Walrasian Equilibrium without Survival: Existence, Efficiency, and Remedial Policy

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Abstract

Standard general equilibrium theory excludes starvation by assuming that everybody can survive without trade. Because trade cannot harm consumers, they can therefore also survive with trade. Here this assumption is abandoned, and equilibria in which not everybody survives are investigated. A simple example is discussed, along with possible policies which might reduce starvation. Thereafter, for economies with a continuum of agents, the usual results are established — existence of equilibrium, the two fundamental efficiency theorems of welfare economics, and core equivalence. Their validity depends on some special but not very stringent assumptions needed to deal with natural non-convexities in each consumer's feasible set.

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Equilibrium without Survival

It is unquestionably true, that in no country of the globe have the government, the distribution of property, and the habits of the people, been such as to call forth, in the most effective manner, the resources of the soil. Consequently, if the most advantageous possible change in all these respects could be supposed at once to take place, it is certain that the demand for labour, and the encouragement to production, might be such as for a short time, in some countries, and for rather a longer time in others, to lessen the operation of the checks to population which have been described.

— Malthus (1830, pp. 247-248 of A. Flew (ed.))

A society in which some people roll in luxury while others live in acute misery can still be Pareto optimal if the agony of the deprived cannot be reduced without cutting into the ecstasy of the affluent.

— Sen (1990)

It would appear that, on the level of individual nations and of international relations, *the free market* is the most efficient instrument for utilizing resources and effectively responding to needs. But this is true only for those needs which are "solvent", insofar as they are endowed with purchasing power, and for those resources which are "marketable", insofar as they are capable of obtaining a satisfactory price. But there are many human needs which find no place on the market. It is a strict duty of justice and truth not to allow fundamental human needs to remain unsatisfied, and not to allow those burdened by such needs to perish. It is also necessary to help these needy people to acquire expertise, to enter the circle of exchange, and to develop their skills in order to make the best use of their capacities and resources. Even prior to the logic of a fair exchange of goods and the forms of justice appropriate to it, there exists *something which is due to man because he is man*, by reason of his lofty dignity. Inseparable from that required "something" is the possibility to survive and, at the same time, to make an active contribution to the common good of humanity.

— Pope John Paul II's Encyclical Letter *Centesimus Annus* (1991), Section 34.

1. Introduction

Students of economics are routinely taught the efficiency of free markets. Mathematical economists, following Arrow (1951), Debreu (1954, 1959), Arrow and Debreu (1954), McKenzie (1954, 1959, 1961) and others, have provided rigorous foundations to these teachings by proving the existence of Walrasian or "competitive" equilibrium, as well as the two fundamental efficiency theorems of welfare economics which relate Pareto efficient allocations to Walrasian equilibria. As Sen (1977, 1981a, 1981b) has pointed out, however, the standard theorems deal with the issue of survival by assuming that all agents can always survive without trade. This we call the "survival assumption." Since trading opportunities cannot then detract from what is possible without trade, all agents can survive with trade too. Thus, according to the standard theory, in any Walrasian equilibrium there is universal survival. Yet the survival assumption is clearly counter to the facts of life and death in almost all the worlds' economies, whatever their stage of development or their economic system. Only a very few peasant societies are truly self-sufficient. Even in those which are, few individual consumers are likely to be able to survive for long on their own.

In fact it is all too evident that, even in wealthy countries, some people do die for want of adequate nourishment, for lack of simple and cheap life-saving drugs, or because they cannot afford to keep warm during bitter winter weather. So the world does not fit this standard model of Walrasian equilibrium. It is of some importance, however, to know whether this is due to the equally evident failure of economies to conform to the Walrasian market model because of monopolies, externalities, taxes, or other "market failures." Or alternatively, whether it is the survival assumption itself that needs to be questioned. For if the personal tragedies of starvation or disease are the result of market failures and/or government intervention in markets, there is that much better a case for *laissez faire* economic policies (including the promotion of competition and remedies for the most serious external diseconomies). On the other hand, if such personal tragedies are in part the result of market forces when the people are unable to survive without, for instance, being able to trade their labour power or skills, then *laissez faire* becomes much less acceptable.

For this reason it seems important to know whether the survival assumption really does play a crucial role in the standard Walrasian theory of competitive markets. This requires a theory of how competitive markets can work in the absence of the survival assumption.

One of our aims, therefore, will be to see what it takes to assure survival — either free markets on their own, or else some redistributive policies such as welfare programs or land reform.

Among past writers, Malthus (1798, 1830) certainly described features of economies in which the growth of population was limited only by drastic scarcity. He also realized that the distribution of wealth was an important influence on the numbers of survivors from one generation to the next. Earlier Turgot, as government representative for the Limousin region of central France, had recorded his impression that the 1770 famine owed much to lack of purchasing power rather than to grain shortages (see Rothschild, 1992). Other references to discussions of starvation by classical economists can be found in Sen (1986). Yet, as far as the modern literature on Walrasian economics is concerned, we have found only the papers by Bergstrom (1971), Moore (1975) and McKenzie (1981) which really consider the implications of abandoning the standard survival assumption, and they do so only to discuss conditions that suffice to ensure the existence of a Walrasian equilibrium in which all survive. We are not aware of anybody who has yet considered formal general models of Walrasian equilibrium in which some individuals do not survive. Koopmans (1957, p. 62), however, did at least pose the question, noting that "...there is considerable challenge to further research on the survival problem ...," and going on to suggest, moreover, that "One 'hard-boiled' alternative would be to assume instantaneous elimination by starvation of those whose resources prove insufficient for survival, and to look for conditions ensuring existence of an 'equilibrium' involving survival of some consumers." This is the key idea that appears not to have been followed up at all thoroughly or systematically as yet, and which will be explored in this paper.

One reason for economists' failure to tackle this problem may be the general belief that standard general equilibrium theory somehow breaks down or does not apply when survival of all agents is not guaranteed. Indeed, in an otherwise illuminating presentation of some examples of entitlement failure, Desai (1989, p. 430) first points out that, "If there is no market failure, then relative price movements should lead to optimal outcomes," which is quite correct if we interpret "optimal" to mean just "Pareto efficient." He then goes on, however, to state and even emphasize that, "while an equilibrium exists, relative prices fail to play an allocative role." It is true that he does not discuss Pareto efficiency directly.

Yet these sentences suggest to us a belief that entitlement failures, while not preventing existence of (competitive) equilibrium, nevertheless do cause the price mechanism to break down in a way which leads to (Pareto) suboptimal or inefficient outcomes. Now, unless entitlement failures are themselves regarded as a form of market failure, this would clearly contradict the most robust result in general equilibrium theory — namely, the first efficiency theorem of welfare economics stating that whenever Walrasian equilibrium allocations exist, they must be Pareto efficient. Yet we shall show that entitlement failures do nothing to create any Pareto inefficiencies, and so cannot possibly be market failures in any normal sense. Another purpose of this paper, therefore, is to correct the apparently common misconception that, because some individuals are starving to death, the usual results of general equilibrium theory somehow do not apply.

So we will reconsider the theory of Walrasian economic equilibrium without imposing the usual survival assumption. Indeed we will consider when and in what sense equilibrium allocations can occur without survival of all individuals. The more technical later sections show that the standard theorems concerning existence and Pareto efficiency of Walrasian equilibrium are still valid, with some modifications. The most major of these arises from the simple observation that the number of surviving people is a discrete variable, so that there are inherent non-convexities, whereas standard Walrasian theory assumes convexity. Yet the proportion of survivors in a large population is, to a very good approximation, a continuous variable. Thus we choose to work with a continuum of agents in the way that Aumann (1964, 1966) pioneered. And we shall also prove a version of the core equivalence theorem for such continuum economies.

For existence proofs in particular, it will also be necessary in general to assume that the distribution of "needs" — of what individuals need in order to be on the margin of survival — is dispersed or, technically, is a non-atomic measure. Then mean demand per individual will be continuous and so existence will be assured under the other usual assumptions.

The formal part of the paper begins in Section 2 with an extended discussion of a particular example. This incorporates a Leontief technology for using land and labour in order to produce a single consumption good. Labour is supplied inelastically by those who survive. Conditions on exogenous parameter values which permit survival are derived. It is shown how the proportion of survivors is determined under a *laissez faire* Walrasian equilibrium when these conditions are not met and when inequality causes some people to starve. Remedial policy is also considered. We show that either lump-sum or land redistribution, or a poll subsidy financed by income taxes, can ensure survival whenever it is physically feasible. Food subsidies, however, are no help at all even in increasing the proportion of survivors.

The example of Section 2 illustrates how it is possible to model the survival issue by adding survival as a good in its own right and then specifying each agent's consumption set as the union of a "survival" set with a disjoint "non-survival" set. Section 3 considers a general continuum economy in which each agent's consumption set C is such a union, which is non-convex, of course. Accordingly we use methods for dealing with such non-convex consumption sets such as those developed by, amongst others, Hildenbrand (1968, 1969), Mas-Colell (1977), Yamazaki (1978, 1981), Coles (1986), Funaki and Kaneko (1986). Thus, Section 3 shows that most of the standard theorems of general equilibrium analysis do still hold — notably, existence of equilibrium, the two fundamental efficiency theorems of welfare economics, and core equivalence.

Finally, Section 4 contains conclusions and discusses some important reservations.

2. An Example

2.1. Description of the Economy

The economy is assumed to consist of a large number (in fact, a continuum) of small farmers who each produce the same crop using identical technologies. Farmers also have identical tastes and consumption sets; only their endowments of land differ.

Each farmer has preferences defined over the three-dimensional consumption set

$$C := \{\, (-t, -h, c) \mid t \geq 0,\ 0 \leq h \leq \ell,\ c \geq \underline{c}\,\}. \tag{1}$$

Here t denotes the (exogenous) amount of land which the farmer has available to supply or use in production, c denotes consumption of the crop, h denotes labour supply, and $\underline{c}$, ℓ are two positive constants representing the minimum subsistence food consumption and the maximum possible supply of labour respectively.

The constant returns to scale Leontief technology to which each farmer has access requires λ units of labour and τ units of land in order to produce each unit of the crop. The cumulative distribution function of land endowments is $F(t)$ which is defined for all $t \geq 0$. The mean holding of land is denoted by $\bar{t}$, and the proportion of farmers who own no land by $f_0 := F(0)$. Assume that $\bar{t} > \tau\,\underline{c}$ and $\ell > \lambda\,\underline{c}$, so there is more land and each person has more labour than the minimum needed to produce subsistence for everyone.

It will be assumed that farmers care only for consumption. Perhaps leisure is a luxury good that poor farmers cannot afford. Contingent upon survival ($c \geq \underline{c}$), each farmer prefers allocations with more consumption to less, and supplies the maximum amount of labour ($h = \ell$) in order to have as much food as possible, whether it is produced directly or else purchased out of wage income.

There are assumed to be perfectly competitive markets for output (corn), labour and land, with corresponding prices denoted by p, w and r respectively. As usual, Walrasian equilibrium requires utility maximization, profit maximization, and market clearing. It is clear from the usual no-pure-profit condition that in any Walrasian equilibrium with positive production the price system must satisfy $p = w\,\lambda + r\,\tau$. Thus $p \geq w\,\lambda \geq 0$ and $p = \lambda\,w$ when $r = 0$. For later reference, we note that the rental-price ratio $r/p \in [0, 1/\tau]$ is determined by the equation

$$r/p = [1 - (w/p)\,\lambda]/\tau \tag{2}$$

for each wage-price ratio w/p in the closed interval of possible values $[0, 1/\lambda]$.

Because production takes place under constant returns to scale, it is also true that in Walrasian equilibrium each farmer is completely indifferent to the allocation of his land and labour between different farming enterprises, including any that he himself may own. Each farmer is also indifferent to how much land and labour he hires from other farmers (cf. Newbery, 1977).

2.2. Equilibrium with Survival

Consumption demand by a farmer who has t units of land must satisfy the budget equation $p\,c = w\,\ell + r\,t$. Equilibrium with positive wages therefore requires that mean consumption per head $\bar{c}$ must satisfy $p\,\bar{c} = w\,\ell + r\,\bar{t}$ and also $\ell = \lambda\,\bar{c}$, $\bar{t} \geq \tau\,\bar{c}$.

When $\bar{t} > \tau\,\ell/\lambda$ there is more land than can possibly be cultivated even if the farmers use all the available labour, so land must be a free good in any equilibrium. This implies that $r = 0$ and $p = \lambda\,w$. The distribution of land holdings is then irrelevant. There is in fact an obvious equilibrium in which every farmer consumes $\bar{c} = \ell/\lambda > \underline{c}$ and supplies ℓ units of labour.

When $\bar{t} = \tau\,\ell/\lambda$ there is just enough labour to farm all the land, and then any wage-price ratio w/p in the interval $[\underline{c}/\ell, 1/\lambda]$ is a potential equilibrium. By (2) it follows that r/p lies in the interval $[0, (\ell-\lambda)\,\underline{c}/\tau\,\ell]$. One has $\bar{c} = \ell/\lambda = \bar{t}/\tau$ and also $p\,\bar{c} = w\,\ell + r\,\bar{t} \geq w\,\ell \geq p\,\underline{c}$. Thus all markets clear and all agents survive, supplying ℓ units of labour. Of course, this case is non-generic and so we shall not have much to say about it in the rest of the paper.

2.3. Non-Survival

Another and for us a much more interesting possibility occurs when there is surplus labour and the per capita land endowment is therefore small, with $\bar{t} < \tau\,\ell/\lambda$. Then, if there are some farmers who own no land, no Walrasian equilibrium exists in the usual sense, with everybody surviving. This is because the only possible equilibrium wage is zero, which does not permit the landless to afford subsistence consumption.

To avoid this non-existence problem, we now modify our concept of equilibrium by allowing that some agents may not survive. To do so, let us add to the commodity space a fourth dimension with an indicator variable $i \in \{-1, 0\}$ representing either survival, if $i = 0$, or non-survival, in case $i = -1$. The number -1 can be thought of as signifying that the agent has to give up his life in order to achieve any feasible plan of net consumption within his budget constraint. Then we replace C in (1) above by the new four-dimensional *survival set*

$$S := \{\,(-t, -h, c, i) \mid t \geq 0,\ 0 \leq h \leq \ell,\ c \geq \underline{c},\ i = 0\,\}. \tag{3}$$

But we also append to the set S another *non-survival set* defined as

$$N := \{\,(-t, -h, c, i) \mid t \geq 0,\ h = 0,\ c \geq 0,\ i = -1\,\}. \tag{4}$$

Note that if $c \geq \underline{c}$, then the individual can choose either survival or non-survival. Of course, we expect survival to be chosen whenever it is feasible. The consumption set then becomes the union $C = S \cup N$ of the extended survival set S in (3) above with this additional non-survival set N. This is depicted in Figure 1, which shows the projection of the two sets S and N onto the three-dimensional set of points $(-h, c, i)$ for which land supply t is

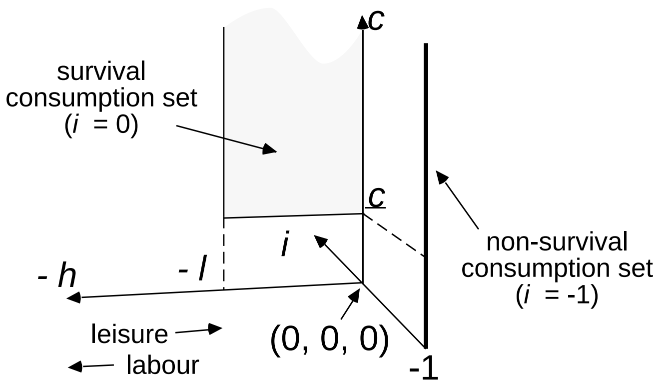

Figure 1

treated as an exogenous constant. Note that, apart from the indivisible good "survival," the consumption set is convex and allows free disposal.

Allocations in which the farmer supplies no labour ($h = 0$) but consumes less than the subsistence level ($0 \leq c < \underline{c}$) are now assumed to be feasible provided that the farmer does not survive. Note how it is assumed that a non-surviving farmer's labour dies with him, as he is too weak or malnourished to be able to add his labour to the pool of surplus labour. Thus his consumption is limited to the proceeds from selling any land endowment. Of course, if a farmer's land endowment is large enough relative to existing prices, he may attain subsistence consumption and sell his labour endowment at the going wage, thereby attaining survival in S.

Assume that survival of any sort is preferred to non-survival. Also, whereas Green (1976, pp. 32 and 36), for instance, postulates that there is no preference ordering over points at which the consumer does not survive, we assume that the preference ordering does extend to points of N. Indeed, we assume that whether the farmer survives or not, more food is always preferred to less food.

Consider now the difficult case where some farmers are landless and also $\bar{t} < \tau\,\ell/\lambda$, so that there would be surplus labour if all farmers were to survive. Then there is no equilibrium in which every farmer can attain the survival portion of the consumption set. Incorporate into the model the augmented consumption set $C = S \cup N$ defined above. Also, allow the possibility of compensated equilibrium (in the sense of Arrow and Hahn, 1971), in which farmers at a minimum expenditure point in the survival section S of the consumption set C may be arbitrarily restricted to an allocation in the non-survival portion N of the consumption set that costs the same. In compensated equilibrium utility is maximized by all agents except some of those at the minimum expenditure level needed to attain the survival portion of the consumption set. These agents on the margin of survival minimize expenditure subject to reaching a given indifference curve, but they may or may not maximize utility. Indeed, those whose utility is not maximized may not even survive, and moreover the survivors among those on the margin are arbitrarily chosen according to the requirements of market equilibrium. It should be noted that Dasgupta and Ray (1986–7) use a similar concept of equilibrium in their efficiency wage explanation of involuntary unemployment. Also, observe how admitting compensated equilibrium in this way resolves the existence problem by closing the graph of the aggregate excess demand correspondence. Then a continuum of agents will guarantee that mean excess demand is convex-valued (see Section 3).

An important feature of this kind of equilibrium is that agents who have some land but do not survive are still allowed to trade in order to maximize their consumption of food. Unlike labour, land does not perish with its owner. Rather, non-survivors sell all their land for its rental value r and use the proceeds to consume what they can afford before they succumb. We are not sure that this faithfully represents Koopmans' (1957, p. 62) "instant elimination" of non-survivors. But it seems more realistic than imposing zero consumption on all non-survivors. It is, moreover, a crucial feature permitting existence and efficiency of equilibrium. For if agents who did not survive were unable to consume, the set C^N defined in (4) would consist only of the half-line through the origin with $t \geq 0$. Then agents' preferences would be locally satiated, since all points of this half-line are equally abhorrent.

Given that there is now no equilibrium with all farmers surviving, we consider three cases.

Case A. $\ell\,\tau\,(1 - f_0) \leq \lambda\,\bar{t} < \ell\,\tau$ *and all those with any land survive.*

In this case there will turn out to be a compensated equilibrium at a subsistence real wage of $w/p = \underline{c}/\ell$, which is just enough to enable the survival of a landless person who works ℓ hours. The excess labour is removed by starvation. Each landless farmer is a marginal survivor in this compensated Walrasian equilibrium. By (2), the rental-price ratio is therefore given by $r/p = (\ell - \lambda \underline{c})/\ell \tau$. A proportion $f_n \leq f_0$ of the population will be unable to survive, however, because they remain involuntarily unemployed; for these $c = 0$ and $h = 0$. All who starve are landless. Yet among the landless farmers, a proportion $f_0 - f_n$ of the total population do manage to survive at the least cost point of the survival portion S of their consumption set; for these $c = \underline{c}$ and $h = \ell$. Of course, all farmers with positive land endowment survive; for these one has $h = \ell$ and $c = \underline{c} + (\ell - \lambda \underline{c})\, t/\tau \ell$. Note how clearing of all markets requires that

$$\bar{h} = (1 - f_n)\, \ell = \lambda \bar{t}/\tau \quad \text{and} \quad \bar{c} = (1 - f_n)\, [\underline{c} + (\ell - \lambda \underline{c})\, \bar{t}/\tau \ell] = \bar{t}/\tau. \tag{5}$$

Both these equations are satisfied when the proportion of survivors is $1 - f_n$, where $f_n := 1 - (\lambda \bar{t}/\ell \tau)$. So the allocation which we have described must be a compensated equilibrium. There is no uncompensated equilibrium.

Note finally that $0 < f_n \leq f_0$ precisely when the inequalities defining this case are both satisfied.

Case B. $\ell \tau [1 - F(\tau \underline{c})] < \lambda \bar{t} < \ell \tau (1 - f_0)$ *and not all farmers with land survive.*

Here there will be a compensated equilibrium with the minimum amount of land needed for survival being given by some number s in the range $0 < s < \tau \underline{c}$. Since $p \underline{c} = w \ell + r s$ and, in addition, (2) must still be true, the corresponding equilibrium wage-price and rental-price ratios must be

$$\frac{w}{p} = \frac{\tau \underline{c} - s}{\tau \ell - \lambda s} \quad \text{and} \quad \frac{r}{p} = \frac{\ell - \lambda \underline{c}}{\tau \ell - \lambda s}. \tag{6}$$

Farmers whose land holdings t exceed the critical value s survive with $h = \ell$ and $c = (w \ell + r t)/p > \underline{c}$. But those with $t < s$ could not afford to consume $\underline{c}$ even if they could somehow supply ℓ units of labour; they are therefore unable either to survive or to work, and so find themselves with $h = 0$ and $c = r t/p < r s/p < \underline{c}$.

Finally, according to the requirements of market clearing, some of the farmers with land endowment exactly equal to s survive, while others do not. Indeed, the proportion q of non-survivors in the total population is determined so that the labour market clearing condition $\lambda \bar{c} = \ell (1 - q)$ is satisfied. Actually, since the land market must clear as well, one must also have $\bar{t} = \tau \bar{c}$ and so $\lambda \bar{t}/\tau = \ell (1 - q)$. This determines a unique value of q which, in the case being considered, must lie in the interval $f_0 < q < F(\tau\, \underline{c})$. The corresponding value of s must then be given by

$$s = \inf\{\, t \mid 1 - F(t) \leq \lambda\, \bar{t}/\ell\, \tau \,\} = \sup\{\, t \mid 1 - F(t) > \lambda\, \bar{t}/\ell\, \tau \,\}. \tag{7}$$

This is the unique value of s which ensures that $F(t) > q$ whenever $t > s$ and that $F(t) < q$ whenever $t < s$. If there is a negligible set of farmers whose land endowment equals s exactly, then $t = s$ will solve the equation $F(t) = q$. In this latter case the compensated equilibrium found here becomes a Walrasian equilibrium, since only agents in a negligible set are failing to maximize their preferences. Generally, however, there is only a compensated equilibrium, in which some farmers whose landholding is s survive, while others do not.

Case C. $\lambda\, \bar{t} \leq \ell\, \tau\, [1 - F(\tau\, \underline{c})]$ *and only the self-sufficient survive.*

In this final case there is so much labour and so little land that the only possible Walrasian equilibrium prices satisfy $w/p = 0$ and $r/p = 1/\tau$. At these prices all agents with $t < \tau\, \underline{c}$ do not survive, since they can only afford to have $h = 0$ and to sell all their land in order to consume at $c = t/\tau < \underline{c}$. But every farmer who has $t \geq \tau\, \underline{c}$ and so has enough land to produce subsistence consumption using his own labour does survive by choosing $h = \ell$ and $c = t/\tau \geq \underline{c}$. Even though the wage is zero, the threat of starvation motivates the self-sufficient farmers to work. Mean consumption per head then satisfies $\bar{c} = \bar{t}/\tau$, while mean labour supply per head is given by $\ell\, [1 - F(\tau\, \underline{c})] \geq \lambda\, \bar{c}$. So markets clear with labour as a free good, and there is a Walrasian equilibrium at these prices.

2.4. Remedial Policy

Despite the potential for non-survival when $\bar{t} < \tau\, \ell/\lambda$, note that a program of land redistribution can always be used to ensure that everybody survives, because of the assumption that $\bar{t} > \tau\, \underline{c}$ and $\ell > \lambda\, \underline{c}$. For if land is redistributed so that all farmers have access to the same amount $\bar{t}$, then each farmer is obviously able to attain survival through production using only his own land and labour.

Moreover, a balanced-budget tax-transfer system can achieve the same effect. To see this, suppose that all rental income from land is taxed at the flat rate θ, with the resulting tax revenues being redistributed equally to all agents by means of a poll subsidy or uniform lump-sum transfer m. Real transfers of $m/p = \underline{c}$ and a tax rate of $\theta = \tau\, \underline{c}/\bar{t} < 1$ will then give rise to an equilibrium with an excess supply of labour (since $\ell > \lambda\, \bar{t}/\tau$) and so with prices given by $w/p = 0$ and $r/p = 1/\tau$. Each farmer's budget constraint takes the form $c \leq \underline{c} + [(1/\tau) - (\underline{c}/\bar{t})]\, t$. Because of our assumption that $\bar{t} > \tau\, \underline{c}$, the right hand side of this constraint exceeds $\underline{c}$ and so every farmer can certainly afford to survive even though there is no wage income. The government's budget balances with $m = \theta\, r\, \bar{t}$.

Such a tax-transfer scheme is also equivalent to a system under which the government uses the proceeds from a rental income tax at rate $\theta = \tau\, \underline{c}/\bar{t}$ in order to purchase $\underline{c}$ units of food per head on the open market and then distribute this amount equally to all farmers. This system is not a food subsidy, of course, but distribution in kind. Notice that wage income is always zero in each of these equilibria, so there is no scope for redistribution financed by a tax on wage income.

In contrast, any system of food subsidies, financed by a tax on landlords or on labour income or on any combination of the two, completely fails to reduce the extent of starvation or even to alter the allocation in any way. For let (p^e, w^e, r^e) denote the (compensated) equilibrium prices in the absence of any taxes or subsidies. Let γ be the *ad valorem* rate of subsidy on food purchases and let ω, ρ denote the *ad valorem* rates of tax on wage income and on land rents respectively. The budget constraint for a farmer with landholding t then becomes

$$p\,(1-\gamma)\, c \leq w\,(1-\omega)\, \ell + r\,(1-\rho)\, t. \tag{8}$$

We will show that when such taxes and subsidies are introduced and the government balances its budget, there is a new equilibrium in which producer prices (p, w, r) adjust so that prices to consumers are still given by

$$p^e = p\,(1-\gamma);\;\; w^e = w\,(1-\omega);\;\; r^e = r\,(1-\rho) \tag{9}$$

exactly as before. Then, since consumer prices are entirely unchanged, so is the demand side of the economy and the entire equilibrium allocation of consumption, work, land, and survival opportunities to all consumers.

Indeed, to show that there is a new equilibrium as described, it suffices to check that, when faced with the new producer prices (p, w, r), producers are still maximizing profits at the same zero level when they choose the same output/land and labour/land ratios $\bar{c}/\bar{t}$

and $\bar{h}/\bar{t}$ as in the original equilibrium. Since both consumers and the government are all balancing their budgets in any new equilibrium, one must have $p^e\,\bar{c} = w^e\,\bar{h} + r^e\,\bar{t}$ or

$$p\,(1-\gamma)\,\bar{c} = w\,(1-\omega)\,\bar{h} + r\,(1-\rho)\,\bar{t} \tag{10}$$

and also

$$p\,\gamma\,\bar{c} = w\,\omega\,\bar{h} + r\,\rho\,\bar{t}. \tag{11}$$

But then adding (10) and (11) gives

$$p\,\bar{c} = w\,\bar{h} + r\,\bar{t} = w\,\lambda\,\bar{c} + r\,\tau\,\bar{c} = (w\,\lambda + r\,\tau)\,\bar{c} \tag{12}$$

where the second equality holds because clearing of the labour and land markets implies that $\bar{h} = \lambda\,\bar{c}$ and $\bar{t} = \tau\,\bar{c}$. Since $\bar{c}$ must be positive in the original equilibrium, the no pure profit condition $p = w\,\lambda + r\,\tau$ is a direct implication of (12). (The fact that the no pure profit condition is satisfied by both the new producer prices and consumer prices is reminiscent of the result presented by Diamond and Mirrlees, 1976). So is the optimality of the output/land and labour/land ratios $\bar{c}/\bar{t}$ and $\bar{\ell}/\bar{t}$.

This means that there is indeed an equilibrium with taxes in which neither consumer prices nor quantities change. In particular, the food subsidy does nothing at all to lower the price of food faced by consumers or to help the starving. Instead the subsidy is entirely passed on to the producers and then all taxed away in order to finance the food subsidies. Of course, this result depends crucially on the fixed coefficients production technology. Otherwise taxes and land and labour would affect the equilibrium labour/land ratio.

3. General Equilibrium Analysis

3.1. Agents' Feasible Sets

Sen (1977, 1981a, b) chose to use an "exchange entitlements" approach to analyse the question of whether individuals could afford to survive. This certainly has a powerful intuitive appeal. Yet, as Srinivasan (1983) has observed, it is not strictly necessary, and essentially the same idea can also be captured, at least for the results to presented below, by the usual kind of budget set within a finite dimensional commodity space $\Re^G$. The issue of whether an individual survives depends on whether this budget set intersects the set of net trade vectors which that individual needs in order to survive. If it does, then the individual will be able to survive by a judicious choice of consumption and production plans. But if the intersection is empty then the economic system condemns the agent to starve.

So we consider an economy in which all agents are consumer/workers who may or may not own land and other primary resources. The typical agent has a survival consumption set C^S, together with a non-survival consumption set C^N. Each vector $c \in C^S \cup C^N$ is a net consumption vector which may have negative components corresponding to the kinds of labour which the agent supplies. The two sets C^S and C^N are both assumed to be closed convex subsets of $\Re^G$ that allow free disposal. The consumption set then becomes the union $C^S \cup C^N$ of two convex sets.

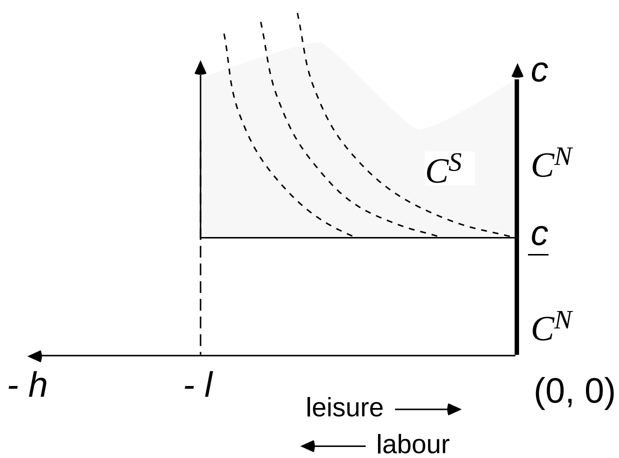

Figure 2

Figure 2 shows the consumption set of Figure 1 after it has been projected onto the space $\Re^2$ of consumption/leisure pairs. As Figure 2 indicates, the two sets C^S and C^N may intersect. Then, if $c \in C^S \cap C^N$, the agent has the choice between surviving and not. Of course, it is presumed that the agent always chooses survival in this case. It is only when $c \in C^N \setminus C^S$ that survival becomes impossible and the agent starves. It is also presumed that $0 \in C^S \cup C^N$, so that no trade is always feasible at least for an agent who does not survive. Usually, of course, $0 \in C^N \setminus C^S$, because few agents are entirely self-sufficient.

It now seems natural to have preferences for the consumer defined over the set $C^S \cup C^N$, and to assume that any point in C^S is preferred to any point in C^N, though within C^N more food is always preferred to less. This approach, however, leads to certain difficulties which are illustrated in Figure 2 above, where possible indifference curves are displayed. Notice that $(-t, 0, \underline{c})$ is strictly preferred to $(-t, -\ell, 0)$. Yet the lower contour set of points which are weakly worse than $(-t, -\ell, 0)$ is not closed, since it includes points of the form $(-t, 0, \underline{c} - \epsilon)$ for all small positive ϵ, but not the limit point $(-t, 0, \underline{c})$.[1] Thus, not only is the consumption set non-convex, but also preferences are typically discontinuous at $(-t, 0, \underline{c})$.

To avoid such discontinuities we shall make use of the simple trick set out in Section 2.3. This involves treating survival as an extra good in its own right, labelled as good 0. So we add to the commodity space $\Re^G$ an extra dimension with an indicator variable $i \in \{-1, 0\}$ representing either survival, if $i = 0$, or non-survival, in case $i = -1$. The extra good can be thought of as life itself, which has to be given up in case the individual does not consume enough to survive. The price of this extra good will always be zero. The two values -1 and 0 have been chosen so that: (i) preferences can still be monotone in the extended space $\Re^{G+1}$, with life preferred to death; (ii) the aggregate excess demand for good 0 could never be positive, thus allowing equilibrium to occur with a non-negative demand even for good 0, whose price is always zero anyway.

[1] In fact earlier versions of this paper used precisely this formulation, and used particular *ad hoc* methods in order to overcome the difficulties created by the resulting discontinuities. In this context, note that Zaman (1986) considers a rather different kind of discontinuity in preferences.

This leads us to define the *consumption set* as the union

$$C := (\{0\} \times C^S) \cup (\{-1\} \times C^N) \subset \{\,-1, 0\,\} \times \Re^G \tag{13}$$

of the two disjoint extended convex sets $\tilde{C}^S := \{0\} \times C^S$ and $\tilde{C}^N := \{-1\} \times C^N$. Note that C is not convex, and even incorporates survival as an indivisible good.

As in the example of Section 2, assume that production is undertaken by many small and individually owned production units. Following Rader (1964), assume that each agent has access to a convex production possibility set $Y \subset \Re^{G+1}$. It is assumed here that Y includes 0, allows free disposal of all goods except 0, and that $y \in Y$ implies $y_0 = 0$ because only labour and not life itself can be used as an input to the production process. Finally, Y is assumed to be bounded above because any individual agent can only control bounded quantities of inputs and these produce bounded outputs.

To survive, the agent requires a net trade vector in the *survival set*

$$X^S := \tilde{C}^S - Y = \{\, x \in \Re^{G+1} \mid \exists c \in \tilde{C}^S; \exists y \in Y : x = c - y \,\}. \tag{14}$$

The corresponding *non-survival set* $X^N := \tilde{C}^N - Y$ is also feasible for the agent. Notice that both X^S and X^N are convex because $\tilde{C}^S$, $\tilde{C}^N$ and Y are all convex. Also, because of (13), the agent's set of feasible net trades is

$$\begin{aligned} X = X^S \cup X^N &= (\tilde{C}^S - Y) \cup (\tilde{C}^N - Y) = (\tilde{C}^S \cup \tilde{C}^N) - Y \\ &= (\{0\} \times C^S) \cup (\{-1\} \times C^N) - Y \subset \{\,-1, 0\,\} \times \Re^G. \end{aligned} \tag{15}$$

To summarize, we have so far assumed that:

(A.1). *Each agent has a feasible set of net trades taking the form $X = C - Y$, where the consumption set $C \subset \{\,-1, 0\,\} \times \Re^G$ satisfies (13), with $C^S, C^N \subset \Re^G$ and the production set $Y \subset \{0\} \times \Re^G$ satisfying the conditions that: (i) both sets C^S and C^N are closed, convex, bounded below, and allow free disposal of all physical commodities $g \in G$; (ii) $0 \in C^S \cup C^N$; (iii) Y is convex, bounded above, and allows free disposal of all goods except 0; (iv) $0 \in Y$.*

These assumptions on the two sets C and Y have implications for the set X which are summarized in the following:

Lemma. *X is a closed subset of the commodity space $\{-1, 0\} \times \Re^G$ such that: (i) $0 \in X$; (ii) X allows free disposal of all goods except 0; (iii) X is bounded below by a vector $\underline{x}$; (iv) X is the union of two disjoint convex sets $X^S \subset \{0\} \times \Re^G$ and $X^N \subset \{-1\} \times \Re^G$, as specified in (15) above.*

Proof: Since all the other properties claimed in the Lemma are obvious, we prove only that X is closed.

Indeed, suppose that $x^\nu \in X$ where $x^\nu = c^\nu - y^\nu$ with $(c^\nu, y^\nu) \in C \times Y$ for $\nu = 1, 2, \ldots$, and suppose that $x^\nu \to x^*$ as $\nu \to \infty$. Then, since C has a lower bound $\underline{c}$ and Y has an upper bound $\bar{y}$, it follows that

$$\underline{c} \leqq c^\nu = x^\nu + y^\nu \leqq x^\nu + \bar{y} \to x^* + \bar{y}$$

$$\text{and} \quad \bar{y} \geqq y^\nu = c^\nu - x^\nu \geqq \underline{c} - x^\nu \to \underline{c} - x^*$$

as $\nu \to \infty$. Hence the sequence of pairs $(c^\nu, y^\nu) \in C \times Y$ must be bounded, and so has a convergent subsequence with a limit (c^*, y^*) which, because the two sets C and Y are both closed, must be a member of $C \times Y$. But now $c^\nu - y^\nu = x^\nu \to x^*$ as $\nu \to \infty$, which is only possible if $x^* = c^* - y^* \in C - Y = X$. ∎

In future a typical vector $z \in \{-1, 0\} \times \Re^G$ will be written in the partitioned form $z = (z_0, z^G)$, where $z_0 \in \{-1, 0\}$ and $z^G \in \Re^G$.

3.2. Agents' Preferences

Each agent is also assumed to have a (complete and transitive) preference ordering R on the consumption set C, and to be unconcerned about production except insofar as it affects consumption and labour supply, etc. It is assumed that these preferences are:

(i) *monotone* in the sense that $c' \geqq c$ implies $c' \; R \; c$ and also that $b \gg b'$ implies $(i, b) \; P \; (i, b')$ whenever $b, b' \in \Re^G$ and $i \in \{-1, 0\}$ (where P denotes the strict preference relation corresponding to R);

(ii) *continuous* in the sense that the upper and lower contour sets $\{c \in C \mid c \; R \; c'\}$ and $\{c \in C \mid c' \; R \; c\}$ are both closed for every $c' \in C$.

Of course, it is assumed that consumers prefer survival, so that $(i, b) \; P \; (i', b')$ whenever $b, b' \in \Re^G$, $i = 0$ and $i' = -1$. In this sense, life is lexicographically prior to physical commodities in each consumer's preference ordering. To summarize:

(A.2). *There is a continuous and monotone preference ordering R defined on the consumption set C with the property that $(0, b)\ P\ (-1, b')$ whenever $b \in C^S$ and $b' \in C^N$.*

The agent's preferences R for consumption can be converted into preferences for net trades. For given any fixed net trade vector $x \in X$, define

$$\gamma(x) := \{\, c \in \tilde{C} \mid \exists \hat{y} \in Y : c = x + \hat{y} \quad \text{and} \quad \forall y \in Y : c\ R\ (x + y) \,\} \tag{16}$$

as the agent's set of optimal consumption vectors. Because Y is compact and the upper contour sets of the preference ordering R are all closed, the set $\gamma(x)$ is indeed non-empty for every $x \in X$. Moreover, if c and c' both belong to $\gamma(x)$ for any $x \in X$, then c and c' must be indifferent. So there is a well defined preference ordering $\succsim$ on X such that

$$x \succsim x' \iff [\forall c \in \gamma(x); \forall c' \in \gamma(x') : c\ R\ c']. \tag{17}$$

Of course, to be an ordering, $\succsim$ must not only be complete — as it obviously is, because R is complete — but it must also be transitive. However, transitivity of $\succsim$ follows readily from transitivity of R.

Notice that when the agent chooses $\hat{x}$ to maximize the preference ordering $\succsim$ over X subject to a budget constraint of the form $p\,x \leq m$, this implies that $c = \gamma(\hat{x})$ maximizes R over $\tilde{C}$ subject to the constraint $p\,c \leq m + p\,\hat{y}$, where $\hat{y}$ is any net output vector which maximizes (net) profits $p\,y$ subject to $y \in Y$. So preference maximization by the agent implies profit maximization.

An immediate implication of our assumptions regarding R is that the preference ordering $\succsim$ constructed above and the corresponding strict preference relation $\succ$ must satisfy the following:

Lemma. *There is a (complete and transitive) preference ordering $\succsim$ on the feasible set X which is monotone and satisfies:*

(i) for every $x' \in X$, the upper contour set $\{\, x \in X \mid x\ R\ x' \,\}$ is closed;

(ii) $x \succ x'$ whenever $x \in X^S$ and $x' \in X^N$.

Note how it is *not* claimed that the lower contour set $\{\, x \in X \mid x'\ R\ x \,\}$ is closed for all $x' \in X$. Indeed, this is generally not true.

The typical agent is therefore characterized by the non-convex but closed consumption set C, the closed production set Y, and the continuous preference ordering R on C. Thus, the space Θ of agents' characteristics will consist of triples (C, Y, R) satisfying (A.1) and (A.2), and this will be given the closed convergence topology for continuous preferences that is described in Hildenbrand (1974, p. 96). Note that, because the corresponding preference ordering $\precsim$ for net trade vectors is not continuous in general, the closed convergence topology cannot be applied to the space consisting only of pairs $(X, \precsim)$. Write $\mathcal{B}(\Theta)$ for the family of Borel measurable sets in Θ with its topology of closed convergence.

3.3. A Continuum Economy

Following Aumann (1964, 1966) and Hildenbrand (1974), assume that there is a non-atomic measure space of agents $(A, \mathcal{A}, \alpha)$. Then a *continuum economy* is a mapping $E : A \to \Theta$ which is measurable with respect to the two σ-algebras $\mathcal{A}$ and $\mathcal{B}(\Theta)$ — i.e., $E^{-1}(H) \in \mathcal{A}$ for every measurable set $H \in \mathcal{B}(\Theta)$. For every $a \in A$, write a's characteristic $E(a)$ as $(X_a, \precsim_a)$. Because each such characteristic must satisfy (A.1), in particular there exists a lower bound $\underline{x}_a$ to X_a. Assume:

(A.3). *The vector function $\underline{x} : A \to \Re^{G+1}$ is measurable and the integral $\int \underline{x}$ over A is finite.*

An *allocation* of net trade vectors in the economy E is a measurable mapping $f : A \to \Re^{G+1}$ such that $f(a) \in X_a$ a.e. and $\int f \leqq 0$.

Define the modified price simplex

$$\Delta := \{ p \in \Re^{G+1} \mid p_0 = 0; \quad p_g \geq 0 \ (g = 1, 2, \ldots, G) \quad \text{and} \quad \sum\nolimits_{g=1}^{G} p_g = 1 \} \tag{18}$$

with relative interior

$$\Delta^0 := \{ p \in \Re^{G+1} \mid p_0 = 0; \quad p_g > 0 \ (g = 1, 2, \ldots, G) \quad \text{and} \quad \sum\nolimits_{g=1}^{G} p_g = 1 \}. \tag{19}$$

As explained above, the price of life is taken to be zero — the right to live cannot be bought or sold, even though some agents may not be able to afford the commodities they need in order to ensure their own survival.

For each agent $a \in A$ and price vector $p \in \Delta$, define:

(i) the *budget set* $B_a(p) := \{ x \in X_a \mid p\,x \leq 0 \}$;

(ii) the *demand set* $\xi_a(p) := \{ x \in B_a(p) \mid x' \succ_a x \Longrightarrow p\,x' > 0 \}$;

(iii) the *compensated demand set* (Arrow and Hahn, 1971)

$$\xi_a^C(p) := \{ x \in B_a(p) \mid x' \succsim_a x \Longrightarrow p\,x' \geq 0 \}$$

which Hildenbrand (1968) had earlier called the "expenditure minimizing" set;

(iv) the *weak demand set* (Khan and Yamazaki, 1981)

$$\xi_a^W(p) := \{ x \in B_a(p) \mid x' \succ_a x \Longrightarrow p\,x' \geq 0 \}.$$

Note that $\xi_a(p) \cup \xi_a^C(p) \subset \xi_a^W(p)$ always, trivially. Because of locally non-satiated preferences, it is easy to see that $\xi_a(p) \subset \xi_a^C(p) = \xi_a^W(p)$ for all $p > 0$. In addition, because $0 \in X_a$, so $B_a(p) \neq \emptyset$ always. Because X_a is bounded below, $B_a(p)$ is compact whenever $p \in \Delta^0$. Because the upper contour sets of $\succsim$ are closed, it follows that $\emptyset \neq \xi_a(p)$ whenever $p \in \Delta^0$.

A *Walrasian equilibrium* (f, p) is an allocation f of net trade vectors, together with a price vector $p \in \Delta$ satisfying $f(a) \in \xi_a(p)$ a.e. in A and $\int p\,f = 0$. Because $p \geq 0$ and $\int f \leqq 0$ for an allocation, it follows from this definition that, for all $g \in \{0\} \cup G$, one has both $\int f_g \leq 0$ and also $p_g = 0$ whenever $\int f_g < 0$ — the usual "rule of free goods."

To show that a Walrasian equilibrium exists, it will be convenient to prove first the existence of a compensated equilibrium (f, p) consisting of an allocation f and a price vector $p \in \Delta$ such that $f(a) \in \xi_a^C(p)$ a.e. in A and $\int p\,f = 0$ (Arrow and Hahn, 1971). As with Walrasian equilibrium, the rule of free goods must be satisfied.

An allocation f is *Pareto efficient* if there is no other (feasible) allocation f' such that $f'(a) \succ_a f(a)$ a.e. in A. An allocation f is in the *core* if there is no *blocking coalition* K with an alternative allocation $f' : K \to \Re^{G+1}$ such that

$$\text{(i) } f'(a) \succ_a f(a) \text{ a.e. in } K;\ \text{(ii) } \int_K f' \leqq 0;\ \text{(iii) } \alpha(K) > 0. \tag{20}$$

The following two results are standard because (A.1) and (A.2) together guarantee that consumers have locally non-satiated preferences:

(1) First Efficiency Theorem. *If (f,p) is a Walrasian equilibrium, then the allocation f is Pareto efficient.*

(2) First Core Theorem. *If (f,p) is a Walrasian equilibrium, then f is in the core.*

Because of the Walrasian equilibria without survival which were exhibited in Sections 2 and 3, these trivial results already establish that Pareto efficiency by no means guarantees survival of all agents. Nor can non-survivors necessarily block an allocation in order to bring about their own survival.

It is much less trivial to prove the other three promised results, namely:

(3) existence of a Walrasian equilibrium;

(4) the second efficiency theorem — i.e., any Pareto efficient allocation is Walrasian after suitable lump-sum taxes and transfers have been made;

(5) core equivalence — i.e., not only is every Walrasian allocation in the core, as in (2), but also every core allocation is Walrasian for some price vector.

Indeed Section 5.7 below introduces extra assumptions in order to ensure that a compensated equilibrium is actually a Walrasian equilibrium. Moreover, because of our insistence that $p_0 = 0$ — that life should be a free good — not even standard proofs for economies with indivisible goods can be applied without a few minor changes.

3.4. Continuity of Compensated Demand

A proof of existence of compensated equilibrium in somewhat different economies with non-convex consumption sets has been provided by Khan and Yamazaki (1981, Proof of Proposition 2, pp. 223–4). Note, however, how they actually prove existence of "weakly competitive allocations," which are not the same because they do not assume local non-satiation of preferences. Also, they do not normalize agents' endowments to zero as we would do in a pure exchange economy. Their proof in turn relies heavily on the work of Hildenbrand (1974) as amplified by Debreu (1982).

A key part of the existence proof relies on the continuity result below, which is of some general interest that goes beyond the specific model being considered here. The Lemma concerns the continuity properties of the typical agent's profit function $\pi(Y,p)$, and of the net output, compensated consumption demand, and compensated net trade demand correspondences $\eta(Y,p)$, $\zeta(C,R,p,m)$, and $\xi(C,Y,R,p)$ respectively. These are defined for every consumption set C, production set Y, and preference ordering R which satisfy (A.1) and (A.2) of Sections 3.1 and 3.2, as well as for every price vector $p \in \Delta^0$ and income level m, as follows:

$$\left.\begin{array}{r}\pi(Y,p) := \arg\max \\ \eta(Y,p) := \max\end{array}\right\} \{\, p\,y \mid y \in Y\,\}; \tag{21}$$

$$\begin{aligned}\zeta(C,R,p,m) &:= \{\, c \in C \mid p\,c \leq m \quad \text{and} \quad [c' \in C,\ c'\ R\ c \Longrightarrow p\,c' \geq m]\,\}; \\ \xi(C,Y,R,p) &:= \zeta(C,R,p,\pi(Y,p)) - \eta(Y,p).\end{aligned} \tag{22}$$

Note how each agent does choose $y \in \eta(Y,p)$, and so faces the budget constraint $p\,c \leq \pi(Y,p)$, because there are no direct preferences over production.

LEMMA. *Under assumptions (A.1) and (A.2) as stated in Sections 3.1 and 3.2 above, the profit function $\pi(Y,p)$ is continuous when the space $\Theta \times \Delta$ is given its product topology, and the three correspondences $\eta(Y,p)$, $\zeta(C,R,p,m)$ and $\xi(C,Y,R,p)$ all have closed graphs relative to the appropriate spaces $\Theta \times \Delta \times \Re^{G+1}$ or $\Theta \times \Delta \times \Re \times \Re^{G+1}$ when each is given its appropriate product topology, and when Θ is given its topology of closed convergence.*

PROOF:

A: *Continuity of profit and of supply.*

Let (Y^ν, p^ν, y^ν) $(\nu = 1,2,\ldots)$ be any infinite sequence of points in the graph of $\eta(Y,p)$ which converges to $(\bar{Y}, \bar{p}, \bar{y})$ as $\nu \to \infty$. Then $\pi(Y^\nu, p^\nu) = p^\nu\, y^\nu \to \bar{p}\,\bar{y}$. Also, because $y^\nu \in Y^\nu \to \bar{Y}$ in the closed convergence topology, so $\bar{y} \in \bar{Y}$.

Now, for any $y = (y_0, y^G) \in \bar{Y}$, consider any other $\underline{y} = (\underline{y}_0, \underline{y}^G) \in \Re^{G+1}$ such that $\underline{y}_0 = y_0 = 0$ and $\underline{y}^G \ll y^G$. Because $Y^\nu \to \bar{Y}$ as $\nu \to \infty$, for each large enough ν there must exist $\tilde{y}^\nu = (\tilde{y}^\nu_0, \tilde{y}^{\nu G}) \in Y^\nu$ such that $\tilde{y}^\nu_0 = y_0 = 0$ and $\tilde{y}^{\nu G} \gg \underline{y}^G$. Since each $p^\nu > 0$, it follows from profit maximization that

$$p^\nu\, y^\nu = \pi(Y^\nu, p^\nu) \geq p^\nu\, \tilde{y}^\nu \geq p^\nu\, \underline{y} \to \bar{p}\, \underline{y}$$

as $\nu \to \infty$. Since $p^\nu\, y^\nu \to \bar{p}\,\bar{y}$ as $\nu \to \infty$, it follows that $\bar{p}\,\bar{y} \geq \bar{p}\,\underline{y}$ whenever $\underline{y}_0 = y_0 = 0$ and $\underline{y}^G \ll y^G$. Therefore $\bar{p}\,\bar{y} \geq \bar{p}\,y$. Since this is true for all $y \in \bar{Y}$, and since $\bar{y} \in \bar{Y}$, it must be true that: (i) $\bar{y} \in \eta(\bar{Y}, \bar{p})$; (ii) $\pi(\bar{Y}, \bar{p}) = \bar{p}\,\bar{y}$. From (i) it follows directly that $\eta(Y,p)$ has a closed graph. From (ii) on the other hand, since $p^\nu\, y^\nu \to \bar{p}\,\bar{y}$ as $\nu \to \infty$, it follows that $\pi(Y^\nu, p^\nu) \to \pi(\bar{Y}, \bar{p})$. So $\pi(Y,p)$ is a continuous function.

B: *Continuity of compensated consumption demand.*

Let $(C^\nu, R^\nu, p^\nu, m^\nu, c^\nu)$ $(\nu = 1, 2, \ldots)$ be any infinite sequence of points in the graph of $\zeta(C, R, p, m)$ which converges to $(\bar{C}, \bar{R}, \bar{p}, \bar{m}, \bar{c})$ as $\nu \to \infty$. Then $p^\nu c^\nu \leq m^\nu$ and $c^\nu \in C^\nu$ for $\nu = 1, 2, \ldots$, so that in the limit as $\nu \to \infty$ one has $\bar{p}\,\bar{c} \leq \bar{m}$ and also $\bar{c} \in \bar{C}$.

Suppose that $c \in \bar{C}$ with $c\ P\ \bar{c}$. Because preferences are continuous and monotone in all goods except 0, and because (A.1) is satisfied, there exists $z = (z_0, z^G) \in \Re^{G+1}$ with $z_0 = \bar{c}_0$ and $z^G \gg \bar{c}^G$ such that $c\ P\ z$. Also, for all large enough ν one must have $z_0 = c_0^\nu$ and $z^G \gg c^{\nu G}$, implying that $z \in C^\nu$ with $z\ P^\nu\ c^\nu$.

Now let $w = (w_0, w^G) \in \Re^{G+1}$ satisfy $w_0 = c_0$ and $w^G \gg c^G$. Then, since $c \in \bar{C}$ while $C^\nu \to \bar{C}$ as $\nu \to \infty$, free disposal implies that $w \in C^\nu$ for all large enough ν. Also monotone preferences imply that $w\ P\ c$. Because we have already shown that $c\ P\ z$, transitive preferences imply that $w\ P\ z$. Thus (z, w) does not lie in the graph of the relation R, and so there must be an infinite sequence of values of ν for which (z, w) does not lie in the graph of R^ν either. But then, since $w, z \in C^\nu$ for all large enough ν, there must be infinitely many values of ν for which $w\ P^\nu\ z$ and so $w\ P^\nu\ z\ P^\nu\ c^\nu$. By transitivity of P^ν, it follows that $w\ P^\nu\ c^\nu$ for all these values of ν. Since $c^\nu \in \zeta(C^\nu, R^\nu, p^\nu, m^\nu)$ for $\nu = 1, 2, \ldots$, this implies that $p^\nu w \geq p^\nu c^\nu = m^\nu$ for infinitely many ν. But $p^\nu \to \bar{p}$ and $m^\nu \to \bar{m}$ as $\nu \to \infty$, so $\bar{p}\, w \geq \bar{m}$. Since this is true whenever $w_0 = c_0$ and $w^G \gg c^G$, it follows that $\bar{p}\, c \geq \bar{m}$.

So we have proved that $c\ P\ \bar{c}$ implies $\bar{p}\, c \geq \bar{m}$. Because preferences are monotone and so locally non-satiated, it follows that $c\ R\ \bar{c}$ implies $\bar{p}\, c \geq \bar{m}$. This confirms that $\bar{c} \in \zeta(\bar{C}, \bar{R}, \bar{p}, \bar{m})$, as required for the compensated demand correspondence $\zeta(C, R, p, m)$ to have a closed graph.

C: *Continuity of compensated net trade demand.*

Finally, let $(C^\nu, Y^\nu, R^\nu, p^\nu, x^\nu)$ $(\nu = 1, 2, \ldots)$ be any infinite sequence of points in the graph of $\xi(C, Y, R, p)$ which converges to $(\bar{C}, \bar{Y}, \bar{R}, \bar{p}, \bar{x})$ as $\nu \to \infty$. By definition of ξ, there exist sequences $c^\nu \in \zeta(C^\nu, R^\nu, p^\nu, \pi(Y^\nu, p^\nu))$ and $y^\nu \in \eta(Y^\nu, p^\nu) \subset Y^\nu$ such that $x^\nu = c^\nu - y^\nu$ $(\nu = 1, 2, \ldots)$.

Now, as $\nu \to \infty$, so $Y^\nu \to \bar{Y}$ and $C^\nu \to \bar{C}$ in the topology of closed convergence. But $\bar{Y}$ is bounded above, while $\bar{C}$ is bounded below, so the two sequences $y^\nu \in Y^\nu$ and $c^\nu \in C^\nu$ must be bounded above and below, respectively. Yet then y^ν is also bounded below, because $y^\nu = c^\nu - x^\nu$ and $x^\nu \to \bar{x}$ as $\nu \to \infty$. Therefore y^ν is actually bounded both above and below, and so must have a convergent subsequence with a limit point $\bar{y} \in \bar{Y}$. But then, since $x^\nu \to \bar{x}$ as $\nu \to \infty$, the corresponding subsequence of c^ν must also have a limit point $\bar{c} \in \bar{C}$ given by $\bar{c} := \bar{x} + \bar{y}$.

Because of the continuity properties of $\pi(Y,p)$, $\eta(Y,p)$, and $\zeta(C,R,p,m)$ which have just been proved in parts A and B above, it now follows that $\bar{y} \in \eta(\bar{Y},\bar{p})$, that $\pi(\bar{Y},\bar{p}) = \bar{p}\,\bar{y}$, and also that $\bar{c} \in \zeta(\bar{C},\bar{R},\bar{p},\pi(\bar{Y},\bar{p}))$. Therefore

$$\bar{x} = \bar{c} - \bar{y} \in \zeta(\bar{C},\bar{R},\bar{p},\pi(\bar{Y},\bar{p})) - \eta(\bar{Y},\bar{p}) = \xi(\bar{C},\bar{Y},\bar{R},\bar{p}),$$

thus confirming that $\xi(C,Y,R,p)$ does have a closed graph. ∎

3.5. Existence of Compensated Equilibrium

Theorem. *Under Assumptions (A.1–A.3) as stated in Sections 3.1–3.3 above, there exists a compensated equilibrium.*

Proof: From the continuity result of Section 3.4, it follows that the correspondence $\hat{\xi} : A \twoheadrightarrow \Re^G$ defined by $\hat{\xi}(a) := \xi(E(a),p)$ has a measurable graph because, by hypothesis, E is a measurable function, and because ξ has a closed and so a measurable graph (Hildenbrand, 1974, p. 59, Prop. 1(b)). Moreover, for each positive integer $k \geq G$ the budget correspondence $B_a(p)$ is integrably bounded on the restricted domain $A \times \Delta_k$, where

$$\Delta_k := \{\, p \in \Delta \mid \forall g \in G : p_g \geq 1/k \,\}. \tag{23}$$

So the mean compensated demand correspondence $\beta_k : \Delta_k \twoheadrightarrow \Re^{G+1}$ defined by $\beta_k(p) := \int_A \xi_a^C(p)\, d\alpha$ has all the relevant properties of Khan and Yamazaki's (1981, pp. 223) mapping $F_k(p)$ — in particular, it has non-empty, compact and convex values, a closed graph, and the range $\beta_k(\Delta_k)$ is also compact. As a result, for each $k = G, G+1, G+2, \ldots$ the correspondence $\psi_k : \Delta_k \times \beta_k(\Delta_k) \twoheadrightarrow \Delta_k \times \beta_k(\Delta_k)$ which is defined throughout its domain by

$$\psi_k(p,z) := \arg\max_{\tilde{p}} \{\, \tilde{p}\, z \mid \tilde{p} \in \Delta_k \,\} \times \beta_k(p)$$

has a fixed point $(p_k, z_k) \in \psi_k(p_k, z_k)$. So there exist infinite sequences of price vectors $p_k \in \Delta_k$, quantity vectors $z_k \in \Re^{G+1}$ and integrably bounded measurable mappings $f_k : A \to \Re^{G+1}$ $(k = G, G+1, G+2, \ldots)$ such that: (i) $f_k(a) \in \xi_a^C(p_k)$ a.e. in A; (ii) $z_k = \int f_k$; (iii) $p\, z_k \leq 0$ for all $p \in \Delta_k$. But then $z_k \geqq \int \underline{x}$. Also, because $(1/G)\,(1,1,\ldots,1) \in \Delta_k$, (iii) above implies that $\sum_{g \in G} z_{kg} \leq 0$ for all $k \geq G$. So the sequence of fixed points (p_k, z_k) always lies in the compact set $\Delta \times Z$, where

$$Z := \{\, z \in \Re^{G+1} \mid z \geqq \int \underline{x} \quad \text{and} \quad \sum\nolimits_{g \in G} z_g \leq 0 \,\}.$$

Hence there must exist some subsequence of (p_k, z_k) $(k = G, G+1, G+2, \ldots)$ which converges to a limit point $(p^*, z^*) \in \Delta \times Z$. Moreover, Fatou's Lemma in many dimensions (see, for instance, Hildenbrand, 1974, p. 69) can now be applied to show that there exists a subsequence $k(m)$ ($m = 1, 2, \ldots$) of $k = G, G+1, G+2, \ldots$, together with some $p \in \Delta$ and some measurable function $f : A \to \Re^{G+1}$ such that: (iv) $\int f \leqq z^*$; and also, as $m \to \infty$, so: (v) $p_{k(m)} \to p^*$; (vi) $\int f_{k(m)} = z_{k(m)} \to z^*$; (vii) $f_{k(m)}(a) \to f(a)$ a.e. in A.

Now, for any positive integers m and r such that $k(m) \geq r$, (iii) implies that $p \int f_{k(m)} \leq 0$ for all $p \in \Delta_r \subset \Delta_{k(m)}$. Because of (vi), taking the limit as $m \to \infty$ gives $p\, z^* \leq 0$ for all $p \in \Delta_r$. Since this is true for any positive integer r, one has $p\, z^* \leq 0$ for all $p \in \Delta^0 = \cup_{r=1}^{\infty} \Delta_r$. But $p_0 = 0$ and so $p^G\, z^{*G} \leq 0$ for all $p^G \gg 0$ satisfying $\sum_{g \in G} p_g = 1$. Hence $z^{*G} \leqq 0$. Moreover, since nobody can ever have a positive demand for good 0 and so $z_0^* \leq 0$, it follows that $z^* \leqq 0$ — and so, by (iv) above, that $\int f \leqq 0$.

Finally, since (i) implies that $f_{k(m)}(a) \in \xi_a^C(p_{k(m)})$ a.e. in A, and since (v) and (vii) above are both true, the closed graph property of the compensated demand correspondence implies that $f(a) \in \xi_a^C(p^*)$ a.e. in A. So (f, p^*) together form a compensated equilibrium. ∎

3.6. Core Allocations Are Compensated Equilibria

Theorem. *Under Assumptions (A.1) and (A.2) as stated in Sections 3.1 and 3.2, any core allocation is a compensated equilibrium.*

Proof: Let $f : A \to \Re^{G+1}$ be any (measurable) allocation in the core. Now define the four correspondences

$$\begin{aligned}
\phi^N(a) &:= \{\, b \in \Re^G \mid (-1, b) \in X_a \quad \text{and} \quad (-1, b) \succ_a f(a) \,\} \\
\phi^S(a) &:= \{\, b \in \Re^G \mid (0, b) \in X_a \quad \text{and} \quad (0, b) \succ_a f(a) \,\} \\
\phi(a) &:= \phi^N(a) \cup \phi^S(a) \\
&= \{\, x^G \in \Re^G \mid \exists x_0 \in \{\, -1, 0 \,\} : (x_0, x^G) \in X_a \quad \text{and} \quad (x_0, x^G) \succ_a f(a) \,\} \\
\psi(a) &:= \phi(a) \cup \{0\}
\end{aligned} \tag{24}$$

on the common domain A, and for the common range space $\Re^G$. Arguing as in Hildenbrand (1974, pp. 133–5), it follows that the two correspondences $\phi^N(a)$ and $\phi^S(a)$ both have measurable graphs in $A \times \Re^G$. So therefore does the correspondence ψ, since its graph is the union of the measurable graphs of the correspondences $\phi^N(a)$ and $\phi^S(a)$ with the measurable set $A \times \{0\}$. Also $\int \psi\, d\alpha$ is a convex subset of $\Re^G$ including 0.

Suppose it were true that $z \in \int \psi\, d\alpha$ for some $z \in \Re^G$ such that $z \ll 0$. Then there must be a measurable set $K \subset A$ for which $z \in \int_K \phi(a)\, d\alpha$, and so K must be a blocking coalition.

Hence, if f is indeed in the core, the two convex sets $\int \psi \, d\alpha$ and $\{ z \in \Re^G \mid z \ll 0 \}$ must be non-empty and disjoint. So they can be separated by a hyperplane $p^G z = 0$ through the origin with $p^G > 0$ and $\sum_{g \in G} p_g = 1$. Thus, $p^G z \geq 0$ whenever $z \in \int \psi \, d\alpha$. Let $p := (0, p^G) \in \Delta$ be the corresponding $G+1$-dimensional price vector in which life is given a price of zero. Then, a.e. in A, it must be true that $x \succ_a f(a)$ implies $p\, x \geq 0$. In particular, a.e. in A, $p\, x \geq 0$ whenever $x^G \gg f^G(a)$. Then, because $f^G(a)$ is the limit of an infinite sequence $(x^{\nu G})$ of points satisfying $x^{\nu G} \gg f^G(a)$, it follows that, a.e. in A, $p\, f(a) \geq 0$. Yet $p > 0$ and so, since feasibility implies that $\int f \leqq 0$, it must be true that $\int p\, f \leq 0$. Now the last two sentences will contradict each other unless $\int p\, f = 0$ and in fact $p\, f(a) = 0$ a.e. in A. Therefore (f, p) is a compensated equilibrium. ∎

3.7. *Pareto Efficient Allocations Are Compensated Competitive*

Theorem. *Under Assumptions (A.1) and (A.2) as stated in Sections 3.1 and 3.2, any Pareto efficient allocation $f : A \to \Re^{G+1}$ is a compensated equilibrium at some price vector $p \in \Delta$ when each consumer $a \in A$ receives the net lump-sum transfer $p\, f(a)$.*

Proof: The separation argument used in Section 3.6 above can be easily be adapted as follows. Indeed, $\int \phi \, d\alpha$ and $\{ z \in \Re^G \mid z \ll 0 \}$ are non-empty convex sets which must be disjoint if f is a Pareto efficient allocation. Now we can follow the argument of Hildenbrand (1974, p. 232) in order to show the existence of a normalized price vector $p^G > 0$ such that $p^G \int f^G = 0$ and also, a.e. in A, $x^G \in \phi(a)$ implies $p^G x^G \geq p^G f^G(a)$. But then, if we let $p := (0, p^G) \in \Delta$, it follows that $p \int f = 0$ and also, a.e. in A, $x \succ_a f(a)$ implies $p\, x \geq p\, f(a)$. ∎

3.8. *The Cheaper Point Lemma*

Lemma. *Suppose that (C, Y, R) satisfy (A.1) and (A.2), while $(X, \succsim)$ are derived as in Sections 3.1 and 3.2. Suppose that $\hat{x} \in X$ is such that, for all $x \in X$, one has $p\, x \geq p\, \hat{x}$ whenever $x \succsim \hat{x}$. Then:*

(a) If $\hat{x} \in X^S$ and there also exists $x^* \in X^S$ for which $p\, x^* < p\, \hat{x}$, then for all $x \in X$ one has $p\, x > p\, \hat{x}$ whenever $x \succ \hat{x}$.

(b) If $\hat{x} \in X^N$ and there also exists $x^* \in X^N$ for which $p\, x < p\, x^*$, then for all $x \in X^N$ one has $p\, x > p\, \hat{x}$ whenever $x \succ \hat{x}$.

Proof: Under the hypothesis of the Lemma, there must exist $\hat{c} \in C$ and $\hat{y} \in Y$ such that $\hat{x} = \hat{c} - \hat{y}$ while $c \; R \; \hat{c} \Longrightarrow p\, x \geq p\, \hat{c}$ and $y \in Y \Longrightarrow p\, y \leq p\, \hat{y}$. Now:

Case (a). $\hat{c} \in \tilde{C}^S$.

In this case there must also exist $c^* \in \tilde{C}^S$ and $y^* \in Y$ for which $x^* = c^* - y^*$ and $p\,x^* < p\,\hat{x}$. Then $p\,y^* \leq p\,\hat{y}$ and so $p\,c^* = p\,x^* + p\,y^* < p\,\hat{x} + p\,\hat{y} = p\,\hat{c}$. By a standard argument for the convex consumption set $\tilde{C}^S$ and the continuous preference ordering R, it now follows that, whenever $c \in \tilde{C}^S$ with $c\ P\ \hat{c}$, then $p\,c > p\,\hat{c}$. Hence, if $x \in X$ with $x \succ \hat{x}$, so that there exist $c \in \tilde{C}^S$ and $y \in Y$ for which $x = c - y$ and $c\ P\ \hat{c}$, then $p\,\hat{y} \geq p\,y$ and so

$$p\,x + p\,y = p\,c > p\,\hat{c} = p\,\hat{x} + p\,\hat{y} \geq p\,\hat{x} + p\,y.$$

Therefore $p\,x > p\,\hat{x}$, as required for the Lemma to be true.

Case (b). $\hat{c} \in \tilde{C}^N$.

As in the proof of case (a) above, here there must exist $c^* \in \tilde{C}^N$ such that $p\,c^* < p\,\hat{c}$. Once again, by a standard argument applied to the convex consumption set $\tilde{C}^N$ and the continuous preference ordering R, it follows that, whenever $c \in \tilde{C}^N$ with $c\ P\ \hat{c}$, then $p\,c > p\,\hat{c}$. Hence, if $x \in X^N$ with $x \succ \hat{x}$, so that there exist $c \in \tilde{C}^N$ and $y \in Y$ for which $x = c - y$ and $c\ P\ \hat{c}$, then $p\,x > p\,\hat{x}$ as in case (a). ∎

3.9. Conditions for Compensated Equilibria to Be Walrasian

First we assume that:

(A.4). *0 is in the interior of the projection of the set $\int_A X_a\,\alpha(da)$ onto the subspace $\Re^G$.*

Effectively (A.4) requires that some trade is possible in every direction of the physical commodity space $\Re^G$, thereby ruling out the "exceptional case" presented by Arrow (1951).

The irreducibility assumption due to McKenzie (1959, 1961, 1981, 1987) has been adapted by Hildenbrand (1972; 1974, p. 143, Problem 8) for a continuum economy. Next we generalize Hildenbrand's version of this assumption somewhat, along the lines discussed in Hammond (1993). There too rather more motivation is provided, along with a further weakening. Here we assume that:

(A.5). *For every allocation f and every measurable partition of A into two sets A_1 and A_2 of positive measure, there exist measurable functions $t : A \to \Re^{G+1}$, $y : A_2 \to \Re^{G+1}$, and a set $A^* \subset A_1$ whose measure is positive, such that: (i) $\int_A t\,d\alpha + \int_{A_2} y\,d\alpha \leqq 0$; (ii) $y(a) \in X_a$ a.e. in A_2; (iii) $t(a) \succsim_a f(a)$ a.e. in A; (iv) $t(a) \succ_a f(a)$ a.e. in A^*.*

Thus (A.5) requires that there exist balanced net trades $t(a)$ $(a \in A)$ and $y(a)$ $(a \in A_2)$ which, if there were duplicates of the agents $a \in A_2$ who could be required to provide the net supply vectors $-y(a)$, would make possible a Pareto improvement with every agent $a \in A$ having the new net trade vector $t(a)$, and with a non-null set A^* of agents $a \in A_1$ becoming strictly better off.

For the usual reasons, (A.4) and (A.5), when combined with the earlier assumptions (A.1) and (A.2), will together ensure that, at any compensated equilibrium price vector $p \in \Delta$, almost all agents a can afford a net trade vector $x \in X_a$ with positive value $p\,x > 0$. However, there may still be a non-negligible set of agents on the margin of survival who create a discontinuity in mean demand and prevent existence of Walrasian equilibrium. For every price vector $p \in \Delta$ and every income level m, it is agents in the set

$$A(p,m) := \{\, a \in A \mid \exists x \in X_a^S : p\,x = m \quad \text{and} \quad \forall x' \in X_a^S : p\,x' \geq m \,\}. \tag{25}$$

who are on the margin of survival when faced with the budget constraint $p\,x = m$. If $a \in A(p,m)$ then in fact m can be regarded as a's minimum subsistence expenditure, because it must be equal to $\min_x \{\, p\,x \mid x \in X_a^S \,\}$.

An assumption which is more than sufficient for our purposes is:

(A.6*). *For every $p > 0$ and $m \in \Re$ one has $\alpha(A(p,m)) = 0$.*

This is a "dispersed needs" version of Yamazaki's (1981) "dispersed endowments" assumption. It states that there can be no atoms in the distribution of subsistence expenditures at any given price vector $p > 0$. An obvious implication of (A.6*) is the much weaker:

(A.6). *For every $p > 0$ one has $\alpha(A(p,0)) = 0$.*

The three extra assumptions (A.4–A.6) will combine with (A.1) and (A.2) to ensure that any compensated equilibrium is Walrasian. Indeed, given the compensated equilibrium (f,p), define the two sets of agents

$$A' := \{\, a \in A \mid \exists x \in X_a : p\,x < 0 \,\}; \quad \bar{A} := A(p,0). \tag{26}$$

The set $\bar{A}$ consists of those agents whose budget constraint $p\,x \leq 0$ leaves them on the margin of survival. By a standard argument, (A.4) implies that A' has positive measure, otherwise it would be true that $p \int x_a \, d\alpha \geq 0$ whenever $x_a \in X_a$ for a.e. $a \in A$. Now we make use of the following two Lemmas:

LEMMA A. *Let (f, p) be a compensated equilibrium in an economy satisfying assumptions (A.1), (A.2), and (A.4–A.6), and let a be any agent in the set $A' \setminus \bar{A}$. Then $x \succ_a f(a)$ implies $p\,x > 0$, and so $f(a) \in \xi_a(p)$.*

PROOF: There are two different cases to consider:

Case S. $f(a) \in X_a^S$.

Then $\tilde{x} = f(a)$ is a member of X_a^S satisfying $p\,\tilde{x} = 0$. Since $a \notin \bar{A}$, there must exist $x^* \in X_a^S$ such that $p\,x^* < 0$. But $x \succ_a f(a)$ is only possible in this case if $x \in X_a^S$.

Case N. $f(a) \in X_a^N$.

In this case $x \in X_a^S$ implies that $x \succ_a f(a)$ and so $p\,x \geq 0$. Since $a \notin \bar{A}$, there can be no $x \in X_a^S$ for which $p\,x = 0$. Hence $x \in X_a^S$ implies $p\,x > 0$. So we need only consider the case when $x \in X_a^N$. Yet by hypothesis $a \in A'$ and so there exists $x^* \in X_a$ such that $p\,x^* < 0$. Of course, this cheaper point must satisfy $x^* \in X_a^N$ because we have already seen that $x^* \in X_a^S$ would imply that $p\,x^* > 0$.

Now note that because preferences are monotone and (f, p) is a compensated equilibrium, $p\,f(a) = 0$ a.e. in A. Let $x \in X_a$ be any feasible net trade vector satisfying $x \succ_a f(a)$. In either case S or N, the Cheaper Point Lemma of Section 3.8 can then be applied to show that $p\,x > p\,f(a) = 0$. ∎

LEMMA B. *Under assumptions (A.1), (A.2), and (A.4–A.6), any compensated equilibrium (f, p) is a Walrasian equilibrium.*

PROOF: Suppose that (f, p) is a compensated equilibrium for which (A.1), (A.2), (A.4) and (A.5) are all satisfied. Let $A_2 := A \setminus A'$ and $A_1 := A'$. By (A.4) it must be true that $\alpha(A_1) > 0$.

Suppose also that $\alpha(A_2) > 0$. Then (A.5) implies that there exist mappings $t : A \to \Re^{G+1}$, $y : A_2 \to \Re^{G+1}$, and a set $A^* \subset A_1$ such that: (i) $\alpha(A^*) > 0$; (ii) $\int_A t\,d\alpha + \int_{A_2} y\,d\alpha \leqq 0$; (iii) $y(a) \in X_a$ a.e. in $A_2 = A \setminus A'$; (iv) $t(a) \succsim_a f(a)$ a.e. in A; and (v) $t(a) \succ_a f(a)$ a.e. in A^*. Then, since $p > 0$ it follows from (ii), (iii), and the definitions of A_2, A' that

$$\int_A p\,t\,d\alpha \leq -\int_{A_2} p\,y\,d\alpha \leq 0.$$

But (iv) implies that $p\,t(a) \geq p\,f(a) = 0$ a.e. in A, and so the above inequality can only be true if $p\,t(a) = 0$ a.e. in A. Now, together with the above Lemma, (v) clearly implies that $A^* \subset A \setminus (A' \setminus \bar{A})$. Yet $A^* \subset A_1 = A'$ and so

$$A^* \subset A' \cap [A \setminus (A' \setminus \bar{A})] = A' \cap [A' \setminus (A' \setminus \bar{A})] = A' \cap \bar{A} \subset \bar{A}.$$

Since $\alpha(A^*) > 0$ by (i) above, this implies that $\alpha(\bar{A}) > 0$. This contradicts (A.6), however.

So all three assumptions (A.4–A.6) can only be satisfied if $\alpha(A_2) = \alpha(A \setminus A') = 0$. Because $\alpha(\bar{A}) = 0$, this clearly implies that the set $A \setminus (A' \setminus \bar{A})$ must also have zero measure. Hence (f, p) must in fact be a Walrasian equilibrium, because of Lemma A above. ∎

Combined with our earlier results, we now have:

Existence Theorem. *Under assumptions (A.1–A.6), there exists a Walrasian equilibrium.*

Core Equivalence Theorem. *Under assumptions (A.1–A.6), the core coincides with the set of Walrasian equilibrium allocations.*

3.10. Second Efficiency Theorem

Here we find when a particular Pareto efficient allocation f is competitive, in the sense that there exists a price vector $p > 0$ such that $p \int f \, d\alpha = 0$ and, a.e. in A, $x \succ_a f(a)$ implies $p\,x > p\,f(a)$. Then f could be decentralized by facing each agent $a \in A$ with the budget constraint $p\,x \leq p\,f(a)$, each agent receiving a net lump-sum transfer equal to $p\,f(a)$.

We have already shown in Section 3.7 that a Pareto efficient allocation is compensated competitive at some price vector $p > 0$ with $p_0 = 0$. To show that it is competitive at p, it suffices to make modified versions of the assumptions (A.4) and (A.5) in Section 3.8 above.

First, for the fixed allocation f, define the sets of survivors and of non-survivors as

$$A^S := \{ a \in A \mid f(a) \in X_a^S \} \quad \text{and} \quad A^N := \{ a \in A \mid f(a) \in X_a^N \} \tag{27}$$

respectively. Then the modified version of (A.4) which we shall use here is:

(A.4′). *0 belongs to the interior of the projection of the set* $\int_{A^S} X_a^S \, d\alpha + \int_{A^N} X_a \, d\alpha$ *onto the subspace* $\Re^G$.

And the modified version of (A.5) is the following "non-oligarchy" condition:

(A.5′). *For the particular allocation f and for every measurable partition of A into two sets A_1 and A_2 of positive measure, there exists a measurable function $t : A \to \Re^{G+1}$ with $t_0(a) = 0$ (all $a \in A$), and a measurable set $A^* \subset A_1$ of positive measure, such that:*

(i) $\int_A t \, d\alpha \leqq 0$*; (ii)* $t(a) \in X_a$ *a.e. in* A_2*; (iii)* $t(a) \in X_a^S$ *a.e. in* $A_2 \cap A^S$*; (iv)* $t(a) \succsim_a f(a)$ *a.e. in* A_1*; (v)* $t(a) \succ_a f(a)$ *a.e. in* A^*.

The above condition requires that there be no "oligarchy" — i.e., no proper subset A_1 of agents who are so well off with the allocation f that collectively they could not possibly ever be made any better off even if they were given access to all the resources which the other agents in the complementary set A_2 could supply. Except for part (iii), it has been taken directly from Hammond (1993). Here part (iii) has been added because, as will be seen, it guarantees that almost no agent is on the boundary of X_a^S. For the survivors in A_2, it restricts the complementary coalition to use only those resources that can be taken without driving them below subsistence.

THE SECOND EFFICIENCY THEOREM. *Let* f *be any Pareto efficient allocation satisfying (A.4′) and (A.5′) in an economy* E *satisfying (A.1), (A.2) and (A.3). Then there exists a price vector* $p > 0$ *at which* f *is competitive — i.e., a.e. in* A, $x \succ_a f(a)$ *implies* $p\,x > p\,f(a)$.

PROOF: Define

$$\begin{aligned} \tilde{A}^S &:= \{\, a \in A^S \mid \exists x \in X_a^S : p\,x < p\,f(a) \,\}; \\ \tilde{A}^N &:= \{\, a \in A^N \mid \exists x \in X_a : p\,x < p\,f(a) \,\}; \\ A' &:= \tilde{A}^S \cup \tilde{A}^N. \end{aligned} \tag{28}$$

Then by the standard argument which we recapitulated previously, $\alpha(A') > 0$ because of (A.4′) and because $\int p\,f = 0$. Also, the Cheaper Point Lemma of Section 3.8 shows that, a.e. in A', $x \succ_a f(a)$ implies $p\,x > p\,f(a)$. So it suffices to show that $\alpha(A \setminus A') = 0$. To this end, note how $\alpha(A \setminus A') > 0$ would allow (A.5′) to be applied with $A_1 = A'$ and $A_2 = A \setminus A'$. A standard argument would then establish a contradiction, so in fact $\alpha(A \setminus A') = 0$. Hence the allocation f must indeed be competitive at prices p. ∎

3.11. A Sufficient Condition for Universal Survival

Finally, we find a sufficient condition like Moore's (1975) and McKenzie's (1981, 1987) for the existence of a Walrasian equilibrium in which all agents survive. We modify the old conditions (A.4) and (A.5) so that they become:

(A.4^S). *0 belongs to the interior of the projection of the set* $\int X_a^S \, d\alpha$ *onto the subspace* $\Re^G$.

(A.5^S). *For every allocation* f *and every measurable partition of* A *into two sets* A_1 *and* A_2 *of positive measure, there exist measurable functions* $t : A \to \Re^{G+1}$ *with* $t_0(a) = 0$ *(all* $a \in A$*),* $y : A_2 \to \Re^{G+1}$ *with* $y_0(a) = 0$ *(all* $a \in A_2$*), and a measurable set* $A^* \subset A_1$ *of positive measure, such that: (i)* $\int_A t \, d\alpha + \int_{A_2} y \, d\alpha \leqq 0$*; (ii)* $y(a) \in X_a^S$ *a.e. in* A_2*; (iii)* $t(a) \succsim_a f(a)$ *a.e. in* A*; (iv)* $t(a) \succ_a f(a)$ *a.e. in* A^*.

Now:

Survival Theorem. *Let* E *be an economy satisfying (A.1–A.3), (A.4*^{S}*), and (A.5*^{S}*). Then there exists a Walrasian equilibrium and, in any Walrasian equilibrium, almost all agents survive. The same is true of the core allocations, and the core coincides with the set of Walrasian equilibrium allocations.*

Proof: Suppose that (f, p) is a compensated Walrasian equilibrium. Let

$$\bar{A}^S := \{\, a \in A \mid \exists x \in X_a^S : p\,x < 0 \,\}. \tag{29}$$

Then (A.4^S) evidently implies that $\alpha(\bar{A}^S) > 0$. By a standard argument, one has $f(a) \in \xi_a(p)$ a.e. in $\bar{A}^S$. The "survival" irreducibility assumption (A.5^S) then establishes that $\alpha(A \setminus \bar{A}^S) = 0$, by another standard argument. So (f, p) must be a Walrasian equilibrium in which all agents in $\bar{A}^S$, and so almost all agents in A, do survive. ∎

4. Conclusions and Reservations

Section 2 considered an example showing how to extend the usual Walrasian model of a competitive market economy in order to deal seriously with the issue of survival, which standard general equilibrium theory is quite unable to discuss. That example was also able to offer considerable support for the Malthusian insight that more equality in the distribution of land would help more people to survive. As Woodham-Smith (1962, p. 20) has written of the Irish Famine:

> All this wretchedness and misery could, almost without exception, be traced to a single source — the system under which land had come to be owned and occupied in Ireland, a system produced by centuries of successive conquests, rebellions, confiscations and punitive legislation.

Also, in that example a system of food subsidies financed by a tax on land, on labour, or on both, was not even able to reduce starvation, much less guarantee complete survival. In another model not presented here, where labour supply is elastic, we were able to show essentially the same results, and also conclude that redistribution permits more people to survive than a welfare program financed by a (distortionary) income tax.

Section 3 then presented a general model of a continuum economy with individual production. With relatively minor modifications, the standard results of general equilibrium theory were shown to apply even when one takes into account the possibility that not all agents survive. The modifications were needed to deal with the inherent non-convexities in each individual's consumption set as one passes between survival and death. Existence of Walrasian equilibrium was proved without any survival assumption when there is a continuum of individuals whose needs for subsistence net trade vectors are dispersed. A Walrasian equilibrium, however, may require that some individuals not survive. Even when it does, as happens in some of the cases presented in Section 2, such an equilibrium is still Pareto efficient; to allow more individuals to survive, for instance, would require sacrifices from some of those who are fortunate enough to be able to survive anyway in equilibrium. And any Pareto efficient allocation, even one without complete survival, is a Walrasian equilibrium for a suitable system of lump-sum transfers. Thus all the classical existence and efficiency theorems apply. Core equivalence holds as well. Non-survivors lack the resources they need to block a Walrasian equilibrium and ensure their own survival. Finally, the paper presented sufficient conditions analogous to McKenzie's (1981) for survival of the whole population.

So the tragedy of starvation can arise in economies characterized by perfect competition. Then starvation is not a result of market failure. Like the involuntary unemployment that arises in Dasgupta and Ray (1986–7), it is not the result either of unnecessary institutional rigidities in the labour market. Instead it is an entirely natural phenomenon of a neoclassical economy with surplus labour. Only after excess labour has been removed through starvation can general equilibrium arise. As Joan Robinson (1946, pp. 189) wrote, "The hidden hand will always do its work, but it may work by strangulation." Or as Benjamin Jowett (formerly master of Balliol College, Oxford) was once moved to remark (according to *The White Plague*, by Frank Herbert), "I have always felt a certain horror of political economists, since I heard one of them say that he feared the famine of 1848 would not kill more than a million people, and that would be scarcely enough to do much good." Even then, (compensated) Walrasian equilibrium with starvation is Pareto efficient. To allow more individuals to survive requires sacrifices from some of those who survive anyway.

The fact that both Walrasian equilibrium and Pareto efficiency do not require all to survive should really be no great surprise. If the analysis seems heartless in the face of human misery, that is a true reflection of the price mechanism in a *laissez faire* economy which general equilibrium theory is intended to model. It also illustrates the ethical inadequacy of the Pareto criterion unless it is supplemented by further value judgments concerning the distribution of income or at least the alleviation of extreme poverty. For starvation may well be Pareto efficient, just as maximizing the preference ordering of a dictator is. Nor do the starving have the economic power to "block" a competitive market allocation in which they starve — they can only hope for non-market remedies, some of which were discussed in Section 2.

We have heard the view expressed that, if people really care enough about such poverty, then private charities will arise to assist the destitute. Yet such charity is effectively a public good, subject to the well known free-rider problem, as has been discussed by Mirrlees (1973), Arrow (1981) and many others. It can also be argued that charity exploits unduly those who have a charitable disposition. While the coercion of a tax-financed welfare program may not necessarily be the best resolution of this particular free-rider problem, more suitable alternatives have yet to be found, and the starving can hardly afford to wait for one.

This is clearly a problem for which a political economist should be able to give useful advice. Blind adherence to *laissez faire* economic policy and neglect of distributional issues lead to starvation (cf. Rashid, 1980). As Sen (1984, p. 31) wrote: "In the past, economic policies regarding food have often been ineffective, or worse, precisely because of concentrating on misleading variables, e.g. total food output, physical transport capacity. Unhappily, these mistakes are still made ..." Similar ideas are also discussed, of course, in Sen (1983), in Drèze and Sen (1989), and elsewhere. In our model, the right kind of government intervention, if it is possible, modifies Walrasian equilibrium to ensure complete survival. If market forces cannot be brought under control, complete survival may be impossible (cf. Hammond, 1987). We hope to have helped in making more economists understand how there are almost no limits to the cruel injustices which are possible in even a "perfect" market economy, and to encourage them to allow more "imperfections" or "distortions" into a market system if those are what even limited distributive justice requires.

The major limitation of our work is that we have considered only static Walrasian equilibria. Yet issues of survival are essentially dynamic. Death from malnutrition is gradual, and anyway is often indirect because it increases susceptibility to diseases which then appear to strike at random. The margin of survival, in the dynamic sense, is not so much a discrete boundary. Famine especially is inherently dynamic, as crops fail, food prices rise rapidly, and populations of whole villages leave their land in a desperate search for something to eat. A realistic dynamic model would be much more complicated, but it is our belief that it would not greatly modify or add to what the simple static model has to teach us concerning the existence and efficiency of Walrasian equilibrium without survival.

One other important question, however, certainly cannot be discussed in our static model. This is the Malthusian issue: if more of the population is enabled to survive today, does this only serve to render intolerable the increased pressure of population tomorrow? We have nothing here to add to this old and much discussed question, except for the common and hopeful observation that going beyond survival and into moderate prosperity appears to limit fertility. We believe, then, that Malthusian arguments do not provide us with any justifiable excuse for not trying to help more of the poorest in the world to survive, going beyond *laissez faire* to do so wherever necessary.

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