The Dynamics of Bidding Markets with Financial Constraints: Supplementary Material[53]

Pablo Beker
University of Warwick

Ángel Hernando-Veciana
Universidad Carlos III de Madrid

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This supplementary material consists of four parts. Section S1 contains the proofs of all the Lemmae of the main text. Section S2 provides a proof for the existence of a fixed point to the operator T defined in (20). Section S3 studies a finite horizon version of our model of Section 4. Section S4 analyses a model with moral hazard and limited liability that endogenizes our function π, see (1).

[53] In this supplementary material we use cross references to equations, definitions, lemmas and propositions in the original paper. The numbering of footnotes is consecutive to the original paper.

S1 Proofs of Lemmae

Proof of Lemma A1

Proof. To get a contradiction, suppose that $w \in (0, \min\{\overline{\nu}^{\beta}, m_i\}]$ belongs to the support of F_i and F_j puts zero probability on $[w-\epsilon, w]$ for some $\epsilon > 0$. We shall argue that Firm i has a profitable deviation when Firm j plays (b^*, F_j). The contradiction hypothesis has two implications. (a) $w - \epsilon$ gives Firm i strictly greater expected payoffs than w since the former saves on the cost of working capital and increases the profit when winning without affecting the probability of winning. (b) Firm i's expected payoffs are continuous in its working capital at w since F_j does not put an atom at w. (a) and (b) mean that there exists an $\epsilon' \in (0, \epsilon)$ such that any working capital in $(w - \epsilon', w + \epsilon')$ gives strictly less expected payoffs than a working capital $w - \epsilon$. The fact that w belongs to the support of F_i means that F_i puts strictly positive probability in $(w - \epsilon', w + \epsilon')$ and thus Firm i has a profitable deviation: shift the probability mass in $(w - \epsilon', w + \epsilon')$ to $w - \epsilon$. ■

Proof of Lemma A2

Proof. To get a contradiction, suppose there exists $w \in [0, \min\{\overline{\nu}^{\beta}, m_j\})$ for which F_j puts strictly positive probability on $[w - \epsilon, w]$ for all $\epsilon > 0$ and F_i has an atom at w. Suppose, first, that $m_j > m_i$ and F_j puts zero probability on $[w - \varepsilon', w)$ for some $\varepsilon' > 0$. Our tie breaking rule in (2) implies that Firm i loses with probability one conditional on both firms choosing w. Thus, Firm i can profitably deviate by shifting the probability that F_i puts in w to $w - \varepsilon'$. Since F_j puts zero probability in $[w - \varepsilon', w)$, this deviation does not affect the probability that i wins the contract but it saves the cost of working capital ε'. We conclude that either (i) $m_j > m_i$ and F_j puts strictly positive probability on $[w - \varepsilon, w)$ for all $\varepsilon > 0$ or (ii) $m_j \leq m_i$ and F_j puts strictly positive probability on $[w - \varepsilon, w)$ for all $\varepsilon > 0$. In each case one can find $\varepsilon' > 0$ small enough such that Firm j can improve by moving the probability that F_j puts in $[w - \epsilon', w)$ in case (i) and in $[w - \epsilon', w]$ in case (ii), to a point slightly above w. Each deviation affects marginally Firm j's cost of working capital and profits conditional on winning but allows the firm to increase from at most $\frac{1}{2}$ to 1 the probability of winning the procurement contract at a strictly positive profit if Firm i plays the atom at w. ■

Proof of Lemma A3

Proof. Lemma A1 and the definition of support implies (i). (ii) and (iii) follow from putting Lemmas A1 and A2 together. (iv) follows from Lemma A2 for $w = 0$. To prove (v), note that Lemma A1 and (i) (with the roles of i and j interchanged) imply that it is sufficient to show that F_j is continuous at w. That follows from (ii) (applied to F_j instead of F_i). The case $w = m_l$ is similar, the only differences are that we use (iii) instead of (ii) and we take $m_i < m_j$ without loss of generality by (i). To prove (vi), note that for $\{i, j\} = \{1, 2\}$ the continuity of F_i in $(0, \nu)$ implies that j's equilibrium expected payoffs of choosing $w \in (0, \nu)$ are equal to (6) for $F = F_i$. Since $(0, \nu)$ belongs to the support of F_j, the usual indifference condition of a mixed strategy equilibrium implies that F_i is a continuous solution of (7) in $(0, \nu)$. Since F_i is a distribution and thus it is right-continuous at 0, the uniqueness result in Theorem 7.1 in Coddington and Levinson (1984), pag. 22, implies that the solutions we seek are characterized by (A1) as can be proved by taking derivatives. ■

Proof of Lemma A4

Proof. It follows from (5) and $\nu^\beta \in [\overline{\nu}^\beta, \theta)$. That $\nu^\beta < \theta$ follows from Definitions 6 and 7. To show that $\nu^\beta \geq \overline{\nu}^\beta$ we use that (A4) and Definition 6 imply:

$$1 - \frac{1-\beta}{\beta} \int_0^{\hat{\nu}^\Psi} \frac{e^{\int_x^{\hat{\nu}^\Psi} \frac{(-\pi'(y))}{\pi(y)+\Psi(y+\underline{m})} dy}}{\pi(x) + \Psi(x + \underline{m})} dx = 0. \tag{S1}$$

Thus, $\nu^\beta \geq \overline{\nu}^\beta$ follows from the facts that the left hand side of (S1) increases in Ψ and decreases in $\hat{\nu}^\Psi$ and that, by Definitions 2 and 7, two solutions to (S1) are $(\Psi, \hat{\nu}^\Psi) = (0, \overline{\nu}^\beta)$ and $(\Psi, \hat{\nu}^\Psi) = (\Psi^\beta, \nu^\beta)$ ■

Proof of Lemma A5

Proof. In this lemma, we use five auxiliary results. First, we use the implication of $\pi(2\underline{m}) > \pi(0) - \pi(\underline{m})$ and π strictly decreasing that $\pi(2\underline{m}) > 0$, and hence $2\underline{m} < \theta$, by Definition 1, which together with Lemma A4 means that for β close to 1,

$$2\underline{m} < \nu^\beta. \tag{S2}$$

Second, note that for $m \leq \nu^\beta$, (22) implies that

$$F^\beta_{l,m}(0) = F^{\Psi^\beta}_m(0) \geq \frac{\pi(m)}{\pi(0)}\left(1 - \frac{1-\beta}{\beta}\frac{m}{\pi(m)}\right), \tag{S3}$$

where the inequality follows from the fact that its right hand side is equal to the right hand side of (A4) for Ψ equal to the zero function and $w = 0$, and that the right hand side of (A4) decreases if the function Ψ shifts downwards.

Third, we use the following implication of (19)-(20) for $m, \underline{m} < \nu^\beta$:

$$\frac{\Psi^\beta(m)}{\Psi^\beta(\underline{m})} = \frac{F^{\Psi^\beta}_m(0)}{F^{\Psi^\beta}_{\underline{m}}(0)}. \tag{S4}$$

Fourth, we use that $F^\beta_{l,m}$ is a solution to (18) in $[w', w''] \subset [0, \min\{m, \nu^\beta\}]$, by (21)-(23), and hence satisfies:

$$F^\beta_{l,m}(w'') - F^\beta_{l,m}(w') = \int_{w'}^{w''} \frac{\frac{1-\beta}{\beta} + (-\pi'(y))F^\beta_{l,m}(y)}{\pi(y) + \Psi(y+\underline{m})}dy. \tag{S5}$$

Fifth, we use that $\underline{m} \leq \nu^\beta$ means that,

$$\begin{aligned}
1 - F^\beta_{l,\underline{m}}(0) &= F^\beta_{l,\underline{m}}(\underline{m}) - F^\beta_{l,\underline{m}}(0) \\
&= \int_0^{\underline{m}} \frac{\frac{1-\beta}{\beta} + (-\pi'(y))F^\beta_{l,\underline{m}}(y)}{\pi(y) + \Psi^\beta(y+\underline{m})}dy \\
&\leq \int_0^{\underline{m}} \frac{\frac{1-\beta}{\beta} + (-\pi'(y))}{\Psi^\beta(2\underline{m})}dy \\
&= \frac{\frac{1-\beta}{\beta}\underline{m} + \pi(0) - \pi(\underline{m})}{\Psi^\beta(2\underline{m})},
\end{aligned} \tag{S6}$$

where the first step follows from $F^\beta_{l,\underline{m}}(\underline{m}) = F^{\Psi^\beta}_{\underline{m}}(\underline{m})$, by (22), and $F^{\Psi^\beta}_{\underline{m}}(\underline{m}) = 1$, by Definition 5, the second step from (S5), the third step from $F^\beta_{l,\underline{m}}(y) \leq 1$, $\pi(y) \geq 0$ and $\Psi^\beta(y + \underline{m}) \geq \Psi^\beta(2\underline{m})$, and the last step from standard algebra.

The lemma is proved using that $\pi(2\underline{m}) > 0$, as explained above, $\lim_{\beta\uparrow 1} F^\beta_{l,\underline{m}}(0) > 0$, by (S3), and the following equalities and inequalities, for β close to 1:

$$\begin{aligned}
(1-\beta)\Psi^\beta(\underline{m}) &= \beta F^\beta_{l,\underline{m}}(0)\left(\pi(0) + \Psi^\beta(\underline{m})\right) - \beta\Psi^\beta(\underline{m}) \\
&= \beta F^\beta_{l,\underline{m}}(0)\left(\pi(0) - \frac{1 - F^\beta_{l,\underline{m}}(0)}{F^\beta_{l,\underline{m}}(0)}\Psi^\beta(\underline{m})\right) \\
&\geq \beta F^\beta_{l,\underline{m}}(0)\left(\pi(0) - \frac{\frac{1-\beta}{\beta}\underline{m} + \pi(0) - \pi(\underline{m})}{F^\beta_{l,2\underline{m}}(0)}\right) \\
&\geq \beta F^\beta_{l,\underline{m}}(0)\pi(0)\frac{\pi(2\underline{m}) + \pi(\underline{m}) - \pi(0) - \frac{1-\beta}{\beta}3\underline{m}}{\pi(2\underline{m}) - \frac{1-\beta}{\beta}2\underline{m}},
\end{aligned}$$

where we use in the first equality (19) and (20); in the second equality, an algebraic transformation; in the first inequality, (S2), (S4) for $m = 2\underline{m}$ and (S6); and in the second inequality, (S2), (S3) for $m = 2\underline{m}$ and some algebra. ■

Proof of Lemma A6

Proof. Direct from (21), (22), (23), and (A5). ■

Proof of Lemma A7

Proof. It is sufficient to show that:

$$Q^\beta(m, [\underline{m}, m']) - F^\beta_{l,m}(0) \leq 2(1 - F^\beta_{l,m'-\underline{m}}(0)). \tag{S7}$$

This can be deduced from $Q^\beta(m, \mathcal{M}) \leq Q^\beta(m, (\underline{m}, m'])$, (A12) for $m'' = \underline{m}$ and $Q^\beta(m, \{\underline{m}\}) = F^\beta_{l,m}(0)$. That $Q^\beta(m, \{\underline{m}\}) = F^\beta_{l,m}(0)$ can be deduced from (26) since $F^\beta_{L,m}(0) = 0$ by (21) and (23) and Definition 6.

If $m' - \underline{m} \geq m$, (S7) follows from $Q^\beta(m, [\underline{m}, m']) = 1$, by (26), and $F^\beta_{l,m}(0) \geq F^\beta_{l,m'-\underline{m}}(0)$, by Lemma A6. If $m' - \underline{m} < m$, (S7) follows from:

$$\begin{aligned}
&Q^\beta(m, [\underline{m}, m']) - F^\beta_{l,m}(0) \\
&\quad = F^\beta_{l,m}(m' - \underline{m}) + F^\beta_{L,m}(m' - \underline{m})(1 - F^\beta_{l,m}(m' - \underline{m})) - F^\beta_{l,m}(0) \\
&\quad = F^\beta_{l,m}(m' - \underline{m}) - F^\beta_{l,m}(0) + (F^\beta_{L,m}(m' - \underline{m}) - F^\beta_{L,m}(0))(1 - F^\beta_{l,m}(m' - \underline{m})) \\
&\quad \leq F^\beta_{l,m}(m' - \underline{m}) - F^\beta_{l,m}(0) + (F^\beta_{L,m}(m' - \underline{m}) - F^\beta_{L,m}(0)) \\
&\quad \leq 2(F^\beta_{l,m}(m' - \underline{m}) - F^\beta_{l,m}(0)) \\
&\quad \leq 2(F^\beta_{l,m'-\underline{m}}(m' - \underline{m}) - F^\beta_{l,m'-\underline{m}}(0)) \\
&\quad = 2(1 - F^\beta_{l,m'-\underline{m}}(0)),
\end{aligned}$$

where the first step uses (A10) which applies by $m' - \underline{m} < m$ and the Lemma's hypothesis; the second step uses again that $F^\beta_{L,m}(0) = 0$; the third step uses that $F^\beta_{l,m} \leq 1$ and that

$F^{\beta}_{L,m}(\cdot)$ is increasing and $m' - \underline{m} \geq 0$; and the fourth step uses that:

$$\begin{aligned} F^{\beta}_{L,m}(m'-\underline{m}) - F^{\beta}_{L,m}(0) &= \int_0^{m'-\underline{m}} \frac{\frac{1-\beta}{\beta} + (-\pi'(y))F^{\beta}_{L,m}(y)}{\pi(y)+\Psi(y+\underline{m})} dy \\ &\leq \int_0^{m'-\underline{m}} \frac{\frac{1-\beta}{\beta} + (-\pi'(y))F^{\beta}_{l,m}(y)}{\pi(y)+\Psi(y+\underline{m})} dy \\ &= F^{\beta}_{l,m}(m'-\underline{m}) - F^{\beta}_{l,m}(0), \end{aligned}$$

where the equalities use $m' - \underline{m} < m$ and the Lemma's hypothesis to apply (S5) and a similar equation in which $F^{\beta}_{L,m}$ replaces $F^{\beta}_{l,m}$, and the inequality follows from Lemma A6 and (21)-(23); in the fifth step, we use the Lemma's hypothesis, $m' - \underline{m} < m$, (S5) for $w'' = m' - \underline{m}$ and $w' = 0$ and Lemma A6; and in the sixth step, that $F^{\beta}_{l,m'-\underline{m}}(m'-\underline{m}) = 1$ by (21) and (22). ■

Proof of Lemma A8

Proof. We use the following implication of $(\pi, \underline{m}) \in \Lambda$, first line, and $\Psi^{\beta} \in \mathcal{P}^{\beta}$, second line:

$$\lim_{\beta \to 1} \Psi^{\beta}(m) = \begin{cases} \infty & \text{if } m < \theta, \\ 0 & \text{if } m \geq \theta. \end{cases} \tag{S8}$$

The definition of $\mathcal{P}^{\beta}$ in (17) implies the second line. The first line can be deduced using Lemma A4, (S4), $\lim_{\beta \to 1} \Psi^{\beta}(\underline{m}) = \infty$, $F^{\Psi^{\beta}}_{\underline{m}}(0) \leq 1$ and (S3).

To prove (i) and (ii), we use that $\lim_{\beta \uparrow 1} \nu^{\beta} = \theta$, by Lemma A4, $\nu^{\beta} < \theta$, by Definitions 6 and 7, and (21)-(23) imply that for β close to 1:

$$F^{\beta}_{l,m}(w) = \begin{cases} F^{\Psi^{\beta}}_{m}(w) & \text{if } m < \theta \\ F^{\Psi^{\beta}}_{\nu^{\beta}}(w) & \text{if } m \geq \theta \end{cases}, \tag{S9}$$

and,

$$F^{\beta}_{L,m}(w) = \; F^{\Psi^{\beta}}_{\nu^{\beta}}(w) \quad \text{if } w \in [0, \min\{m, \theta\}). \tag{S10}$$

The limit, as β goes to 1, of the first line of the right hand side of (S9) is equal to the corresponding expressions in the first three lines of the right of (A13). This is because: (a) the two terms of the product in the right hand side of (A4) evaluated at Ψ^{β} have the necessary finite limits as β goes to 1; and (b) the product of (finite) limits equals the limit of the product. Result (a) can be shown by using the the bounded convergence theorem

(Royden (1988), page 81), denoted BCT hereafter,[54] and (S8). That the limit, as β goes to 1, of the second line of the right of (S9) is equal to the right of the fourth line of (A13) is a consequence of the integral in the last line of the right hand side of (A6) being bounded for $\Psi = \Psi^\beta$. Since this argument does not require neither $(\pi, \underline{m}) \in \Lambda$ nor $\theta > 2\underline{m}$, it also implies the limit in (ii).

To prove (iii), we start with the case $m = \underline{m}$. We use that for β close to 1, $F^\beta_{l,\underline{m}}(0) = F^{\Psi^\beta}_{\underline{m}}(0)$ by (22) and $\underline{m} < \nu^\beta$, a consequence of $3\underline{m} < \theta$ and Lemma A4. Thus, Definition 7 means that:

$$\begin{aligned}(1-\beta)\Psi^\beta(\underline{m}) &= \beta F^\beta_{l,\underline{m}}(0)(\pi(0) + \Psi^\beta(\underline{m})) - \beta\Psi^\beta(\underline{m}) \\ &= \beta F^\beta_{l,\underline{m}}(0)\pi(0) - \beta\Psi^\beta(\underline{m})(1 - F^\beta_{l,\underline{m}}(0)), \end{aligned} \tag{S11}$$

where the first term on the second line of the right hand side of (S11) tends to $\pi(0)$ since $\lim_{\beta\uparrow 1} F^\beta_{l,\underline{m}}(0) = 1$ by the first line of the right of (A13) and $3\underline{m} > \theta$. We use that $F^\beta_{l,\underline{m}}(\underline{m}) = 1$, by (22) and Definition 5, and (S5) to rewrite the last term of (S11) as

$$\beta\Psi^\beta(\underline{m}) \int_0^{\underline{m}} \frac{\frac{1-\beta}{\beta} + (-\pi'(z))F^\beta_{l,\underline{m}}(z)}{\pi(z) + \Psi^\beta(z+\underline{m})} dz = \beta \int_0^{\underline{m}} \frac{\frac{1-\beta}{\beta} + (-\pi'(z))F^\beta_{l,\underline{m}}(z)}{\frac{\pi(z)}{\Psi^\beta(\underline{m})} + \frac{F^\beta_{l,z+\underline{m}}(0)}{F^\beta_{l,\underline{m}}(0)}} dz. \tag{S12}$$

where the equality in (S12) follows from (S4). By application of the BCT, the limit of this last term when β tends to 1 is equal to $\pi(0) - \pi(\underline{m})$, as required, since $\Psi^\beta(\underline{m})$ diverges to infinity and $F^\beta_{l,m}(w)$ tends to 1 for $w \le \underline{m}$ and $m \in [\underline{m}, 2\underline{m}]$. The former follows from the assumption that $(\pi, \underline{m}) \in \Lambda$ and Definition A1, and the latter follows from the first line of the right hand side of (A13) since we assume that $3\underline{m} < \theta$ which means that $2\underline{m} < \theta - \underline{m}$.

The result for a general m can be deduced from the result for $m = \underline{m}$ and (S4) and the limit results for $F^\beta_{l,m}$ in part (i) of this lemma. ■

Proof of Lemma A9

Proof. To show (i), note that (21) and (23) and Lemma A4 imply that $F_{L,m}(w) = F^{\Psi^\beta}_{\nu^\beta}(w)$ for $w < \min\{\theta, m\}$ and β close to 1. Thus, we can deduce (i) from (A6), (S8), and the BCT.

[54] The BCT applies because each integrand in (A4) is uniformly bounded along the sequence since Ψ being non negative implies that $|\frac{-\pi'(y)}{\pi(y)+\Psi(y+\underline{m})}| \le \frac{\pi'(y)}{\pi(y)}$ and $0 \le \frac{1}{\pi(x)+\Psi(x+\underline{m})} \le \frac{1}{\pi(x)}$.

To show (ii), note that (21) and Lemma A4 imply that $F^{\beta}_{L,\theta}(\theta - 2\underline{m} - \epsilon) = F^{\Psi^\beta}_{\nu^\beta}(\theta - 2\underline{m} - \epsilon)$ for β close to 1. Thus, we can deduce from (A6) and Definition 7 that:

$$\frac{F^{\beta}_{L,\theta}(\theta - 2\underline{m} - \epsilon)}{(1-\beta)^2} = \frac{1}{\beta}\int_0^{\theta-2\underline{m}-\epsilon} \frac{e^{\int_x^{\theta-2\underline{m}-\epsilon} \frac{(-\pi'(y))}{\pi(y)+\Psi^\beta(y+\underline{m})}dy}}{(1-\beta)\pi(x)+(1-\beta)\Psi^\beta(x+\underline{m})}dx.$$

We can deduce (ii) from this equation, applying the limits in (S8), the first line of the right of (A14) and the BCT. The BCT can be applied to the integral in x because, first, its integrand is non negative and bounded above by:

$$\frac{e^{\int_0^{\theta-2\underline{m}-\epsilon} \frac{(-\pi'(y))}{\pi(y)}dy}}{(1-\beta)\Psi^\beta(\theta-\underline{m}-\epsilon)}, \tag{S13}$$

since $\pi(x)$ is non-negative for $x \in [0,\theta]$ and $\Psi^\beta(z+\underline{m})$ is decreasing and non-negative in x; and second, expression (S13) is uniformly bounded above for any β in $(1/2,1)$ as a consequence of the first line of the right of (A14).

To show (iii), note that for β close to 1:

$$\begin{aligned}
\lim_{\beta\uparrow 1}\frac{1-F^{\beta}_{l,\theta-\underline{m}-\epsilon}(0)}{1-\beta} &= \lim_{\beta\uparrow 1}\frac{F^{\beta}_{l,\theta-\underline{m}-\epsilon}(\theta-\underline{m}-\epsilon)-F^{\beta}_{l,\theta-\underline{m}-\epsilon}(0)}{1-\beta}\\
&= \lim_{\beta\uparrow 1}\int_0^{\theta-\underline{m}-\epsilon}\frac{\frac{1-\beta}{\beta}+(-\pi'(z))F^{\beta}_{l,\theta-\underline{m}-\epsilon}(z)}{(1-\beta)\pi(z)+(1-\beta)\Psi^\beta(z+\underline{m})}dz\\
&= \int_0^{\theta-\underline{m}-\epsilon}\lim_{\beta\uparrow 1}\left(\frac{\frac{1-\beta}{\beta}+(-\pi'(z))F^{\beta}_{l,\theta-\underline{m}-\epsilon}(z)}{(1-\beta)\pi(z)+(1-\beta)\Psi^\beta(z+\underline{m})}\right)dz\\
&= \int_0^{\theta-\underline{m}-\epsilon}\frac{(-\pi'(z))\min\left\{\frac{\pi(m)}{\pi(\theta-\underline{m})},1\right\}}{\pi(\underline{m})\min\left\{\frac{\pi(z+\underline{m})}{\pi(\theta-\underline{m})},1\right\}}dz,
\end{aligned}$$

where first equality follows from Lemma A4, (22), and $F^{\Psi^\beta}_{m}(m)=1$; the second equality follows by Lemma A4 and (S5); in the third equality we apply, once again, the BCT using that the integrand is nonnegative and bounded above by:

$$\frac{\frac{1-\beta}{\beta}+\gamma}{(1-\beta)\pi(z)+(1-\beta)\Psi^\beta(z+\underline{m})},$$

which is uniformly bounded above in β for $\beta \in (1/2,1)$ and $z \in [0,\theta-\underline{m}-z]$ as can be shown by applying Lemma A8(iii) to the denominator; and finally, in the fourth equality we use the following three properties: (a) the numerator converges to $(-\pi'(w))\min\left\{\frac{\pi(m)}{\pi(\theta-\underline{m})},1\right\}$ since the first and third line of the right hand side of (A13) implies that $\lim_{\beta\uparrow 1} F^{\beta}_{l,m}(z) =$

$\min\left\{\frac{\pi(m)}{\pi(\theta-\underline{m})},1\right\}$ because $z\leq\theta-\underline{m}-\epsilon$, (b) the denominator converges to $\pi(\underline{m})\min\left\{\frac{\pi(w+\underline{m})}{\pi(\theta-\underline{m}-\epsilon)},1\right\}>0$ since Lemma A8(iii) implies that $\lim_{\beta\uparrow 1}(1-\beta)\Psi^{\beta}(z+\underline{m})=\pi(\underline{m})\min\left\{\frac{\pi(z+\underline{m})}{\pi(\theta-\underline{m})},1\right\}$ because $z+\underline{m}<\theta$ and (c) the limit of the ratio equals the ratio of the limits when the denominator has a positive limit. ■

Proof of Lemma A10

Proof. For $\mathcal{M}=(\underline{m},\theta]$, (27) implies that $\mu^{\beta}(\mathcal{M})=\int Q^{\beta}(m,\mathcal{M})\mu^{\beta}(dm)$, which is less than $2(1-F^{\beta}_{l,\theta-\underline{m}}(0))$ for β close to 1 since Lemma A7 for $m'=\theta$ can be applied by Lemma A4 and since μ^{β} is a probability measure. Thus, the lemma follows from the third line on the right hand side of (A13). ■

S2 Existence of Fixed Points of the Operator T

To simplify the notation, we adopt the convention that

$$(a)^{+}\equiv\max\{a,0\}. \tag{S14}$$

We also find convenient to compute a bound for $\hat{\nu}^{\Psi}$ for any $\Psi\in\mathcal{P}^{\beta}$.

Definition S1. $\overline{x}^{\beta}$ is the unique $w\in[(\theta-\underline{m})^{+},\theta]$ that solves:

$$\beta\pi(w)=(1-\beta)(w-(\theta-\underline{m})^{+}).$$

Note the similarity between the equation that defines $\overline{x}^{\beta}$ and (4). Indeed, as next lemma shows, $\overline{x}^{\beta}$ is an upper bound to the support of the randomizations that firms play over working capitals in the models of Section 4 and Section S3. In this sense, it is an analogue of $\overline{\nu}^{\beta}$ in the static model.

Lemma S1. $\hat{\nu}^{\Psi}\leq\overline{x}^{\beta}$ for any $\Psi\in\mathcal{P}^{\beta}$.

Proof. By Definition 5, $F^{\Psi}_{\hat{\nu}^{\Psi}}(\hat{\nu}^{\Psi})=1$. Besides, the right hand side of (A6) is increasing in w for $w<\theta$. Thus, it is sufficient to show the right hand side of (A6) is greater than 1 at

$w = \overline{x}^{\beta}$. This expression is equal to:

$$\begin{aligned}\frac{1-\beta}{\beta}\int_0^{\overline{x}^{\beta}} \frac{e^{\int_x^{\overline{x}^{\beta}} \frac{(-\pi'(y))}{\pi(y)+\Psi(y+\underline{m})}dy}}{\pi(x)+\Psi(x+\underline{m})}dx &\geq \frac{1-\beta}{\beta}\int_{(\theta-\underline{m})^+}^{\overline{x}^{\beta}} \frac{e^{\int_x^{\overline{x}^{\beta}} \frac{(-\pi'(y))}{\pi(y)+\Psi(y+\underline{m})}dy}}{\pi(x)+\Psi(x+\underline{m})}dx \\ &= \frac{1-\beta}{\beta}\int_{(\theta-\underline{m})^+}^{\overline{x}^{\beta}} \frac{e^{\int_x^{\overline{x}^{\beta}} \frac{(-\pi'(y))}{\pi(y)}dy}}{\pi(x)}dx \\ &= \frac{1-\beta}{\beta}\cdot\frac{\overline{x}^{\beta}-(\theta-\underline{m})^+}{\pi(\overline{x}^{\beta})} \\ &= 1,\end{aligned}$$

where we use that the integrand is positive in the first step, that $\Psi(m) = 0$ for $m \geq \theta$ in the second step, standard algebra in the third step and Definition S1 in the last step. ■

We prove that T has a fixed point applying Schauder Fixed-Point Theorem. For this purpose, we restrict the domain of Ψ to $[0, \theta + \underline{m}]$. Define $\tilde{\mathcal{P}}^{\beta}$ as:

$$\left\{\Psi\colon [0,\theta+\underline{m}] \to \left[0, \frac{\beta}{1-\beta}\pi(0)\right] \text{ is continuous, decreasing and } \Psi(m) = 0\,\forall m \geq \theta\right\}. \quad \text{(S15)}$$

For any $\tilde{\Psi} \in \tilde{\mathcal{P}}^{\beta}$, we define its (unique) extension to $\mathcal{P}^{\beta}$ by the function Ψ that satisfies $\Psi(m) = \tilde{\Psi}(m)$ if $m \in [0, \theta + \underline{m}]$ and $\Psi(m) = 0$ if $m \in (\theta + \underline{m}, \infty)$. For any $\tilde{\Psi} \in \tilde{\mathcal{P}}^{\beta}$, let $F_m^{\tilde{\Psi}} \equiv F_m^{\Psi}$ where Ψ is the extension of $\tilde{\Psi}$ to $\mathcal{P}^{\beta}$. For any $\tilde{\Psi} \in \tilde{\mathcal{P}}^{\beta}$, let $\tilde{T}(\tilde{\Psi})(m) \equiv T(\Psi)(m)$ for $m \in [0, \theta + \underline{m}]$, where T is defined in (20) and Ψ is the extension of $\tilde{\Psi}$ to $\mathcal{P}^{\beta}$.

Lemma S2. The operator $\tilde{T} : \tilde{\mathcal{P}}^{\beta} \to \tilde{\mathcal{P}}^{\beta}$ has a fixed point. Moreover, the extension of any fixed point of $\tilde{T}$ is a fixed point of the operator $T : \mathcal{P}^{\beta} \to \mathcal{P}^{\beta}$ and any fixed point of T is the extension of some fixed point of $\tilde{T}$.

Proof. We endow $\tilde{\mathcal{P}}^{\beta}$ with the sup-norm, that we denote by $||\cdot||$, and check that the operator $\tilde{T}$ on $\tilde{\mathcal{P}}^{\beta}$ satisfies all the conditions of Schauder Fixed-Point Theorem, see Stokey and Lucas (1999), Theorem 17.4, page 520.

$\tilde{\mathcal{P}}^{\beta}$ is a nonempty, closed, bounded and convex subset of the set of continuous functions on $[0, \theta + \underline{m}]$. Furthermore, it is easy to see that $T : \mathcal{P}^{\beta} \to \mathcal{P}^{\beta}$, see Footnote 37, implies that $\tilde{T} : \tilde{\mathcal{P}}^{\beta} \to \tilde{\mathcal{P}}^{\beta}$. We show below that $\tilde{T}$ is continuous and that the family $\tilde{T}(\tilde{\mathcal{P}}^{\beta})$ is equicontinuous, as desired.

Claim 1: $\tilde{T}$ is continuous.

We assume an arbitrary sequence $\{\Psi_n\} \to \Psi$ in $\tilde{\mathcal{P}}^\beta$ and show that $\tilde{T}(\Psi_n) \to \tilde{T}(\Psi)$. Note that $||\tilde{T}(\Psi_n) - \tilde{T}(\Psi)||$ is equal to:

$$
\begin{aligned}
& \sup_{m\in[0,\overline{x}^\beta]} \left|\beta(F_m^{\Psi_n}(0))^+(\pi(0)+\Psi_n(\underline{m})) - \beta(F_m^{\Psi}(0))^+(\pi(0)+\Psi(\underline{m}))\right| \\
= & \sup_{m\in[0,\overline{x}^\beta]} \left|((F_m^{\Psi_n}(0))^+ - (F_m^{\Psi}(0))^+)\beta(\pi(0)+\Psi(\underline{m})) + \beta(F_m^{\Psi_n}(0))^+(\Psi_n(\underline{m}) - \Psi(\underline{m}))\right| \\
\leq & \sup_{m\in[0,\overline{x}^\beta]} \left\{\left|(F_m^{\Psi_n}(0))^+ - (F_m^{\Psi}(0))^+\right|\beta(\pi(0)+\Psi(\underline{m})) + \beta(F_m^{\Psi_n}(0))^+\left|\Psi_n(\underline{m}) - \Psi(\underline{m})\right|\right\} \\
\leq & \sup_{m\in[0,\overline{x}^\beta]} \left\{\left|(F_m^{\Psi_n}(0))^+ - (F_m^{\Psi}(0))^+\right|\frac{\beta}{1-\beta}\pi(0) + \beta\left|\Psi_n(\underline{m}) - \Psi(\underline{m})\right|\right\} \\
\leq & \frac{\beta}{1-\beta}\pi(0)\sup_{m\in[0,\overline{x}^\beta]} \left|F_m^{\Psi_n}(0) - F_m^{\Psi}(0)\right| + \beta\sup_{m\in[0,\overline{x}^\beta]} \left|\Psi_n(\underline{m}) - \Psi(\underline{m})\right|,
\end{aligned}
$$

where the first step follows from (20), Lemma S1 and because (S14) implies that $(F_m^{\Psi}(0))^+ = F_m^{\Psi}(0)$ if $m \leq \hat{\nu}^\Psi$ and[55] $(F_m^{\Psi}(0))^+ = 0$ if $m > \hat{\nu}^\Psi$; in the second step we add and subtract $\beta\Psi(\underline{m})(F_m^{\Psi_n}(0))^+$; the third step follows from $|A+B| \leq |A|+|B|$; the fourth step because $\Psi(m) \leq \frac{\beta}{1-\beta}\pi(0)$ and $(F_m^{\Psi_n}(0))^+ \leq 1$; and the fifth step from the property that $\sup_x\{a(x)+b(x)\} \leq \sup_x a(x) + \sup_x b(x)$, and that $|(A)^+ - (B)^+| \leq |A-B|$.

Definition S1 implies that $\overline{x}^\beta < \theta + \underline{m}$ and so,

$$
\sup_{m\in[0,\overline{x}^\beta]} |\Psi_n(\underline{m}) - \Psi(\underline{m})| \leq \sup_{m\in[0,\theta+\underline{m}]} |\Psi_n(m) - \Psi(m)| = ||\Psi_n - \Psi||.
$$

Since we assume that $||\Psi_n - \Psi||$ converges to zero, it only remains to be shown that:

$$
\lim_{n\to\infty}\sup_{m\in[0,\overline{x}^\beta]} \left|F_m^{\Psi_n}(0) - F_m^{\Psi}(0)\right| = 0. \tag{S16}
$$

Let $\underline{\Psi}_n(w) \equiv (\Psi(w) - \epsilon_n)^+$ and $\overline{\Psi}_n(w) \equiv \Psi(w) + \epsilon_n$ for $\epsilon_n \equiv \sup_{\tilde{n}\geq n} ||\Psi_{\tilde{n}} - \Psi||$. With a slight abuse of notation, we denote by $F_m^{\underline{\Psi}_n(w)}(w)$ and $F_m^{\overline{\Psi}_n(w)}(w)$ the right hand side of (A4) at $\Psi = \underline{\Psi}_n$ and $\Psi = \overline{\Psi}_n$, respectively. Thus, that $\Psi(w)$ and $\Psi_n(w)$ belong to $[\underline{\Psi}_n(w), \overline{\Psi}_n(w)]$ and that $F_m^{\Psi} \geq F_m^{\tilde{\Psi}}$ if $\Psi(w) \geq \tilde{\Psi}(w) \geq 0$ for any w, by (A4), imply that $F_m^{\Psi(w)}(0)$ and $F_m^{\Psi_n(w)}(0)$ belong to $[F_m^{\underline{\Psi}_n(w)}(0), F_m^{\overline{\Psi}_n(w)}(0)]$. Thus:

$$
|F_m^{\Psi_n}(0) - F_m^{\Psi}(0)| \leq \left|F_m^{\overline{\Psi}_n}(0)) - F_m^{\underline{\Psi}_n}(0)\right|. \tag{S17}
$$

Note that,

$$
\lim_{n\to\infty}\sup_{m\in[0,\overline{x}^\beta]} \left|F_m^{\overline{\Psi}_n}(0) - F_m^{\underline{\Psi}_n}(0)\right| = 0, \tag{S18}
$$

[55] Since $F_m^{\Psi}(0)$ is decreasing in m, see (A4), and $F_{\hat{\nu}^\Psi}^{\Psi}(0) = 0$, see Definition 6.

by an application of Theorem 7.13 in Rudin (1976), pag. 150. We can apply this theorem, because (A4) implies that: (a) $F_m^{\overline{\Psi}_n}(0)$ and $F_m^{\underline{\Psi}_n}(0)$ are continuous in $m \in [0, \overline{x}^\beta]$, (b) $\{F_m^{\overline{\Psi}_n}(0) - F_m^{\underline{\Psi}_n}(0)\}_n$ is a decreasing sequence since $\{\overline{\Psi}_n\}_n$ and $\{\underline{\Psi}_n\}_n$ are respectively decreasing and increasing sequences, and (c) $F_m^{\overline{\Psi}_n}(0) - F_m^{\underline{\Psi}_n}(0)$ converges to zero pointwise in $m \in [0, \overline{x}^\beta]$ since $\lim_{n\to\infty} ||\overline{\Psi}_n - \underline{\Psi}_n|| = 0$ and the BCT[56] applies.

Equations (S17) and (S18) imply (S16) as desired.

Claim 2: the family $\tilde{T}(\tilde{\mathcal{P}}^\beta)$ is equicontinuous.

(20), $\Psi(\underline{m}) \in \left[0, \frac{\beta}{1-\beta}\pi(0)\right]$ for $\Psi \in \tilde{\mathcal{P}}^\beta$ and Lemma S1 means that it is sufficient to show that there exists a finite κ such that:

$$\left|\frac{\partial F_m^\Psi(0)}{\partial m}\right| \leq \kappa \text{ for any } \Psi \in \tilde{\mathcal{P}}^\beta \text{ and } m \in [0, \overline{x}^\beta]. \tag{S19}$$

Equation (A5) evaluated at $w = 0$ implies that:

$$\left|\frac{\partial F_m^\Psi(0)}{\partial m}\right| = \frac{e^{\int_0^m \frac{\pi'(y)}{\pi(y)+\Psi(y+\underline{m})}dy}}{\pi(m) + \Psi(m + \underline{m})} \left|\pi'(m) - \frac{1-\beta}{\beta}\right| \text{ for } m \in [0, \overline{x}^\beta]. \tag{S20}$$

Note that $e^{\int_0^m \frac{\pi'(y)}{\pi(y)+\Psi(y+\underline{m})}dy} \leq 1$ because $\frac{\pi'(y)}{\pi(y)+\Psi(y+\underline{m})} < 0$ and that $\pi(m) + \Psi(m + \underline{m}) \geq \pi(\overline{x}^\beta)$ because π is a decreasing function and $\Psi(m + \underline{m}) \geq 0$. Besides, since $-\pi'$ is continuous, there exists a finite $\gamma \geq 0$ such that:

$$-\pi'(z) \leq \gamma, \text{ for all } z \in [0, \theta]. \tag{S21}$$

These arguments imply the condition in (S19) for

$$\kappa = \frac{1}{\pi(\overline{x}^\beta)} \left|\gamma + \frac{1-\beta}{\beta}\right|.$$

■

Lemma S2 implies the following proposition (and no proof is required).

Proposition S1. The operator T defined in (20) has a fixed point in $\mathcal{P}^\beta$.

[56]The same arguments as in Footnote 54 can be used to show that each integrand in (A4) is uniformly bounded in β.

S3 A Finite Horizon Model

In this section, we show that the unique equilibrium of a finite period version of the model in Section 4 converges as the number of periods tend to infinity to one of the equilibria that we analyse in Section 4. As Athey and Schmutzler (2001) have argued:

> If there are multiple equilibria, one equilibrium of particular interest (if it exists) is an equilibrium attained by taking the limit of first-period strategies as the horizon T approaches infinity.

We consider a $\bar{t}+1$-period model with periods denoted by $t \in \{1, 2, \ldots, \bar{t}+1\}$. All the periods but the last one are as in the model of Section 4. In the last period, all the firm's cash is consumed. The dynamic link between periods and the firms' objective functions, adapted to the finite time horizon, are also as in Section 4. Both firms start with identical cash larger than θ and we assume Assumption 1. We study the subgame perfect equilibria of the game.

Definition S2. We let Ψ_t, $t = 2, \ldots, \bar{t}+1$, be defined recursively (starting from $\bar{t}+1$) by $\Psi_{\bar{t}+1} \equiv \mathbf{0}$, and $\Psi_t \equiv T(\Psi_{t+1})$, where T is the operator defined in (20). We let ν_t, $t = 1, \ldots, \bar{t}$, denote $\hat{\nu}^{\Psi_{t+1}}$.

For $t = 1, \ldots, \bar{t}$, let,

$$F_{t,l,m}(w) = F_{t,L,m}(w) = F_{\nu_t}^{\Psi_{t+1}}(w) \qquad \text{if } w \leq \nu_t \leq m \tag{S22}$$

$$F_{t,l,m}(w) = F_{m}^{\Psi_{t+1}}(w) \qquad \text{if } w \leq m < \nu_t \tag{S23}$$

$$F_{t,L,m}(w) \equiv \begin{cases} F_{\nu_t}^{\Psi_{t+1}}(w) & \text{if } w < m < \nu_t \\ 1 & \text{if } w = m < \nu_t \end{cases} \tag{S24}$$

Let Ω denote (as in Section 4) the cash vectors that can arise along the game tree. For any $t \in \{1, 2, ..., \bar{t}\}$ and $(m, m') \in \Omega$, let:

$$\sigma_t^*(w|m, m') \equiv \begin{cases} F_{t,l,m}(w) & \text{if } m \leq m' \\ F_{t,L,m'}(w) & \text{if } m > m', \end{cases} \tag{S25}$$

and,

$$W_t(m, m') \equiv \begin{cases} m + \underline{m} \sum_{\tau=1}^{\bar{t}+1-t} \beta^\tau & \text{if } m \leq m' \\ m + \underline{m} \sum_{\tau=1}^{\bar{t}+1-t} \beta^\tau + \Psi_t(m') & \text{if } m > m'. \end{cases} \tag{S26}$$

Note that W_t is continuous in Ω because the only conflicting point is when $m = m'$ and Assumption 1 implies that in this case $m' \geq \theta$ and hence $\Psi_t(m') = 0$ by (17). We also let $W_{\bar{t}+1}(m, m') \equiv m$, for any $(m, m') \in \Omega$.

Proposition S2. There is a unique subgame perfect equilibrium of the game. In this equilibrium, and at any period $t \in \{1, 2, ..., \bar{t}\}$, both firms randomize their working capital according to σ_t^*, bid according to b^* and have W_t expected continuation payoffs at the beginning of period t.

Proof: We prove the proposition using backward induction. It is trivial that the continuation payoffs at the beginning of period $\bar{t}+1$ are described by $W_{\bar{t}+1}$ in period $\bar{t}+1$. We can then apply recursively the following claim:

Claim: *There is a unique equilibrium in the reduced game defined by period t and the continuation payoffs βW_{t+1}. In this equilibrium both firms use the strategy (b^*, σ_t^*), and get expected equilibrium payoffs described by W_t.*

To prove the claim, note that the argument in the first paragraph of the proof of Proposition 4 also applies here using W_{t+1}, Ψ_{t+1}, (S26) and Definition S2 instead of W^*, Ψ^β, (25) and Definition 7, respectively. Thus, in equilibrium both firms use b^*. Under this assumption, the expected payoffs of a firm with cash m and working capital $w \geq \theta - \underline{m}$ that faces a rival that randomizes according to F are equal to:

$$\begin{aligned} m - w + \beta \int_0^w W_t(w + \pi(w) + \underline{m}, \tilde{w} + \underline{m}) F(d\tilde{w}) + \beta \int_w^\infty W_t(w + \underline{m}, \tilde{w} + \pi(\tilde{w}) + \underline{m}) F(d\tilde{w}) \\ = m - (1-\beta)w + \underline{m} \sum_{\tau=1}^{\bar{t}+1-t} \beta^\tau + \beta\pi(w)F(w) + \beta \int_0^{\min\{w, \overline{x}^\beta - \underline{m}\}} \Psi_{t+1}(\tilde{w} + \underline{m}) F(d\tilde{w}) \end{aligned} \tag{S27}$$

since $\Psi_{t+1}(\tilde{w} + \underline{m}) = 0$ for $\tilde{w} + \underline{m} \geq \overline{x}^\beta$ by (20) and Lemma S1.

We use (S27) to show, first, that σ_t^* is an equilibrium with equilibrium expected payoffs given by W_t and, second, that there is no other equilibrium.

To show that σ_t^* is optimal when the other firm uses σ_t^* we distinguish whether the laggard's cash is larger than ν_t. If this is the case, the proof follows by an adaptation of the proof of the case $m, m' \geq \nu^\beta$ in Proposition 4, but using ν_t, $F_{\nu_t}^{\Psi_{t+1}}(\cdot)$, Ψ_{t+1}, W_{t+1}, (S22) and Definition S2 instead of ν^β, $F^{\Psi^\beta}_{\nu^\beta}(\cdot)$, Ψ^β, W^*, (21) and Definition 7, respectively. Otherwise, the proof follows by an adaptation of the proof of the case $m < m'$ and $m \in [0, \nu^\beta)$ and the case $m > m'$ and $m' \in [0, \nu^\beta)$ in Proposition 4, but using ν_t, $F_{t,L,m}$, $F_{t,l,m}$, Ψ_{t+1}, W_{t+1} and (S22), (S23) and (S26) instead of ν^β, $F^\beta_{L,m}$, $F^\beta_{l,m}$, Ψ^β, W^*, and (21), (22) and (25), respectively.

Similarly, one can apply the argument in the proof of Proposition 4 that the value function of the Bellman equation is equal to W^* to show that the expected equilibrium payoffs are equal to W_t.

To prove uniqueness of our equilibrium we explain how to adapt the proof of Propositions 1 and 2. First, we show that one can restrict to working capitals in $[0, \overline{x}^\beta]$ because for the expected payoff function in (S27), a working capital $w \geq \overline{x}^\beta$ is strictly dominated by a working capital of $(\theta - \underline{m})^+$. To see why, note that one can deduce from (S27) and the definition of $\overline{x}^\beta$ in Definition S1 that the difference in expected payoffs between working capital $w > \overline{x}^\beta$ and working capital $(\theta - \underline{m})^+$ is equal to:

$$
\begin{aligned}
&\beta\pi(w)F(w) - \beta\pi((\theta-\underline{m})^+)F((\theta-\underline{m})^+) \\
&\qquad\qquad -(1-\beta)(w-(\theta-\underline{m})^+) \;\leq\; \beta\pi(w) - (1-\beta)(w-(\theta-\underline{m})^+) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad <\; \beta\pi(\overline{x}^\beta) - (1-\beta)(\overline{x}^\beta-(\theta-\underline{m})^+) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad =\; 0,
\end{aligned}
$$

as desired.

We can show that a version of Lemmas A1, A2 and A3 in which $\overline{\nu}^\beta$ is replaced by $\overline{x}^\beta$ and (A1) by:

$$
F_i(w) = e^{\int_0^w \frac{-\pi'(y)}{\pi(y)+\Psi_{t+1}(y+\underline{m})}dy} F_i(0) + \frac{1-\beta}{\beta}\int_0^w \frac{e^{\int_x^w \frac{(-\pi'(y))}{\pi(y)+\Psi_{t+1}(y+\underline{m})}dy}}{\pi(x)+\Psi_{t+1}(x+\underline{m})} \,\forall w \in [0,\nu). \qquad \text{(S28)}
$$

is satisfied by the game defined by the expected payoffs in (S27). We can proceed as in the proof of Propositions 1 and 2 to prove uniqueness when the laggard's cash is greater than ν_t, and when the laggard's cash is less than ν_t, respectively. The only difference is that we use $\overline{\nu}_t$ and $\overline{x}^\beta$ instead of $\overline{\nu}^\beta$, Definition S1 instead of Definition 2, (18) for $\Psi = \Psi_{t+1}$ instead of (7), $F^{\Psi_{t+1}}_{\nu_t}$ instead of F^β, $F_{t,l,m}$ instead of $F_{l,m}$, and $F_{t,L,m}$ instead of $F_{L,m}$, and thus (S22)-(S24) instead of (8)-(10). ■

Finally, we show that the limit of the equilibrium of this finite game as the number of periods goes to infinity is equal to the equilibrium of the model in Section 4. To get this result, we abuse a little bit of the notation and denote by $\sigma^*_{t,\bar{t}}$, $W_{t,\bar{t}}$, $\Psi_{t,\bar{t}}$ and $\nu_{t,\bar{t}}$ the functions σ^*_t, W_t, Ψ_t and ν_t, respectively, to make the dependence in the length of the time horizon of the game explicit.

Lemma S3. $\{\Psi_{t,\bar{t}}\}_{\bar{t}=t+1}^{\infty}$ is an increasing sequence in $\mathcal{P}^\beta$ with limit $\Psi_\infty \in \mathcal{P}^\beta$. Besides, Ψ_∞ is a fixed point of T, where T is defined in (20).

Proof. We first note that by Definition S2, $\Psi_{t,\bar{t}} = T^{\bar{t}-t}(\Psi)$ for Ψ the zero function in $\mathcal{P}^\beta$, and $T^n : \mathcal{P}^\beta \to \mathcal{P}^\beta$ an operator defined recursively by $T^1 = T$ and $T^n = T^{n-1}$ for $n > 1$.

The operator T is monotone in the sense that $\Psi \geq \Psi'$ implies that $T(\Psi) \geq T(\Psi')$. This is a consequence of (20) because (A4) implies that $F_m^{\Psi}(0) \geq F_m^{\Psi'}(0)$. Furthermore, the operator T is continuous and the set $T(\mathcal{P}^\beta)$ is equicontinuous by analogous arguments to Claim 1 and Claim 2, respectively, in the proof of Lemma S2. Thus, $\{\Psi_{t,\bar{t}}\}_{\bar{t}=t+1}^{\infty}$ is an increasing sequence in an equicontinuous set $T(\mathcal{P}^\beta)$. Consequently, $\{\Psi_{t,\bar{t}}\}_{\bar{t}=t+1}^{\infty}$ has a limit in $\mathcal{P}^\beta$ that we denote by Ψ_∞. By continuity of T and the definitions of Ψ_∞ and $\Psi_{t,\bar{t}}$:

$$T(\Psi_\infty) = T(\lim_{\bar{t}\to\infty} \Psi_{t,\bar{t}}) = \lim_{\bar{t}\to\infty} T(\Psi_{t,\bar{t}}) = \lim_{\bar{t}\to\infty} \Psi_{t-1,\bar{t}} = \Psi_\infty.$$

Thus Ψ_∞ is a fixed point of T as desired. ■

Denote by σ^* and W^* the equilibrium of Proposition 4 that corresponds to the function $\Psi^\beta = \Psi_\infty$, where Ψ_∞ is defined in Lemma S3.

Proposition S3. For any $(m, m') \in \Omega$, $\sigma^*_{t,\bar{t}}(\cdot|m, m')$ converges weakly to $\sigma^*(\cdot|m, m')$ and $W_{t,\bar{t}}(m, m')$ converges uniformly to $W^*(m, m')$ as $\bar{t}$ goes to infinity.

Proof. The uniform convergence of $W_{t,\bar{t}}(m, m')$ to $W^*(m, m')$ is a straightforward consequence of Lemma S3 and the definitions of $W^*(m, m')$ and $W_{t,\bar{t}}(m, m')$ in (25) and (S26).

The convergence of $\sigma^*_{t,\bar{t}}$ follows the definitions in (21)-(24) and in (S22)-(S25) and the limits:

$$F_m^{\Psi_{t,\bar{t}}}(w) \xrightarrow{\bar{t}\to\infty} F_m^{\Psi^\beta}(w) \tag{S29}$$

$$F_{\nu_{t,\bar{t}}}^{\Psi_{t,\bar{t}}}(w) \xrightarrow{\bar{t}\to\infty} F_{\nu^\beta}^{\Psi^\beta}(w) \tag{S30}$$

$$\nu_{t,\bar{t}} \xrightarrow{\bar{t}\to\infty} \nu^\beta, \tag{S31}$$

for $\nu_{t,\bar{t}} = \hat{\nu}^{\Psi_{t,\bar{t}}}$, $\Psi^\beta = \Psi_\infty$ and $\nu^\beta = \hat{\nu}^{\Psi_\infty}$.

Equation (S29) follows from Lemma S3 and the application of the property that the limit of the product is equal to the product of the limits if finite and the BCT to (A4) for $\Psi = \Psi_{t,\bar{t}}$. BCT can applied because the same arguments as in Footnote 54 mean here that the integrands of the corresponding limits are uniformly bounded in $\bar{t}$. Equation (S30) follows from the application of Lemma S3 and the BCT to (A6) for $\Psi = \Psi_{t,\bar{t}}$. Equation (S31) follows from the application of the property that the sequence of unique solutions $x_{\bar{t}}$ to the sequence of equations $\Upsilon_{\bar{t}}(x_{\bar{t}}) = 1$ converges to the unique solution x of the limit equation $\lim_{\bar{t}\to\infty} \Upsilon_{\bar{t}}(x) = 1$ when $\Upsilon_{\bar{t}}$ is a strictly increasing function and $\{\Upsilon_{\bar{t}}\}$ converges to a strictly increasing function. This property applies to $x_{\bar{t}} = \nu_{t,\bar{t}}$ and $x = \nu^\beta$ because (a) $\nu_{t,\bar{t}}$ and ν^β are the unique solutions in w to 1 equal to the last line of the right hand side of (A6) for $\Psi = \Psi_{t,\bar{t}}$ and $\Psi = \Psi_\infty$, respectively; (b) the last line of the right hand side of (A6) is strictly increasing in w; and (c) the last line of the right hand side of (A6) evaluated at $\Psi = \Psi_{t,\bar{t}}$ converges to the last line of the right hand side of (A6) evaluated at $\Psi = \Psi_\infty$ by application of the BCT to the two integrals and Lemma S3. ■

S4 A Model of Financial Constraints

In this section, we endogenize the function π in a model in which moral hazard and limited liability restrict the set of acceptable bids. In this model, the firm who wins the procurement contract can divert some funds at the cost of jeopardizing the success of the procurement contract. The main implication is that the minimum acceptable bid for a firm with working capital w is given by an endogenous function π which under natural assumptions is strictly decreasing.

We endogenize the set of acceptable bids in the model[57] of Section 4 by assuming that a bid b is acceptable if and only if the firm has incentives to comply with the procurement contract in case of winning. We begin by formalizing this incentive compatibility constraint, and later we discuss institutional frameworks that enforce it.

Suppose the same game tree as in Section 4 with an additional stage each period after a firm wins the procurement contract and before the firm complies with the contract. To

[57] It is straightforward how to adapt this variation to the static model of Section 3.

describe this new stage, we use a differentiable function $\alpha : [0,\infty)^2 \to [0,\infty)$. As before, the total funds of the firm that wins the procurement contract are equal to its working capital w. We assume that either the working capital is sufficient to pay for the cost c, i.e. $w \geq c$, or the firm gets a loan $d = c - w$ at zero interest rate.[58] In both cases, the firm can choose between complying with the procurement contract and the loan, if any, or defaulting.[59] If the firm complies with both, it starts next period with cash equal to $w+d$, minus the production cost c, plus the procurement price b, minus the loan repayment d and plus the exogenous cash flow $\underline{m}$, i.e.[60]

$$w + b - c + \underline{m}. \tag{S32}$$

If the same firm defaults, it starts next period with cash equal to $\alpha(d,w) \in [0, d+w]$ plus the exogenous cash flow $\underline{m}$, i.e.

$$\alpha(d,w) + \underline{m}, \tag{S33}$$

Thus, $\alpha(d,w)$ denotes the funds of the firm that cannot be expropriated after default.

This model is realistic. For instance, it is a common practice that an entrepreneur who participates in a procurement contest uses a limited liability company (LLC) that can be liquidated in case of default. The entrepreneur could divert the funds from the LLC to its personal account before defaulting. $\alpha(d,w)$ represents the diverted funds that cannot be expropriated even after litigation and $w + d - \alpha(d,w)$ represents the funds that the entrepreneur cannot keep after default because they are either used to pay a compensation to the sponsor, spent on litigation costs or recouped by the lender. In this case, the limited liability status of the LLC implies that neither the lender nor the sponsor can seize any future revenue of the entrepreneur. Finally, default may not restrict the entrepreneur's future ability to borrow if there are other lenders who are willing to lend to the entrepreneur. Formally, this would happen in a model in which firms and lenders are matched only once and lenders do not observe the outcome of past matches.

The comparison of (S32) and (S33) means that a firm that borrows $d = \max\{c-w, 0\}$

[58] A zero interest rate is consistent with a competitive banking sector, no discounting between the moment the money is transferred to the firm and when it pays back and the fact that acceptable bids are risk free.

[59] For simplicity, we do not allow for default only in the loan or only in the procurement contract.

[60] Formally, we identify the case $w \geq c$ and no loan with the case of borrowing $d = 0$.

prefers to comply rather than default if and only if:

$$w + b - c \geq \alpha(\max\{c - w, 0\}, w). \tag{S34}$$

Thus, and no proof is necessary:

Proposition S4. If the firm's continuation value W is increasing in its own cash, only bids greater than $\alpha(\max\{c-w, 0\}, w)+c-w$ are acceptable for a firm with working capital $w \in [0, \infty)$. This implies that the set of acceptable bids is characterized by a function $\pi(w) = \alpha(\max\{c - w, 0\}, w) - w$.

The endogenous function π is strictly decreasing under natural assumptions. For instance, the derivative of π at[61] $w < c$ is negative if substituting one unit of working capital for one unit of debt increases the amount of non-expropriable funds by no more than one unit. Furthermore, one can find meaningful economic conditions under which the linear case, $\pi(w) = \theta - w$, arises. For instance, if $\frac{\partial \alpha(d,w)}{\partial d} = \frac{\partial \alpha(d,w)}{\partial w}$, for $d = c - w$, i.e. if the amount of non-expropriable funds does not change when debt is substituted by working capital.

In reality, the incentive compatibility constraint that we analyse above is usually enforced by different institutional frameworks. A natural example in procurement is the requirement of a surety bond. Indeed, the role of sureties[62] is to guarantee that the firm will comply with the procurement contract. An alternative explanation for the particular case $w < c$ is that the sponsor requires proof of the availability of the required external financing and banks are happy to provide it only if they do not expect the firm to default.

[61] The relevant case is $w < c$ because of two reasons. First, $\theta \leq c$ since θ solves $\pi(\theta) = \alpha(\max\{c - w, 0\}, \theta) - \theta = 0$ and $\pi(c) = \alpha((0, c) - c \leq 0$ because $\alpha(0, w) \leq w$. Second, the analysis in Section 4 only requires characterizing π in the domain $[0, \theta)$, since firms have no incentive to carry more working capital than θ. To see why, recall that a firm with working capital w bids $b^*(w|m, m')$ and note that $w > \theta$ implies that $\pi(w) < 0$, and hence $b^*(w|m, m') < c$. If continuation values are increasing in the firm's cash, a firm does not have incentives to bid below c if the rival does not do it, as in our proposed equilibrium. Bidding less than c does not increase the cases in which the firm wins but reduces the profits in case of winning.

[62] See Footnote 5.