_{\tau+1} &= M^{*\tau}\left(z^1 = ...z^\tau = z_2, \left(0,m'_{\beta^2_u}\right)\right) = \left(0, \min\left\{m'_{\beta^2_u} + \tau\phi_{\beta^2_u}, \hat{\mu}_{\beta^2_u}(z_2)\right\}\right) \\ &\geq M^{*\tau}\left(z^1 = ...z^\tau = z_2, \left(0,m_{\beta^2_u}\right)\right) = \left(0, \min\left\{m_{\beta^2_u} + \tau\phi_{\beta^2_u}, \hat{\mu}_{\beta^2_u}(z_2)\right\}\right) \\ &= m_{\tau+1} \end{aligned}$$

and thus,

$$\begin{aligned} & M^{*\tau+1}\left(z^1 = ...z^\tau = z_2, z^{\tau+1} = z_1, \left(0,m'_{\beta^2_u}\right)\right) \\ = \quad & \left(\phi_{\beta^1_u}, \min\left\{m'_{\beta^2_u} + \tau\phi_{\beta^2_u}, \hat{\mu}_{\beta^2_u}(z_2)\right\} + \phi_{\beta^2_u}\right) \\ \geq \quad & M^{*\tau+1}\left(z^1 = ...z^\tau = z_2, z^{\tau+1} = z_1, \left(0,m_{\beta^2_u}\right)\right) \\ = \quad & \left(\phi_{\beta^1_u}, \min\left\{m_{\beta^2_u} + \tau\phi_{\beta^2_u}, \hat{\mu}_{\beta^2_u}(z_2)\right\} + \phi_{\beta^2_u}\right) \end{aligned}$$

implying, by Proposition 10:

$$\begin{aligned} & \rho\left(z^1 = z_2...z^\tau = z_2, z^{\tau+1} = z_1, M^{*\tau}\left(z^1 = z_2...z^\tau = z_2, z^{\tau+1} = z_1, \left(0,m'_{\beta^2_u}\right)\right)\right) \\ \leq \quad & \rho\left(z^1 = z_2...z^\tau = z_2, z^{\tau+1} = z_1, M^{*\tau}\left(z^1 = z_2...z^\tau = z_2, z^{\tau+1} = z_1, \left(0,m_{\beta^2_u}\right)\right)\right) \end{aligned}$$

By induction, the same relation applies as long as $M^{*\tau+\kappa} = \left(M^{*\tau+1}_{\beta^1_u} + \kappa\phi_{\beta^1_u}, M^{*\tau+1}_{\beta^2_u} + \phi_{\beta^2_u}\right)$, i.e.,

$$\begin{aligned} & \rho\left(z^1 = ...z^\tau = z_2, z^{\tau+1} = ...z^{\tau+\kappa} = z_1, M^{*\tau}\left(z^1 = ...z^\tau = z_2, z^{\tau+1} = ...z^{\tau+\kappa} = z_1, \left(0,m'_{\beta^2_u}\right)\right)\right) \\ \leq \quad & \rho\left(z^1 = ...z^\tau = z_2, z^{\tau+1} = ...z^{\tau+\kappa} = z_1, M^{*\tau}\left(z^1 = ...z^\tau = z_2, z^{\tau+1} = ...z^{\tau+\kappa} = z_1, \left(0,m_{\beta^2_u}\right)\right)\right) \end{aligned}$$

Furthermore, for any sequence of realizations of z, $(z^1...z^t)$ such that: $z^1 = z_2$, and such that for any period $\tilde{\tau}$ at which $z^{\tilde{\tau}} = z_1$,

$$M^{*\tilde{\tau}}\left(z^1...z^{\tilde{\tau}} = z_1, \left(0,m'_{\beta^2_u}\right)\right) = \begin{pmatrix} M^{*\tilde{\tau}-1}_{\beta^1_u}\left(z^1...z^{\tilde{\tau}} = z_1, \left(0,m'_{\beta^2_u}\right)\right) + \phi_{\beta^1_u}, \\ M^{*\tilde{\tau}-1}_{\beta^2_u}\left(z^1...z^{\tilde{\tau}} = z_1, \left(0,m'_{\beta^2_u}\right)\right) + \phi_{\beta^2_u} \end{pmatrix} \quad (66)$$

and

$$M^{*\tilde{\tau}}\left(z^1...z^{\tilde{\tau}}=z_1,\left(0,m_{\beta_u^2}\right)\right)=\left(\begin{array}{c} M^{*\tilde{\tau}-1}_{\beta_u^1}\left(z^1...z^{\tilde{\tau}}=z_1,\left(0,m_{\beta_u^2}\right)\right)+\phi_{\beta_u^1}, \\ M^{*\tilde{\tau}-1}_{\beta_u^2}\left(z^1...z^{\tilde{\tau}}=z_1,\left(0,m_{\beta_u^2}\right)\right)+\phi_{\beta_u^2}\end{array}\right) \quad (67)$$

we can combine the two arguments above to state that:

$$\begin{aligned} & M^{*\tilde{\tau}}\left(z^1...z^{\tilde{\tau}}=z_1,\left(0,m'_{\beta_u^2}\right)\right) \\ \geq \quad & M^{*\tilde{\tau}}\left(z^1...z^{\tilde{\tau}}=z_1,\left(0,m_{\beta_u^2}\right)\right) \end{aligned} \quad (68)$$

and

$$\begin{aligned} & \rho\left(z^1=z_2,z^2...z^t,M^{*t}\left(z^1=z_2,z^2...z^t,\left(0,m'_{\beta_u^2}\right)\right)\right) \\ \leq \quad & \rho\left(z^1=z_2,z^2...z^t,M^{*t}\left(z^1=z_2,z^2...z^t,\left(0,m_{\beta_u^2}\right)\right)\right) \end{aligned} \quad (69)$$

Note that for any such $(z^1...z^t)$, we have

$$M^{*t}_{\beta_u^1}\left(z^1=z_2,z^2...z^t,\left(0,m_{\beta_u^2}\right)\right)=M^{*t}_{\beta_u^1}\left(z^1=z_2,z^2...z^t,\left(0,m'_{\beta_u^2}\right)\right) \quad (70)$$

Now we consider the first instance $\tilde{\tau}$ in which $z^{\tilde{\tau}}=z_1$ and is such that (66) is violated. (Note that (67) can only be violated if (66) is as well). By (70), we have:

$$k_2\left(M^{*\tilde{\tau}-1}_{\beta_u^1}\left(z^1=z_2...z^{\tilde{\tau}-1},\left(0,m'_{\beta_u^2}\right)\right)\right)=k_2\left(M^{*\tilde{\tau}-1}_{\beta_u^1}\left(z^1=z_2...z^{\tilde{\tau}-1},\left(0,m_{\beta_u^2}\right)\right)\right)$$

Thus, two cases are possible. Either

$$m_{\beta_u^2}+\phi_{\beta_u^2}\leq k_2\left(M^{*\tilde{\tau}-1}_{\beta_u^1}\left(z^1...z^{\tilde{\tau}-1},\left(0,m'_{\beta_u^2}\right)\right)\right)<m'_{\beta_u^2}+\phi_{\beta_u^2}, \quad (71)$$

or

$$k_2\left(M^{*\tilde{\tau}-1}_{\beta_u^1}\left(z^1...z^{\tilde{\tau}-1},\left(0,m'_{\beta_u^2}\right)\right)\right)<m_{\beta_u^2}+\phi_{\beta_u^2} \quad (72)$$

In both cases, we have that

$$M^{*\tilde{\tau}}_{\beta_u^2}\left(z^1=z_2,z^2...z^{\tilde{\tau}}=z_1,\left(0,m_{\beta_u^2}\right)\right)\leq M^{*\tilde{\tau}}_{\beta_u^2}\left(z^1=z_2,z^2...z^{\tilde{\tau}}=z_1,\left(0,m'_{\beta_u^2}\right)\right)$$

while

$$M^{*\tilde{\tau}}_{\beta_u^1}\left(z^1=z_2,z^2...z^{\tilde{\tau}}=z_1,\left(0,m_{\beta_u^2}\right)\right)=M^{*\tilde{\tau}}_{\beta_u^1}\left(z^1=z_2,z^2...z^{\tilde{\tau}}=z_1,\left(0,m'_{\beta_u^2}\right)\right)$$

and thus,

$$\begin{aligned} & \rho\left(z^1=z_2,z^2...z^{\tilde{\tau}}=z_1,M^{*\tilde{\tau}}\left(z^1=z_2,z^2...z^{\tilde{\tau}}=z_1,\left(0,m'_{\beta_u^2}\right)\right)\right) \\ \leq \quad & \rho\left(z^1=z_2,z^2...z^{\tilde{\tau}}=z_1,M^{*\tilde{\tau}}\left(z^1=z_2,z^2...z^{\tilde{\tau}}=z_1,\left(0,m_{\beta_u^2}\right)\right)\right) \end{aligned} \quad (73)$$

The same argument applied inductively gives for any κ,

$$M^{*\tilde{\tau}}_{\beta_u^1}\left(z^1=z_2,z^2...z^{\tilde{\tau}+\kappa}=z_1,\left(0,m_{\beta_u^2}\right)\right)=M^{*\tilde{\tau}}_{\beta_u^1}\left(z^1=z_2,z^2...z^{\tau}=...z^{\tilde{\tau}+\kappa}=z_1,\left(0,m'_{\beta_u^2}\right)\right)$$

$$M^{*\tilde{\tau}}_{\beta^2_u}\left(z^1 = z_2, z^2 ... = z^\tau = ... z^{\tilde{\tau}+\kappa} = z_1, \left(0, m_{\beta^2_u}\right)\right) \leq M^{*\tilde{\tau}}_{\beta^2_u}\left(z^1 = z_2, z^2 ... = z^\tau = ... z^{\tilde{\tau}+\kappa} = z_1, \left(0, m'_{\beta^2_u}\right)\right)$$

and thus,

$$\begin{aligned} & \rho\left(z^1 = z_2, z^2 ... z^{\tilde{\tau}} = ... z^{\tilde{\tau}+\kappa} = z_1, M^{*\tilde{\tau}}\left(z^1 = z_2, z^2 ... z^{\tilde{\tau}} = ... z^{\tilde{\tau}+\kappa} = z_1, \left(0, m'_{\beta^2_u}\right)\right)\right) \\ \leq \quad & \rho\left(z^1 = z_2, z^2 ... z^{\tilde{\tau}} = ... z^{\tilde{\tau}+\kappa} = z_1, M^{*\tilde{\tau}}\left(z^1 = z_2, z^2 ... z^{\tilde{\tau}} = ... z^{\tilde{\tau}+\kappa} = z_1, \left(0, m_{\beta^2_u}\right)\right)\right) \end{aligned} \tag{74}$$

Finally, adding to such a sequence a state z_2, we have:

$$\begin{aligned} & M^{*\tilde{\tau}}\left(z^1 = z_2, z^2 ... z^{\tilde{\tau}} = ... z^{\tilde{\tau}+\kappa} = z_1, z^{\tilde{\tau}+\kappa+1} = z_2 \left(0, m'_{\beta^2_u}\right)\right) \\ = \quad & \left(0, \min M^{*\tilde{\tau}+\kappa}_{\beta^2_u}\left(z^1 = z_2, z^2 ... z^\tau = ... z^{\tilde{\tau}+\kappa} = z_1, \left(0, m'_{\beta^2_u}\right) + \phi_{\beta^2_u}\right), \hat{\mu}_{\beta^2_u}(z_2)\right) \\ \geq \quad & \left(0, \min M^{*\tilde{\tau}+\kappa}_{\beta^2_u}\left(z^1 = z_2, z^2 ... z^\tau = ... z^{\tilde{\tau}+\kappa} = z_1, \left(0, m_{\beta^2_u}\right) + \phi_{\beta^2_u}\right), \hat{\mu}_{\beta^2_u}(z_2)\right) \\ = \quad & M^{*\tilde{\tau}}\left(z^1 = z_2, z^2 ... z^{\tilde{\tau}} = ... z^{\tilde{\tau}+\kappa} = z_1, z^{\tilde{\tau}+\kappa+1} = z_2 \left(0, m_{\beta^2_u}\right)\right) \end{aligned}$$

and thus,

$$\begin{aligned} & \rho\left(z^1 = z_2, z^2 ... z^{\tilde{\tau}} = ... z^{\tilde{\tau}+\kappa} = z_1, z^{\tilde{\tau}+\kappa+1} = z_2, M^{*\tilde{\tau}}\left(z^1 = z_2, z^2 ... z^{\tilde{\tau}} = ... z^{\tilde{\tau}+\kappa} = z_1, z^{\tilde{\tau}+\kappa+1} = z_2 \left(0, m'_{\beta^2_u}\right)\right)\right) \\ \leq \quad & \rho\left(z^1 = z_2, z^2 ... z^{\tilde{\tau}} = ... z^{\tilde{\tau}+\kappa} = z_1, z^{\tilde{\tau}+\kappa+1} = z_2, M^{*\tilde{\tau}}\left(z^1 = z_2, z^2 ... z^{\tilde{\tau}} = ... z^{\tilde{\tau}+\kappa} = z_1, z^{\tilde{\tau}+\kappa+1} = z_2, \left(0, m_{\beta^2_u}\right)\right)\right) \end{aligned} \tag{75}$$

Thus, starting with $m' > m$ at z_2 implies weakly lower prices on all future sequences of state realizations.

Let

$$\begin{aligned} Z^2_{2,m} &= \left\{ \begin{array}{c} \left(z^1 = z_2 ... z^\tau\right) \in \mathbb{Z}^{\mathbb{N}} \mid \text{if } z^\kappa = z_1, \text{ then} \\ k_2\left(M^{*\kappa-1}_{\beta^1_u}\left(z^1 ... z^{\kappa-1}, \left(0, m_{\beta^2_u}\right)\right)\right) > M^{*\kappa-1}_{\beta^2_u}\left(z^1 ... z^{\kappa-1}, \left(0, m_{\beta^2_u}\right)\right) + \phi_{\beta^2_u} \end{array} \right\} \\ Z^2_{2,m'} &= \left\{ \begin{array}{c} \left(z^1 = z_2 ... z^\tau\right) \in \mathbb{Z}^{\mathbb{N}} \mid \text{if } z^\kappa = z_1, \text{ then} \\ k_2\left(M^{*\kappa-1}_{\beta^1_u}\left(z^1 ... z^{\kappa-1}, \left(0, m'_{\beta^2_u}\right)\right)\right) > M^{*\kappa-1}_{\beta^2_u}\left(z^1 ... z^{\kappa-1}, \left(0, m'_{\beta^2_u}\right)\right) + \phi_{\beta^2_u} \end{array} \right\} \end{aligned}$$

be the set of all sequences of state realizations which start at z_2 and such that if the state realization is z_1, all β^2_u-type firms enter the market, respectively for an initial market composition m and m'. Clearly, $Z^2_{2,m} \supseteq Z^2_{2,m'}$. Note that by the definition of k_2, the continuation value of β^2_u-type firms is 0 after for any sequence $\left(z^1 ... z^\tau, z^{\tau+1}\right)$ such that $\left(z^1 ... z^\tau\right) \in Z^2_{2,m}$, but $\left(z^1 ... z^\tau, z^{\tau+1}\right) \notin Z^2_{2,m}$ when the initial market composition is m:

$$v^*\left(z_1, M^{*\tau}\left(z^1 ... z^\tau, \left(0, m_{\beta^2_u}\right)\right), \beta^2_u\right) = 0$$

and similarly, for any sequence $\left(z^1 ... z^\tau, z^{\tau+1}\right)$ such that $\left(z^1 ... z^\tau\right) \in Z^2_{2,m'}$, but $\left(z^1 ... z^\tau, z^{\tau+1}\right) \notin Z^2_{2,m'}$ when the initial market composition is m':

$$v^*\left(z_1, M^{*\tau}\left(z^1 ... z^\tau, \left(0, m'_{\beta^2_u}\right)\right), \beta^2_u\right) = 0$$

Now consider the continuation value of β_u^2-type firms in state z_2. This continuation value for m and for m' is given by:

$$\begin{aligned} v^*\left(z_2, m', \beta_u^2\right) &= \max\left\{\sum_{z^\tau \in Z^2_{2,m'}} q\left(z^\tau\right)\pi\left(\beta_u^2, \rho\left(z^1...z^\tau, M^{*\tilde{\tau}}\left(z^1...z^\tau, \left(0, m'_{\beta_u^2}\right)\right)\right)\right), 0\right\} \\ v^*\left(z_2, m, \beta_u^2\right) &= \max\left\{\sum_{z^\tau \in Z^2_{2,m}} q\left(z^\tau\right)\pi\left(\beta_u^2, \rho\left(z^1...z^\tau, M^{*\tilde{\tau}}\left(z^1...z^\tau, \left(0, m_{\beta_u^2}\right)\right)\right)\right), 0\right\} \end{aligned}$$

Now, because $Z^2_{2,m'} \subseteq Z^2_{2,m}$, because, as shown above, for each $z^\tau \in Z^2_{2,m'}$, we have (69), and since by Proposition 10, profits are strictly decreasing in price, we conclude that:

$$v^*\left(z_2, m', \beta_u^2\right) \leq v^*\left(z_2, m, \beta_u^2\right)$$

whenever $m' > m$.

The monotonicity of $v^*\left(z_2, m, \beta_u^2\right)$ in $m_{\beta_u^2}$ in particular implies that $k_2\left(z_1, m\right)$ is a decreasing function of $m_{\beta_u^1}$.

Next, consider the continuation value of β_u^1-type firms in state z_2. We have by (75):

$$\begin{aligned} v^*\left(z_2, m', \beta_u^1\right) &= \max\left\{\sum_{z^\tau \in \mathbb{Z}^{\mathbb{N}}} q\left(z^\tau\right)\pi\left(\beta_u^1, \rho\left(z^1...z^\tau, M^{*\tau}\left(z^1...z^\tau, \left(0, m'_{\beta_u^2}\right)\right)\right)\right), 0\right\} \\ &\leq \\ v^*\left(z_2, m, \beta_u^1\right) &= \max\left\{\sum_{z^\tau \in \mathbb{Z}^{\mathbb{N}}} q\left(z^\tau\right)\pi\left(\beta_u^1, \rho\left(z^1...z^\tau, M^{*\tau}\left(z^1...z^\tau, \left(0, m_{\beta_u^2}\right)\right)\right)\right), 0\right\} \end{aligned}$$

In particular, since condition (33) holds, we have

$$v^*\left(z_2, (0,0), \beta_u^1\right) \geq v^*\left(z_2, m, \beta_u^1\right) = 0$$

for all m.

Next consider sequence realizations of states starting at $z^1 = z_1$.

Let $m' > m$. Note that since $M^*\left(z, m\right)$ is weakly increasing in m,

$$M^{*\kappa-1}_{\beta_u^1}\left(z^1 = z_1...z^{\kappa-1}, m'\right) \geq M^{*\kappa-1}_{\beta_u^1}\left(z^1 = z_1...z^{\kappa-1}, m\right) \tag{76}$$

for any sequence realization.

Let

$$\begin{aligned} Z^1_{2,m} &= \left\{ \begin{array}{c} \left(z^1 = z_1...z^\kappa\right) \in \mathbb{Z}^{\mathbb{N}} \mid \text{if } z^\kappa = z_1 \text{, then} \\ k_2\left(M^{*\kappa-1}_{\beta_u^1}\left(z^1...z^{\kappa-1}, m\right)\right) > M^{*\kappa-1}_{\beta_u^2}\left(z^1...z^{\kappa-1}, m\right) + \phi_{\beta_u^2} \end{array} \right\} \\ Z^1_{2,m'} &= \left\{ \begin{array}{c} \left(z^1 = z_1...z^\kappa\right) \in \mathbb{Z}^{\mathbb{N}} \mid \text{if } z^\kappa = z_1 \text{, then} \\ k_2\left(M^{*\kappa-1}_{\beta_u^1}\left(z^1...z^{\kappa-1}, m'\right)\right) > M^{*\kappa-1}_{\beta_u^2}\left(z^1...z^{\kappa-1}, m'\right) + \phi_{\beta_u^2} \end{array} \right\} \end{aligned}$$

By the monotonicity of $k_2\left(\cdot\right)$, combined with (76),

$$k_2\left(M^{*\kappa-1}_{\beta_u^1}\left(z^1...z^{\kappa-1}, m\right)\right) \geq k_2\left(M^{*\kappa-1}_{\beta_u^1}\left(z^1...z^{\kappa-1}, m'\right)\right) \tag{77}$$

Thus, since $m'_{\beta_u^2} \geq m_{\beta_u^2}$, $Z_{2,m}^1 \supseteq Z_{2,m'}^1$.

Note that for all $z^\kappa \in Z_{2,m'}^1$, we have:

$$\rho\left(z^1 = z_1 ... z^\kappa, M^{*\kappa}\left(z^1...z^\kappa, m\right)\right) \geq \rho\left(z^1 = z_1...z^\kappa, M^{*\kappa}\left(z^1...z^\kappa, m'\right)\right)$$

Furthermore, if $\left(z^1...z^\tau\right) \in Z_{2,m}^1$, but $\left(z^1...z^{\tau+1} = z^1\right) \notin Z_{2,m}^1$, we have:

$$v^*\left(\beta_u^2, z^1, M^{*\kappa-1}\left(z^1...z^{\tau-1}, m\right)\right) = 0$$

We conclude that

$$\begin{aligned} v^*\left(z_1, m', \beta_u^2\right) &= \max\left\{\sum_{z^\tau \in Z_{2,m'}^1} q\left(z^\tau\right)\pi\left(\beta_u^2, \rho\left(z^1...z^\tau, M^{*\tau}\left(z^1...z^\tau, m'\right)\right)\right), 0\right\} \\ &\leq \\ v^*\left(z_1, m, \beta_u^2\right) &= \max\left\{\sum_{z^\tau \in Z_{2,m}^1} q\left(z^\tau\right)\pi\left(\beta_u^2, \rho\left(z^1...z^\tau, M^{*\tau}\left(z^1...z^\tau, m\right)\right)\right), 0\right\} \end{aligned}$$

Now consider a sequence of state realizations $\left(z^1 = z_1...z^\kappa, z^{\kappa+1} = ...z^\tau = z_1\right)$, where $\left(z^1...z^\kappa\right) \in Z_{2,m'}^1$, but $\left(z^1...z^{\kappa+1}\right) \notin Z_{2,m'}^1$. We now show that for each such sequence,

$$\rho\left(z^1...z^\tau, M^{*\tau}\left(z^1...z^\tau, m\right)\right) \geq \rho\left(z^1...z^\tau, M^{*\tau}\left(z^1...z^\tau, m'\right)\right)$$

Assume in a manner of contradiction that

$$\rho\left(z^1...z^\tau, M^{*\tau}\left(z^1...z^\tau, m\right)\right) < \rho\left(z^1...z^\tau, M^{*\tau}\left(z^1...z^\tau, m'\right)\right)$$

Therefore,

$$\pi\left(\beta_u^2, \rho\left(z^1...z^\tau, M^{*\tau}\left(z^1...z^\tau, m'\right)\right)\right) > \pi\left(\beta_u^2, \rho\left(z^1...z^\tau, M^{*\tau}\left(z^1...z^\tau, m'\right)\right)\right)$$

Note that by (77), and since as shown above, $v^*\left(z_2, m, \beta_u^2\right)$ is decreasing in $m_{\beta_u^2}$, we have

$$v^*\left(z_2, 0, k_2\left(M_{\beta_u^1}^{*\tau-1}\left(z^1...z^{\tau-1}, m\right)\right), \beta_u^2\right) \leq v_2\left(z_2, 0, k_2\left(M_{\beta_u^1}^{*\tau-1}\left(z^1...z^{\tau-1}, m'\right)\right), \beta_u^2\right)$$

It follows that

$$\begin{aligned} 0 &= \pi\left(\beta_u^1, \rho\left(z_1, \min\left\{m_{\beta_u^1} + \phi_{\beta_u^1}, \hat{\mu}_{\beta_u^1}\left(z_1\right)\right\}, k_2\left(z_1, m\right)\right)\right) - c + \delta\left(1-q\right)v^*\left(z_2, 0, k_2\left(z_1, m\right), \beta_u^2\right) \\ &> \pi\left(\beta_u^1, \rho\left(z_1, \min\left\{\tilde{m}_{\beta_u^1} + \phi_{\beta_u^1}, \hat{\mu}_{\beta_u^1}\left(z_1\right)\right\}, k_2\left(z_1, \tilde{m}\right)\right)\right) - c + \delta\left(1-q\right)v^*\left(z_2, 0, k_2\left(z_1, \tilde{m}\right), \beta_u^2\right) \geq 0 \end{aligned}$$

a contradiction. Thus, the equilibrium price in state z_1 is decreasing in m.

Since, by definition, $v^*\left(z_2, m, \beta_u^1\right) = 0$, it follows that

$$\begin{aligned} v^*\left(z_1, m', \beta_u^2\right) &= \max\left\{\sum_{\left\{\left(z^\tau, z^{\tau+1} = ... z^{\tau+\kappa} = z_1\right) | z^\tau \in Z_{2,m'}^1\right\}} q\left(z^\tau\right)\pi\left(\beta_u^1, \rho\left(z^1...z^{\tau+\kappa}, M^{*\tau}\left(z^1...z^{\tau+\kappa}, m'\right)\right)\right), 0\right\} \\ &\leq \\ v^*\left(z_1, m, \beta_u^2\right) &= \max\left\{\sum_{\left\{\left(z^\tau, z^{\tau+1} = ... z^{\tau+\kappa} = z_1\right) | z^\tau \in Z_{2,m}^1\right\}} q\left(z^\tau\right)\pi\left(\beta_u^1, \rho\left(z^1...z^{\tau+\kappa}, M^{*\tau}\left(z^1...z^{\tau+\kappa}, m\right)\right)\right), 0\right\} \end{aligned}$$

or, $v^*(z_1, m, \beta_u^1)$ is decreasing in m.

To show that the so-defined M^* constitutes a Markov equilibrium, we have to show that expectations are consistent: optimal entry decisions (18) based on the continuation values v^* determined from M^* as in (19) and satisfying the conditions of Definition 17 induce the market composition mapping M^* as in (20). This is equivalent to showing that:

$$M^*_{\beta_u^i}(z,m) = \begin{cases} m_{\beta_u^i} + \phi_{\beta_u^i} & \text{if } \pi\left(\beta_u^i, \rho\left(z, M^*(z,m)\right)\right) - c + \delta E_q v^*\left(z, M^*(z,m), \beta_u^i\right) > 0 \\ \in \left[0, m_{\beta_u^i} + \phi_{\beta_u^i}\right] & \text{if } \pi\left(\beta_u^i, \rho\left(z, M^*(z,m)\right)\right) - c + \delta E_q v^*\left(z, M^*(z,m), \beta_u^i\right) = 0 \\ 0 & \text{if } \pi\left(\beta_u^i, \rho\left(z, M^*(z,m)\right)\right) - c + \delta E_q v^*\left(z, M^*(z,m), \beta_u^i\right) < 0 \end{cases}$$

We first consider z_2. Take β_u^1-firms. Note that at $m = (0,0)$,

$$\begin{aligned} v^*\left(z_2, (0,0), \beta_u^1\right) \quad = \quad & \max\{\pi\left(\beta_u^1, \rho\left(z_2, M^*(z_2, 0, 0)\right)\right) - c + \\ & +\delta q v^*\left(z_1, M^*(z_2, 0, 0), \beta_u^1\right) + \\ & +\delta(1-q) v^*\left(z_2, M^*(z_2, 0, 0), \beta_u^1\right), 0\} \end{aligned}$$

and since $v^*(z_2, m)$ is non-increasing in m and $M^*(z_2, 0, 0) = \left(0, \phi_{\beta_u^2}\right) > (0,0)$, we have that

$$v^*\left(z_2, (0,0), \beta_u^1\right) \geq v^*\left(z_2, M^*(z_2, 0, 0), \beta_u^1\right)$$

Thus, if

$$\pi\left(\beta_u^1, \rho\left(z_2, M^*(z_2, 0, 0)\right)\right) - c + \delta q v^*\left(z_1, M^*(z_2, 0, 0), \beta_u^1\right) \leq 0$$

we obtain:

$$v^*\left(z_2, (0,0), \beta_u^1\right) = v^*\left(z_2, M^*(z_2, 0, 0), \beta_u^1\right) = 0$$

Similarly, since $M^*(z_2, m) = \left(0, \min\left\{\hat{\mu}_{\beta_u^2}(z_2), m_{\beta_u^2} + \phi_{\beta_u^2}\right\}\right)$, we have that $M^*(z_2, m) > M^*(z_2, 0, 0)$, it follows that

$$v^*\left(z_2, M^*(z_2, m), \beta_u^1\right) \leq v^*\left(z_2, (0,0), \beta_u^1\right) = 0$$

We conclude that if condition (33) is satisfied,

$$\begin{aligned} \pi\left(\beta_u^1, \rho\left(z, M^*(z_2, m)\right)\right) - c + \delta E_q v^*\left(z, M^*(z,m), \beta_u^1\right) \quad & \leq \\ \pi\left(\beta_u^1, \rho\left(z, M^*\left(z_2, 0, \phi_{\beta_u^2}\right)\right)\right) - c + \delta v^*\left(z_1, M^*(z,m), \beta_u^1\right) \quad & \leq \quad 0 \end{aligned}$$

and thus,

$$M^*\left(z_2, m, \beta_u^1\right) = 0$$

guarantees that expectations are consistent for β_u^1-type firms in state z_2.

Next consider β_u^2-firms in state z_2. Note that at $m = \left(m_{\beta_u^1}, \hat{\mu}_{\beta_u^2}(z_2)\right)$,

$$\begin{aligned} v^*\left(z_2, m, \beta_u^2\right) \quad = \quad & \max\{\pi\left(\beta_u^2, \rho\left(z_2, M^*\left(z_2, m_{\beta_u^1}, \hat{\mu}_{\beta_u^2}(z_2)\right)\right)\right) - c + \\ & +\delta q v^*\left(z_1, M^*\left(z_2, m_{\beta_u^1}, \hat{\mu}_{\beta_u^2}(z_2)\right), \beta_u^2\right) + \\ & +\delta(1-q) v^*\left(z_2, M^*\left(z_2, m_{\beta_u^1}, \hat{\mu}_{\beta_u^2}(z_2)\right), \beta_u^2\right), 0\} \end{aligned}$$

$$\begin{aligned} = \quad & \max\{\pi\left(\beta_u^2, \rho\left(0, \hat{\mu}_{\beta_u^2}(z_2)\right)\right) - c + \\ & +\delta q v^*\left(z_1, \left(0, \hat{\mu}_{\beta_u^2}(z_2)\right), \beta_u^2\right) + \\ & +\delta(1-q) v^*\left(z_2, \left(0, \hat{\mu}_{\beta_u^2}(z_2)\right), \beta_u^2\right), 0\} \end{aligned}$$

Since

$$M^*\left(z_2, m_{\beta_u^1}, m_{\beta_u^2}\right) = M^*\left(z_2, m_{\beta_u^1}, \hat{\mu}_{\beta_u^2}(z_2)\right) = \left(0, \hat{\mu}_{\beta_u^2}(z_2)\right)$$

for any $m_{\beta_u^2} \geq \hat{\mu}_{\beta_u^2}(z_2) - \phi_{\beta_u^2}$, we have:

$$v^*\left(z_2, \left(0, \hat{\mu}_{\beta_u^2}(z_2)\right), \beta_u^2\right) = v^*\left(z_2, m, \beta_u^2\right)$$

Thus, whenever condition (35) is satisfied at $\mu = \hat{\mu}_{\beta_u^2}(z_2)$, we have that:

$$v^*\left(z_2, m, \beta_u^2\right) = v^*\left(z_2, \left(0, \hat{\mu}_{\beta_u^2}(z_2)\right), \beta_u^2\right) = 0$$

for any $m \geq \left(0, \hat{\mu}_{\beta_u^2}(z_2) - \phi_{\beta_u^2}\right)$. Thus, at $m \geq \left(0, \hat{\mu}_{\beta_u^2}(z_2) - \phi_{\beta_u^2}\right)$, condition (35) implies

$$\pi\left(\beta_u^2, \rho\left(z, M^*\left(z_2, m\right)\right)\right) - c + \delta E_q v^*\left(z, M^*\left(z_2, m\right), \beta_u^2\right) = 0$$

and thus, $M^*_{\beta_u^2}(z_2, m) = \hat{\mu}_{\beta_u^2}(z_2)$ guarantees that expectations are consistent for β_u^2-type firms in state z_2 as long as $m \geq \left(0, \hat{\mu}_{\beta_u^2}(z_2) - \phi_{\beta_u^2}\right)$.

Since by Proposition 10, ρ and π are strictly decreasing in m on the relevant price range, by Lemma 36 v^* is decreasing in m, and since $M^*_{\beta_u^1}(z_1, m) = 0$ for any m, we furthermore have for $m_{\beta_u^2} < \hat{\mu}_{\beta_u^2}(z_2) - \phi_{\beta_u^2}$:

$$\begin{aligned} & \pi\left(\beta_u^2, \rho\left(z_2, M^*\left(z_2, m\right)\right)\right) - c + \delta q v^*\left(z_1, M^*\left(z_2, m\right), \beta_u^2\right) \\ > \quad & \pi\left(\beta_u^2, \rho\left(z_2, M^*\left(z_2, \left(0, \hat{\mu}_{\beta_u^2}(z_2)\right)\right)\right)\right) - c + \delta q v^*\left(z_1, M^*\left(z_2, \left(0, \hat{\mu}_{\beta_u^2}(z_2)\right)\right), \beta_u^2\right) \end{aligned}$$

It follows that for $m_{\beta_u^2} < \hat{\mu}_{\beta_u^2}(z_2)$,

$$\pi\left(\beta_u^2, \rho\left(z_2, M^*\left(z_2, m\right)\right)\right) - c + \delta q v^*\left(z_1, M^*\left(z_2, m\right), \beta_u^2\right) > 0$$

and thus,

$$\pi\left(\beta_u^2, \rho\left(z_2, M^*\left(z_2, m\right)\right)\right) - c + \delta E_q v^*\left(z, M^*\left(z_2, m\right), \beta_u^2\right) > 0$$

Therefore, $M^*_{\beta_u^2}(z_2, m) = \min\left\{m_{\beta_u^2} + \phi_{\beta_u^2}, \hat{\mu}_{\beta_u^2}(z_2)\right\}$ guarantees that expectations are consistent for β_u^2-type firms in state z_2 and any m.

Next consider state z_1 and type β_u^1. We already showed that $v^*\left(z_2, m, \beta_u^1\right) = 0$ for any m. Thus,

$$v^*\left(z_1, m, \beta_u^1\right) = \max\left\{\pi\left(\beta_u^1, \rho\left(z_1, M^*\left(z_1, m\right)\right)\right) - c + \delta q v^*\left(z_1, M^*\left(z_1, m\right), \beta_u^2\right), 0\right\} \quad (78)$$

From the proof of Lemma 36, we know that

$$\rho\left(z_1, M^*_{\beta_u^1}(z_1, m), k_2(z_1, m)\right) \leq \rho\left(z_1, M^*_{\beta_u^1}(z_1, \tilde{m}), k_2(z_1, \tilde{m})\right) \quad (79)$$

whenever $\tilde{m} \leq m$ with $\tilde{m}_{\beta_u^1} < m_{\beta_u^1} < \hat{\mu}_{\beta_u^1}$.

Since by Lemma 36 $v^*(z_1, m, \beta_u^1)$ is weakly decreasing in $m_{\beta_u^1}$, $v^*(z_2, m, \beta_u^1) = 0$, and by (78), we have that at $m = \left(\hat{\mu}_{\beta_u^1}(z_1), \check{\mu}_{\beta_u^2}(z_1)\right)$, $M^*(z_1, m) = m$ and, since $\rho(z_1, m) = r^c_{\beta_u^1}$, $\pi(\beta_u^1, \rho(z_1, m)) = c$ and thus, $v^*(z_1, m, \beta_u^1) = 0$. Thus, at m, $M^*_{\beta_u^1}(z_1, m) = \hat{\mu}_{\beta_u^1}(z_1)$ guarantees that expectations are consistent for β_u^1-firms at state z_1. Furthermore, since as shown above, $\rho\left(z_1, M^*_{\beta_u^1}(z_1, m), k_2(z_1, m)\right)$ is decreasing in $m_{\beta_u^1}$, and profits are increasing in price, we have that

$$\begin{aligned} \pi\left(\beta_u^1, \rho\left(z_1, M^*(z_1, m)\right)\right) - c &\geq 0 \\ v^*\left(z_1, M^*(z_1, m), \beta_u^1\right) &\geq v^*\left(z_1, \left(\hat{\mu}_{\beta_u^1}(z_1), \check{\mu}_{\beta_u^2}(z_1)\right), \beta_u^1\right) \end{aligned}$$

whenever $m_{\beta_u^1} \leq \hat{\mu}_{\beta_u^1}(z_1)$. Finally, since $M^*(z_1, m) = \left(\hat{\mu}_{\beta_u^1}(z_1), \check{\mu}_{\beta_u^2}(z_1)\right)$ whenever $m_{\beta_u^1} > \hat{\mu}_{\beta_u^1}(z_1)$, we have

$$\begin{aligned} \pi\left(\beta_u^1, \rho\left(z_1, M^*(z_1, m)\right)\right) - c &= 0 \\ v^*\left(z_1, M^*(z_1, m), \beta_u^1\right) &= v^*\left(z_1, \left(\hat{\mu}_{\beta_u^1}(z_1), \check{\mu}_{\beta_u^2}(z_1)\right), \beta_u^1\right) \end{aligned}$$

This in turn implies that at any m, $M^*_{\beta_u^1}(z_1, m) = \min\left\{m_{\beta_u^1} + \phi_{\beta_u^1}\hat{\mu}_{\beta_u^1}(z_1)\right\}$ guarantees consistency of expectations for β_u^1-type firms in state z_1.

Finally, consider β_u^2-firms in state z_2. As explained above, $k_2(z_1, m)$ as well as $\rho(z_1, M^*(z_1, m))$ are decreasing in $m_{\beta_u^1}$. Thus, we can consider two cases. First, if for some m,

$$\pi\left(\beta_u^2, \rho\left(z_1, M^*_{\beta_u^1}(m), m_{\beta_u^2} + \phi_{\beta_u^2}\right)\right) \geq c$$

we have that the r.h.s. of (34) is strictly positive at $k_2(z_1, m) = m_{\beta_u^2} + \phi_{\beta_u^2}$, and thus, $k_2(z_1, m) = m_{\beta_u^2} + \phi_{\beta_u^2}$. In turn, the r.h.s. (34) is a lower bound for the discounted expected profits of β_u^2-firms (for the case when $v^*(z_1, M^*(z_1, m), \beta_u^2) = 0$) and since this term is non-negative, we have that the entry-decision given by $M^*_{\beta_u^2}(z_1, m) = k_2(z_1, m)$ ensures consistency of expectations for β_u^2-type firms in state z_1 for this case.

Second, suppose that for a given m,

$$\pi\left(\beta_u^2, \rho\left(z_1, M^*_{\beta_u^1}(m), m_{\beta_u^2} + \phi_{\beta_u^2}\right)\right) < c$$

This, in turn implies that

$$\pi\left(\beta_u^2, \rho\left(z_1, M^*_{\beta_u^1}(m), k_2(z_1, m)\right)\right) \leq c$$

(Indeed, if the reverse were true, then the r.h.s. of (34) would be strictly positive, in contradiction to the definition of $k_2(z_1, m)$). As shown above, this implies that

$$\rho\left(z_1 ... z_1, M^{*t}\left(M^*_{\beta_u^1}(m), k_2(z_1, m)\right)\right) \leq r^c_{\beta_u^2}$$

for any t and any consequent t-period realization of state z_1. Thus,

$$v\left(z_1, M^*_{\beta_u^1}(m), k_2(z_1, m), \beta_u^2\right) = 0$$

and the r.h.s. of (34) represents the expected discounted profits of β_u^2-type firms in state z_1 given m. Thus, setting $M^*_{\beta_u^2}(z_1, m) = k_2(z_1, m)$ ensures consistency of expectations for β_u^2-types firms in state z_1 also for this case.

We conclude that M^* and v^* constitute a Markovian equilibrium of the economy.

To show that an invariant distribution exists, we first show that both states $\left(z_1, \hat{\mu}_{\beta_u^1}(z_1), \check{\mu}_{\beta_u^2}(z_1)\right)$ and $\left(z_2, 0, \hat{\mu}_{\beta_u^2}(z_2)\right)$ are accessible atoms of the Markov chain on $\mathbb{Z} \times \mathbb{M}$ defined by q and M^*. Denote this Markov chain by Φ and let $\tilde{Q}$ denote its one-step ahead transition probability:

$$\tilde{Q}\left((z', m') \mid (z, m)\right) = \begin{cases} q & \text{for } z' = z_1,\ m' = M^*(z_1, m) \\ (1-q) & \text{for } z' = z_2,\ m' = M^*(z_2, m) \end{cases}$$

Take any initial state (z, m) and consider a sequence of t repetitions of state z_2. We have shown above that for $t \geq \frac{\hat{\mu}_{\beta_u^1}(z_1)}{\phi_{\beta_u^1}}$, $M^{*t}(z_1 ... z_1, m) = \left(\hat{\mu}_{\beta_u^1}(z_1), \check{\mu}_{\beta_u^2}(z_1)\right)$ obtains. Since a sequence of $\left\lceil \frac{\hat{\mu}_{\beta_u^1}(z_1)}{\phi_{\beta_u^1}} \right\rceil$ repetitions of state z_1 occurs a.s. in finite time on the set of paths of the process, we conclude that starting from any initial state, the Markov chain reaches $\left(z_1, \hat{\mu}_{\beta_u^1}(z_1), \check{\mu}_{\beta_u^2}(z_1)\right)$ a.s. in finite time. Furthermore, this state is reached with strictly positive probability of $q^{\left\lceil \frac{\hat{\mu}_{\beta_u^1}(z_1)}{\phi_{\beta_u^1}} \right\rceil}$ in $\left\lceil \frac{\hat{\mu}_{\beta_u^1}(z_1)}{\phi_{\beta_u^1}} \right\rceil$ periods. This implies that $\left(z_1, \hat{\mu}_{\beta_u^1}(z_1), \check{\mu}_{\beta_u^2}(z_1)\right)$ is an accessible atom of the chain.

Similarly, for any initial state (z, m), consider a sequence of t repetitions of state z_1. For $t \geq \frac{\hat{\mu}_{\beta_u^2}(z_2)}{\phi_{\beta_u^2}}$, $M^{*t}(z_2 ... z_2, m) = \left(0, \hat{\mu}_{\beta_u^2}(z_2)\right)$ obtains. Since a sequence of $\left\lceil \frac{\hat{\mu}_{\beta_u^2}(z_2)}{\phi_{\beta_u^2}} \right\rceil$ repetitions of state z_2 occurs a.s. in finite time on the set of paths of the process, we conclude that starting from any initial state, the Markov chain reaches $\left(z_2, 0, \hat{\mu}_{\beta_u^2}(z_2)\right)$ a.s. in finite time. Furthermore, this state is reached with strictly positive probability of $(1-q)^{\left\lceil \frac{\hat{\mu}_{\beta_u^2}(z_2)}{\phi_{\beta_u^2}} \right\rceil}$ in $\left\lceil \frac{\hat{\mu}_{\beta_u^2}(z_2)}{\phi_{\beta_u^2}} \right\rceil$ periods.

It follows that the Markov chain Φ satisfies Doeblin's condition, Stokey and Lucas (1989, p. 345). In particular, set

$$\psi(B) = \begin{cases} 1 & \text{if } \left(z_1, \hat{\mu}_{\beta_u^1}(z_1), \check{\mu}_{\beta_u^2}(z_1)\right) \in B,\ \left(z_2, 0, \hat{\mu}_{\beta_u^2}(z_2)\right) \in B \\ q & \text{if } \left(z_1, \hat{\mu}_{\beta_u^1}(z_1), \check{\mu}_{\beta_u^2}(z_1)\right) \in B,\ \left(z_2, 0, \hat{\mu}_{\beta_u^2}(z_2)\right) \notin B \\ 1-q & \text{if } \left(z_1, \hat{\mu}_{\beta_u^1}(z_1), \check{\mu}_{\beta_u^2}(z_1)\right) \notin B,\ \left(z_2, 0, \hat{\mu}_{\beta_u^2}(z_2)\right) \in B \\ 0 & \text{if } \left(z_1, \hat{\mu}_{\beta_u^1}(z_1), \check{\mu}_{\beta_u^2}(z_1)\right) \notin B,\ \left(z_2, 0, \hat{\mu}_{\beta_u^2}(z_2)\right) \notin B \end{cases}$$

Let $\tilde{\epsilon} = \min\left\{ q^{\left\lceil \frac{\hat{\mu}_{\beta_u^1}(z_1)}{\phi_{\beta_u^1}} \right\rceil + 1}, 1-q \right\}$, $\tilde{N} = \left\lceil \frac{\hat{\mu}_{\beta_u^1}(z_1)}{\phi_{\beta_u^1}} \right\rceil + 1$ and note that $\psi(B) < \tilde{\epsilon}$ implies $\left(z_1, \hat{\mu}_{\beta_u^1}(z_1), \check{\mu}_{\beta_u^2}(z_1)\right) \notin B$. Furthermore, as shown above, we have

$$\tilde{Q}^{\tilde{N}}\left(\left(z_1, \hat{\mu}_{\beta_u^1}(z_1), \check{\mu}_{\beta_u^2}(z_1)\right) \mid (z', m')\right) \geq \tilde{\epsilon}$$

for any (z', m'), and thus,

$$\tilde{Q}^{\tilde{N}}\left(B \mid (z', m')\right) \leq 1 - \tilde{\epsilon}$$

for any B such that $\psi(B) < \tilde{\epsilon}$. It follows that the measure ψ satisfies the Doeblin condition for the Markov chain Φ.

Note, furthermore, that for any B with $\psi(B) > 0$, and any $(z, m) \in \mathbb{Z} \times \mathbb{M}$, $P^n(B \mid z, m) > 0$ where n is chosen to be $\left\lceil \frac{\hat{\mu}_{\beta_u^1}(z_1)}{\phi_{\beta_u^1}} \right\rceil$ if $\left(z_1, \hat{\mu}_{\beta_u^1}(z_1), \check{\mu}_{\beta_u^2}(z_1)\right) \in B$ and $\left\lceil \frac{\hat{\mu}_{\beta_u^2}(z_2)}{\phi_{\beta_u^2}} \right\rceil$ if $\left(z_2, 0, \hat{\mu}_{\beta_u^2}(z_2)\right) \in B$ (and the minimum of the two if both states are in B). Thus, the Markov chain Φ together with the measure ψ satisfies the conditions of Theorem 11.10 in Stokey and Lucas (1989, p. 348). We conclude that Φ has a unique ergodic set (this is the set of all states $(z, m) \in \mathbb{Z} \times \mathbb{M}$ which are reachable from $\left(z_1, \hat{\mu}_{\beta_u^1}(z_1), \check{\mu}_{\beta_u^2}(z_1)\right)$) and a unique invariant measure Q.

Furthermore, we can verify that the Markov chain Φ satisfies condition M in Stokey and Lucas (1989, p. 348):

Condition M, Stokey and Lucas, 1989 There exists an $\epsilon > 0$ and $N \geq 1$ such that for any $B \subseteq \mathbb{Z} \times \mathbb{M}$, either $\tilde{Q}^N(B \mid (z, m)) \geq \epsilon$ for all $(z, m) \in \mathbb{Z} \times \mathbb{M}$, or $\tilde{Q}^N(\neg B \mid (z, m)) \geq \epsilon$ for all $(z, m) \in \mathbb{Z} \times \mathbb{M}$.

To see that this condition holds, set $\epsilon = q^{\left\lceil \frac{\hat{\mu}_{\beta_u^1}(z_1)}{\phi_{\beta_u^1}} \right\rceil}$, $N = \left\lceil \frac{\hat{\mu}_{\beta_u^1}(z_1)}{\phi_{\beta_u^1}} \right\rceil$ and note that we have:

$$\tilde{Q}^N(B \mid (z, m)) \geq \epsilon$$

whenever $\left(z_1, \hat{\mu}_{\beta_u^1}(z_1), \check{\mu}_{\beta_u^2}(z_1)\right) \in B$ and

$$\tilde{Q}^N(\neg B \mid (z, m)) \geq \epsilon$$

if $\left(z_1, \hat{\mu}_{\beta_u^1}(z_1), \check{\mu}_{\beta_u^2}(z_1)\right) \notin B$, which implies the condition.

We can thus invoke Theorem 11.2 in Stokey and Lucas (1989, p. 350) which implies strong convergence of the process to the invariant measure Q.

Proof of Proposition 20:

By Example **??**, the overemployment factors for the Cobb-Douglas production function, $\gamma_i(\alpha, \beta_u, \bar{\beta}_s)$ do not depend on the price of output. Furthermore, by Example 4, condition (41) implies (38) and thus, (39). Recall that by the Definition 14 of a regular Markovian equilibrium, in a recession, z_2, only β_u^1-type firms are present in the market, whereas β_u^2-type firms enter during a boom, z_1. Thus, by (40), the average labor wedge for type i labor in state z_2 is given by $\tau^i(\beta_u^1)$, whereas in the initial periods of a boom, z_1, when both firms have a strictly positive mass, it is strictly between $\tau^i(\beta_u^1)$ and $\tau^i(\beta_u^2)$. We conclude that the according to the invariant distribution Q derived in Proposition 15, the labor wedge for skilled labor satisfies:

$$\tau_t^s(\sigma_0) > \sum_{\sigma \in \{\sigma_1 \ldots \sigma_n\}} Q(\sigma)\, \tau_t^s(\sigma)$$

whereas the labor wedge for unskilled labor satisfies:

$$\tau_t^u(\sigma_0) < \sum_{\sigma \in \{\sigma_1 \ldots \sigma_n\}} Q(\sigma) \tau_t^u(\sigma)$$

Proof of Proposition 21:
Recall that the overemployment factor for labor type $i \in \{u, s\}$ is

$$\gamma_i(\beta, p) = \frac{\bar{w}_i}{p \frac{\partial f(\ell_u(\beta,p), \ell_s(\beta,p))}{\partial \ell_i}}$$

Clearly, $\gamma_i(\beta, p)$ is increasing in p iff $p\frac{\partial f(\ell_u(\beta,p),\ell_s(\beta,p))}{\partial \ell_i}$ is decreasing in p. In particular, we know from Proposition 34 that for the generalized CES-function satisfying the conditions of Proposition **21**, $p\frac{\partial f(\ell_u(\beta,p),\ell_s(\beta,p))}{\partial \ell_u}$ is increasing in p and thus, $\gamma_u(\beta, p)$ is decreasing in p. Since $\tau^u(\beta, p) = 1 - \gamma_u(\beta, p)$, we obtain the result of the Proposition for τ^u and γ_u.

To show the result for γ_s, we proceed in two steps. First, we formulate a condition (similar to Condition B) which implies that $\gamma_s(\beta, p)$ is increasing in p. We then show that this condition is indeed satisfied under the conditions of the Proposition.

Condition B-S: Suppose that marginal productivity of skilled labor s satisfies for some $\psi < 0$ for any $p \in \mathbb{R}^+$

$$\frac{\partial f(p\ell_u, p\ell_s)}{\partial \ell_s} \leq p^{\psi} \frac{\partial f(\ell_u, \ell_s)}{\partial \ell_s}$$

Furthermore, let optimal demand for the two types of labor satisfy for any $p \in \mathbb{R}^+$

$$\begin{aligned} \ell_u^*(p, \beta_u) &= p^{\chi_u(p)} \ell_u^*(1, \beta_u) \\ \ell_s^*(p, \beta_u) &= p^{\chi_s(p)} \ell_s^*(1, \beta_u) \end{aligned}$$

for some $\chi_s(p) \geq \chi_u(p) > 0$ such that $\max_{p \in \mathbb{R}^+} \chi_u(p) \psi < -1$.

Proposition 37 *If Condition B Decreasing optimal bargaining power is satisfied then* $\gamma_s(\beta, p)$ *is increasing in p and thus,* $\tau^s(\beta, p)$ *is decreasing in p (countercyclical).*

Proof of Proposition 37:
If Condition B-S is satisfied, then there are ψ and for any $p \in \mathbb{R}^+$, $\chi_s(p) \geq \chi_u(p) > 0$ such that:

$$\begin{aligned} \ell_u^*(p, \beta_u) &= p^{\chi_u(p)} \ell_u^*(1, \beta_u) \\ \ell_s^*(p, \beta_u) &= p^{\chi_s(p)} \ell_s^*(1, \beta_u) \end{aligned}$$

and such that $\max_{p\in\mathbb{R}^+} \chi_u(p)\psi < -1$. Thus, for any $p > 1$,

$$\begin{aligned}\frac{\partial f\left(\ell_u^*(p,\beta_u),\ell_s^*(p,\beta_u)\right)}{\partial \ell_s} &= \frac{\partial f\left(p^{\chi_u(p)}\ell_u^*(1,\beta_u),p^{\chi_s(p)}\ell_s^*(1,\beta_u)\right)}{\partial \ell_s} \\ &\leq \frac{\partial f\left(p^{\chi_u(p)}\ell_u^*(1,\beta_u),p^{\chi_u(p)}\ell_s^*(1,\beta_u)\right)}{\partial \ell_s} \\ &\leq p^{\chi_u(p)\psi}\frac{\partial f\left(\ell_u^*(1,\beta_u),\ell_s^*(1,\beta_u)\right)}{\partial \ell_s} \\ &\leq p^{\max\chi_u(p)\psi}\frac{\partial f\left(\ell_u^*(1,\beta_u),\ell_s^*(1,\beta_u)\right)}{\partial \ell_s} \\ &< \frac{1}{p}\frac{\partial f\left(\ell_u^*(1,\beta_u),\ell_s^*(1,\beta_u)\right)}{\partial \ell_s}\end{aligned}$$

as required.

We can now complete the proof of Proposition 21. By Proposition 32, we can set $\psi = -1 - \frac{1}{\alpha_s}$ to obtain

$$\frac{\partial f(p\ell_u,p\ell_s)}{\partial \ell_s} \leq p^{\psi}\frac{\partial f(\ell_u,\ell_s)}{\partial \ell_s} = p^{-1-\frac{1}{\alpha_s}}\frac{\partial f(\ell_u,\ell_s)}{\partial \ell_u}$$

By Proposition 33, whenever the parameters satisfy the conditions of Proposition 21, optimal demand for the two types of labor satisfies for any $p \in \mathbb{R}^+$

$$\begin{aligned}\ell_u^*(p,\beta_u) &= p^{\chi_u(p)}\ell_u^*(1,\beta_u) \\ \ell_s^*(p,\beta_u) &= p^{\chi_s(p)}\ell_s^*(1,\beta_u)\end{aligned}$$

for some $\chi_s(p) \geq \chi_u(p) > 0$ such that $\chi_i(p) \in \left[\frac{\alpha_s}{1+\alpha_s},\frac{\alpha_u}{1+\alpha_u}\right]$, $i \in \{u,s\}$ and $p \in \mathbb{R}^+$. Thus,

$$\max_{p\in\mathbb{R}^+}\chi_u(p)\psi = \max_{\chi_u\in\left[\frac{\alpha_s}{1+\alpha_s},\frac{\alpha_u}{1+\alpha_u}\right]}\chi_u\psi = -\frac{\alpha_u}{1+\alpha_u}\frac{\alpha_s+1}{\alpha_s}$$

We know that

$$\begin{aligned}\frac{\alpha_u}{1+\alpha_u} &\geq \frac{\alpha_s}{1+\alpha_s}, \text{ or} \\ \frac{\alpha_u}{1+\alpha_u}\frac{\alpha_s+1}{\alpha_s} &\geq 1\end{aligned}$$

and thus,

$$\max_{p\in\mathbb{R}^+}\chi_u(p)\psi = \max_{\chi_u\in\left[\frac{\alpha_s}{1+\alpha_s},\frac{\alpha_u}{1+\alpha_u}\right]}\chi_u\psi = -\frac{\alpha_s}{1+\alpha_s}\frac{1+\alpha_u}{\alpha_u} \leq -1$$

We, therefore, obtain

$$\begin{aligned}\frac{\partial f\left(\ell_u^*(p,\beta_u),\ell_s^*(p,\beta_u)\right)}{\partial \ell_s} &= \frac{\partial f\left(p^{\chi_u(p)}\ell_u^*(1,\beta_u),p^{\chi_s(p)}\ell_s^*(1,\beta_u)\right)}{\partial \ell_s} \\ &\leq \frac{\partial f\left(p^{\chi_u(p)}\ell_u^*(1,\beta_u),p^{\chi_u(p)}\ell_s^*(1,\beta_u)\right)}{\partial \ell_s} \\ &\leq p^{\chi_u(p)\psi}\frac{\partial f\left(\ell_u^*(1,\beta_u),\ell_s^*(1,\beta_u)\right)}{\partial \ell_s} \\ &\leq p^{\max\chi_u(p)\psi}\frac{\partial f\left(\ell_u^*(1,\beta_u),\ell_s^*(1,\beta_u)\right)}{\partial \ell_s} \\ &\leq \frac{1}{p}\frac{\partial f\left(\ell_u^*(1,\beta_u),\ell_s^*(1,\beta_u)\right)}{\partial \ell_s}\end{aligned}$$

with strict inequality, whenever $\alpha_s > \alpha_u$.

Similarly to the proof of Proposition 34, we can replace $p_0 = 1$ by an arbitrary $p_0 < p$ and obtain the same result, obtaining that $p\frac{\partial f(\ell_u(\beta,p),\ell_s(\beta,p))}{\partial \ell_u}$ is decreasing in p and thus, $\gamma_u(\beta,p)$ is increasing in p implying that $\tau^s(\beta,p)$ is countercyclical.

8 Appendix B

Derivations for Example 4:

From equation (9) in Cahuc et al. (2008), the wage of a type i worker given a hiring decision (ℓ_s, ℓ_u) at a price p of the output is:

$$\begin{aligned} w_i(\ell_s,\ell_u) &= (1-\beta_i)\bar{w}_i + p\beta_i \int_0^1 \frac{\alpha_i}{\beta_i} z^{\frac{1-\beta_i}{\beta_i}} (z\ell_i)^{\alpha_i - 1}\left(z^{\frac{\beta_{-i}}{1-\beta_{-i}}\frac{1-\beta_i}{\beta_i}}\ell_{-i}\right)^{\alpha_{-i}} dz \\ &= (1-\beta_i)\bar{w}_i + p\alpha_i\beta_i \ell_i^{\alpha_i-1}\ell_{-i}^{\alpha_{-i}} \int_0^1 \frac{1}{\beta_i} z^{\frac{1-\beta_i}{\beta_i}+\alpha_i-1+\alpha_{-i}\frac{\beta_{-i}}{1-\beta_{-i}}\frac{1-\beta_i}{\beta_i}} dz \end{aligned}$$

Integrating, one obtains:

$$\begin{aligned} & \int_0^1 \frac{1}{\beta_i} z^{\frac{1-\beta_i}{\beta_i}+\alpha_i-1+\alpha_{-i}\frac{\beta_{-i}}{1-\beta_{-i}}\frac{1-\beta_i}{\beta_i}} dz \\ = & \left. \frac{z^{\frac{1-\beta_i}{\beta_i}+\alpha_i+\alpha_{-i}\frac{\beta_{-i}}{1-\beta_{-i}}\frac{1-\beta_i}{\beta_i}}}{\beta_i\left(\frac{1-\beta_i}{\beta_i}+\alpha_i+\alpha_{-i}\frac{\beta_{-i}}{1-\beta_{-i}}\frac{1-\beta_i}{\beta_i}\right)}\right|_0^1 \\ = & \frac{1-\beta_{-i}}{1+(1-\alpha_i-\alpha_{-i})\beta_i\beta_{-i}-(1-\alpha_i)\beta_i-(1-\alpha_{-i})\beta_{-i}} \end{aligned}$$

Thus, the stable wage schedule for the Cobb-Douglas case is given by:

$$w_i(\ell_s,\ell_u) = (1-\beta_i)\bar{w}_i + \beta_i\gamma_i(\alpha,\beta)p\frac{\alpha_i \ell_i^{\alpha_i}\ell_{-i}^{\alpha_{-i}}}{\ell_i} = (1-\beta_i)\bar{w}_i + \beta_i\gamma_i(\alpha,\beta)p\frac{\partial f(\ell_s,\ell_u)}{\partial \ell_i} \tag{80}$$

where,

$$\begin{aligned} \gamma_i(\alpha,\beta) &= \frac{\bar{w}_i}{p\frac{\partial f(\ell_u(\beta,p),\ell_s(\beta,p))}{\partial \ell_i}} = \int_0^1 \frac{1}{\beta_i} z^{\frac{1-\beta_i}{\beta_i}+\alpha_i-1+\alpha_{-i}\frac{\beta_{-i}}{1-\beta_{-i}}\frac{1-\beta_i}{\beta_i}} dz \\ &= \frac{1-\beta_{-i}}{1+(1-\alpha_i-\alpha_{-i})\beta_i\beta_{-i}-(1-\alpha_i)\beta_i-(1-\alpha_{-i})\beta_{-i}} \end{aligned}$$

is the overemployment factor of Cahuc et al. (2008, p. 950).

We now check how overemployment depends on the bargaining powers. We have:

$$\begin{aligned} \frac{\partial \gamma_i(\alpha,\beta)}{\partial \beta_i} &= -\frac{(1-\beta_{-i})\left[(1-\alpha_i-\alpha_{-i})\beta_{-i}-(1-\alpha_i)\right]}{\left[1+(1-\alpha_i-\alpha_{-i})\beta_i\beta_{-i}-(1-\alpha_i)\beta_i-(1-\alpha_{-i})\beta_{-i}\right]^2} \\ &= \frac{(1-\beta_{-i})\left[(1-\alpha_i)(1-\beta_{-i})+\alpha_{-i}\beta_{-i}\right]}{\left[1+(1-\alpha_i-\alpha_{-i})\beta_i\beta_{-i}-(1-\alpha_i)\beta_i-(1-\alpha_{-i})\beta_{-i}\right]^2} > 0 \end{aligned} \tag{81}$$

and

$$\frac{\partial\gamma_i(\alpha,\beta)}{\partial\beta_{-i}} = -\frac{\left[1+(1-\alpha_i-\alpha_{-i})\,\beta_i\beta_{-i}-(1-\alpha_i)\,\beta_i-(1-\alpha_{-i})\,\beta_{-i}\right]}{\left[1+(1-\alpha_i-\alpha_{-i})\,\beta_i\beta_{-i}-(1-\alpha_i)\,\beta_i-(1-\alpha_{-i})\,\beta_{-i}\right]^2} - \frac{(1-\beta_{-i})\left[(1-\alpha_i-\alpha_{-i})\,\beta_i-(1-\alpha_{-i})\right]}{\left[1+(1-\alpha_i-\alpha_{-i})\,\beta_i\beta_{-i}-(1-\alpha_i)\,\beta_i-(1-\alpha_{-i})\,\beta_{-i}\right]^2} \tag{82}$$

$$= -\frac{\alpha_{-i}\,(1-\beta_i)}{\left[1+(1-\alpha_i-\alpha_{-i})\,\beta_i\beta_{-i}-(1-\alpha_i)\,\beta_i-(1-\alpha_{-i})\,\beta_{-i}\right]^2} < 0$$

Thus, overemployment of labor of type i increases in i's own bargaining power β_i and decreases in the bargaining power of the other type of labor β_{-i}.

Using the definition of overemployment elasticities in (9), we obtain:

$$\eta_{ii}\,(\beta,p) = \beta_i \frac{\frac{\partial\gamma_i(\beta,p)}{\partial\beta_i}}{\gamma_i\,(\beta,p)} = \beta_i \frac{\left[(1-\alpha_i)\,(1-\beta_{-i})+\alpha_{-i}\beta_{-i}\right]}{\left[1+(1-\alpha_i-\alpha_{-i})\,\beta_i\beta_{-i}-(1-\alpha_i)\,\beta_i-(1-\alpha_{-i})\,\beta_{-i}\right]} > 0$$

$$\eta_{i,-i}\,(\beta,p) = \beta_i \frac{\frac{\partial\gamma_{-i}(\beta,p)}{\partial\beta_i}}{\gamma_{-i}\,(\beta,p)} = -\beta_i \frac{\alpha_i\,(1-\beta_{-i})}{(1-\beta_i)\left[1+(1-\alpha_i-\alpha_{-i})\,\beta_i\beta_{-i}-(1-\alpha_i)\,\beta_i-(1-\alpha_{-i})\,\beta_{-i}\right]} < 0$$

Thus, the own overemployment elasticity is positive, whereas the cross-overemployment elasticity is negative, as implied by Proposition 3 for the case of complements.

Combining the result of Proposition 3 with (83) and (88), we have that for a given β_{-i}, the profit-maximizing bargaining power β_i satisfies:

$$\frac{\beta_i}{\beta_{-i}} = -\frac{\eta_{i,-i}\,(\beta,p)}{\eta_{ii}\,(\beta,p)}\frac{\bar{w}_{-i}\ell_{-i}\,(\beta,p)}{\bar{w}_i\ell_i\,(\beta,p)}$$

or

$$\frac{\beta_i}{(1-\beta_i)} = \frac{\alpha_{-i}}{1-\alpha_i}\frac{\beta_{-i}}{1-\beta_{-i}}$$

Thus,

$$\frac{\eta_{i,-i}\,(\beta,p)}{\eta_{ii}\,(\beta,p)} = -\frac{1-\beta_{-i}}{1-\beta_i}\frac{\alpha_i}{(1-\alpha_i)\,(1-\beta_{-i})+\alpha_{-i}\beta_{-i}}. \tag{83}$$

The profit maximization problem of the firm is to choose ℓ_u and ℓ_s to maximize profits given an output price p and given the already determined in (80) stable wage schedule $w\,[\beta,p]$:

$$\max_{(\ell_u,\ell_s)\in\Re^2_+} p\ell_u^{\alpha_u}\ell_s^{\alpha_s} - \left((1-\beta_u)\,\bar{w}_u + p\beta_u\gamma_u\,(\alpha,\beta)\,\frac{\alpha_u\ell_u^{\alpha_u}\ell_s^{\alpha_s}}{\ell_u}\right)\ell_u - \left((1-\beta_s)\,\bar{w}_s + p\beta_s\gamma_s\,(\alpha,\beta)\,\frac{\alpha_s\ell_u^{\alpha_u}\ell_s^{\alpha_s}}{\ell_s}\right)\ell_s$$

$$\Updownarrow$$

$$\max_{(\ell_u,\ell_s)\in\Re^2_+} p\hat{\gamma}\,(\alpha,\beta)\,\ell_u^{\alpha_u}\ell_s^{\alpha_s} - (1-\beta_u)\,\bar{w}_u\ell_u - (1-\beta_s)\,\bar{w}_s\ell_s$$

where $\hat{\gamma}\,(\alpha,\beta)$ is defined as:

$$\hat{\gamma}\,(\alpha,\beta) \equiv \left[1-\alpha_u\beta_u\gamma_u(\alpha,\beta)-\alpha_s\beta_s\gamma_s(\alpha,\beta)\right] = \frac{(1-\beta_u)\,(1-\beta_s)}{1+(1-\alpha_u-\alpha_s)\,\beta_u\beta_s-(1-\alpha_u)\,\beta_s-(1-\alpha_s)\,\beta_u}$$

Note that for $i \in \{u, s\}$,

$$\hat{\gamma}(\alpha,\beta) = \gamma_i(\alpha,\beta)(1-\beta_i) > 0 \tag{84}$$

$$\frac{\partial \hat{\gamma}(\alpha,\beta)}{\partial \beta_i} = -\alpha_i \left(\frac{\hat{\gamma}(\alpha,\beta)}{1-\beta_i}\right)^2 < 0 \tag{85}$$

The necessary and sufficient first-order conditions for an interior solution are:

$$p\hat{\gamma}(\alpha,\beta)\alpha_i \frac{\ell_u^{\alpha_u}\ell_s^{\alpha_s}}{\ell_i} = (1-\beta_i)\bar{w}_i, i \in \{u,s\} \tag{86}$$

and the unique solution is:

$$\ell_i(\beta,p) = (\hat{\gamma}(\alpha,\beta)p)^{\frac{1}{1-\alpha_u-\alpha_s}} \left(\frac{\alpha_i}{(1-\beta_i)\bar{w}_i}\right)^{\frac{1-\alpha_{-i}}{1-\alpha_u-\alpha_s}} \left(\frac{\alpha_{-i}}{(1-\beta_{-i})\bar{w}_{-i}}\right)^{\frac{\alpha_{-i}}{1-\alpha_u-\alpha_s}}$$

where $\hat{\gamma}(\alpha,\beta) = \gamma_i(\alpha,\beta)(1-\beta_i)$. It follows that

$$\begin{aligned}\frac{\bar{w}_{-i}\ell_{-i}(\beta,p)}{\bar{w}_i\ell_i(\beta,p)} &= \frac{\bar{w}_{-i}(\hat{\gamma}(\alpha,\beta)p)^{\frac{1}{1-\alpha_u-\alpha_s}}\left(\frac{\alpha_i}{(1-\beta_i)\bar{w}_i}\right)^{\frac{1-\alpha_{-i}}{1-\alpha_u-\alpha_s}}\left(\frac{\alpha_{-i}}{(1-\beta_{-i})\bar{w}_{-i}}\right)^{\frac{\alpha_{-i}}{1-\alpha_u-\alpha_s}}}{\bar{w}_i(\hat{\gamma}_i(\alpha,\beta)p)^{\frac{1}{1-\alpha_u-\alpha_s}}\left(\frac{\alpha_i}{(1-\beta_i)\bar{w}_i}\right)^{\frac{1-\alpha_{-i}}{1-\alpha_u-\alpha_s}}\left(\frac{\alpha_{-i}}{(1-\beta_{-i})\bar{w}_{-i}}\right)^{\frac{\alpha_{-i}}{1-\alpha_u-\alpha_s}}} \qquad (87)\\ &= \frac{\bar{w}_{-i}\left(\frac{\alpha_{-i}}{(1-\beta_{-i})\bar{w}_{-i}}\right)^{\frac{1-\alpha_u-\alpha_s}{1-\alpha_u-\alpha_s}}}{\bar{w}_i\left(\frac{\alpha_i}{(1-\beta_i)\bar{w}_i}\right)^{\frac{1-\alpha_u-\alpha_s}{1-\alpha_u-\alpha_s}}} = \frac{\alpha_{-i}}{\alpha_i}\frac{1-\beta_i}{1-\beta_{-i}}\end{aligned}$$

Therefore,

$$\frac{\bar{w}_{-i}\ell_{-i}(\beta,p)}{\bar{w}_i\ell_i(\beta,p)} = \frac{\alpha_{-i}(1-\beta_i)}{\alpha_i(1-\beta_{-i})} \tag{88}$$

To obtain the profit-maximizing bargaining power, we set, by Proposition 3

$$\frac{\beta_i}{\beta_{-i}} = -\frac{\eta_{i,-i}(\beta,p)}{\eta_{ii}(\beta,p)}\frac{\bar{w}_{-i}\ell_{-i}(\beta,p)}{\bar{w}_i\ell_i(\beta,p)}$$

or, by (83) and (87),

$$\frac{\beta_i}{\beta_{-i}} = \frac{\alpha_i(1-\beta_{-i})}{(1-\beta_i)}\frac{1}{[(1-\alpha_i)(1-\beta_{-i})+\alpha_{-i}\beta_{-i}]}\frac{\alpha_{-i}(1-\beta_i)}{\alpha_i(1-\beta_{-i})}$$

which reduces to:

$$\frac{\beta_i}{(1-\beta_i)} = \frac{\alpha_{-i}}{1-\alpha_i}\frac{\beta_{-i}}{1-\beta_{-i}}$$

We conclude that the profit-maximizing bargaining power $\beta_i^*(\bar{\beta}_{-i},p)$ is determined by:

$$\beta_i^*(\bar{\beta}_{-i},p) = \beta_i^*(\bar{\beta}_{-i}) = \frac{\alpha_{-i}\bar{\beta}_{-i}}{(1-\alpha_i)(1-\bar{\beta}_{-i})+\alpha_{-i}\bar{\beta}_{-i}} \tag{89}$$

Note that for decreasing returns to scale, $\beta_i^*(\bar{\beta}_{-i}) \leq \bar{\beta}_{-i}$ obtains (with strict inequality except at $\bar{\beta}_{-i} = 0$).

Overemployment of labor of type i

$$\gamma_i(\alpha,\beta) > 1$$

is equivalent to

$$\frac{(1-\beta_{-i})}{1+(1-\alpha_i-\alpha_{-i})\beta_i\beta_{-i}-(1-\alpha_i)\beta_i-(1-\alpha_{-i})\beta_{-i}} > 1$$

$$0 > (1-\alpha_i-\alpha_{-i})\beta_i\beta_{-i}-(1-\alpha_i)\beta_i+\alpha_{-i}\beta_{-i}$$

$$(1-\alpha_i)\beta_i(1-\beta_{-i}) > \alpha_{-i}\beta_{-i}(1-\beta_i)$$

$$\frac{\beta_i}{1-\beta_i} > \frac{\alpha_{-i}}{(1-\alpha_i)}\frac{\beta_{-i}}{(1-\beta_{-i})}$$

Thus, factor i is underemployed, $\gamma_i(\alpha,\beta) < 1$, if and only if $\frac{\partial\pi(\beta,p)}{\partial\beta_i} < 0$.

It follows that

$$\frac{\partial\pi(\beta,p)}{\partial\beta_i} > 0 \Leftrightarrow \frac{\beta_i}{\beta_{-i}} < -\frac{\eta_{i,-i}(\beta,p)}{\eta_{ii}(\beta,p)}\frac{\bar{w}_{-i}\ell_{-i}(\beta,p)}{\bar{w}_i\ell_i(\beta,p)} \Leftrightarrow \frac{\beta_i}{1-\beta_i} < \frac{\alpha_{-i}}{1-\alpha_i}\frac{\beta_{-i}}{1-\beta_{-i}} \Leftrightarrow \gamma_i(\alpha,\beta) < 1,$$

Notably, simultaneous underemployment for both labor inputs cannot occur if there are decreasing returns to scale, $\alpha_u+\alpha_s < 1$. Indeed, underemployment of both factors requires:

$$\begin{aligned}\frac{\beta_i}{1-\beta_i} &< \frac{\alpha_{-i}}{(1-\alpha_i)}\frac{\beta_{-i}}{(1-\beta_{-i})}\\ \frac{\beta_{-i}}{1-\beta_{-i}} &< \frac{\alpha_i}{(1-\alpha_{-i})}\frac{\beta_i}{(1-\beta_i)}\end{aligned}$$

which can only be satisfied if

$$\frac{(1-\alpha_{-i})\beta_{-i}}{\alpha_i(1-\beta_{-i})} < \frac{\beta_i}{1-\beta_i} < \frac{\alpha_{-i}}{(1-\alpha_i)}\frac{\beta_{-i}}{(1-\beta_{-i})}$$

and therefore,

$$\frac{(1-\alpha_{-i})}{\alpha_i} < \frac{\alpha_{-i}}{(1-\alpha_i)}$$

But this is equivalent to $1-\alpha_{-i}-\alpha_i < 0$, a contradiction.

Thus,

$$\frac{\partial\pi(\beta,p)}{\partial\beta_i} > 0 \Longrightarrow \frac{\partial\pi(\beta,p)}{\partial\beta_{-i}} < 0. \tag{90}$$

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