GENERAL ASSET MARKETS, PRIVATE CAPITAL FORMATION, AND THE EXISTENCE OF TEMPORARY WALRASIAN EQUILIBRIUM

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1. Introduction

Economists have become accustomed by now to recognizing that the set of markets on which trade can be carried out is incomplete. As Arrow and Debreu have taught us, a complete system of markets would allow trade in all future commodities and, when there is uncertainty, in all contingent commodities as well. It is patently obvious that the markets in any actual economy are much less numerous than this. Accordingly, economic theorists have become interested in incomplete markets. In particular, they have developed the theory of temporary equilibrium. This theory is extensively described in Hicks' Value and Capital. More recent contributions are admirably surveyed in Grandmont [1982]. There are, it is assumed, complete markets in commodities for current delivery at any one time. There may also be markets for transferring purchasing power between periods, e.g., a money market, bond markets, future commodity markets.

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I owe an obvious and substantial debt to the published work of Green [1973] especially and of Hart [1975] and Milne [1976]. I am also grateful to Frank Milne for discussions which originally stimulated this work, and to Roy Bailey, Volker Böhm, Jacques Drèze, Douglas Gale, Jean-Michel Grandmont, Frank Hahn, Oliver Hart, Robert Kast, Mrudula Patel and Dale Stahl for their further encouragement and help, some of which I have not still yet been able to exploit as I would wish.

Existing studies of temporary equilibrium have tended to assume a particular limited set of markets for transferring purchasing power. Hicks himself, and later Grandmont [1974] assumed that there was a single asset, be it money or bonds, which enabled purchasing power to be transferred forwards or backwards in time. Sondermann [1974] allowed firms to issue shares. Green [1973] has complete markets in future commodities. Radner [1972] has a general system of incomplete contingent commodity markets, together with stock markets, but assumes that agents have correct expectations about prices in markets which are going to open in the future. This assumption of correct expectations rules out the kind of speculation which can occur in Green's economy, and which he had to deal with by his "common expectations" assumption. Hart [1974] has a general set of asset markets, but only in a one-commodity economy. Jordan [1976] has no asset at all, only private capital formation.

The main purpose of this paper, then, is to allow a rather general system of asset or security markets. Following Green [1973], we shall find conditions on agents' expectations which will guarantee the existence of a temporary competitive equilibrium. Green's model will, therefore, be extended to allow what may seem to be a more realistic structure of asset markets. In addition, I shall allow for exogenous uncertainty, since agents cannot be sure what will be their future production possibilities and endowments. Also the assumptions I shall present will allow a somewhat more general form of consumption set than the nonnegative orthant, including one which is consistent with certain domestic production activities.

With general asset markets, an obstacle to the existence of a Walrasian equilibrium is the absence of any natural bounds to traders' asset transactions. In particular, a trader can take an arbitrarily large short position in any one asset market. Many papers circumvent this problem by disallowing short sales. Though in practice institutions may enforce limitations on a trader's short position, such limitations are a form of rationing which is entirely foreign to the pure Walrasian economy. One of the achievements of Green's paper was to discover sufficient conditions for such rationing to be unnecessary because there will exist a pure Walrasian equilibrium even when traders are allowed to take up unlimited short positions. Essentially, in a Walrasian equilibrium the relative prices of various assets must reflect what each trader believes are possible relative returns to these assets: for this to be possible at all, there must be some measure of agreement between all the traders concerning what are possible relative returns to different assets. This is what will be meant here by "common expectations", a notion to be made precise later in Assumption A.11 of Section 3. This condition, we shall see, enables us to put a lower bound on the return to each trader's portfolio, in some eventualities at least. This lower bound, moreover, still applies even as the current price vector tends to one where a trader does want to undertake unbounded asset transactions, and, in particular, to have an unbounded short position in some assets. The condition is significantly weaker than Green's Assumption (4.2) which would require in addition that traders should have expectations which are not "too elastic" --specifically, there should be a fixed common set of future asset values which each trader regards as possible. Further discussion of this aspect of Green's contribution is given in Appendix 1.

I shall now present the formal model of the economy which I shall be using. The outline of the remainder of the paper is contained at the end of the next section.

2. An Economy With General Asset Markets

2.A. Traders and Physical Commodities

The model of the economy is an extension of Green's [1973], and I shall use corresponding notation as far as possible.

In period 1, there are ℓ_1 physical commodities which are traded in current markets. There are also ℓ_2 assets--financial assets, or future commodities, including possibly contingent commodities--which are traded in period 1 as well. In addition, there are ℓ_3 kinds of physical and human capital. If these are traded, they are also included in the ℓ_1 traded physical commodities. In period 2, there are m physical commodities which are all traded in current markets in period 2.

The economy is an exchange economy with a finite set of traders. In period 1, the typical trader buys and sells both physical commodities and assets, to achieve a net demand vector $x^1 \in R^{\ell_1}$ and a net holding of assets $b \in R^{\ell_2}$. He also leaves himself with a stock vector $k \in R^{\ell_3}$ of physical and human capital. Thus his action in period 1 is the triple $a := (x^1,b,k) \in R^{\ell_1+\ell_2+\ell_3}$.

Trade in period 1 takes place at a price vector $p = (p^1, p^2) \in R^\ell$, where $\ell := \ell_1 + \ell_2$, $p^1 \in R^{\ell_1}$ is the vector of prices of physical commodities in period 1, and $p^2 \in R^{\ell_2}$ is the asset price vector. Assume, as usual, that $p > 0$, and normalize the price vector p so that $p \in \Delta$, where $\Delta^\ell := \{p \in R^\ell_+ | \Sigma_g p^1_g + \Sigma_a p^2_a = 1\}$ (g denotes a typical good, and a a typical asset).

Thus, the budget constraint the trader faces in period 1 is: $p^1 x^1 + p^2 b \leqq 0$.

Finally, let X^1 denote the "real" feasible set of the trader in period 1 --i.e. the set of possible triples (c^1, x^1, k) of consumption, net demand and capital stock vectors, which are the "real" variables, as opposed to the financial variables in b. It is assumed that there is a fixed consumption set $C^1 \subseteq R^{\ell_1}$ which represents both the consumption requirements and the labour supply possibilities of the trader. In addition, there is a private production set $Y^1 \subseteq R^{\ell_1} \times R^{\ell_3}_+$ consisting of pairs (y^1, k) where y^1 is the net output of current physical commodities and k is the (nonnegative) capital stock vector. Y^1 may include an endowment ω^1 of current physical commodities. (If there is an endowment k^o of capital goods, it serves to determine the production possibility set $Y^1(k^o)$ but has no other role). Then the trader's feasible set X^1 takes the form:

$$X^1 = \{(c^1, x^1, k) | \exists\, y^1 \in R^{\ell_1} \text{ s.t. } c^1 \in C^1,\ (y^1, k) \in Y^1,\ x^1 = c^1 - y^1\}$$

In period 2 the trader has a production possibility set $Y^2(k,s)$ which depends upon the capital stock vector k, but is also random, and so depends on the state of the world s. Let S denote the set of possible states of the world. The trader's second period feasible set is then $X^2(k,s)$ which is given by:

$$X^2(k,s) = \{(c^2,x^2) \mid c^2 \in C^2(k,s) \text{ and } \exists\, y^2 \in Y^2(k,s) \text{ s.t. } x^2 = c^2 - y^2\}$$

Here $C^2(k,s)$ is the trader's (random) second period consumption set. It is allowed to depend on k because k includes human capital components, for instance.

2.B. Asset Valuation Functions

In addition to his physical goods, the trader also carries over to period 2 the vector of net asset holdings $b \varepsilon R^{\ell_2}$ which he was left with after trade in period 1. The value of these assets depends upon the nature of each asset. But, for a wide class of assets, including bonds, money, contingent securities, future or contingent commodities, and even shares of firms, the value of asset a can be written as a function $r_a(q,s)$ of q, the price vector in period 2, and of s, the state of the world in period 2.

In fact, some examples of assets which can be described in this way are the following:

a. Money: Here $r_{money}(q,s) = q_{money}$

b. Riskless Bonds. For these $r_{bond}(q,s) = (1 + \rho)q_{money}$ where ρ is the coupon rate of interest. One could also have index-linked bonds, with ρ a function $\rho(I(q))$ of a price index $I(q)$.

c. Contingent Securities. If a security pays one unit of money in the set of states of the world E, then:

$$r_{security}(q,s) = \begin{cases} q_{money} & (\text{if } s \in E) \\ 0 & (\text{if } s \notin E) \end{cases}$$

d. Future Commodities. If the asset a is one unit of good g to be delivered in period 2, then:

$$r_a(q,s) = q_g$$

e. Futures Contracts with Delayed Payments. As Stahl [1981] points out, the future commodities considered by Green [1974] and others as in (d) above are unrealistic insofar as they require immediate current payment. It is more usual in practice to allow at least part of the payment to be made later when the future commodity is actually delivered.

Suppose that all of the payment can be put off until the second period. Then formally the price of the contract in the first period is zero. In the second period, the return to the futures contract is:

$$r_a(q,s) = q_g - \gamma_g q_{money}$$

where g is the good to be delivered, and γ_g is the contract price of the future commodity.

Such futures contracts cannot be accomodated directly within my framework. They can however be accommodated indirectly, provided that there is a market for riskless bonds as in (b) above. For then a futures contract with a delayed payment is equivalent to one with current payment together with borrowing whatever is needed for the current payment. If γ_g is the contract price as above, this is equivalent to paying $\gamma_g/(1 + \rho)$ now, borrowing (issuing a riskless bond for) this amount, and then repaying the loan with interest at a cost of γ_g.

f. Contingent future commodities. Just as contingent security is contingent future money, we have here:

$$r_a(q,s) = \begin{cases} q_g & (\text{if } s \in E) \\ 0 & (\text{if } s \notin E) \end{cases}$$

where a is an asset which is one unit of good g to be delivered in period 2 contingent upon event E occurring.

g. Shares in firms. Suppose that the asset a is a share in a firm f which pays a divident in period 2 on each share related to its profit $\pi_f(q,s)$ in period 2. Assume that this profit on each share is independent of the trade which is undertaken in period 1. Then $r_a(q,s) = \pi_f(q,s)$.

h. Options. Let a be any asset whose valuation function is $r_a(q,s)$. Then an option on the asset a can take one of two forms-- a call option or a put option.

Take first a call option. This is the right to buy the asset a at the start of the second period at a specified price $\bar{r}$, say. The option is then an asset a' whose valuation function $r_{a'}(q,s)$ takes the form:

$$r_{a'}(q,s) := \max \{r_a(q,s) - \bar{r}, 0\}$$

For, if the value of asset a falls below $\bar{r}$, the owner of the option will clearly not exercise his right to buy it at $\bar{r}$, so the option is worthless. But if $r_a(q,s) > \bar{r}$, then exercising the option and liquidating the asset a yields a return of $r_a(q,s) - \bar{r}$ on the option.

A put option is the right to sell an asset a at a fixed price $\bar{r}$. So a put option for asset a is a new asset a' whose valuation function $r_{a'}(q,s)$ takes the form:

$$r_{a'}(q,s) := \max \{\bar{r} - r_a(q,s), 0\}$$

So, let $q \in \Delta^m$ denote a possible normalized price vector. For each $q \in \Delta^m$ and each $s \in S$, let $r(q,s)$ denote the vector of values of the various assets, so that $r(q,s) \in R_+^{\ell_2}$ --assuming, as I do, that no asset ever has a negative value. Then $r: \Delta^m \times S \to R_+^{\ell_2}$ is the <u>asset value function</u>.

2.C. <u>General Asset Markets</u>

Although there are certainly many assets which can be described by an asset valuation function $r_a(q,s)$ as described above, there are also some which cannot. In practice, some assets will yield returns which are unlikely to be <u>known</u> functions of the state of the world. A company may default on its bonds in a well-defined set of states of the world. But this set is unlikely to be generally known by all agents. Indeed, it may not be known by the company itself. More seriously still, the occurrence of bankruptcy in the second period depends not only on prices q and the state of the world s --it is also likely to depend crucially on the financial transactions which the company undertook in the first period. Thus, the economy described so far cannot handle bankruptcy or related phenemona which affect the returns to various assets, particularly financial ones. Moreover, an attempt to do so is likely to be difficult, since bankruptcy tends to lead to nonconvex budget sets which destroy existence of a competitive equilibrium.

The difficulties which potential bankruptcy creates will have to be considered later. For the moment, I propose to allow somewhat more general asset markets than those which can be described by asset valuation functions. To do so, I shall follow a suggestion I owe to Jean-Michel Grandmont and his comments on Hammond [1977]. I shall regard the vector $r \in R_+^{\ell_2}$ of non-negative returns to each of the m assets as a random variable and postulate a joint distribution or the set of triples (q,r,s) of commodity market prices, asset returns, and the state of the world. The case when q and s together suffice to determine r is then just an important special case. Let e denote a typical triple (q,r,s).

2.D. Second Period Demands

The trader's budget constraint in period two can now be written simply as:

$$qx^2 \leq rb$$

where $x^2 \in R^m$ is the trader's net demand vector in period two, and $b \in R^{\ell_2}$ is the vector of net asset holdings carried over from period one. Each trader has a utility function $u(c^1,k,c^2,s)$ which depends upon the consumption vectors c^1, c^2 in each of the two periods. It also depends on capital k which includes human capital, it should be remembered. Finally, utility also depends upon the state of the world s in general. This is because the consumer's state of health, the ease with which the consumer can supply various kinds of labour services, and the consumer's capacity to enjoy particular commodities, can all be affected by the state of the world. They can also be affected by the consumer's holding of capital stocks, both human and non-human.

Suppose that the trader is in the second period after the first period action $a := (c^1,b,k)$ has already been chosen. Then the trader is free to choose c^2, x^2 and y^2 subject to the feasibility constraints:

$$c^2 \in C^2(k,s) \quad , \quad y^2 \in Y^2(k,s) \quad , \quad x^2 = c^2 - y^2$$

as well as to the budget constraint:

$$qx^2 \leq rb$$

which is equivalent to:

$$qc^2 \leq rb + qy^2 \quad .$$

It follows that the trader will choose $\hat{y}^2(q;k,s)$ in order to maximize (net) profit qy^2 subject to $y^2 \in Y^2(k,s)$; let $\pi^2(q;k,s)$ denote the resulting net profit. Then the trader chooses $\hat{c}^2$ in order to maximize $u(c^1,k,c^2;s)$ with respect to c^2, subject to the constraint $c^2 \in \beta(b,k;q,r,s)$ where:

$$\beta(b,k;q,r,s) := \{c^2 \in C^2(k,s) \mid qc^2 \leq rb + \pi^2(q;k,s)\}$$

is the trader's second period (consumption) budget set given his previous transactions b, k and given the outcome $e = (q,r,s)$. The maximum utility the trader can enjoy if his action in period one is $a = (c^1,b.k)$ and if the price vector, asset returns, and state of the world in period two turn out to be (q,r,s) is given by:

$$\phi(a;e) \equiv \phi(c^1,b,k;q,r,s) := \max_{c^2} \{u(c^1,k,c^2;s) \mid c^2 \in \beta(b,k;q,r,s)\}$$

The associated utility maximizing plan in period two consists of a triple $(\hat{c}^2,\hat{x}^2,\hat{y}^2)$ where $\hat{y}^2$ is chosen to maximize profit, $\hat{c}^2$ to maximize utility and $\hat{x}^2 = \hat{c}^2 - \hat{y}^2$ is chosen to make up the difference. Write $\xi^2(c^1,b,k;q,r,s) \equiv \xi^2(a;e)$ for the set of optimal choices of the net demand vector x^2 given $(c^1,b,k;q,r,s)$ --then $\xi^2(\ )$ is the trader's second period net demand correspondence.

2.E. First Period Constraint

In the first period the trader is typically uncertain about the outcome $e = (q,r,s) \in \Delta^m \times R_+^{\ell_2} \times S =: E$. This uncertainty is represented, I assume, by a probability measure μ. In fact S is taken to be a metric space, so μ is taken to be defined on the Borel sets of the product space E. The probability measure $\mu(p)$ will be allowed to depend on the current price vector $p \in \Delta^{\ell}$ because the prices of both current physical commodities p^1 and the prices of assets p^2 are assumed to convey to the trader some information about both likely future commodity prices q and future asset returns r. They may even condition expectations about the state of the world s as well if different traders are known to have different information concerning the likely state of the world. Thus the probability measure for the trader is $\mu(p)$ where $\mu\colon \Delta^{\ell} \to \mathcal{M}(E)$ is a mapping from the price simplex Δ^{ℓ} to the set $\mathcal{M}(E)$ of Borel probability measures on the space E.

Now, in period one the trader must be certain that, no matter what happens in period two, with probability one he will be able to afford some consumption vector c^2 in his consumption set $C^2(k,s)$ given the state of the world s and his earlier choice of capital stock k. Thus, his net purchases of assets b and his holdings of capital k must be such that, for almost every possible (q,r,s), there exists some $c^2(q,r,s) \in C^2(k,s)$ which satisfies the budget constraint $qc^2 \leq rb + \pi^2(q;k,s)$ --i.e., the set $\beta(b,k;q,r,s)$ must be non-empty.

Define $E(b,k) := \{(q,r,s) \in E \mid \beta(b,k;q,r,s) \neq \phi\}$ as the set of outcomes (q,r,s) which, given his earlier choices (b,k), allow the trader to get inside his consumption set $C^2(k,s)$. The condition which the trader must meet can then be expressed as:

$$\mu(p)(E(b,k)) = 1$$

Thus, in period one, the trader faces the above constraint, as well as the budget constraint, and the first period possibility constraints. His budget set of possible actions a in period one is therefore:

$$A(p) := \{(c^1,b,k) | c^1 \in C^1,\ \mu(p)(E(b,k)) = 1 \text{ and } \exists\, x^1,y^1$$
$$\text{s.t. } (y^1,k) \in Y,\ x^1 = c^1 - y^1,\ p^1x^1 + p^2b \leq 0\}$$

The budget constraint can be rewritten as:

$$p^1c^1 + p^2b \leq p^1y^1$$

and so the trader will choose $\hat{y}^1(p^1;k)$ in order to maximize first period net profit p^1y^1 subject to $(y^1,k) \in Y^1$, taking the investment plan k as fixed. Let $\pi^1(p^1;k)$ denote the maximized net profit. Then the set $A(p)$ takes the form:

$$A(p) := \{(c^1,b,k) | c^1 \in C^1,\ \mu(p)(E(b,k)) = 1 \text{ and}$$
$$p^1c^1 + p^2b \leq \pi^1(p^1;k)\}$$

2.F. First Period Objective

For a given action $a = (c^1,b,k)$ in period one, the trader is also uncertain what will be his eventual utility, since this is:

$$\phi(a;e) \equiv \phi(c^1,b,k;q,r,s) = \max_{c^2} \{u(c^1,k,c^2;s) | c^2 \in \beta(b,k;q,r,s)\}$$

and so a function of the uncertain triple $e = (q,r,s)$. Assume that the trader's risk preferences are represented by the expected value of the function $u(\cdot,\cdot,\cdot;s)$ over the various possible outcomes $e = (q,r,s)$.

Thus u is taken to be the trader's state-dependent von Neumann-Morgenstern utility function. Then, given the action a as well as the price vector p, the trader's expected utility is:

$$v(a,p) := \int_E \phi(a;e)d\mu(p)$$

where the integration is performed with respect to the (Borel) probability measure $\mu(p)$ on the metric space E of triples $e = (q,r,s)$. v is the trader's "Bellman function". In the first period, the trader will choose an action $\hat{a}$ in order to maximize $v(a,p)$ subject to $a \in A(p)$, where $A(p)$ is the trader's budget set of possible actions as discussed in 2.E. Let $\alpha(p)$ denote the set of such optimal actions--the trader's "action correspondence".

Given $a = (c^1,b,k)$, let $z := (c^1,b)$ denote the corresponding pair of vectors which describes the trader's market transactions. Then the trader's first period demand correspondence $\xi(p)$ consists, for each $p \in \Delta^\ell$, of trade vectors z for which there exists a corresponding $\hat{k} \in R_+^{\ell 3}$ such that $\hat{a} := (\hat{z},\hat{k})$ maximizes $v(a,p)$ subject to $a \in A(p)$.

2.G. Outline of Paper

We shall proceed to study an exchange economy in period one in which each trader maximizes his "Bellman" utility function $v(a,p)$ subject to the constraint $a \in A(p)$. In particular, we shall find conditions which are sufficient to ensure the existence of a temporary Walrasian equilibrium. Both assumptions and proof are, for the most part, natural extensions of Green's.

The first step is to establish when the expected utility function $v(a,p)$ is continuous as a function of both a and p. This is done in Section 4. The second step is to establish for what values of p the trader will be able to find an optimal action a which maximizes $v(a,p)$ subject to $a \in A(p)$. A necessary and sufficient condition for such an $\hat{a}$ to exist will be that $A(p)$ is compact. This is done in Section 5. The third step is to establish when and where the correspondence $A: \Delta^{\ell} \to P(R^{\ell})$ is continuous in the price vector p, so that, using Berge's maximum theorem we know that the trader's first period demand $\xi: \Delta^{\ell} \to P(R^{\ell})$ is upper hemi-continuous. This is done in Section 6. Section 6 also establishes certain properties of a trader's demand correspondence as the price vector p tends to the boundary of points p for which the $A(p)$ is compact, and the demand set $\xi(p)$ is non-empty. The existence of a Walrasian exchange temporary equilibrium is then proved in Section 7 by adapting an argument due originally to Grandmont [1977]. Section 8 provides an example of non-existence which illustrates the role of the assumptions regarding expectations. Section 9 considers overlapping expectations as an alternative sufficient condition for existence. Finally, Section 10 discusses possible extensions and limitations.

I shall mention the assumptions required for each result at the appropriate point, so that the role of each separate assumption can be highlighted. The assumptions are first collected together and discussed in Section 3, which follows immediately.

3. Summary and Discussion of the Assumptions

The first set of assumptions, from A.1 to A.4, concern the form of the model, and most are reasonably standard and unexceptionable.

Assumption 1: The set S of states of the world is a compact metric space.

A.1 does not exclude the possibility that S is finite, nor do any of the later assumptions.

Assumption 2:

(a) In each state of the world $s \in S$, any trader's first and second period consumption sets C^1, $C^2(k,s)$ are closed, convex and bounded below. Moreover, each second period consumption set $C^2(k,s)$ is strictly convex. And the consumption set $C(s) := \{(c^1,k,c^2) \in R^{\ell_1} \times R_+^{\ell_3} \times R^m | c^1 \in C^1,\ c^2 \in C^2(k,s)\}$ is convex.

(b) In each state of the world $s \in S$, any trader's first and second period production sets Y^1, $Y^2(k,s)$ are compact and convex. The overall production set for both periods:

$$Y(s) := \{(y^1,k,y^2) \in R^{\ell_1} \times R_+^{\ell_3} \times R^m | (y^1,k) \in Y^1 \text{ and } y^2 \in Y^2(k,s)\}$$

is also compact and convex, Y^1 admits free disposal of both commodities and capital, while $Y^2(k,s)$ admits free disposal of commodities.

(c) The correspondences $C^2\colon\ R_+^{\ell 3} \times S \to P(R^m)$ and $Y^2\colon\ R_+^{\ell 3} \times S \to P(R^m)$ are both jointly continuous in k and s. And the correspondence $Y^1(k) := \{y^1 \in R^{\ell 1} | (y^1, k) \in Y^1\}$ is continuous in k.

In part (a) of A.2, I have assumed that the second period consumption set $C^2(k,s)$ is always <u>strictly</u> convex, which means that if c^2, $\tilde{c}^2$ both belong to $C^2(k,s)$, $c^2 \neq \tilde{c}^2$, and λ is a number such that $0 < \lambda < 1$, then $\lambda c^2 + (1 - \lambda)\tilde{c}^2 \in \text{int } C^2(k,s)$. This will avoid discontinuity problems on the lower boundary of the consumption set of a kind I propose to discuss elsewhere. It would suffice instead to have $C^2(k,s)$ as an orthant of the form $\{c^2 | c^2 \geqq \underline{c}^2(k,s)\}$ but such an assumption does not seem realistic. After all, one can supply one form of labour or another, or combinations of the two, but one cannot have two completely full-time occupations. More generally, what is really required is that the minimum wealth correspondence:

$$\gamma(q; k,s) := \{\hat{c}^2 \in C^2(k,s) | c^2 \in C^2(k,s) \text{ implies } qc^2 \geq q\hat{c}^2\}$$

should be single valued for all strictly positive price vectors q. Both strict convexity and having $C^2(k,s)$ in the form of an orthant do suffice to ensure this, but so does having no linear segments to the lower boundary of $C^2(k,s)$, except perhaps linear segments parallel to the axes.

In part (b) of A.2, the assumption that the sets Y^1, $Y^2(k,s)$ and $Y(s)$ are all compact and so, in particular, are bounded, may seem unduly restrictive. A partial justification may be offered as follows.

In the first period, even if Y^1 is not bounded, we can first of all follow Debreu ([1959], ch. 5) and make assumptions which suffice to ensure that the set of attainable allocations of commodities in the first period is bounded. Then it is no loss to assume that, for any trader, the set:

$$Z := \{y^1 \in R^{\ell_1} | \exists\, k \in R_+^{\ell_3} \text{ s.t. } (y^1, k) \in Y^1\}$$

of feasible net commodity output vectors in the first period is bounded too, because the space of attainable allocations of commodities is bounded. Second, we can go on to assume, as do Sondermann ([1974], p. 245, (T.3)), Chetty and Dasgupta ([1978], p. 37, Assumption (T.1b)), that the trader can only produce bounded outputs of capital k if the net output vector y^1 is restricted to a bounded set, and so if the inputs are bounded. Diewert ([1978], p. 90, fn. 4) mentions a similar assumption but then goes on to assume, as I do, that the production set is bounded--see Diewert ([1978], p. 90, Assumption 3.1(d)). In Hammond ([1975], p. 5, Assumption (A.3)) I made the formally weaker assumption that fixed inputs allow only bounded outputs. However, if we also assume that the production correspondence mapping input vectors to feasible output vectors is upper hemi-continuous, then it will also follow that bounded inputs allow bounded outputs.

Thus, if we assume both that the set of attainable allocations of commodities in the first period is bounded, and that bounded inputs produce bounded outputs, it follows that it is no loss to assume that Y^1 is bounded. But then the set of feasible capital stock vectors $k \in R_+^{\ell_3}$

is also bounded. Repeating our assumption that bounded inputs produce bounded outputs it follows that each set $Y^2(k,s)$ is bounded and also that the two-period production set $Y(s)$ is bounded.

Let me conclude this discussion by remarking that it is not sufficient to assume, as Radner [1972] does in effect, that the set of actually attainable allocations is bounded. The reason is that, in a general temporary equilibrium, when markets in the future may not clear, the fact that attainable allocations are actually bounded does not suffice to bound the allocations which different agents believe are attainable, because their beliefs are likely to be inconsistent. Radner [1972], of course, postulated "common expectations", so that markets in the future did have to clear.

The next assumption concerns the traders' utility functions:

<u>Assumption 3</u>: In each state of the world $s \in S$ any trader has a state-dependent von Neumann-Morgenstern utility function $u(c^1,k,c^2,s)$ which is bounded and continuous. It is also concave as a function of (c^1,k,c^2) for each $s \in S$, and strictly monotone as a function of (c^1,c^2) for each $k \in R_+^{\ell_3}$ and each $s \in S$, throughout its domain of definition, which is the Cartesian product of the sets:

$$C(s) := \{(c^1,k,c^2) | c^1 \in C^1,\ c^2 \in C^2(k,s)\}$$

for each $s \in S$.

The next three assumptions concern the traders' expectation measures $\mu(p)$:

Assumption 4: For every current price vector $p \in \Delta^{\ell}$ trader's beliefs about the possible values of the triple $e = (q,r,s)$ of commodity prices, asset returns and the state of the world in period two are represented by a probability measure $\mu(p)$ on the Borel sets of the metric space $E := \Delta^m \times R_+^{\ell_2} \times S$. The probability that $q > 0$ is always equal to one.

Notice that A.4 requires a trader to believe that the probability of a physical commodity having a zero price in period 2 is zero--since all traders have strictly monotone preferences, we know that any equilibrium price vector in period 2 must be strictly positive, so this assumption appears reasonable.

Define $E^* := \text{int } \Delta^m \times R_+^{\ell_2} \times S$. Then A.4 requires that, for all $p \in \Delta^{\ell}$, $\mu(p) \in \mathcal{M}(E^*)$, the set of Borel measures on E^*.

To prove continuity of expected utility, one extra assumption is needed, which says that expectations must be continuous:

Assumption 5: The Borel probability measure $\mu(p)$ on E is weakly continuous as a function of p.

This is the standard continuity assumption of Grandmont [1972], [1977], Green [1973], and Jordan [1977].

In order to prove that all assets are desirable, the following assumption rules out assets which are clearly worthless:

Assumption 6: For every asset a, there is a positive probability that $r_a > 0$.

Next, define the marginal probability measure $\psi(p)$ on the space $R_+^{\ell_2}$ of asset returns r so that, for every Borel set $F \subseteq R_+^{\ell_2}$, $\psi(p)(F) := \mu(p)\{(q,r,s) \in E^* | r \in F\}$. Since it is relative security prices which matter, define $K(p)$ as the convex cone generated by the support of the measure $\psi(p)$. So $K(p)$ is the smallest closed convex cone for which $r \in K(p)$ with probability one.

Following Green's Assumption 2.3 (i) ([1973], p. 1105), but a little weaker:

Assumption 7: For every $p \in \text{int } \Delta^\ell$, $K(p)$ has a non-empty interior, relative to R^{ℓ_2}.

This rules out point expectations, of course. It also implies that for every $p \in \text{int } \Delta^\ell$, no group of assets is stochastically linearly dependent. In Section 5 it is shown that the above assumptions guarantee that a trader's action correspondence $\alpha(p)$ is non-empty valued if and only if $p \in D$, where:

$$D := \{p \in \text{int } \Delta^\ell | p > 0 \text{ and } p^2 \in \text{int } K(p)\}$$

$p > 0$ is necessary because the consumer desires both physical commodities and assets. The condition $p^2 \in \text{int } K(p)$, in combination with A.1, A.2 and especially A.7, is both necessary and sufficient to rule out unbounded speculative transactions in the asset markets. Notice that $p^2 \in K(p)$ means that the consumer regards asset prices r proportional to p^2 as "possible", roughly speaking.

To ensure upper hemi-continuity of the action correspondence $\alpha(p)$, the following assumptions will be made:

Assumption 8: There exist $\tilde{c}^1 \in C^1$, and $(\tilde{y}^1, \tilde{k}) \in Y^1$ such that $\tilde{c}^1 < \tilde{y}^1$ and then for each $s \in S$, there exist $\tilde{c}^2(s) \in C^2(\tilde{k}, s)$ and $\tilde{y}^2(s) \in Y^2(\tilde{k}, s)$ such that $\tilde{c}^2(s) < \tilde{y}^2(s)$.

This means that there is a strictly negative net trade vector in each period and in each state of the world which still enables the consumer to survive. It generalizes the usual assumption that endowments are in the interior of the trader's consumption set, and is clearly needed, unless one goes on to consider resource-relatedness and its generalizations as in McKenzie [1981].

Assumption 9: The correspondence supp μ: $\Delta^{\ell} \to E$ is upper hemi-continuous.

This assumption is somewhat obscure--cf. Green ([1973], (3.2), p. 1109). Example 6.5 may help to explain why it is needed.

The final assumptions apply to the group of traders I as a whole, rather than each trader i separately. They guarantee existence of a temporary Walrasian equilibrium. The prefix i denotes an object referring to trader i.

Assumption 10: The set $P := \bigcap_{i \in I} {}^{i}D$ is non-empty and convex. (where ${}^{i}D$ is defined after A.7)

It will become evident in Section 5 that trader i's demands for assets are unbounded unless $p \in {}^iD$, which justifies the need to assume that P is non-empty. The assumption that P is convex, on the other hand, is rather obscure. What is required really is that P be a set on which we can apply a suitable fixpoint theorem such as Kakutani's. Example 8.2 is offered as a justification for this assumption, which corresponds to Green's ([1973], Assumption 3.4, p. 1110). A perhaps more transparent version of A.10 can be stated as:

Assumption 10*: P is non-empty. Also, whenever $i \in I$ and $\lambda(1)p^2(1) \in \text{supp } {}^i\psi(p(1))$, $\lambda(2)p^2(2) \in \text{supp } {}^i\psi(p(2))$, while $(p_g/p_h) = [(p_g(1))/(p_h(1))]^\theta[(p_g(2))/(p_h(2))]^{1-\theta}$ for some θ satisfying $0 \leq \theta \leq 1$ and for all $g, h = 1$ to $\ell_1 + \ell_2$, then $\lambda p^2 \in \text{supp } {}^i\psi(p)$. This is a kind of "log-convexity" which may be more meaningful insofar as relative prices mean more than absolute prices.

Define $K^*(p) := \bigcap_{i \in I} {}^iK(p)$.

$K^*(p)$ is the convex cone generated by the set of relative asset values which, at price p, all traders in common believe are possible. Notice that

$$P = \{p \in \text{int } \Delta^\ell \mid p^2 \in \bigcap_{i \in I} \text{int } {}^iK(p)\}$$

$$= \{p \in \text{int } \Delta^\ell \mid p^2 \in \text{int } K^*(p)\} \quad .$$

Under A.9, the correspondence ${}^iK(p)$ is upper hemi-continuous, and so therefore is $K^*(p)$. As we shall see in Section 6 (Lemma 6.8) the set

${}^iD$ is open, for each $i \in I$. So therefore is P. So, if $\bar{p} \in \text{bd } P$, then $\bar{p}^2 \notin \text{int } K^*(p)$. But, because $K^*(p)$ is upper hemi-continuous, $\bar{p}^2 \in K^*(\bar{p})$. Thus $\bar{p} \in \text{bd } P$ if and only if $\bar{p}^2 \in \text{bd } K^*(\bar{p})$. Then the last assumption which has a counterpart in Green's work is:

Assumption 11: If $\bar{p} \in \text{bd } K^*(\bar{p})$, then there exists $p \in P$ such that $p^2 \in \text{int } K^*(\bar{p})$.

This is the "boundary expectations" assumption which is used to prove existence in Section 7--see Theorem 7.2. It is an assumption which is notably weaker than Green's, who requires some degree of "inelastic" expectations as well. The counter-part of his Assumption (4.2) for the general asset economy of this paper, is:

Assumption 11*: There exists a fixed open set $C \subset R_+^{\ell 2}$ such that, for all $p \in \Delta^\ell$, $C \subseteq K^*(p)$.

The final assumption is an alternative to A.10 which then enables us to dispense completely with an assumption like A.11. It is an assumption which I have called "overlapping expectations" (see Hammond [1980]) and is the dual of Hart's existence condition [1974]).

Assumption 10' (overlapping expectations): For all $p \in \Delta^\ell$, the convex cone $K^*(p)$ has a non-empty interior.

In fact it is obvious that A.10' is actually implied directly by Green's "common expectations" assumption A.11*, and so, incidentally, it follows that A.11* is sufficient to prove existence of equilibrium (in combination with Assumption A.1 to A.9, of course) without any need to assume A.10. In particular, as shown in Hammond [1980], A.10' implies that P is non-empty (in combination with the earlier assumptions).

All of these last assumptions, of course, require traders to have common expectations only as regards relative asset values--they say nothing about the value of assets relative to physical commodities, unless some of the assets happen to be future physical commodities.

4. Continuity of Expected Utility

Green ([1973], Lemma 3.4, part (i), p. 1112) was able to assert, quite correctly, that continuity of the expected utility function $v(a,p)$ could be demonstrated using the method of Grandmont [1972, 1974] or Sondermann [1974]. Yet Green's economy circumvents a major difficulty by assuming that the consumption set is the non-negative orthant. This means that, even if the consumer chooses to be on the lower boundary of the consumption set in the second period, the budget set $\beta(b,k;q,r,s)$ will still be lower hemi-continuous in (q,r,s) even where it collapses to a single point. This, however, is just not true when there is a more general consumption set. If $\beta(b,k;q,r,s)$ consists only of boundary points, then β may fail to be lower hemi-continuous. Normally, in demand theory, such cases are ruled out by assuming that the consumer's endowment is in the interior of his consumption set. But, in the model of temporary equilibrium, even if this assumption is made, the trader may prefer first period consumption enough to drive him to the lower boundary of his second period consumption set, and there bring him to a discontinuity.

An example illustrating this possibility will be presented elsewhere. Here, I have circumvented this problem by assuming that the second period consumption set $C^2(k,s)$ is strictly convex, though in fact I still have to prove that this does guarantee continuity. This I propose to do next.

Define $E^* := \text{int}\ \Delta^m \times R_+^{\ell_3} \times S$ and define the set Γ by:

$$\Gamma := \{(b,k;q,r,s) \in R_+^{\ell_3} \times E^* | \beta(b,k;q,r,s) \neq \phi\}$$

<u>Lemma 4.1</u>: The budget correspondence $\beta: \Gamma \to P(R^m)$ is continuous and compact-valued under Assumptions 1 and 2.

<u>Proof</u>:

(1) Recall the definition of β, which is:

$$\beta(b,k;q,r,s) := \{c^2 \in C^2(k,s) | qc^2 \leq rb + \pi^2(q;k,s)\}$$

where

$$\pi^2(q;k,s) := \max \{qy^2 | y^2 \in Y^2(k,s)\}$$

Because the correspondence $Y^2: R_+^{\ell_3} \times S \to P(R^m)$ is continuous and compact valued, the function $\pi^2(.;.,.)$ is well-defined and continuous. Because $q > 0$ throughout Γ, and because $C^2(k,s)$ is bounded below, β is compact valued. It also has a closed graph, because π^2 is continuous and so is the correspondence $C^2(.,.)$.

(2) Let $(b^\nu, k^\nu; q^\nu, r^\nu, s^\nu)$, $(\nu = 1,2,\ldots)$ be any sequence in Γ which converges to $(\bar{b}, \bar{k}; \bar{q}, \bar{r}, \bar{s})$, also in Γ. Then define:

$$w^\nu := r^\nu b^\nu + \pi^2(q^\nu; k^\nu, s^\nu) \quad , \quad (\nu = 1,2,\ldots)$$

$$\bar{w} := \bar{r}\bar{b} + \pi^2(\bar{q}; \bar{k}, \bar{s})$$

(3) Suppose that $c^{2\nu} \in \beta(b^\nu, k^\nu; q^\nu, r^\nu, s^\nu)$, $(\nu = 1,2,\ldots)$. To prove upper hemi-continuity of β, it now suffices to show that $c^{2\nu}$ has a convergent subsequence. In fact, because $C^2(\bar{k}, \bar{s})$ is bounded below and C^2 is a continuous correspondence, there exists $\underline{c}^2$ such that $c^{2\nu} \geq \underline{c}^2$ for all large ν. So, for any good $g = 1$ to m:

$$c_g^{2\nu} \leq \underline{c}_g^2 + \left[\frac{w^\nu - \sum_{h \neq g} q_h^\nu \underline{c}_h^2}{q_g^\nu}\right]$$

Because $w^\nu \to \bar{w}$ and $q^\nu \to \bar{q} > 0$ as $\nu \to \infty$ it follows that, for any $\varepsilon > 0$:

$$c_g^{2\nu} \leq \underline{c}_g^2 + \left[\frac{\bar{w} - \sum_{h \neq g} \bar{q}_h \underline{c}_h^2}{\bar{q}_g}\right] + \varepsilon$$

for all large ν. So the sequence $c^{2\nu}$ is bounded above and is also bounded below by $\underline{c}^2$ for all large ν. Therefore it certainly has a convergent subsequence.

(4) To prove lower hemi-continuity of β, consider any $\bar{c}^2 \in \beta(\bar{b},\bar{k};\bar{q},\bar{r},\bar{s})$. Then $\bar{q}\bar{c}^2 \leq \bar{w}$. We must find a sequence $c^{2\nu} \in \beta(b^\nu,k^\nu;q^\nu,r^\nu,s^\nu)$, $(\nu = 1,2,\ldots)$ which converges to $\bar{c}^2$.

(5) Because $C^2(.,.)$ is lower hemi-continuous, there exists a sequence $\bar{c}^{2\nu} \in C^2(k^\nu,s^\nu)$, $(r = 1,2,\ldots)$ which converges to $\bar{c}^2$.

(6) If $\bar{q}\bar{c}^2 < \bar{w}$ then, for large ν, $q^\nu\bar{c}^{2\nu} < w^\nu$ and so $\bar{c}^{2\nu} \in \beta(b^\nu,k^\nu;q^\nu,r^\nu,s^\nu)$, as required. So suppose that $\bar{q}\bar{c}^2 = \bar{w}$. Then, whenever $q^\nu\bar{c}^{2\nu} \leq w^\nu$ we can again choose $\bar{c}^{2\nu} \in \beta(b^\nu,k^\nu;q^\nu,r^\nu,s^\nu)$, so we shall suppose that, after selection of a subsequence if necessary, $q^\nu\bar{c}^{2\nu} > w^\nu$ for all ν.

(7) Because $(b^\nu,k^\nu;q^\nu,r^\nu,s^\nu) \in \Gamma$, $(\nu = 1,2,\ldots)$ there is a sequence of points $\tilde{c}^{2\nu} \in \beta(b^\nu,k^\nu;q^\nu,r^\nu,s^\nu)$, $(\nu = 1,2,\ldots)$. If $\tilde{c}^{2\nu}$ converges to $\bar{c}^2$, this is the required sequence. So suppose that, for every such sequence $\tilde{c}^{2\nu}$, there is always some subsequence which converges to a point $\tilde{c}^2 \neq \bar{c}^2$. From now on, we assume that the subsequence is the sequence itself.

(8) Notice that $\tilde{c}^2 \in \beta(\bar{b},\bar{k};\bar{q},\bar{r},\bar{s})$ because β has a closed graph. Thus both $\tilde{c}^2$ and $\bar{c}^2$ are members of $C^2(\bar{k},\bar{s})$ which is a strictly convex set. Therefore the point $(1/2)(\tilde{c}^2 + \bar{c}^2)$ is in the interior of $C^2(\bar{k},\bar{s})$. But $(1/2)\bar{q}(\tilde{c}^2 + \bar{c}^2) \leq \bar{w}$ because $\tilde{c}^2$ and $\bar{c}^2$ both are members of $\beta(\bar{b},\bar{k};\bar{q},\bar{r},\bar{s})$. So there exists $\hat{c}^2 < (1/2)(\tilde{c}^2 + \bar{c}^2)$ such that $\hat{c}^2 \in C^2(\bar{k},\bar{s})$ and then $\bar{q}\hat{c}^2 < \bar{w}$.

(9) Because $C^2(.,.)$ is lower hemi-continuous and $\hat{c}^2 \in C^2(\bar{k},\bar{s})$, there must exist a sequence $\hat{c}^{2\nu} \in C^2(k^\nu,s^\nu)$, $(\nu = 1,2,\ldots)$ which converges to $\hat{c}^2$. Then, as $\bar{q}\hat{c}^2 < \bar{w}$, it follows that $q^\nu\hat{c}^{2\nu} < w^\nu$ for all large ν.

(10) Define $\lambda^{\nu} := (q^{\nu}\bar{c}^{2\nu} - w^{\nu})/(q^{\nu}\bar{c}^{2\nu} - q^{\nu}\hat{c}^{2\nu})$, $(\nu = 1,2,\ldots)$. Then $0 < \lambda^{\nu} < 1$. Also, because $q^{\nu}\bar{c}^{2\nu} - w^{\nu} \to \bar{q}\bar{c}^{2} - \bar{w} = 0$ and because $q^{\nu}\bar{c}^{2\nu} - q^{\nu}\hat{c}^{2\nu} \to \bar{q}\bar{c}^{2} - \bar{q}\hat{c}^{2} = \bar{w} - \bar{q}\hat{c}^{2} > 0$ it follows that $\lambda^{\nu} \to 0$ as $\nu \to \infty$.

(11) Now construct $c^{2\nu} := \lambda^{\nu}\hat{c}^{2\nu} + (1 - \lambda^{\nu})\bar{c}^{2\nu}$, $(\nu = 1,2,\ldots)$. Because $C^{2}(k^{\nu},s^{\nu})$ is convex, it always includes $c^{2\nu}$. Because of the definition of λ^{ν}, $q^{\nu}c^{2\nu} = w^{\nu}$. Therefore $c^{2\nu} \in \beta(b^{\nu},k^{\nu}; q^{\nu},r^{\nu},s^{\nu})$. Because $\lambda^{\nu} \to 0$, $c^{2\nu} \to \bar{c}^{2}$. So $c^{2\nu}$ is a sequence which converges to $\bar{c}^{2}$, contradicting (7). Q.E.D.

The rest of the proof that the expected utility function $v(a,p)$ is jointly continuous in a and p is somewhat delicate. Recall that $v(a,p)$ was defined in Section 2F as:

$$v(a,p) := \int_{E} \phi(a; q,r,s)d\mu(p)$$

where:

$$\phi(a; e) \equiv \phi(c^{1},b,k; q,r,s) := \max_{c^{2}} \{u(c^{1},k,c^{2}; s) | c^{2} \in \beta(b,k; q,r,s)\} .$$

It follows that ϕ is only well defined for $(c^{1},b,k,q,r,s) \in C^{1} \times \Gamma$ where

$$\Gamma := \{(b,k,q,r,s) \in R^{\ell_2} \times R_{+}^{\ell_3} \times E^{*} | \beta(b,k; q,r,s) \neq \phi\}$$

as before. Thus, for $v(a,p)$ to be well-defined, we need to be sure that:

$$\mu(p)(E^*(b,k)) = 1$$

where

$$E^*(b,k) := E(b,k) \cap E^* = \{(q,r,s) \in \text{int}\ \Delta^m \times R_+^{\ell_2} \times S \mid (b,k,q,r,s) \in \Gamma\}$$
$$= \{(q,r,s) \in E^* \mid \beta(b,k;q,r,s) \neq \phi\}$$

Then

$$v(a,p) = \int_{E^*(b,k)} \phi(c^1,b,k;q,r,s)d\mu(p) \quad .$$

Let $B(p) := \{(b,k) \in R^{\ell_2} \times R_+^{\ell_3} \mid \mu(p)(E^*(b,k)) = 1\}$. Then $v(a,p)$ is well defined provided that $a \in X^1 \times B(p)$ where X^1 is as defined in Section 2.A. Notice that $B(p)$ is empty if $\mu(p)(\text{int}\ \Delta^m \times R_+^{\ell_2} \times S) < 1$; A.4 rules out this possibility.

Because the domain of integration of the function $\phi(c^1,b,k;q,r,s)$ is $E^*(b,k)$ which depends on b and k, it is not possible to apply here the arguments of Grandmont [1972] to prove continuity of the function $v(a,p)$. If it were true that the expectations measure $\mu(\cdot)$ satisfied the stronger continuity property:

<u>Assumption 5*</u>: For every Borel set $F \subseteq E$, the measure $\mu(p)(F)$ of F is a continuous function p on Δ^m.

as postulated by Delbaen [1974] and by Sondermann [1974], then it is possible to demonstrate that $v(a,p)$ is continuous, even though the domain $E^*(b,k)$ is varying. This was done in Hammond [1977]. However, given that the budget correspondence $\beta(b,k;\ q,r,s)$ is now continuous (which it was not in Hammond [1977]), it is possible to use an ingenious argument due to Jordan [1977] in order to prove that $v(a,p)$ is continuous even with the weaker A.5, that $\mu(p)$ is continuous in the topology of weak convergence of probability measures. Jordan suggests, in the context of the present paper, considering the space of Borel probability measures:

$$M := \mathcal{M}(R_+^{\ell_3} \times R^m \times E^*)$$

on the product space of capital stock vectors $k \in R_+^{\ell_3}$, second period consumption vectors $c^2 \in R^m$, and uncertain outcomes $e = (q,r,s) \in E^*$. Taking into account the constraints imposed by the trader's consumption set, the trader must choose a Borel probability measure $\pi \in \mathcal{M}(R_+^{\ell_3} \times r^m \times E^*)$ in order to satisfy the constraint:

$$\pi[\{(k,c^2;\ q,r,s)\,|\,c^2 \in C^2(k,s)\}] = 1 \quad .$$

or else:

$$\text{supp}\ \pi \subseteq \hat{C}^2$$

where $\hat{C}^2$ is the "extended" second period consumption set:

$$\hat{C}^2 := \{(k,c^2;\ q,r,s) \in R_+^{\ell_3} \times R^m \times E^* | c^2 \in C^2(k,s)\}$$

and supp π is the smallest set F which is closed relative to $R_+^{\ell_3} \times R^m \times E^*$ and for which $\pi(F) = 1$.

The trader's objective function can now be expressed in the form:

$$U\ (c^1,\pi) := \int_{\hat{C}^2} u(c^1,k,c^2;\ s)d\pi$$

where the integration takes place over the fixed set $\hat{C}^2$, which always contains the support of the measure π.

Lemma 4.2: Under A.3, the objective function $U(c^1,\pi)$ is jointly continuous in the variables (c^1,π), when the space M is given the topology of weak convergence.

Proof: By A.3, the utility function $u(c^1,k,c^2;\ s)$ is continuous and is also bounded for each s on the domain $C^1 \times \hat{C}^2$. This suffices to establish that U is continuous--see, for example, Grandmont ([1972], Theorem A.3, p.56). Q.E.D.

The conditions for the Borel probability measure $\pi \in M$ to be feasible, given the first period action $a = (c^1,b,k)$, are essentially a special case of those discussed in Jordan ([1977], Section 2.5). First, the marginal probability measure which π induces on E^* must be the same as $\mu(p)$, so that, for all Borel sets $F \subseteq E^*$:

$$\pi(R_+^{\ell_3} \times R^m \times F) = \mu(p)(F) \quad .$$

Second, π must attach probability one to the points $(k,c_2; q,r,s)$ which lie in the graph of the budget correspondence with $\bar{b}$ and $\bar{k}$ fixed:

$$\pi[\{(k,c^2,q,r,s) \in R_+^{\ell_3} \times R^m \times E^* | k = \bar{k} \text{ and } c^2 \in \beta(\bar{b},\bar{k};\, q,r,s)\}] = 1$$

or

$$\text{supp } \pi \subseteq H^2(\bar{b},\bar{k})$$

where

$$H^2(\bar{b},\bar{k}) := \{(k,c^2,q,r,s) \in R_+^{\ell_3} \times R^m \times E^* | k = \bar{k} \text{ and } c^2 \in \beta(\bar{b},\bar{k};\, q,r,s)\} .$$

Notice that $H^2(\bar{b},\bar{k}) \subseteq \hat{C}^2$ so that the chosen measure π will certainly lie in the extended consumption set provided that supp $\pi \subseteq H^2(\bar{b},\bar{k})$.

For each $(b,k,p) \in R^{\ell_2} \times R_+^{\ell_3} \times \Delta^\ell$, let $\Pi(b,k,p)$ denote the set of all Borel probability measures $\pi \in M(R_+^{\ell_3} \times R^m \times E^*)$ which satisfy these two conditions. Then $\Pi(b,k,p)$ is the trader's constraint set when he comes to choose a probability measure π with his first period action $a = (c^1,b,k)$ already determined. It is then evident that:

$$v(a,p) \equiv v(c^1,b,k,p) = \max_{\pi} \{\int_{\hat{C}^2} u(c^1,k,c^2;\, s)d\pi | \pi \in \Pi(b,k,p)\}$$

<u>Lemma 4.3</u>: Under assumptions A1, A2, A3, A4 and A5, the correspondence Π: $R^{\ell_2} \times R_+^{\ell_3} \times \Delta^\ell \to P(M(R_+^{\ell_3} \times R^m \times E^*))$ defined above is compact valued and continuous when the space of Borel measures $M(R_+^{\ell_3} \times R^m \times E^*)$ is given the the weak topology.

Proof: Define the correspondence $\Pi^*: R^{\ell_2} \times R_+^{\ell_3} \times \Delta^{\ell} \to P(M(R^m \times E^*))$ by

$$\Pi^*(b,k,p) = \{\pi \mid \pi(R^m \times F) = \mu(p)(F) \text{ for all Borel sets } F \subseteq E^* \text{ and } \operatorname{supp} \pi \in \{(c^2,e) \mid c^2 \in \beta(b,k,e)\}\} .$$

Then $\Pi(b,k,p) = \{\pi \in M(R^{\ell_3} \times E^*) \mid \exists\, \pi^* \in \Pi^*(b,k,p)$ such that $\pi[\{k\} \times F] = \pi^*(F)$ and $\pi[K \times F] = 0$ unless $k \in K$ for all Borel sets $K \subseteq R_+^{\ell_3}$, $F \subseteq R^m \times E^*\}$.

Thus $\Pi(b,k,p) = \{k\} \times \Pi^*(b,k,p)$, in effect. It follows from Hildenbrand ([1974], Prop. 4, p. 25 and Prop. 8, p. 27), for example, that Π will be compact valued and continuous provided that Π^* is compact valued and continuous. But Jordan ([1977], Lemmas 5.2 and 5.7) effectively proves that Π^* is indeed compact valued and continuous, so no further proof is necessary. Q.E.D.

Theorem 4.4: Under assumptions A.1, A.2, A.3, A.4 and A.5 the utility function $v(a,p)$ is jointly continuous in a and p.

Proof: This follows readily from Berge's maximum theorem. The objective $U(c^1,\pi)$ is continuous when the space of Borel measures is given the topology of weak convergence, and so is the correspondence Π. Q.E.D.

5. Existence of Expected Utility Maximizing Demands

In this section, we shall follow Green ([1973], Theorem 2.1, p. 1107) and characterize those price vectors $p \in \Delta^{\ell}$ for which there exists a non-empty set $\alpha(p)$ of first period actions $a = (c^1,b,k)$ which maximize the expected utility function $v(a,p)$ over the budget set $A(p)$. First, notice that $u(c^1,k,c^2; s)$ is strictly monotone in (c^1,c^2), by A.3. Then, if A.6 is also assumed, obviously:

<u>Lemma 5.1</u>: Under assumptions A.2, A.3, A.5 and A.6, the expected utility function $v(a,p) \equiv v(c^1,b,k,p)$ is strictly monotone in (c^1,b), for every $p \in \Delta^{\ell}$.

Equally obviously:

<u>Corollary 5.2</u>: If $v(a,p)$ is strictly monotone in (c^1,b), and if it has a maximum with respect to a over the budget set $A(p)$, then $p > 0$. .

As in Section 3, for every Borel set $F \subseteq R_+^{\ell_2}$, define

$$\psi(p)(F) := \mu(p)(\text{int } \Delta^m \times E \times S)$$

as the marginal probability measure on R^{ℓ_2}, and define $K(p)$ as the convex cone generated by the support of the measure $\psi(p)$. Recall A.7, which states that $K(p)$ has a non-empty interior, for every $p \in \Delta^{\ell}$.

<u>Lemma 5.3</u>: Under assumptions A.2, A.3, A.5, A.6 and A.7, if $p > 0$ but $p^2 \notin \text{int } K(p)$, then $v(a,p)$ has no maximum with respect to a over the budget set $A(p)$.

Proof: The proof is close to that of Green ([1973], Lemma 2.5, p. 1108).

(1) Because $p^2 > 0$ and $p^2 \notin \text{int } K(p)$, there exists $t \in R^{\ell_2}$ such that $p^2 t \leq 0$ and, for all $y \in \text{int } K(p)$, $yt > 0$, while $yt \geq 0$ for all $y \in K(p)$.

(2) Suppose $yt \leqq 0$ for all $y \in \text{supp } \psi(p)$. Then $yt \leqq 0$ for all $y \in \text{co supp } \psi(p)$, and for all $y \in K(p)$. But there exists $y^0 \in \text{int } K(p)$ by A.7, and then $y^0 t > 0$, by (1), which is a contradiction. So there exists $r^* \in \text{supp } \psi(p)$ such that $r^* t > 0$. It follows that there is an open ball $N(r^*)$ with centre r^* such that $rt > 0$ for all $r \in N(r^*)$.

(3) Suppose $\psi(p)(N(r^*)) = 0$. Let $V_1 := \text{supp } \psi(p) \cap N(r^*)$, $V_2 := \text{supp } \psi(p) \backslash N(r^*)$. Then V_2 is closed, and $r^* \in \text{supp } \psi(p) \backslash V_2$. Also $\psi(p)(V_2) = 1$. This contradicts the definition of $\text{supp } \psi(p)$. Therefore $\psi(p)(N(r^*)) > 0$, which implies that $\psi(p)(V_1) > 0$ because $\psi(p)(V_1) = \psi(p)(N(r^*))$.

(4) Define $\bar{V} := \{r \in \text{supp } \psi(p) \mid rt > 0\}$. Then $V_1 \subseteq \bar{V}$ and so $\psi(p)(\bar{V}) \geqq \psi(p)(V_1) > 0$.

(5) Suppose it were true that $\hat{a} = (\hat{c}^1, \hat{b}, \hat{k})$ maximized $v(a,p)$ with respect to a subject to $a \in A(p)$. Choose $a = (\hat{c}^1, \hat{b} + t, \hat{k})$ instead. Since, by (1), $p^2 t \leqq 0$, it follows that $p^1 \hat{c}^1 + p^2(\hat{b} + t) \leq \pi^1(p^1; \hat{k})$. Also, for all $e = (q,r,s) \in \text{supp } \mu(p)$, $r \in \text{supp } \psi(p)$, and so, by (1), $rt \geqq 0$. It follows that $\beta(\hat{b} + t, \hat{k}; e) \supseteq \beta(\hat{b}, \hat{k}; e)$ for all $e \in \text{supp } \mu(p)$, and in particular, that $\beta(\hat{b} + t, \hat{k}, e) \neq \phi$. So $a \in A(p)$.

(6) Finally, let $V^* := \{e = (q,r,s) \in \text{supp}\ \mu(p) | rt > 0\}$. Then $\mu(p)(V^*) = \psi(p)(\bar{V}) > 0$. Now the function c^2: supp $\mu(p) \to R^m$ can be chosen so that:

$$\left.\begin{array}{ll} \text{(i)} & c^2(e) \geq \hat{c}^2(e) \\ \text{(ii)} & qc^2(e) \leq r(\hat{b} + t) + \pi^2(q;\ \hat{k},s) \end{array}\right\} \text{ all } e = (q,r,s) \in \text{supp}\ \mu(p)$$

$$\text{(iii)}\ c^2(e) > \hat{c}^2(e),\ (\text{all } e \in V^*)$$

where $\hat{c}^2(e)$ maximizes $u(\hat{c}^1,\hat{k},c^2;\ s)$ with respect to c^2 subject to $c^2 \in \beta(\hat{b},\hat{k};\ e)$. This is true because $rt \geq 0$ for all $e \in$ supp $\mu(p)$ by (5), and $rt > 0$ for all $e \in V^*$.

(7) Because u is strictly monotone and $\mu(p)(V^*) > 0$:

$$v(a,p) = v(\hat{c}^1,\hat{b} + t,\hat{k},p) \geq \int_{E^*} (c^1,k,c^2(e);\ s)d\mu(p)$$

$$> \int_{E^*} u(\hat{c}^1,\hat{k},\hat{c}^2(e);\ s)d\mu(p)$$

$$= v(\hat{a},p)\quad ,\ \text{as required}\ .\qquad \text{Q.E.D.}$$

Thus $p > 0$ and $p^2 \in \text{int}\ K(p)$ are necessary conditions for the trader's expected utility function $v(a,p)$ to have a maximum over the set $A(p)$ for a given price vector $p \in \Delta^\ell$. We shall prove that they are also sufficient, because they ensure that the budget set $A(p)$ is compact. Rather than use Green's [1973] argument involving sequences, or Grandmont's [1977] suggestion of using asymptotic cones, I shall base a proof on the following simple result:

Lemma 5.4: Suppose that T is any subset of R^{ℓ_2}, and T_o is any subset of T for which, for some $\varepsilon > 0$, $N_\varepsilon(T_o) \subseteq T$. Then, for any fixed scalars δ, m, the set $B(T,T_o) := \{b \in R^{\ell_2} | \forall\, r \in T: \; rb + \delta \geq 0$ and $\exists\, \bar{r} \in T_o: \bar{r}b \leq m\}$ is a bounded set.

Proof: Suppose $N_\varepsilon(T_o) \subseteq T$ and $b \in B(T,T_o)$. Then, for some $\bar{r} \in T_o$, $\bar{r}b \leq m$. Also, for all $r \in T$, $rb + \delta \geq 0$. Since, for $a = 1, 2, \ldots, \ell_2$, the vectors $\bar{r} + \varepsilon e_a$ and $\bar{r} - \varepsilon e_a$ are members of T (where e_a is the a^{th} unit vector), it follows that $(r + \varepsilon e_a)b + \delta \geq 0$, $(r - \varepsilon e_a)b + \delta \geq 0$. Recalling that $\bar{r}b \leq m$, we have:

$$m + \delta + \varepsilon b_a \geq 0 \quad , \quad m + \delta - \varepsilon b_a \geq 0 \quad ,$$

or $m + \delta \geq \varepsilon b_a \geq -(m + \delta)$, $(a = 1,2,\ldots,\ell_2)$. Therefore $B(T,T_0)$ is bounded. Q.E.D.

Lemma 5.5: Under assumptions A.1, A.2, A.4, for every $p \in \text{int}\, \Delta^\ell$ such that $p^2 \in \text{int}\, K(p)$, $A(p) \subseteq H(p) \times B(p) \times J$ where, for some constants δ and $\lambda > 0$ and some $\bar{y}^1$, $\underline{c}^1 \in R^{\ell_1}$:

$$H(p) := \{c^1 \in C^1 | p^1 c^1 \leq p^1 \bar{y}^1 + \frac{\delta}{\lambda}\}$$

$$B(p) := \bigcap_{r \in \text{co supp}\, \psi(p)} B(p,r)$$

$$B(p,r) := \{b \in R^{\ell_2} | p^2 b \leq p^1(\bar{y}^1 - \underline{c}^1),\; rb + \delta \geq 0\}$$

$$J := \{k \in R_+^{\ell_3} | \exists\, y^1 \in R^{\ell_1} \text{ s.t. } (y^1,k) \in Y^1\}$$

Moreover, the sets $H(p)$, $B(p)$, J are all bounded, and so $A(p)$ is also bounded.

Proof:

(1) By A.2, the consumption set C^1 has a lower bound $\underline{c}^1$ and the first-period production set Y^1 has an upper bound $(\bar{y}^1,\bar{k}) \in R^{\ell_1} \times R_+^{\ell_3}$. Because of A.1 and because the correspondences $C^2(k,s)$ and $Y^2(k,s)$ are both continuous, there exists a uniform lower bound $\underline{c}^2$ for the second period consumption set $C^2(k,s)$ and a uniform upper bound $\bar{\pi}^2$ for the second period profit function $\pi^2(q;\, k,s)$ as q varies over the compact set Δ^m and (k,s) vary over their compact sets of possible values.

(2) Suppose $c^2 \in \beta(b,k;\, e)$. Then $q\underline{c}^2 \leq qc^2 \leq rb + \pi^2(q;\, k,s) \leq rb + \bar{\pi}^2$ and so $rb + \bar{\pi}^2 - q\underline{c}^2 \geq 0$ or $rb + \delta(q) \geq 0$ where $\delta(q) := \bar{\pi}^2 - q\underline{c}^2$. Letting $\delta := \max\, \{\delta(q) | q \in \Delta^m\}$ it follows that $rb + \delta \geq 0$ where δ is independent of p or e.

(3) Define:

$$A(p,e) := \{(c^1,b,k) | c^1 \in C^1,\ p^1c^1 + p^2b \leq \pi^1(p^1,k),\ \beta(b,k,e) \neq \phi\} \quad .$$

Then $A(p) = \bigcap_{e \in \operatorname{supp} \mu(p)} A(p,e)$. But, by (1), $\pi^1(p^1,k) \leq p^1\bar{y}^1$ for all $p^1 \in \Delta^{\ell}$ and all possible $k \in R_+^{\ell_3}$. And, by (2), $\beta(b,k,e) \neq \phi$ implies that $rb + \delta \geq 0$. Therefore, $A(p,e) \subseteq A^*(p,r)$ where:

$$A^*(p,r) := \{(c^1,b,k) | c^1 \in C^1,\ p^1c^1 + p^2b \leq p^1\bar{y}^1,\ rb + \delta \geq 0\} \quad .$$

(4) Notice that $[rb + \delta \geq 0$ for all $r \in \operatorname{supp} \psi(p)]$ is equivalent to $[rb + \delta \geq 0$ for all $r \in \operatorname{co} \operatorname{supp} \psi(p)]$. Therefore:

$$A(p) = \bigcap_{e \in \text{supp}\mu(p)} A(p,e) \subseteq \bigcap_{r \in \text{supp}\psi(p)} A^*(p,r)$$

$$= \bigcap_{r \in \text{co supp } \mu(p)} A^*(p,r)$$

(5) If $a = (c^1,b,k) \in A^*(p,r)$, then $c^1 \geqq \underline{c}^1$, because $c^1 \in C^1$, and so $p^2b \leq p^1\bar{y}^1 - p^1c^1 \leq p^1(\bar{y}^1 - \underline{c}^1)$. It follows that $b \in B(p,r)$ where $B(p,r)$ is as defined in the statement of the lemma. Moreover, if $a \in A(p)$, then $b \in \bigcap_{r \in \text{co supp}\psi(p)} B(p,r)$, as required.

(6) By hypothesis, $p^2 \in \text{int } K(p)$ and so there exists $\lambda > 0$ such that $\lambda p^2 \in \text{int co supp } \psi(p)$. Therefore $A(p) \subseteq A^*(p,\lambda p^2)$ and so, if $a = (c,b,k) \in A(p)$, then $a \in A^*(p,\lambda p^2)$ which implies that $c^1 \in C^1$, $p^1c^1 + p^2b \leq p^1\bar{y}^1$, and $\lambda p^2 b + \delta \geq 0$. Therefore $c^1 \in C^1$ and:

$$p^1c^1 \leq p^1\bar{y}^1 - p^2b \leq p^1\bar{y}^1 + \frac{\delta}{\lambda}$$

It follows that $c^1 \in H^1(p)$, as required.

(7) It is evident that $k \in J$ is implied by $a \in A(p)$ because there must be some production plan (y^1,k) which is feasible in the first period.

(8) Because C^1 is bounded below and $p > 0$, by hypothesis, it is evident that $H(p)$ is bounded.

(9) Because Y^1 is bounded by A.2, J in particular must be bounded.

(10) In the statement of Lemma 5.4, take $T := \text{co supp } \psi(p)$, and $T_o := \{\lambda p^2\}$ where $\lambda > 0$ is such that $\lambda p^2 \in \text{int co supp } \lambda(p)$. Take $m := \lambda p^1(\bar{y} - \underline{c}^1)$. Then there exists $\varepsilon > 0$ such that $N_\varepsilon(T_0) \subseteq T$ and so, by Lemma 5.4, the set $B(T,T_0) = B(p)$ is bounded. Q.E.D.

Define the set $D := \{p \in \Delta^{\ell} | p > 0$ and $p^2 \in \text{int } K(p)\}$.

Theorem 5.6: Under assumptions A.1, A.2, A.3, A.4, A.5, A.6 and A.7, the trader's first period action correspondence $\alpha(p)$ is non-empty valued if and only if $p \in D$. On the set D, $\alpha(p)$ is convex and compact valued, and satisfies the budget exhaustion condition that, for all $(\hat{c}^1, \hat{b}, \hat{k}) \in \alpha(p)$, $p^1\hat{c}^1 + p^2\hat{b} = \pi^1(p^1; \hat{k})$.

Proof:

(1) By Corollary 5.2 and Lemma 5.3, $\alpha(p) = \phi$ whenever $p \notin D$.

(2) But if $p \in D$ then, by Lemma 5.5, $A(p)$ is bounded.

(3) Also, if $p \in D$, the set

$$A(p) = \{a = (c^1,b,k) | c^1 \in C^1,\ p^1c^1 + p^2b \leq \pi^1(p^1; k),$$
$$\beta(b,k,e) \neq \phi \text{ for all } e \in \text{supp } \mu(p)\}$$

is evidently closed because C^1 is closed, $\pi^1(p^1; k)$ is continuous in k, and the correspondence β is continuous and compact valued.

(4) It follows that $A(p)$ is compact in $R^{\ell_1} \times R^{\ell_2} \times R_+^{\ell_3}$ for $p \in D$.

(5) By Theorem 4.4, the trader's utility function $v(a,p)$ is certainly continuous in a. So $\alpha(p)$ is non-empty and compact valued for $p \in D$.

(6) Because the overall production set $Y^2(s)$ is convex for each $s \in S$, it follows that the profit function $\pi^2(q; k,s)$ is concave in k.

(7) Then, because u is concave in (c^1,k,c^2) and because the set $C^2(s) := \{(c^1,k,c^2)|c^1 \in C^1, c^2 \in C^2(k,s)\}$ is convex, by A.2, it follows that the function $\phi(a; e)$ which is given by:

$$\phi(c^1,b,k;\ q,r,s) = \max_{c^2} \{u(c^1,k,c^2;\ s)|c^2 \in C^2(k,s) \quad , \quad qc^2 \leq rb^2 + \pi^2(q;\ k,s)\}$$

must be concave in $(c^1,b,k) = a$ throughout $A(p)$. So therefore is $v(a,p) = \int_{E^*} \phi(a;\ e)d\mu(p)$.

(8) Suppose that $c^2 \in \beta(b,k;\ e)$, $\tilde{c}^2 \in \beta(\tilde{b},\tilde{k};\ e)$. Let $b^* := \lambda b + \mu\tilde{b}$, $k^* := \lambda k + \mu\tilde{k}$ where $\lambda + \mu = 1$, $\lambda, \mu \geq 0$. Then $qc^2 \leq rb + \pi^2(q;\ k,s)$, $q\tilde{c}^2 \leq r\tilde{b} + \pi^2(q;\ \tilde{k},s)$ and so $q(\lambda c^2 + \mu\tilde{c}^2) \leq rb^* + \lambda\pi^2(q;\ k,s) + \mu\pi^2(q;\ \tilde{k},s) \leq rb^* + \pi^2(q;\ k^*,s)$ because π^2 is concave in k, by (6). Also, because $C(s)$ is convex and $c^2 \in C^2(k,s)$, $\tilde{c}^2 \in C^2(\tilde{k},s)$, it follows that $\lambda c^2 + \mu\tilde{c}^2 \in C^2(k^*,s)$. Therefore $\lambda c^2 + \mu\tilde{c}^2 \in \beta(b^*,k^*;\ e)$.

(9) It follows that the first period budget set $A(p)$ is convex because, Y^1 being convex, the profit function $\pi^1(p;\ k)$ is concave in k, and because of (8) above.

(10) Since $v(a,p)$ is concave in a and $A(p)$ is convex it follows that $\alpha(p)$ is convex-valued.

(11) By Corollary 5.2, $v(a,p)$ is strictly monotone in (c^1,b) and so, if $(\hat{c}^1,\hat{b}) \in \alpha(p)$ then evidently:

$$p^1\hat{c}^1 + p^2\hat{b} = \pi^1(p^1;k) \quad . \qquad \text{Q.E.D.}$$

6. Continuity Properties of a Trader's Demand Correspondence

In this section, upper hemi-continuity of the trader's demand correspondence is proved. This follows in a standard way once it is shown that the budget correspondence $A(p)$ is continuous. To show that $A(p)$ is continuous, a number of lemmas are proved. Two extra assumptions are also needed. One (A.8) ensures that the trader can stay away from the lower boundary of his consumption set. The second (A.9) imposes a stronger continuity condition on expectations. An example shows the need for A.9.

First, though, the following lemmas are useful:

Lemma 6.1: Under Assumption 5, the correspondence $\text{supp}\ \mu(p)$ is lower hemi-continuous in p.

Proof: (Green, Remark 3.1. p. 1110, proves a very similar result).

If $\text{supp}\ \mu(p)$ is not lower hemi-continuous at some $\bar{p} \in \Delta^{\ell}$, there exists a sequence $p^{\nu} \in \Delta^{\ell}$ such that $p^{\nu} \to \bar{p}$, a point $\bar{e} \in \text{supp}\ \mu(\bar{p})$, and an open neighbourhood N of $\bar{e}$, such that $N \cap \text{supp}\ \mu(p^{\nu}) = \phi$ for $\nu = 1,2,\ldots$. But now $E^{*}\setminus N$ is a relatively closed set, and $\mu(p^{\nu})[E^{*}\setminus N] = 1$, $(\nu = 1,2\ldots.)$. Because of A.5, it follows that $\mu(\bar{p})[E^{*}\setminus N] = 1$. Therefore $\text{supp}\ \mu(\bar{p}) \subseteq E^{*}\setminus N$, which contradicts $\bar{e} \in N \cap \text{supp}\ \mu(\bar{p})$. Q.E.D.

Lemma 6.2: Let $\bar{\sigma}$ be any convex valued correspondence which is lower hemi-continuous at $\bar{p}$. Suppose that $p^{\nu} \to \bar{p}$ as $\nu \to \infty$ and that $\bar{r} \in \text{int}\ \bar{\sigma}(\bar{p})$. Then there exists $\varepsilon > 0$ and ν_{o} such that, for all

$\nu \geq \nu_o$, $N_\varepsilon(\bar{r}) \subseteq \bar{\sigma}(p^\nu)$.

Proof:

(1) If this lemma is false, there is a sequence $\tilde{r}^\nu \to \bar{r}$ such that, for all ν, $\tilde{r}^\nu \notin \bar{\sigma}(p^\nu)$. Since $\bar{\sigma}(p^\nu)$ is convex, we can separate $\tilde{r}^\nu$ from $\bar{\sigma}(p^\nu)$ by a hyperplane. That is, we can find $t^\nu \neq 0$ such that $r^\nu t^\nu \leq \tilde{r}^\nu t^\nu$ for all $r^\nu \in \bar{\sigma}(p^\nu)$. Moreover, we can take $||t^\nu|| = 1$, and then assume too that $t^\nu \to \bar{t}$, where $||\bar{t}|| = 1$.

(2) Because $\bar{r} \in \text{int } \bar{\sigma}(\bar{p})$, there exists $\hat{r}$ near $\bar{r}$ so that $\hat{r} \in \bar{\sigma}(\bar{p})$ and $\hat{r}\bar{t} > \bar{r}\bar{t}$. Because $\bar{\sigma}$ is lower hemi-continuous at $\bar{p}$, there is a sequence r^ν such that $r^\nu \in \bar{\sigma}(p^\nu)$, $(\nu = 1,2,\ldots)$ and $r^\nu \to \hat{r}$. Then $r^\nu t^\nu \leq \tilde{r}^\nu t^\nu$, by (1), and so, taking limits, $\hat{r}\bar{t} \leq \bar{r}\bar{t}$, a contradiction. Q.E.D.

Given a correspondence $\sigma(p)$ defined on D, say that $\sigma(p)$ is locally bounded if, whenever $p^\nu \in D$, $(k = 1,2,\ldots)$, $p^\nu \to \bar{p}$, and $\bar{p} \in D$, then $\bigcup_{\nu=1}^{\infty} \sigma(p^\nu)$ is a bounded set. Notice that a correspondence is upper hemi-continuous if it is locally bounded and has a closed graph.

Lemma 6.3: Under assumptions A.1, A.2, A.4 and A.5, the trader's budget correspondence $A(p)$ is locally bounded.

Proof:

(1) By Lemma 5.5, $A(p) \subseteq H(p) \times B(p) \times J$ for all $p \in D$. Since J is constant and bounded, it suffices to prove that $H(p)$ and $B(p)$ are both locally bounded on D.

(2) Suppose that $p^\nu \in D$, $(\nu = 1,2,\ldots)$, $p^\nu \to \bar{p}$, and $\bar{p} \in D$. Because of A.5 and Lemma 6.1, supp $\mu(p)$ is a lower hemi-continuous correspondence, and so therefore is supp $\psi(p)$, and also the correspondence co supp $\psi(p)$. Now $\lambda\bar{p}^2 \in$ int co supp $\psi(\bar{p})$, for some $\lambda > 0$. So, by Lemma 6.2, there exists $\varepsilon > 0$ and ν_o such that, such that, for all $\nu \geq \nu_o$:

$$N_{2\varepsilon}(\lambda\bar{p}^2) \subseteq \text{co supp } \psi(p^\nu)$$

(3) In particular, there exists ν_1 such that, for all $\nu \geq \nu_1$,

$$\lambda p^{2\nu} \in \text{int co supp } \psi(p^\nu) \quad .$$

So, as in the proof of Lemma 5.5 we can take:

$$H(p^\nu) = \{c^1 \in C^1 | p^{1\nu}c^1 \leq p^{1\nu}\bar{y}^1 + \tfrac{\delta}{\lambda}\}$$

where δ and λ are independent of ν, and $\lambda > 0$. Since $p^{1\nu} > 0$ $(\nu = 1,2,\ldots)$, $p^{1\nu} \to \bar{p}^1 \in D$, and, by A.2, C^1 is bounded below, a standard argument establishes that $\bigcup_{\nu=1}^{\infty} H(p^\nu)$ is bounded.

(4) Also, $B(p^\nu) = \bigcap_{r \in \text{co supp } \psi(p^\nu)} B(p^\nu, r)$

$$\subseteq \bigcap_{r \in N_{2\varepsilon}(\lambda\bar{p}^2)} B(p,r) \quad , \quad \text{for all } \nu \geq \nu_o \quad .$$

In Lemma 5.4, take $T := N_{2\varepsilon}(\lambda\bar{p}^2)$, $T_o^\nu := \{\lambda p^{2\nu'} | \nu' \geq \nu\}$. Then, there exists ν_2 such that, for all $\nu \geq \nu_2$, $N_\varepsilon(T_o^\nu) \subseteq N_{2\varepsilon}(\lambda\bar{p}^2) = T$, and so $B(p^{\nu}) \subseteq B(T,T_o^{\nu}) \subseteq B(T,T_o^{\nu_2})$ which is a bounded set independent of ν.

Q.E.D.

Write $z := (c^1,b)$ and $pz := p^1c^1 + p^2b$.

Lemma 6.4: Under assumptions A.1, A.2, A.4, A.5, and A.7, the trader's budget correspondence $A(p)$ has a closed graph on Δ^{ℓ}.

Proof:

(1) Suppose $p^{\nu} \in \Delta^{\ell}$, and $a^{\nu} \in A(p^{\nu})$ $(\nu = 1,2,\ldots)$, and that $(p^{\nu},a^{\nu}) \to (\bar{p},\bar{a})$. We must show that $\bar{a} \in A(\bar{p})$.

(2) Because $c^{1\nu} \in C^1$, $p^{\nu}z^{\nu} \leqq \pi^1(p^{1\nu};\, k^{\nu})$, C^1 is closed, and the correspondence $Y^1(k)$ is continuous, π^1 is continuous too and so:

$$\bar{c}^1 \in C^1,\ \bar{p}\bar{z} \leq \pi^1(\bar{p}^1;\, \bar{k}) \quad .$$

(3) Suppose that $\bar{e} \in \operatorname{supp} \mu(\bar{p})$. Then, because $\operatorname{supp} \mu(p)$ is a lower hemi-continuous correspondence, by Lemma 6.1, there exists a sequence $e^{\nu} \in \operatorname{supp} \mu(p^{\nu})$ $(\nu = 1,2\ldots)$ such that $e^{\nu} \to \bar{e}$. Because $a^{\nu} \in A(p^{\nu})$ it follows that there exists a sequence $c^{2\nu} \in \beta(b^{\nu},k^{\nu};\, e^{\nu})$ $(\nu = 1,2,\ldots)$. Then, because the correspondence β is continuous and compact valued, by Lemma 4.1, it follows that a subsequence of $c^{2\nu}$ converges to a point $\bar{c}^2 \in \beta(\bar{b},\bar{k};\, \bar{e})$.

(4) Therefore $\beta(\bar{b},\bar{k};\, \bar{e})$ is non-empty for all $\bar{e} \in \operatorname{supp} \mu(\bar{p})$.

(5) From (2) and (4) it follows that $\bar{a} \in A(\bar{p})$. Q.E.D.

To prove lower hemi-continuity of the first period budget correspondence, I shall invoke the interior point assumption A.8, and also A.9, which requires the correspondence supp $\mu(p)$ to be upper hemi-continuous. Only lower hemi-continuity of supp $\mu(p)$ follows from the weak continuity of $\mu(p)$. The following example shows how supp $\mu(p)$ can fail to be upper hemi-continuous even if $\mu(p)$ is not only weakly continuous, but actually has the stronger property A.5* that $\mu(p)(F)$ be continuous in p for every Borel $F \subseteq E^*$.

Example 6.5: Suppose that $\ell_1 = \ell_2 = m = 2$, there is no capital or exogenous uncertainty, and that we have complete future markets, so that $r_1 = q_1$, $r_2 = q_2$ with probability one. Suppose a trader has the consumption set $C^1 \times C^2$ where $C^1 = C^2 = R^2_+$, and has fixed endowments $\omega^1 = \omega^2 = (1,1)$ in each period. Suppose the trader's expectations $\psi(p)$ on Δ^2 correspond to a distribution function $F(p^2)(q_1)$ defined for $0 \leq q_1 \leq 1$. Write

$$\pi_1 = \frac{p^2{}_1}{p_1^2 + p_2^2} \quad , \quad \pi_2 = \frac{p^2{}_2}{p_1^2 + p_2^2}$$

and suppose that this distribution function takes the following specific form, as illustrated in Figure 1 (at the end of the paper):

$$F(\pi_1)(q_1) = \begin{cases} \left.\begin{cases} (1 - 2\pi_1)q_1 & (0 \leq q < \frac{1}{3}) \\ 1 - 2(1 + \pi_1)(\frac{2}{3} - q_1) & (\frac{1}{3} \leq q_1 < \frac{2}{3}) \\ 1 & (\frac{2}{3} \leq q_1 < 1) \end{cases}\right\} & (\pi_1 < \frac{1}{2}) \\ \left.\begin{cases} 0 & (0 \leq q < \frac{1}{3}) \\ 3(q_1 - \frac{1}{3}) & (\frac{1}{3} \leq q_1 < \frac{2}{3}) \\ 1 & (\frac{2}{3} \leq q_1 < 1) \end{cases}\right\} & (\pi_1 = \frac{1}{2}) \\ \left.\begin{cases} 0 & (0 \leq q_1 < \frac{1}{3}) \\ (4 - 2\pi_1)(q_1 - \frac{1}{3}) & (\frac{1}{3} \leq q_1 < \frac{2}{3}) \\ 1 - (2\pi_1 - 1)(1 - q_1) & (\frac{2}{3} \leq q_1 < 1) \end{cases}\right\} & (\pi_1 > \frac{1}{2}) \end{cases}$$

The trader now satisfies assumptions A.1, A.2, A.4, A.5, A.6, A.7 and A.8. Indeed, the strong form A.5* of A.5 is satisfied because $F(\pi_1)(q_1)$ is continuous in π_1. Notice however that A.9 is not satisfied because:

$$\text{supp } \psi(p) = \begin{cases} \Delta^2 \cap \{q | 0 \leq q_1 \leq \frac{2}{3}\} & (\pi_1 < \frac{1}{2}) \\ \Delta^2 \cap \{q | \frac{1}{3} \leq q_1 \leq \frac{2}{3}\} & (\pi_1 = \frac{1}{2}) \\ \Delta^2 \cap \{q | \frac{1}{3} \leq q_1 \leq 1\} & (\pi_1 > \frac{1}{2}) \end{cases}$$

and this not upper hemi-continuous when $p_1^2 = p_2^2$; in fact, the graph of supp $\psi(p)$ is not closed.

The trader's budget set $A(p)$ is now:

$$A(p) = \{(c^1,b) \in R^2_+ \times R^2 | p^1c^1 + p^2b \leqq p^1\omega^1, b \in B(p)\}$$

where

$$B(p) = \{b \in R^2 | \beta(b,q) \neq \phi \quad \text{for all} \quad q \in \text{supp } \psi(p)\}$$

and

$$\beta(b,q) = \{c^2 \in R^2_+ | qc^2 \leq q(\omega^2 + b)\} \quad .$$

In this example:

$$B(p) = \{b \in R^2 | qb + 1 \geq 0 \quad \text{for all} \quad q \in \text{supp } \psi(p)\}$$

and so:

$$B(p) = B(\pi) = \begin{cases} \{b \in R^2 | b_2 + 1 \geq 0,\ 2b_1 + b_2 + 3 \geq 0\} & (\pi_1 < \frac{1}{2}) \\ \{b \in R^2 | b_1 + 2b_2 + 3 \geq 0,\ 2b_1 + b_2 + 3 \geq 0\} & (\pi_1 = \frac{1}{2}) \\ \{b \in R^2 | b_1 + 2b_2 + 3 \geq 0,\ b_1 + 1 \geq 0\} & (\pi_1 > \frac{1}{2}) \end{cases} \quad .$$

This is not lower hemi-continuous at $\pi_1 = 1/2$ – for example:

$$(-1\tfrac{1}{2},0) \in B(\pi) \quad \text{for} \quad \pi_1 \leq \frac{1}{2} \quad , \quad \text{but not for} \quad \pi_1 > \frac{1}{2} \quad .$$

$$(0,-1\tfrac{1}{2}) \in B(\pi) \quad \text{for} \quad \pi_1 \geq \frac{1}{2} \quad , \quad \text{but not for} \quad \pi_1 < \frac{1}{2} \quad .$$

Nor, therefore, is $A(p)$ lower hemi-continuous. It is also possible to find a utility function $u(c^1,c^2)$ such that $\hat{b}(\pi_1) \to (-1,1)$ as $\pi_1 \to (1/2)-$ and $\hat{b}(\pi_1) \to (0,-1\frac{1}{2})$ as $\pi_1 \to (1/2)+$. Since

$\pi_1\hat{b}_1(\pi_1) + \pi_2\hat{b}_2(\pi_1)$ is discontinuous, so also is consumption expenditure $p_1^1\hat{c}_1^1 + p_2^1\hat{c}_2^1$ and the vector $(\hat{c}_1^1, \hat{c}_2^1)$.

Lemma 6.6: Under assumptions A.1, A.2, A.4, A.5, A.7, A.8 and A.9 the trader's budget correspondence $A(p)$ is lower hemi-continuous over Δ^ℓ.

Proof:

(1) Suppose $p^\nu \in \Delta^\ell$ $(\nu = 1,2,\ldots)$ is a sequence of points which converges to $\bar{p}$, and $\bar{a} \in A(\bar{p})$. We must find a sequence of actions $a^\nu \in A(p^\nu)$ $(\nu = 1,2,\ldots)$ such that $a^\nu \to \bar{a}$.

(2) By A.8, there exist $\tilde{c}^1$, $\tilde{y}^1$, $\tilde{k}^1$ and $\tilde{c}^2(s)$, $\tilde{y}^2(s)$ $(s \in S)$ such that $\tilde{c}^1 \in C^1$, $(\tilde{y}^1, \tilde{k}^1) \in Y^1$, $\tilde{c}^1 < \tilde{y}^1$ and, for each $s \in S$, $\tilde{c}^2(s) \in C^2(\tilde{k},s)$, $y^2(s) \in Y^2(\tilde{k},s)$, $\tilde{c}^2(s) < \tilde{y}^2(s)$.

(3) Let $\tilde{a} := (\tilde{c}^1, 0, \tilde{k})$. Then evidently $\tilde{a} \in A(p)$ for all $p \in \Delta^\ell$, and in particular, $\tilde{a} \in A(\bar{p})$.

(4) For any real λ such that $0 \leq \lambda < 1$, define:

$$a(\lambda) := \lambda\bar{a} + (1 - \lambda)\tilde{a}$$

so that $c^1(\lambda) = \lambda\bar{c}^1 + (1 - \lambda)\tilde{c}^1, b(\lambda) = \lambda\bar{b}$, $k(\lambda) = \lambda\bar{k} + (1 - \lambda)\tilde{k}$.

(5) Evidently $c^1(\lambda) \in C^1$ because C^1 is convex.

(6) Because $\bar{a} \in A(p)$, $\bar{p}^1\bar{c}^1 + \bar{p}^2\bar{b} \leq \pi^1(\bar{p}^1; \bar{k})$. Also $\bar{p}^1\tilde{c}^1 < \bar{p}^1\tilde{y}^1 \leq \pi^1(\bar{p}^1; \tilde{k})$. It follows that, whenever $0 \leq \lambda < 1$:

$$\bar{p}^1c^1(\lambda) + \bar{p}^2b(\lambda) < \lambda\pi^1(\bar{p}^1; \bar{k}) + (1 - \lambda)\pi^1(\bar{p}\ ; \tilde{k})$$

$$\leq \pi^1(\bar{p}^1; k(\lambda))$$

because Y^1 is convex and so π^1 is concave in k. Because π^1 is continuous in p it follows that, for each λ, there exists $\nu_1(\lambda)$ such that, whenever $\nu \geq \nu_1(\lambda)$, then:

$$p^{1\nu}c^1(\lambda) + p^{2\nu}b(\lambda) \leq \pi^1(p^{1\nu};\ k(\lambda)) \quad .$$

(7) Define the minimum wealth function:

$$\underline{w}^2(q;\ k,s) := \min_{c^2} \{qc^2 | c^2 \in C^2(k,s)\} \quad .$$

This is well defined for all $q \in \Delta^m$ because $C^2(k,s)$ is bounded below. $\underline{w}^2$ is also continuous because qc^2 is continuous in q and the correspondence $C^2(k,s)$ is continuous by A.2. In addition, $\underline{w}^2$ is a convex function of k because the overall consumption set is convex, by A.2.

(8) Define the function:

$$f(b,k;\ e) := rb + \pi^2(q;\ k,s) - \underline{w}^2(q;\ k,s) \quad .$$

Notice that it is continuous because π^2 and $\underline{w}^2$ are both continuous, and that it is concave in (b,k) because π^2 is concave in k and $\underline{w}^2$ is convex in k. Notice too that $\beta(b,k;\ e) \neq \emptyset$ if and only if $f(b,k;\ e) \geq 0$ and so that $a \in A(p)$ only if $f(b,k;\ e) \geq 0$ for all $e \in \text{supp}\ \mu(p)$.

(9) Let $\overline{\text{supp}}\ \mu(p)$ denote the closure in E of $\text{supp}\ \mu(p) \subseteq E^*$ where $E = \Delta^m \times R_+^{\ell_2} \times S$ and $E^* = \text{int}\ \Delta^m \times R_+^{\ell_2} \times S$. Notice that $f(b,k;\ e)$ is well defined and continuous for all $e \in E$. Also, if $a \in A(p)$, then $f(b,k;\ e) \geq 0$ for all $e \in \text{supp}\ \mu(p)$ and so the

function:

$$g(p;\ b,k) := \min_{e} \{f(b,k;\ e) | e \in \overline{\text{supp}}\ \mu(p)\}$$

is well-defined. g is also continuous because, by Lemma 6.1 and A.5, supp $\mu(p)$ is a continuous correspondence and so therefore is $\overline{\text{supp}}\ \mu(p)$. And g is concave in (b,k) because f is.

(10) Because $\bar{a} \in A(\bar{p})$ it follows that $g(\bar{p};\ \bar{b},\bar{k}) \geq 0$.

(11) Because $\tilde{c}^2(s) < \tilde{y}^2(s)$ for all $s \in S$, so, for all $e \in E$:

$$f(0,k;\ e) = \pi^2(q;\ k,s) - \underline{w}^2(q;\ k,s)$$

$$\geq q[\tilde{y}^2(s) - \tilde{c}^2(s)] > 0 \quad .$$

Therefore $g(\bar{p};\ 0,\tilde{k}) > 0$ from (9) above.

(12) Because g is concave in (b,k), it follows that:

$$g(\bar{p};\ b(\lambda),k(\lambda)) \geq \lambda g(\bar{p};\ \bar{b},\bar{k}) + (1 - \lambda)g(\bar{p};\ 0,\tilde{k}) > 0$$

whenever $0 \leq \lambda < 1$. Then, because g is continuous, it also follows that, for each λ $(0 \leq \lambda < 1)$ there exists $\nu_2(\lambda)$ such that, whenever $\nu \geq \nu_2(\lambda)$, then $g(p^{\nu};\ b(\lambda),k(\lambda)) \geq 0$.

(13) Combining (5), (6) and (12) with (8), it follows that $a(\lambda) \in A(p^{\nu})$ whenever $\nu \geq \nu(\lambda) := \max\ \{v_1(\lambda),\nu_2(\lambda)\}$.

(14) Let λ_n $(n = 0,1,2,\ldots)$ be any strictly increasing sequence of nonnegative real numbers such that $\lambda_n \to 1$ as $n \to \infty$. Define the sequence ν_n $(n = 0,1,2,\ldots)$ of positive integers recursively by $\nu_0 := 1$ and:

$$\nu_n := \max \{\nu_{n-1} + 1, \nu(\lambda_n)\}$$

where $\nu(\lambda_n)$ is given by (13). Then define the sequence a^ν $(\nu = 0,1,2,\ldots)$ by:

$$a^\nu := a(\lambda_n) \quad (\nu = \nu_n \text{ to } \nu_{n+1} - 1;\ n = 0,1,2,\ldots) \quad .$$

Since $\nu_n \geq \nu(\lambda_n)$ it follows that $a^\nu \in A(p^\nu)$ for $\nu = 0,1,2,\ldots$. Because $\lambda_n \to 1$ and $\nu_n \to \infty$ as $n \to \infty$ it follows that $a^\nu \to \bar{a}$ as $\nu \to \infty$. Q.E.D.

Theorem 6.7: Under assumptions A.1 to A.9 the trader's first period action correspondence $\alpha(p)$ is non-empty, convex and compact valued, as well as upper hemi-continuous, on the restricted domain:

$$D := \{p \in \Delta^\ell | p > 0, p^2 \in \text{int } K(p)\} \quad .$$

Also, $p^1\hat{c}^1 + p^2\hat{b} = \pi^1(p^1; \hat{k})$ for all $\hat{a} = (\hat{c}^1, \hat{b}, \hat{k}) \in \alpha(p)$ whenever $p \in D$.

Proof: To Theorem 5.6 I have added just upper hemi-continuity of α. This follows from Theorem 4.4 which ensures continuity of the utility function $v(a,p)$ and from Lemmas 6.3, 6.4 and 6.6 which ensure continuity of the correspondence $A(p)$. (Notice that $A(p)$ is compact valued on D by part (4) of the proof of Theorem 5.6). Q.E.D.

This is the main theorem of Section 6, but the next three results will be useful for the existence proof in Section 7.

Lemma 6.8: Under A-5, the set $D := \{p \in \Delta^\ell | p > 0, p^2 \in \text{int } K(p)\}$ is open (Green [1973], Lemma 3.8, p. 1112)).

Proof:

(1) Define $D^* := \{p \in \Delta^\ell | p^2 \in \text{int } K(p)\}$. Since $D = D^* \cap \text{int } \Delta^\ell$ and $\text{int } \Delta^\ell$ is open, it suffices to prove that D^* is open.

(2) Suppose $\bar{p} \in \Delta^\ell$ and $\bar{p}^2 \in \text{int } K(\bar{p})$. Then, for some $\varepsilon > 0$, $N_{2\varepsilon}(\bar{p}^2) \subseteq K(\bar{p})$. Also, if $p^2 \in N_\varepsilon(\bar{p}^2)$, and $\tilde{p}^2 \in N_\varepsilon(p^2)$, then $\tilde{p}^2 \in K(\bar{p})$. That is, if $p^2 \in N_\varepsilon(\bar{p}^2)$, then $N_\varepsilon(p^2) \subseteq K(\bar{p})$.

(3) Because the correspondence supp $\mu(p)$ is lower hemi-continuous (Lemma 6.1), so are the correspondences supp $\mu(p)$, co supp $\psi(p)$, and $K(p)$ in turn. So there exists $\delta^1 > 0$ such that, if $p \in N_{\delta^1}(\bar{p})$, then $K(\bar{p}) \subseteq N_{\varepsilon/2}(K(p))$.

(4) Choose $\delta := \min \{\delta^1, \varepsilon\}$. Suppose $p \in N_\delta(\bar{p})$. Then $N_\varepsilon(p^2) \subseteq K(\bar{p}) \subseteq N_{\varepsilon/2}(K(p))$. Because $K(p)$ is a convex set, it follows from Green ([1973], Lemma 3.7, p. 1112) that $N_{\varepsilon/2}(p^2) \subseteq K(p)$. Thus, if $p \in N_\delta(\bar{p})$, then $p^2 \in \text{int } K(p)$. This proves that D^* is open.

Q.E.D.

Lemma 6.9: Under assumptions A.1 to A.9, if $p^\nu \in D$, $a^\nu \in \alpha(p^\nu)$ $(\nu = 1,2,\ldots)$, $p^\nu \to \bar{p}$ and $\bar{p} \in \text{bd } D$, then $||a^\nu|| \to \infty$.

Proof: Suppose $||a^\nu||$ remains bounded. Then we may as well assume that $a^\nu \to \bar{a}$. Now $a^\nu \in A(p^\nu)$ and, by Lemma 6.4, the correspondence $A(p)$ has a closed graph on Δ^ℓ, so $\bar{a} \in A(\bar{p})$. Also, because, $v(a,p)$ is continuous (Theorem 4.4), $v(a^\nu,p^\nu) \to v(\bar{a},\bar{p})$.

Take any $\tilde{a} \in A(\bar{p})$. By Lemma 6.6, the correspondence $A(p)$ is lower hemi-continuous on Δ^ℓ, and so there is a sequence $\tilde{a}^\nu$ such that $\tilde{a}^\nu \in A(p^\nu)$, $(\nu = 1,2,\ldots)$ and $\tilde{\alpha}^\nu \to \tilde{a}$. Since $a^\nu \in \alpha(p^\nu)$,

$v(a^\nu,p^\nu) \geq v(\tilde{a}^\nu,p^\nu)$. Using continuity of v again, $v(\bar{a},\bar{p}) \geq v(\tilde{a},\bar{p})$. So $\bar{a} \in \alpha(\bar{p})$. Yet, by Theorem 5.6, $\alpha(\bar{p})$ is empty, because D is open and $\bar{p} \in \text{bd } D$. This is the required contradiction. Q.E.D.

Lemma 6.10: Under assumptions A.1, and A.3 to A.9, if $p^\nu \in D$ $(\nu = 1,2,\ldots)$ and $p^\nu \to \bar{p}$, then, for each $\bar{r} \in \text{int } K(\bar{p})$, there exist $\varepsilon > 0$, $\lambda > 0$, δ and ν_o such that, for all $\nu \geq \nu_o$ and all $r \in N_\varepsilon(\bar{r})$:

$$\lambda rb + \delta \geq 0 \quad \text{whenever } a \in A(p^\nu) \quad .$$

Proof: Suppose $\bar{r} \in \text{int } K(\bar{p})$. Then, for some $\lambda > 0$, $\lambda\bar{r} \in \text{int co supp } \psi(\bar{p})$. By Lemma 6.2, and because it follows from Lemma 6.1 that $\text{co supp } \psi(p)$ is a lower hemi-continuous correspondence, there exist $\varepsilon > 0$ and ν_o such that, for all $\nu \geq \nu_o$, $N_{\lambda\varepsilon}(\lambda\bar{r}) \subseteq \text{co supp } \psi(p^\nu)$. Now, if $r \in N_\varepsilon(\bar{r})$, then $\lambda r \in N_{\lambda\varepsilon}(\lambda\bar{r})$ and so, for all $\nu \geq \nu_o$, $\lambda r \in \text{co supp } \psi(p^\nu)$. By Lemma 5.5, it follows that, for all $r \in N_\varepsilon(\bar{r})$ and all $\nu \geq \nu_o$:

$$B(p^\nu) \subseteq B(p^\nu,\lambda r) \subseteq \{b \in R^{\ell_2} | \lambda rb + \delta \geq 0\}.$$

Thus, if $a \in A(p^\nu)$, then by Lemma 5.5, $\lambda rb + \delta \geq 0$. Q.E.D.

7. Existence of a Temporary Walrasian Equilibrium

So far, we have been dealing with a single trader in isolation. Now consider an economy with a finite set I of traders. Let each $i \in I$ have a consumption set ${}^iC(s)$, production set ${}^iY(s)$, von Neumann-Morgenstern utility function ${}^iu({}^ic^1,{}^ik,{}^ic^2,s)$, expected utility function ${}^{i}v({}^{i}a,p)$, probability measures ${}^{i}\mu(p)$ on E^* and ${}^{i}\psi(p)$ on R^{ℓ_2}, budget correspondence ${}^{i}A(p)$, action correspondence ${}^{i}\alpha(p)$, etc. Define the aggregate excess demand correspondence $\zeta(p) := \sum_i {}^{i}\zeta(p)$, where ${}^{i}\zeta(p) := \{({}^{i}\hat{c}^1, {}^{i}\hat{b}) \mid \exists\, {}^{i}\hat{k} \text{ s.t. } ({}^{i}\hat{c}^1, {}^{i}\hat{b}, {}^{i}\hat{k}) \in {}^{i}\alpha(p)\}$. We shall assume that Assumptions 1 to 9 apply to each trader $i \in I$. To prove existence, we shall also need the following assumptions, which were discussed in Section 3.

Assumption 10: Let ${}^{i}D := \{p \in \Delta^{\ell} \mid p > 0 \text{ and } p^2 \in \text{int } {}^{i}K(p)\}$. Assume that the set $P := \bigcap_{i \in I} {}^{i}D$ is non-empty and convex.

Assumption 11: If $\bar{p} \in \text{bd } P$, then there exists $p \in P$ such that $p^2 \in \bigcap_{i \in I} \text{int } {}^{i}K(\bar{p})$.

To prove existence of a temporary Walrasian equilibrium, we shall use the following general existence lemma, which generalizes that of Grandmont ([1977], Lemma 1, p. 543).

Lemma 7.1: Let P be an open convex subset of Δ^{ℓ}. For each $i \in I$, let ${}^{i}\zeta(p)$ be a demand correspondence defined on P which is compact and convex valued, as well as upper hemi-continuous and satisfying budget exhaustion $p\,{}^{i}\zeta(p) = 0$ (all $p \in P$). Suppose that the following boundedness assumptions are satisfied, whenever $p^{\nu} \in P$ $(\nu = 1,2,\ldots)$, $p^{\nu} \to \bar{p}$ and $\bar{p} \in \text{bd } P$:

B.1: For some $j \in I$, if ${}^{j}z^{\nu} \in {}^{j}\zeta(p^{\nu})$ $(\nu = 1,2,\ldots)$, then $||{}^{j}z^{\nu}|| \to \infty$.

B.2: There are an open set $P^* \subseteq P$ and lower bounds ${}^{i}\gamma$ $(i \in I)$

such that, whenever ν is large and ${}^iz \in {}^i\zeta(p^\nu)$, then $p\,{}^iz \geq {}^i\gamma$ for all $p \in P^*$.

Then there exists an equilibrium price vector p^* such that $0 \in \zeta(p^*)$.

Proof:

(1) As in Grandmont [1977] or Green [1973], we have $P = \bigcup_{n=1}^{\infty} P^n$ where each P^n is closed and convex, and $P^1 \subseteq P^2 \subseteq \cdots \subseteq P$. Also, for each n, we can show that there exist $p^n \in P^n$ and $z^n \in \zeta(p^n)$ such that $p^n z^n = 0 \geq pz^n$ for all $p \in P^n$ -- cf. Debreu ([1959], p. 82). Let $z^n = \sum_i {}^iz^n$ where ${}^iz^n \in {}^i\zeta(p^n)$, $(n = 1,2,\ldots)$, (all $i \in I$).

(2) Since $p^n \in P$ and P is a subset of the compact set Δ^ℓ, we can assume that $p^n \to p^* \in \Delta^\ell$ as $n \to \infty$.

(3) Suppose $p^* \in \text{bd}\, P$. By B.2, there is an open set $P^* \subseteq P$ such that $p\,{}^iz^n \geq {}^i\gamma$ for all $i \in I$, all $p \in P^*$ and all large n.

(4) Fix any $\bar{p} \in P^*$. Then for large n, $\bar{p} \in P^n$ and so $\bar{p}z^n \leq 0$. By B.1, $||{}^jz^n|| \to \infty$, for some $j \in I$. But $\bar{p}\,{}^jz^n \leq -\sum_{i\neq j}\bar{p}\,{}^iz^n \leq -\sum_{i\neq j}\bar{p}\,{}^i\gamma$. Also $p\,{}^jz^n \geq {}^j\gamma$ for all $p \in P^*$. As in Lemma 5.4, this shows that ${}^jz^n$ lies in a bounded set, because P^* is open and $\bar{p} \in P^*$. This contradicts B.1. So $p^* \in P$.

(5) Since $\zeta(p)$ is upper hemi-continuous on P, we can now also assume that $z^n \to z^*$ where $z^* \in \zeta(p^*)$. Since $p^n z^n = 0$, for all n, it follows that $p^* z^* = 0$.

(6) Take any $p \in P$. For large n, $p \in P^n$ and so $pz^n \leq 0$. It follows that $pz^* \leq 0$ for all $p \in P$. Since P is open, this implies that $z^* = 0$, i.e., $0 \in \zeta(p^*)$. Q.E.D.

Theorem 7.2: Under Assumptions 1 to 11, there exists a temporary Walrasian equilibrium.

Proof:

(1) It is only necessary to verify the conditions of Lemma 7.1. Apart from B.1 and B.2, these are all obvious consequences of Assumption 10, Theorem 6.7 and Lemma 6.8. It remains to use Lemmas 6.9, 6.10 and A.11 to verify B.1 and B.2.

(2) Suppose $p^\nu \in P$ $(\nu = 1,2,\ldots)$, $p^\nu \to \bar{p}$ and $\bar{p} \in \text{bd } P$.

(3) Then, for some $j \in I$, $\bar{p} \in \text{bd } {}^jD$. Now, if ${}^ja^\nu \in {}^j\alpha(p^\nu)$ $(\nu = 1,2,\ldots)$, $||{}^ja^\nu|| \to \infty$ by Lemma 6.9. But the set of feasible k is bounded, so $||{}^jz^\nu|| \to \infty$. This verifies B.1.

(4) By A.11, there exists $\tilde{p} \in P$ such that, for all $i \in I$, $\tilde{p}^2 \in \text{int } {}^iK(\bar{p})$. By Lemma 6.10, since I is a finite set, there exist $\varepsilon > 0$, ${}^i\lambda > 0$, ${}^i\delta$ $(i \in I)$ and ν_o such that, for all $\nu \geq \nu_o$, all $i \in I$, and all $r \in N_\varepsilon(\tilde{p}^2)$, ${}^i\lambda r {}^ib + {}^i\delta \geq 0$ whenever ${}^ia \in {}^iA(p^\nu)$. Also, if ${}^ia \in {}^iA(p^\nu)$, then $p^1\, {}^ic^1 \geq p^1\, {}^i\underline{c}^1$ where ${}^i\underline{c}^1$ is a lower bound to ${}^iC^1$. Define $P^* := \{p \in P | p^2 \in N_\varepsilon(\tilde{p}^2)\}$, an open set with $\tilde{p}$ as a member. Then for all $p \in P^*$, if ${}^ia \in {}^iA(p^\nu)$ then $p^1\, {}^ic^1 \geq p^1\, {}^i\underline{c}^1$ and, for large enough ν, $p^2\, {}^ib \geq -{}^i\delta/(\eta\, {}^i\lambda)$. So, for all large ν,

$$p\, {}^iz \geq p^1\, {}^i\underline{c}^1 - \frac{{}^i\delta}{\eta\, {}^i\lambda}$$

whenever ${}^ia \in {}^iA(p^\nu)$. Since $(p^1,p^2) \in \Delta^\ell$ and ${}^i\underline{c}^1$ is fixed, $p^1\, {}^i\underline{c}^1$ is bounded below, and so therefore is p^iz. This confirms B.2.

Q.E.D.

8. Two Examples

The following two examples illustrate the role played by the Assumption in A.10 that the set P is convex, and by the boundary condition A.11, in ensuring the existence of a temporary Walrasian equilibrium. Both examples involve an economy in which there are no consumption or capital goods in the first period, but only financial assets. There is only a single consumption good in the second period, moreover, and each agent's objective is simply to maximize his expected wealth in terms of this second period consumption good. Also, each agent has a fixed endowment $\omega^2 = 1$ of the second period consumption good, and a fixed second period consumption set $C^2 = R_+ = \{c^2 | c^2 \geq 0\}$.

Example 8.1: There are two agents and two assets. Each of the two traders, i = 1, 2, chooses his net demands for financial assets ${}^i\hat{b} = ({}^i\hat{b}_1, {}^i\hat{b}_2)$ in order to maximize his expected wealth in the second period:

$$\int (r_1 b_1 + r_2 b_2) d\,{}^i\psi(p)$$

subject to the budget constraint:

$$p_1 b_1 + p_2 b_2 \leq 0$$

and subject to the feasibility constraint:

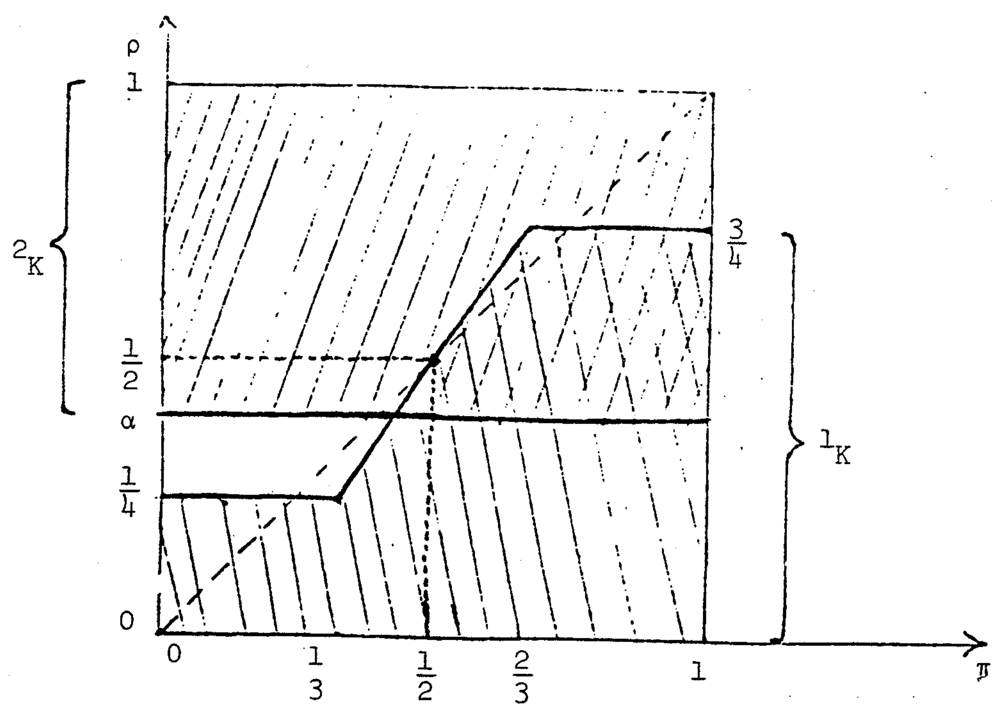
$$r_1 b_1 + r_2 b_2 + 1 \geq 0 \quad \text{for all} \quad r \in \operatorname{supp} {}^i\psi(p) \quad .$$

Take $p \in \Delta^2$ and write $(p_1, p_2) = (\pi, 1 - \pi)$. Also, assume that

$r \in \Delta^2$ always and write $(r_1, r_2) = (\rho, 1-\rho)$ where $0 \leq \rho \leq 1$. Assume that the traders' expectations are such that the supports of ${}^1\psi(p)$, ${}^2\psi(p)$ and so the cones ${}^1K(p)$, ${}^2K(p)$ correspond to the intervals $0 \leq \rho \leq {}^1\rho(\pi)$, ${}^2\rho(\pi) \leq \rho \leq 1$ respectively defined as follows:

$$
{}^1\rho(\pi) := \begin{cases} \frac{1}{4} & (0 \leq \pi \leq \frac{1}{3}) \\ \frac{1}{4}(6\pi - 1) & (\frac{1}{3} < \pi < \frac{2}{3}) \\ \frac{3}{4} & (\frac{2}{3} \leq \pi \leq 1) \end{cases}
$$

and ${}^2\rho(\pi) := \alpha \quad (0 \leq \pi \leq 1)$. (as illustrated below).



Define the expected values of ρ as:

$$^{i}\bar{\rho}(\pi) := \int \rho d^{i}\psi(p) \quad , \quad (i = 1,2) \quad .$$

It is simpler if we assume that $^{1}\bar{\rho}(\pi) < \pi$ for all $\pi \geq \frac{1}{4}$ and that $^{2}\bar{\rho}(\pi) > \pi$ for all $\pi \leq \frac{3}{4}$. Consider first <u>consumer 1</u>, who maximizes $^{1}\bar{\rho}(\pi)b_1 + (1 - {}^{1}\bar{\rho}(\pi))b_2$ subject to $\pi b_1 + (1 - \pi)b_2 \leq 0$, $b_2 + 1 \geq 0$ and:

$$\frac{1}{4} b_1 + \frac{3}{4} b_2 + 1 \geq 0 \qquad (0 \leq \pi \leq \frac{1}{3})$$

$$\frac{1}{4} (6\pi - 1)b_1 + \frac{1}{4}(5 - 6\pi)b_2 + 1 \geq 0 \quad (\frac{1}{3} \leq \pi \leq \frac{2}{3})$$

$$\frac{3}{4} b_1 + \frac{1}{4} b_2 + 1 \geq 0 \quad (\frac{2}{3} \leq \pi \leq 1) \quad .$$

<u>Consumer 1</u>, then, has a linear programming problem. If $(1/4) \leq \pi \leq (1/2)$, there is no solution: consumer 1 wants to make b_2 very large and b_1 very small - the same is also true if $\pi \geq (3/4)$. This leaves two cases:

(A) <u>$\frac{1}{2} < \pi \leq \frac{2}{3}$.</u> Now $(6\pi - 1)/4 > \pi$, and it is easy to see that consumer 1 wants to satisfy the following two constraints with equality:

$$\pi b_1 + (1 - \pi)b_2 \leqq 0, \ \frac{1}{4}(6\pi - 1)b_1 + \frac{1}{4}(5 - 6\pi)b_2 + 1 \geqq 0$$

so

$$^{1}b_1 = \frac{-4(1 - \pi)}{2\pi - 1} \quad , \quad ^{1}b_2 = \frac{4\pi}{2\pi - 1}$$

(B) <u>$\frac{2}{3} \leq \pi < \frac{3}{4}$.</u> Now consumer 1 wants to satisfy the following two constraints with equality:

$$\pi b_1 + (1-\pi)b_2 \leqq 0,\ \frac{3}{4}b_1 + \frac{1}{4}b_2 + 1 \geq 0$$

and so

$$^1b_1 = -\frac{1-\pi}{\frac{3}{4}-\pi}\ ,\quad ^1b_2 = \frac{\pi}{\frac{3}{4}-\pi}$$

Consumer 2 will maximize $^2\bar{\rho}(\pi)b_1 + (1 - {}^2\bar{\rho}(\pi))b_2$ subject to $\pi b_1 + (1-\pi)b_2 \leq 0$, $b_1 + 1 \geq 0$, $\alpha b_1 + (1-\alpha)b_2 + 1 \geq 0$. So consumer 2 also has a linear programming problem. If $\pi \leq \alpha$ there is no solution: consumer 2 then wants to make b_1 very large and b_2 very small. But if $\alpha < \pi \leq 3/4$ then (since $^2\bar{\rho}(\pi) > \pi$) consumer 2 will want to satisfy the following two constraints with equality:

$$\pi b_1 + (1-\pi)b_2 \leq 0,\ \alpha b_1 + (1-\alpha)b_2 + 1 \geq 0$$

Thus $^2b_1 = (1-\pi)/(\pi-\alpha)$, $^2b_2 = -\pi/(\pi-\alpha)$.

Now a number of cases are possible, depending upon the value of α.

(1) $\underline{\frac{1}{4} < \alpha \leq \frac{1}{2}}$. Now the set P corresponds to values of π in the interval $(1/2, 3/4)$. $\pi = 1/2$ is a boundary point, and at this boundary point, although the interval $(\alpha, 1/2)$ corresponds to the interior of $K^*(1/2,1/2)$, no point of P is in the interior. In fact, there is also no equilibrium, because $^1b_1 + {}^2b_1 < 0$ and $^1b_2 + {}^2b_2 > 0$ for every $p \in P$, as is easily checked. Notice that A.11 is violated.

(2) $\underline{\frac{1}{2} < \alpha < \frac{3}{4}}$. Now P corresponds to values of π in the interval $(\alpha, 3/4)$. $\pi = \alpha$ is a boundary point, and there is a point of P in the interior of $K^*(\alpha, 1-\alpha)$, as well as in $K^*(3/4,1/4)$. So A.11 is satisfied, and we expect there to be an equilibrium. In fact, it is easy to check that the equilibrium value of is given by:

$$\pi^* = \begin{cases} 2\alpha - \frac{1}{2} & (\frac{1}{2} < \alpha \leq \frac{7}{12}) \\ \frac{1}{2}(\alpha + \frac{3}{4}) & (\frac{7}{12} \leq \alpha < \frac{3}{4}) \end{cases}$$

(3) $\alpha < \frac{1}{4}$. Now P corresponds to values of π in the interval $(\alpha, 1/4)$. A.11 is clearly satisfied. It follows that an equilibrium will exist in this case too.

Example 8.2: The second example illustrates the need to assume that the set P is convex, as in A.10. Notice that if there are only two assets, then each cone ${}^iK(p)$, when asset values are normalized, becomes a subset of $\Delta^2 \subseteq R^2_+$, in effect. Then, since $p^2 \in \Delta^2$ as well, we see that P is either convex or else the union of disconnected convex sets, namely intervals. Then it is possible to apply the boundary condition A.11 to each interval of the set P separately. In fact, it seems that there will always be some interval of P on which A.11 is satisfied and so a temporary Walrasian equilibrium will exist.

Accordingly, to produce an example of non-existence, I shall assume that there are three assets. I shall also maintain symmetry in the example by assuming that there are also three agents, each of whom has more restrictive beliefs about the possible value of one of the assets in which he is a "specialist".

Each agent, i = 1, 2, 3, chooses his net demand vector for financial assets ${}^i\hat{b} = ({}^i\hat{b}_1, {}^i\hat{b}_2, {}^i\hat{b}_3)$ in order to maximize expected wealth in the second period:

$$\int (r_1 b_1 + r_2 b_2 + r_3 b_3) d\,{}^i\psi(p)$$

subject to the budget constraint:

$$p_1b_1 + p_2b_2 + p_3b_3 \leq 0$$

and subject to the feasibility constraint:

$$r_1b_1 + r_2b_2 + r_3b_3 + 1 \geq 0 \quad \text{for all} \quad r \in \text{supp}\ {}^i\psi(p) \quad .$$

Assume that $r \in \Delta^3$ always and that each agent's expectations measure ${}^i\psi(p)$ is a uniform distribution on the triangular support of ${}^i\psi(p)$ which is given by:

$${}^i\psi(p) = \{r \in \Delta^3 | \rho(p) \leq r_i \leq 1\} \quad (i = 1,2,3) \quad .$$

where $\rho(p)$ is the same function for each agent, and is given by:

$$\rho(p) := 18p_1p_2p_3 \quad (\text{all} \quad p \in \Delta^3) \quad .$$

This function $\rho(p)$ achives a unique maximum over the triangle Δ^3 at the mid-point $p_1 = p_2 = p_3 = 1/3$, where its value is $2/3$. The condition for $p \in P$ is that $\rho(p) < p_i$ $(i = 1,2,3)$. Assume, without loss of generality (given the symmetry of the example) that $p_1 \leq p_2 \leq p_3$. Then $p \in P$ if and only if $18p_2p_3 < 1$. The constraints $p_1 \leq p_2 \leq p_3$ and $18p_2p_3 < 1$ can be satisfied simultaneously if and only if $0 \leq p_1 < (3 - \sqrt{5})/12$. So P consists of points which are near the edge of Δ^3; points in a neighbourhood of the mid-point $(1/3,1/3,1/3)$ are excluded from P. Thus P is non-empty but it is not convex nor is it contractible or even simply connected.

Consider trader 1's demands first; the others' are derived by an obvious permutation. Trader 1's triangular support of ${}^1\psi(p)$ has corners at $(1,0,0)$, $(\rho, 1-\rho, 0)$ and $(\rho, 0, 1-\rho)$ (where I have used ρ to abbreviate the notation $\rho(p)$). His expected wealth is:

$$\frac{1}{3}\{(1+2\rho)b_1 + (1-\rho)b_2 + (1-\rho)b_3\}$$

which he seeks to maximize subject to the budget constraint:

$$p_1b_1 + p_2b_2 + p_3b_3 \leq 0$$

and the feasibility constraints:

$$b_1 + 1 \geq 0$$

$$\rho b_1 + (1-\rho)b_2 + 1 \geq 0$$

$$\rho b_1 + (1-\rho)b_3 + 1 \geq 0$$

(It suffices to consider feasibility at only the extreme points of supp ${}^1\psi(p)$ because the constraints are linear).

This linear programming problem has a solution provided that $p_1 > \rho$, so that $p \in {}^1D$. It solution depends on which is the smallest of the three numbers $p_1 - \rho$, p_2, p_3. In fact, the mean return to asset 1 is $(1/3)(1+2\rho) - p_1$, whereas the mean return to assets 2 and 3 are $(1/3)(1-\rho) - p_2$, $(1/3)(1-\rho) - p_3$ and what matters is which asset has the highest return.

Case A. $p_1 - \rho < \min\{p_2,p_3\}$

Now asset 1 has the highest return, and the trader holds a positive amount of it, choosing the corner $(1 - p_1,-p_1,-p_1)/(p_1 - \rho)$ of the set ${}^1A(p)$ of feasible actions. (Recall that $p_2 + p_3 = 1 - p_1$).

Case B. $p_2 < \min\{p_1-\rho,p_3\}$

Now asset 2 has the highest return and the trader chooses the corner $(-1,-1 + \frac{1}{p_2},-1)$ of ${}^1A(p)$ with ${}^1\hat{b}_2 > 0$.

Case C. $p_3 < \min\{p_1 - \rho,p_2\}$

Now the trader chooses the corner $(-1,-1,-1 + (1/p_3))$ of ${}^1A(p)$.

In intermediate cases, where the minimum of the set $\{p_1 - \rho,p_2,p_3\}$ is not unique, the demands are suitable convex combinations of two corners of ${}^1A(p)$.

Trader 2's and trader 3's demands are derived by an obvious cyclic permutation.

I shall show that this economy has no temporary Walrasian equilibrium by disposing of a number of possible types of potential equilibria.

First, however, if we continue to assume that $p_1 \leq p_2 \leq p_3$, then $p_1 - \rho < p_1 \leq p_2 \leq p_3$ whenever $p \in \text{int}\,\Delta^3$ and so trader 1 is always in Case A.

Type I. $p_3 - \rho < p_1 \leq p_2 \leq p_3$

Then $p_2 - \rho < p_1$ a fortiori, and so each trader has demands as in Case A, with trader i buying asset i and selling the other two. The aggregate demands of the three traders together are given by:

$$b_1 = \frac{1 - 3p_1}{p_1 - \rho} \quad , \quad b_2 = \frac{1 - 3p_2}{p_2 - \rho} \quad , \quad b_3 = \frac{1 - 3p_3}{p_2 - \rho}$$

So, if there were an equilibrium of type I, it would have to be at the mid-point $p_1 = p_2 = p_3 = (1/3)$, but there $\rho > (1/3)$ and so in fact no trader has a well-defined demand. Thus there is no equilibrium of Type I.

Type II. $p_2 - \rho \leq p_1 < p_3 - \rho, \; p_1 \leq p_2 \leq p_3$

Now trader 2 is in Case A (or possibly Case C, buying asset 1, if $p_2 - \rho = p_1$) as well as trader 1, but trader 3 is in Case B and buys asset 1 (or possibly asset 2 if $p_1 = p_2$). Every trader sells asset 3 and so there is no equilibrium, because equilibrium prices all have to be positive which is inconsistent with excess supply of any asset.

Type III $p_1 < p_2 - \rho < p_2 \leq p_3$

Then $p_1 - \rho < p_2 \leq p_3$ and $p_1 < p_2 - \rho \leq p_3 - \rho$. Now every trader demands asset 1 and so no equilibrium can exist.

This exhausts the types where no trader is on the boundary between two cases. Now we must deal with some remaining boundary cases.

Type IV. $p_2 - \rho < p_3 - \rho = p_1 \leq p_2 < p_3$

Traders 1 and 2 remain in Case A but now trader 3 is on the boundary between Case A and Case B, with demands given by convex combinations of the form:

$$\frac{\lambda}{p_3 - \rho}(-p_3, -p_3, 1 - p_3) + (1 - \lambda)(-1 + \frac{1}{p_1}, -1, -1) \quad \text{with} \quad 0 \leq \lambda \leq 1.$$

Aggregate demands are given by:

$$b_1 = \frac{1 - p_1}{p_1 - \rho} - \frac{p_2}{p_2 - \rho} - \frac{\lambda p_3}{p_3 - \rho} + (1 - \lambda)(\frac{1}{p_1} - 1)$$

$$b_2 = \frac{- p_1}{p_1 - \rho} + \frac{1 - p_2}{p_2 - \rho} - \frac{\lambda p_3}{p_3 - \rho} - (1 - \lambda)$$

$$b_3 = \frac{- p_1}{p_1 - \rho} - \frac{p_2}{p_2 - \rho} + \frac{\lambda(1 - p_3)}{p_3 - \rho} - (1 - \lambda)$$

A necessary condition for equilibrium is that:

$$0 = b_2 - b_3 = \frac{1}{p_2 - \rho} - \frac{\lambda}{p_3 - \rho}$$

and so $\lambda = (p_3 - \rho)/(p_2 - \rho) > 1$, a contradiction. So there is no equilibrium of Type IV.

<u>Type V.</u> <u>$p_2 - \rho = p_1 = p_3 - \rho,\ p_1 < p_2 = p_3$</u>

Trader 1 is in Case A, as always. But now both traders 2 and 3 are on the boundary between their own respective Case A and the case where they demand Asset 1. Trader 2's demands are convex combinations of the form:

$$\frac{\lambda}{p_2 - \rho}(-p_2,\ 1 - p_2, -p_2) + (1 - \lambda)(\frac{1}{p_1} - 1, -1, -1)$$

and trader 3's demands are convex combinations of the form:

$$\frac{\mu}{p_3 - \rho}(-p_3, -p_3,\ 1 - p_3) + (1 - \mu)(\frac{1}{p_1} - 1, -1, -1)$$

where $0 \leq \lambda \leq 1$ and $0 \leq \mu \leq 1$. Aggregate demands are:

$$b_1 = \frac{1 - p_1}{p_1 - \rho} - \frac{\lambda p_2}{p_2 - \rho} - \frac{\mu p_3}{p_3 - \rho} + (2 - \lambda - \mu)(\frac{1}{p_1} - 1)$$

$$b_2 = \frac{-p_1}{p_1 - \rho} + \frac{\lambda(1 - p_2)}{p_2 - \rho} - \frac{\mu p_3}{p_3 - \rho} - (2 - \lambda - \mu)$$

$$b_3 = \frac{-p_1}{p_1 - \rho} - \frac{\lambda p_2}{p_2 - \rho} + \frac{\mu(1 - p_3)}{p_3 - \rho} - (2 - \lambda - \mu)$$

A necessary condition for equilibrium is:

$$0 = b_2 - b_3 = \frac{\lambda}{p_2 - \rho} - \frac{\mu}{p_3 - \rho}$$

and so $\lambda = \mu$ because $p_2 = p_3$.

But

$$b_3 = \frac{\mu(1 - \rho)}{p_3 - \rho} - \frac{\lambda\rho}{p_2 - \rho} - \frac{p_1}{p_1 - \rho} - 2 = 0 \text{ in equilibrium} .$$

So, recognizing that $p_2 = p_3$ and that $\lambda = \mu$, it follows that:

$$\lambda = \mu = \left(2 + \frac{p_1}{p_1 - \rho}\right) \frac{p_2 - \rho}{1 - 2\rho} = \frac{(3p_1 - 2\rho)(p_2 - \rho)}{(1 - 2\rho)(p_1 - \rho)} .$$

But $p_2 = p_3 = p_1 + \rho$ and $p_1 + p_2 + p_3 = 1$, so that

$$p_1 = \frac{1 - 2\rho}{3} , \quad p_2 = p_3 = \frac{1 + \rho}{3} .$$

Therefore

$$\lambda = \mu = \frac{1 - 4\rho}{1 - 2\rho} \cdot \frac{1 - 2\rho}{1 - 5\rho} = \frac{1 - 4\rho}{1 - 5\rho} > 1$$

because $\rho > 0$ for all $p \in \text{int } \Delta^3$. This is a contradiction and shows there can be no equilibrium of type V either.

This exhausts all the possible types with $p_1 \leq p_2 \leq p_3$, which is an assumption that loses no generality because of the symmetry of the example. Therefore there is no temporary Walrasian equilibrium.

9. Overlapping Expectations

I claim that an alternative sufficient condition for existence of temporary Walrasian equilibrium, in place of A.10 and A.11, is the following overlapping expecations condition:

Assumption 10': For all $p \in \Delta^{\ell}$, the interior of $K^*(p) := \bigcap_{i \in I} {}^{i}K(p)$ is non-empty.

This condition is effectively dual to Hart's ([1974], Theorem 3.3) sufficient condition for existence of equilibrium in a securities model, as I pointed out in Hammond [1980]. Here I propose to show that it is sufficient for existence in the more general economy with many physical commodities and private capital formation. Before doing so, however, let me note that there are certainly economies in which agents' expectations do not overlap for all $p \in \Delta^{\ell}$ and yet there exists a temporary Walrasian equilibrium. In Example 8.1, for instance, an equilibrium exists in the case when $1/2 < \alpha < 3/4$ and then the set P corresponds to values of π in the open interval $(\alpha, 3/4)$. But if $\pi \leq 1/3$, for instance, then ${}^{1}K(p)$ corresponds to the interval $[0, 1/4)$ and ${}^{2}K(p)$ corresponds to the interval $[\alpha, 1]$, and these two intervals certainly fail to overlap.

It follows that A.10 and A.11 together can certainly be weaker than the overlapping expectations Assumption A.10'. The question then arises whether it is sufficient to have overlapping expectations (with int $K^*(p)$ non-empty) holding on some subset of Δ^ℓ, such as the closure of P, or some neighbourhood of the closure of P. Example 8.1 with $1/4 < \alpha < 1/2$ shows that it is not sufficient, because expectations overlap throughout the set $(4\alpha + 1)/6 < \pi \leq 1$ whose relative interior includes the closure of the interval $(1/2. 3/4)$ which corresponds to P.

Now the promised sufficiency result:

<u>Theorem 9.1</u>. If Assumptions A.1 to A.9 are all satisfied, and if, in addition, the overlapping expectations condition A.10' is met at every $p \in \Delta^\ell$, then there exists a temporary Walrasian equilibrium.

<u>Proof</u>:

(1) Recall the definition of the budget set ${}^iA(p)$, for each $i \in I$ and each $p \in \Delta^\ell$, as in Section 2.E. For each $i \in I$ and each positive integer t, define the restricted budget set:

$${}^iA_t(p) := \{({}^i\alpha^1, {}^ib, {}^ik) \in {}^iA(p) \mid {}^ib \geqq -(t,t,t,\ldots,t)\}$$

(2) Then, for each finite t, the budget set ${}^iA(p)$ is bounded below because the first period consumption set ${}^iC$ is bounded below. So, by the usual arguments--e.g., Debreu [1959]--in the restricted economy there exists a temporary Walrasian equilibrium price vector $p_t \in \Delta^\ell$ for each t and associated actions ${}^i\hat{a}_t = ({}^i\hat{c}^1_t, {}^i\hat{b}_t, {}^i\hat{k}_t) \in {}^i\alpha(p_t)$

which maximize each agent i's expected utility function ${}^i v({}^i a, p_t)$ over the restricted budget set ${}^i A_t(p_t)$, and which satisfy $\sum_i {}^i\hat{z}_t \leqq 0$ where ${}^i\hat{z}_t = ({}^i\hat{c}^1_t, {}^i\hat{b}_t)$.

(3) By Lemma 5.1, all agents' utility functions ${}^i v({}^i a, p_t)$ are strictly monotone in $({}^i c, {}^i b) \in R^\ell$ and so it must be true that $p_t > 0$ (all t).

(4) Suppose that, for some finite t, ${}^i\hat{b}_t > -(t,t,t,\ldots,t)$ for all $i \in I$ and yet p_t is not a temporary Walrasian equilibrium price vector (for the unrestricted economy). Then there exists $j \in I$ and ${}^j a^* \in {}^j A(p_t)$ such that ${}^j v({}^j a^*, p_t) > {}^j v({}^j\hat{a}_t, p_t)$.

(5) For any λ between 0 and 1, define the strict convex combination

$$ {}^j a(\lambda) := \lambda {}^j a^* + (1 - \lambda) {}^j\hat{a}_t \quad . $$

Because ${}^j A(p_t)$ is convex, ${}^j a(\lambda) \in {}^j A(p_t)$. Because ${}^j v({}^j a, p_t)$ is concave in ${}^j a$, ${}^j v({}^j a(\lambda), p_t) > {}^j v({}^j\hat{a}_t, p_t)$.

(6) When λ is small and positive, so that ${}^j a(\lambda)$ is close to ${}^j\hat{a}_t$, then ${}^j b(\lambda) = \lambda {}^j b^* + (1 - \lambda) {}^j\hat{b}_t > -(t,t,t,\ldots,t)$ and so ${}^j a(\lambda) \in {}^j A_t(p_t)$.

(7) Together (5) and (6) contradict the fact that ${}^j\hat{a}_t$ maximizes ${}^j v({}^j a, p_t)$ subject to ${}^j a \in {}^j A_t(p_t)$, and so it must be true after all that p_t is a temporary Walrasian equilibrium price vector for the unrestricted economy.

(8) It remains to be shown that there exists some finite t such that ${}^{i}\hat{b}_t > -(t,t,t,\dots,t)$ for all $i \in I$. Suppose not. Then for each t, there exists $j_t \in I$ such that ${}^{j_t}\hat{b}_t \not> -(t,t,t,\dots,t)$ and in particular $||{}^{j_t}\hat{b}_t|| \geq t$, so that $\sum_{j\in I}||{}^{j}\hat{b}_t|| \geq t$.

(9) Because p_t is a sequence in the compact set Δ^{ℓ}, and because, for each $i \in I$, ${}^{i}\hat{z}_t/\sum_{j\in I}||{}^{j}\hat{z}_t||$ is in the compact unit ball of R^{ℓ}, we can assume, after taking appropriate subsequences, that p_t converges to p^* and that, for all $i \in I$, ${}^{i}\hat{a}_t/\sum_{j\in I}||{}^{j}\hat{z}_t||$ converges to ${}^{i}a^* = ({}^{i}c^{1*}, {}^{i}b^*, {}^{i}k^*) = ({}^{i}z^*, {}^{i}k^*)$ where ${}^{i}k^* = 0$ because ${}^{i}\hat{k}_t$ is bounded, and $||{}^{j}\hat{z}_t|| \geq ||{}^{j}\hat{b}_t|| \to \infty$.

(10) Because of market clearing, $\sum_{i\in I}{}^{i}\hat{z}_t = 0$ for all t and so $\sum_{i\in I}{}^{i}z^* = 0$ also.

(11) For every $i \in I$ and every $t(t=1,2,\dots)$ we know that there exist ${}^{i}\tilde{c}^{1} \in {}^{i}C^{1}$ and ${}^{i}\tilde{k} \in R_{+}^{\ell_3}$ such that $({}^{i}\tilde{c}^{1}, 0, {}^{i}\tilde{k}) \in {}^{i}A(p_t)$ because of A.8.

(12) Take any scalar $\lambda \geq 0$. It follows from (8) above that, whenever $t \geq \lambda$, then:

$$\sum_{j\in I}||{}^{j}\hat{z}_t|| \geq \sum_{j\in I}||{}^{j}\hat{b}_t|| \geq t \geq \lambda \quad .$$

(13) Define ${}^{i}\tilde{a} := ({}^{i}\tilde{c}^{1}, 0, {}^{i}\tilde{k})$ (all $i \in I$) and, for each $t = 1, 2, \dots$, let

$${}^{i}\tilde{a}_t := \frac{\lambda\ {}^{i}\hat{a}_t}{\sum_{j\in I}||{}^{j}\hat{z}_t||} + \left(1 - \frac{\lambda}{\sum_{j\in I}||{}^{j}\hat{z}_t||}\right){}^{i}\tilde{a} \quad .$$

(14) Because ${}^{i}A(p_t)$ is convex we see that ${}^{i}\tilde{a}_t \in {}^{i}A(p_t)$ for all $t \geq \lambda$, and so, taking limits as $t \to \infty$ it follows that, for all $\lambda \geq 0$ and all $i \in I$:

$$ {}^{i}\tilde{a} + \lambda {}^{i}a^* \in {}^{i}A(p^*) $$

because ${}^{i}\tilde{a}_t$ converges to ${}^{i}\tilde{a} + \lambda {}^{i}a^*$ and because the correspondence ${}^{i}A(\cdot)$ has a closed graph by Lemma 6.4.

(15) Because ${}^{i}\tilde{a} + \lambda {}^{i}a^* \in {}^{i}A(p^*)$ and ${}^{i}k^* = 0$, ${}^{i}\tilde{b} = 0$ it follows that for all $e \in \text{supp } {}^{i}\mu(p^*)$ there exists ${}^{i}c^2(e) \in {}^{i}C^2({}^{i}\tilde{k},s)$ for which $q\,{}^{i}c^2(e) \leq r\lambda {}^{i}b^* + {}^{i}\pi^2(q;\ {}^{i}\tilde{k},s)$. But $q\,{}^{i}c^2(e) - {}^{i}\pi^2(q;\ {}^{i}\tilde{k},s)$ is uniformly bounded below over E^* and so $\lambda r\,{}^{i}b^*$ is uniformly bounded below, for all $\lambda \geq 0$. This is only possible if $r\,{}^{i}b^* \geq 0$ for all $r \in \text{supp } {}^{i}\psi(p^*)$ and so for all $r \in {}^{i}K(p^*)$.

(16) By Assumption A.10' (overlapping exectations), the set $K^*(p) := \bigcap_{i \in I} {}^{i}K(p^*)$ is non-empty and has a non-empty interior. Then, by (15), $r\,{}^{i}b^* \geq 0$ for all $i \in I$ and all $r \in K^*(p^*)$. But, by (10), $\sum_{i \in I} {}^{i}b^* = 0$ and so $\sum_{i \in I} r\,{}^{i}b^* = 0$ for all $r \in K^*(p^*)$ This is only possible if $r\,{}^{i}b^* = 0$ for all $i \in I$ and all $r \in K^*(p^*)$. Because $K^*(p^*)$ has a non-empty interior, this in turn implies that ${}^{i}b^* = 0$ for all $i \in I$.

(17) So, from (9), (14) and (16) it follows that ${}^{i}\tilde{a} + \lambda {}^{i}a^* \in {}^{i}A(p^*)$ for all $i \in I$ and all $\lambda \geq 0$ where ${}^{i}\tilde{b} = 0$ and ${}^{i}b^* = 0$, ${}^{i}k^* = 0$. Thus, because ${}^{i}C^1$ is bounded below, and ${}^{i}\tilde{c}^1 + \lambda {}^{i}c^{*1} \in {}^{i}C^1$ for all $\lambda \geq 0$, it follows that ${}^{i}c^{*1} \geqq 0$ for all $i \in I$. But by (10), $\sum_{i \in I} {}^{i}c^{*1} = 0$. Therefore ${}^{i}c^{*1} = 0$ for all $i \in I$. So ${}^{i}z^* = ({}^{i}c^{*1}, {}^{i}b^*) = 0$ for all $i \in I$. This contradicts (9), because ${}^{i}\hat{z}_t / \sum_{j \in I} \| {}^{j}\hat{z}_t \| \to {}^{i}z^*$ and so:

$$0 = \sum_{i\in I} \| {}^i z^* \| = \lim_{t\to\infty} \sum_i \left\| \frac{{}^i\hat{z}_t}{\sum_{j\in I} \| {}^j\hat{z}_t \|} \right\| = 1 \quad ,$$

which establishes the contradiction.

(18) Therefore there does exist some t such that $\| {}^i\hat{b}_t \| < t$ for all $i \in I$, which implies existence of a temporary Walrasian equilibrium price vector, from (4) and (8) above. Q.E.D.

10. Possible Extensions and Limitations

The paper has set out a model of an economy with rather general asset markets and private capital formation, and has proved existence of a temporary Walrasian equilibrium under certain assumptions, notably on agents' expectations. Thus, Green's existence results continue to hold in much more general market situations. Nevertheless, there are some serious limitations. One is the absence of production by joint stock companies. Another is the absence of any monetary policy of the kind considered by Grandmont and Laroque [1975], for example.

Yet another serious limitation is that the set of asset markets has been taken as both exogenous and fixed. In practice, however, markets are set up only if it appears likely that the benefits from trading on them outweigh the transaction costs. Now, both the anticipated benefits and also the transactions costs will depend upon the price vector p in the markets which are already functioning. So a complete theory would have to allow for markets opening and closing in response to price changes. Quite apart from the additional complications this would cause, it leads fairly evidently to another problem with nonconvexities.

Appendix

My claim that assumption A.11 in combination with the other assumptions is sufficient for an equilibrium to exist is a clear contradiction of Green [1973], who asserted that his example 5.2 (pp. 1119-21) showed that the kind of weaker common expectations assumption I have used is insufficient to ensure existence of a Walrasian equilibrium, even in an economy with a single trader. Green's example, however, violates part (i) of his assumption (2.3) (p. 1105), and my Assumption 7. Normalizing so that $q_1 \equiv 1$, $p_1^1 \equiv 1$, the measure $\mu(p)$ Green used in his example is the uniform distribution for q_2 over the interval $[\theta_1(p_2^2), \theta_1(p_2^2) + 1]$. θ_1 here is taken to be 1 for $0 \leq p_2^2 \leq 1 - \log 2$. For $p_2^2 \geq 1 - \log 2$, $\theta_1(p_2^2)$ satisfies the differential equation

$$\frac{d\theta_1}{dp_2^2} = \frac{1}{2}(2 + 3\theta_1 + \theta_1^2) \quad .$$

The solution to this equation satisfies:

$$2[\log (\frac{1 + \theta_1}{2 + \theta_1}) - \log (\frac{2}{3})] = p_2^2 - 1 + \log 2$$

So $\theta_1 \to \infty$ as $p_2^2 \to 1 - \log 2 + 2 \log \frac{3}{2}$. Thus, renormalizing so that $q \in \Delta^2$, we see that

$$\text{supp } \psi(p) = \{(0,1)\} \text{ when } p_2^2 = 1 - \log 2 + 2 \log \frac{3}{2} \quad .$$

This is a crucial violation of Green's assumption (2.3), or my A.7. In fact, $\theta_1(p_2^2)$ is not even defined when $p_2^2 \geq 1 - \log 2 + 2 \log 3/2$.

Because A.10' is automatically implied by A.7 in any economy with a single consumer, it is actually impossible to produce any example of non-existence in a single consumer economy satisfying assumptions A.1 to A.9.

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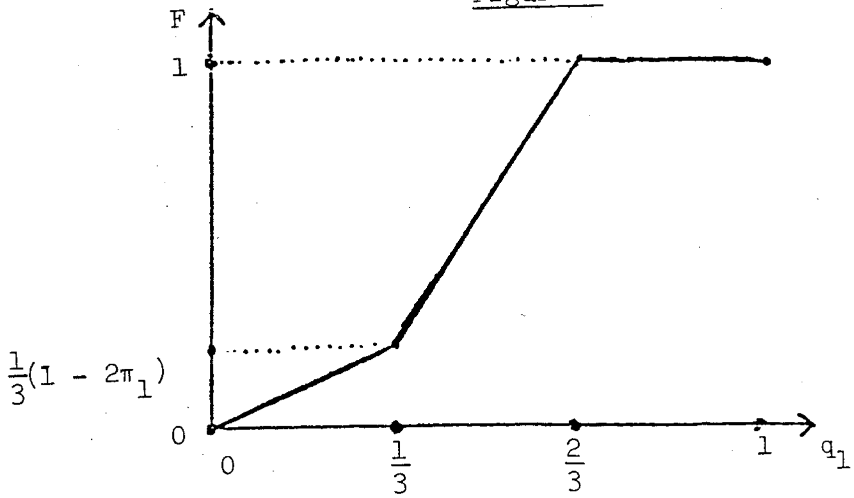
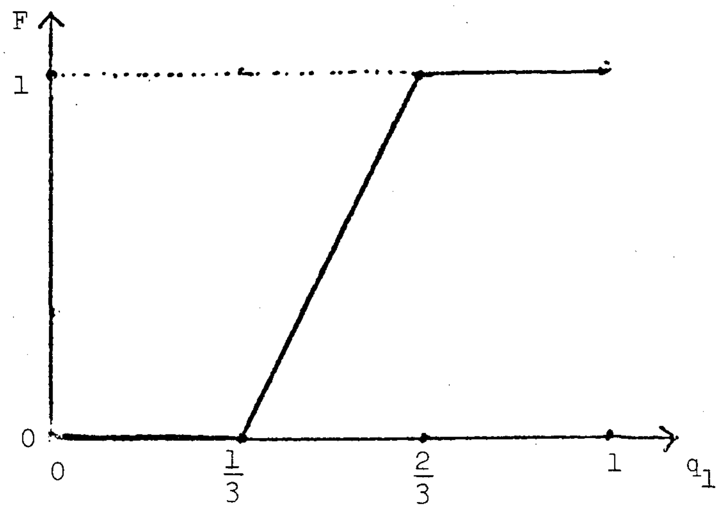
Figure 1



$\pi_1 < \frac{1}{2}$



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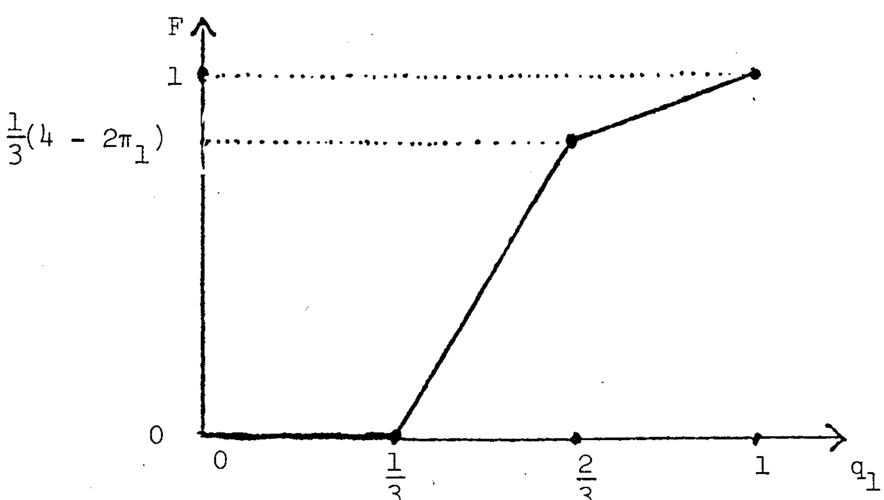

$\pi_1 > \frac{1}{2}$