1 Introduction and Outline

The St. Petersburg game was a problem posed in 1713 by Nicholas Bernoulli to Montfort, another mathematician. The game involves tossing a fair coin repeatedly until the first time it lands "heads". If this happens on the kth toss, the prize is 2^k ducats. How much is it worth paying to be allowed to play? Note that the expected winnings are $\sum_{k=1}^{\infty} 2^{-k}.2^k = +\infty$. But, as Gabriel Cramer (1728) wrote to Daniel Bernoulli, "no reasonable man would be willing to pay 20 ducats as equivalent".[1] Samuelson (1977, 1983) and Shapley (1977) offer some cogent reasons for this, including the obvious fact that there is an upper bound to the real value of what can actually be paid as a prize.

The expected utility (or EU) hypothesis was formulated in Cramer's (1728) suggestion for resolving this "St. Petersburg paradox". The hypothesis is that one lottery affecting wealth is preferred to another prospect iff its expected utility of wealth or "moral expectation" is higher. This generalizes the earlier hypothesis that a lottery is preferred iff expected (monetary) wealth is greater. For the St. Petersburg game, if a potential player has utility of wealth level w given by $v(w)$ and starts with initial wealth $w_0 > 0$, then the amount a that the player is willing to pay to play the game must satisfy

$$v(w_0) = \sum_{k=1}^{\infty} 2^{-k}\, v(w_0 + 2^k - a) \tag{1}$$

Cramer suggested taking $v(w) = \min\{\, w, 2^{24} \,\}$ or $v(w) = \sqrt{w}$; Bernoulli (1738) suggested $v'(w) = 1/w$, implying that $v(w) = \ln w + \text{constant}$. For the last two of these utility functions, as well as for the first provided that $w_0 + 2 - a < 2^{24}$, the right hand side of (1) clearly converges and is strictly decreasing as a function of a. So the solution is well defined and finite.[2]

The general EU hypothesis is stated formally in Section 2. Let $v(y)$ be any utility function defined on the consequence domain Y. Let a, b, c be any three consequences in Y, no two of which are indifferent. In this case, it is proved that the ratio $\dfrac{v(a) - v(c)}{v(b) - v(c)}$ of utility differences must equal the marginal rate of substitution between shifts in probability from consequence c to a and shifts in probability from c to b. In Section 2, this result is shown to imply that the familiar result that the utility function is determined up to a unique cardinal equivalence class.

[1] See Bernoulli (1954, p. 33)

[2] For further useful discussion of the St. Petersburg paradox, see Sinn (1983) and Zabell (1987). Also, some of the ideas of EU theory appear earlier, but less explicitly, in the work of Blaise Pascal and his contemporaries in Paris during the 1660s. See Hacking (1975) for a more modern perspective and interpretation.

Expected utility was little used in economics, and little understood by economists, before von Neumann and Morgenstern (1944, 1953). Their axiomatic treatment was intended to ensure that preferences would have an expected utility representation. This treatment constitutes what they and Savage (1954, p. 97) regarded as a digression from their main work on game theory. According to Leonard (1995, p. 753), their axiomatic formulation apparently occupied not much more than two hours in the first place — see also Morgenstern (1979, p. 181). Nevertheless, their fundamental contribution soon led to an explosion of work in economics and related disciplines making use of the EU hypothesis in order to consider decisions under risk. Since their pioneering work, a utility function whose expected value is to be maximized has generally come to be called a *von Neumann–Morgenstern utility function* (here called an NMUF for short).

Fairly soon Marschak (1950), Nash (1950), Samuelson (1952), Malinvaud (1952) and others noticed that actually von Neumann and Morgenstern had left implicit a rather important independence axiom, as has been pointed out by Fishburn (1989) and by Fishburn and Wakker (1995) in particular. Also Marschak (1950), followed by Herstein and Milnor (1953), moved toward a much more concise system of, in the end, just three axioms. These axioms are, respectively, ordinality (O), independence (I), and continuity (C). Different versions of these axioms are introduced in Section 3, and the strongest variants are shown to be necessary for the EU hypothesis. Weaker variants are then used in Section 4 to demonstrate expected utility maximization, following standard proofs such as those to be found in Blackwell and Girshick (1954), Jensen (1967), and Fishburn (1970, 1982), as well as in most of the original works that have already been cited.[3]

Not much later, expected utility theory, especially the independence axiom, was criticized by Allais (1953) — see also Allais (1979a, b). Partly in response to this and many succeeding attacks, but also in order to justify the existence of a preference ordering, it seemed natural to consider the implications of the hypothesis that behaviour in decision trees should have consequences that are independent of the tree structure — see Hammond (1977, 1983, 1988a, b). This hypothesis plays a prominent role in this chapter, and differentiates it from the many other surveys of expected utility theory. In fact the "consequentialist foundations" are taken up in Sections 5 and 6. First, Section 5 concerns the existence of a preference ordering, and then Section 6 shows how the indepen-

[3] Marschak (1950) has a rather different proof, based on the fact that indifference surfaces are parallel hyperplanes.

dence axiom can be deduced. The third continuity axiom is the subject of Section 7.

Next, Section 8 turns to Blackwell and Girshick's extension of EU theory to accommodate countable lotteries. Obviously, no framework excluding these permits consideration of the problem that originally motivated EU theory, namely the St. Petersburg paradox. Blackwell and Girshick imposed an additional dominance axiom. Following their arguments as well as similar ideas due to Menger (1934), Arrow (1965, pp. 28–44; 1971, ch. 2; 1972), and Fishburn (1967, 1970), it is then shown that generalizations of the St. Petersburg paradox can only be avoided if each possible von Neumann–Morgenstern utility function is bounded both above and below, as Cramer's first suggestion $v(w) = \min\{ w, 2^{24} \}$ is if wealth is bounded below. Conversely, it will also be shown that boundedness implies dominance in the presence of the other conditions (O), (I), and (C). Finally, it will be shown that a stronger continuous preference condition (CP) can replace dominance or boundedness as a sufficient condition.

Section 9 is the only part of the chapter that relies on measure theory. It briefly discusses the extension of EU theory to general probability measures on the space of consequences, showing that this extension is possible if and only if all upper and lower preference sets and all singleton sets are measurable. Section 9 also considers the continuity of expected utility w.r.t. changes in the probability measure, and shows the sufficiency of a particular continuous preference axiom, in combination with other standard conditions. Actually, since the results of Section 8 are subsumed in those of Section 9, the main reason for discussing countable lotteries separately is to allow the most important complications caused by unbounded utilities to be discussed without the use of measure theory.

The brief final Section 10 contains a summary and a few concluding remarks.

2 The Expected Utility Hypothesis

2.1 Simple Lotteries

Let Y denote an arbitrary set of possible *consequences*. A typical *simple lottery* or probability distribution λ is a mapping $\lambda : Y \to [0, 1]$ with the properties that:

(i) there is a finite *support* $K \subset Y$ of λ such that $\lambda(y) > 0$ for all $y \in K$ and $\lambda(y) = 0$ for all $y \in Y \setminus K$;[4]

(ii) $\sum_{y \in K} \lambda(y) = \sum_{y \in Y} \lambda(y) = 1$.

Let $\Delta(Y)$ denote the set of all such simple lotteries. Given any pair $\lambda, \mu \in \Delta(Y)$ and any $\alpha \in [0,1]$, define the *convex combination, compound lottery*, or *mixture* $\alpha\lambda + (1-\alpha)\mu$ by

$$[\alpha\lambda + (1-\alpha)\mu](y) := \alpha\lambda(y) + (1-\alpha)\mu(y)$$

for all $y \in Y$. Note then that $\alpha\lambda + (1-\alpha)\mu$ also belongs to $\Delta(Y)$. Because it is convex, $\Delta(Y)$ is said to be a *mixture space*.[5]

Given any consequence $y \in Y$, let $1_y \in \Delta(Y)$ denote the degenerate simple lottery in which y occurs with probability one. Then each $\lambda \in \Delta(Y)$ can be expressed in the form

$$\lambda = \sum_{y \in Y} \lambda(y)\, 1_y \tag{2}$$

where, because λ has finite support, the sum on the right-hand side of (2) has only finitely many non-zero terms.

2.2 Expected Utility Maximization

A standard model of choice is that due to Arrow (1959, 1963), Sen (1970), Herzberger (1973), etc. In this model, a *feasible set* F is any set in a domain $\mathcal{D}$ of non-empty subsets of a given *underlying set* or *choice space* Z. For each $F \in \mathcal{D}$, the *choice set* $C(F)$ is a subset of F, and the mapping $F \mapsto C(F)$ on the domain $\mathcal{D}$ is the *choice function*. It is typically assumed that $\mathcal{D}$ includes all finite non-empty subsets of Z and that $C(F)$ is non-empty for all such sets. But, in case Z is an infinite set, $\mathcal{D}$ can also include some or even all infinite subsets of Z. Also, for some of these feasible sets, $C(F)$ could be empty.

The *expected utility* (EU) *hypothesis* applies to the choice space of lotteries $\Delta(Y)$. It requires the existence of a *von Neumann–Morgenstern utility function*

[4] Throughout the chapter, $\subset$ will denote the weak subset relation, so that $P \subset Q$ does not exclude the possibility that $P = Q$.

[5] As discussed by Wakker (1989, pp. 136–7) and especially by Mongin (1996), every convex set is a mixture space, but not every mixture space is a convex set. For this reason, it would be more logical in many ways to focus on the convexity of $\Delta(Y)$. But, following Herstein and Milnor (1953), it has become traditional to regard $\Delta(Y)$ as a mixture space.

(or NMUF) $v : Y \to \mathbb{R}$ such that, given any feasible set $F \subset \Delta(Y)$, the choice set is

$$C(F) = \arg\max_{\lambda} \left\{ \sum\nolimits_{y \in Y} \lambda(y) v(y) \mid \lambda \in F \right\} \tag{3}$$

That is, $C(F)$ consists of those lotteries $\lambda \in F$ which maximize the *expected utility function* (EUF) defined by

$$U(\lambda) := \mathbb{E}_\lambda \, v := \sum\nolimits_{y \in Y} \lambda(y) v(y) \tag{4}$$

Notice how (4) implies that

$$U(\alpha\lambda + (1-\alpha)\mu) = \sum\nolimits_{y \in Y} [\alpha\lambda(y) + (1-\alpha)\mu(y)] v(y) = \alpha U(\lambda) + (1-\alpha) U(\mu)$$

So $U(\cdot)$ satisfies the *mixture preservation* (MP) property

$$U(\alpha\lambda + (1-\alpha)\mu) = \alpha U(\lambda) + (1-\alpha) U(\mu) \tag{5}$$

for all $\lambda, \mu \in \Delta(Y)$ and all $\alpha \in [0,1]$. That is, the utility of any mixture of two lotteries is equal to the corresponding mixture of the utilities of the lotteries. Conversely, because of (2), for any utility function U satisfying (MP) on the domain $\Delta(Y)$ it must be true that $U(\lambda) = \sum_{y \in Y} \lambda(y) U(1_y)$ for all $\lambda \in \Delta(Y)$. Then (4) follows if one defines $v(y) := U(1_y)$ for all $y \in Y$. Hence U must be an EUF.

2.3 Ratios of Utility Differences

Suppose that behaviour does satisfy the EU hypothesis, but the NMUF is unknown. What features of the NMUF can be inferred from behaviour, or from revealed preferences?

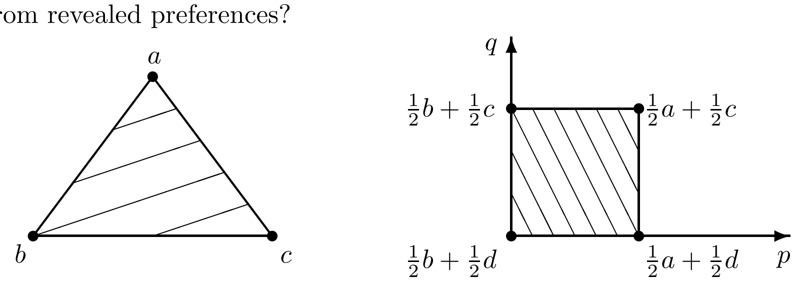

Figure 1 Two Indifference Maps for Constant Ratios of Utility Differences

In fact, suppose that $a, b, c \in Y$ are any three consequences. Consider the *Marschak triangle* (Marschak, 1950) in the left-hand part of Figure 1, which consists of all simple lotteries attaching probability one to the set $\{a, b, c\}$. Obviously the typical member of this triangle can be expressed as $p\,1_a + q\,1_b + (1-p-q)\,1_c$, where $p, q \geq 0$ and $p+q \leq 1$. Under the EU hypothesis, preferences within this triangle give rise to an indifference map in which each indifference curve takes the form

$$p\,v(a) + q\,v(b) + (1-p-q)\,v(c) = \text{constant}$$

Thus, all indifference curves in the Marschak triangle are parallel straight lines. Their common constant slope is

$$-\frac{dq}{dp} = \frac{v(a) - v(c)}{v(b) - v(c)} \tag{6}$$

provided that this expression is well defined because $v(b) \neq v(c)$. In fact, this ratio of utility differences is the constant marginal rate of substitution (MRS) between: (i) any increase in the probability p of getting a that is offset by a compensating decrease in the probability $1-p-q$ of getting c, with q fixed; (ii) any increase in the probability q of getting b that is offset by a compensating decrease in the probability $1-p-q$ of getting c, with p fixed. For each $a, b, c \in Y$, this common constant ratio can therefore be inferred from hypothetical behaviour.

Equation (6) involves a three-way ratio of utility differences. But the four-way ratio $[v(a) - v(b)]/[v(c) - v(d)]$ can also be interpreted as a constant MRS whenever $a, b, c, d \in Y$ are four consequences with $v(c) \neq v(d)$. For consider the square in the right-hand half of Figure 1, which represents the two-dimensional set

$$\{\tfrac{1}{2}p\,1_a + \tfrac{1}{2}(1-p)\,1_b + \tfrac{1}{2}q\,1_c + \tfrac{1}{2}(1-q)\,1_d \mid (p,q) \in [0,1] \times [0,1]\}$$

of lotteries in which the sets $\{a, b\}$ and $\{c, d\}$ each have fixed probability $\frac{1}{2}$. Under the EU hypothesis, the indifference curves in this square are parallel straight lines with

$$\tfrac{1}{2}p\,v(a) + \tfrac{1}{2}(1-p)\,v(b) + \tfrac{1}{2}q\,v(c) + \tfrac{1}{2}(1-q)\,v(d) = \text{constant}$$

So their common constant slope is

$$-\frac{dq}{dp} = \frac{v(a) - v(b)}{v(c) - v(d)}$$

This is the constant MRS between shifts in probability from consequence b to consequence a and shifts in probability from consequence d to consequence c among lotteries giving all four consequences positive probability.

Thus, ratios of utility differences correspond to constant MRSs between appropriate shifts in probability. As will be shown next, no other features of the NMUF can be inferred from expected utility maximizing behaviour alone.

2.4 Cardinally Equivalent Utility Functions

Behaviour satisfying the EU hypothesis does not determine the corresponding NMUF uniquely. Indeed, let $v : Y \to \mathbb{R}$ and $\tilde{v} : Y \to \mathbb{R}$ be any pair of NMUFs which, for some additive constant δ and multiplicative constant $\rho > 0$, are related by the identity

$$\tilde{v}(y) \equiv \delta + \rho v(y) \tag{7}$$

which holds throughout Y. Then, given any pair of lotteries $\lambda, \mu \in \Delta(Y)$, one has

$$\begin{aligned}\sum\nolimits_{y\in Y}[\lambda(y) - \mu(y)]\tilde{v}(y) &= \sum\nolimits_{y\in Y}[\lambda(y) - \mu(y)][\delta + \rho v(y)] \\ &= \rho \sum\nolimits_{y\in Y}[\lambda(y) - \mu(y)]v(y)\end{aligned}$$

because $\sum_{y\in Y} \lambda(y) = \sum_{y\in Y} \mu(y) = 1$. But $\rho > 0$ and so

$$\sum\nolimits_{y\in Y} \lambda(y)\tilde{v}(y) \geq \sum\nolimits_{y\in Y} \mu(y)\tilde{v}(y) \iff \sum\nolimits_{y\in Y} \lambda(y)v(y) \geq \sum\nolimits_{y\in Y} \mu(y)v(y)$$

For this reason, the pair of NMUFs satisfying (7) are said to be *cardinally equivalent*. Furthermore, (7) obviously implies that

$$\frac{\tilde{v}(a) - \tilde{v}(c)}{\tilde{v}(b) - \tilde{v}(c)} = \frac{v(a) - v(c)}{v(b) - v(c)} \tag{8}$$

and so the MRS given by (6) is also the same for all NMUFs in the same cardinal equivalence class. Clearly, any four-way ratio of utility differences of the form $[v(a) - v(b)]/[v(c) - v(d)]$ with $v(c) \neq v(d)$ is also preserved by the transformation (7) from $v(y)$ to $\tilde{v}(y)$.

Conversely, if v and $\tilde{v}$ are NMUFs whose expected values represent the same preferences, then both must satisfy (6) for the same constant MRS $-dq/dp$. Hence (8) is satisfied for all $a, b, c \in Y$. But then

$$\frac{\tilde{v}(a) - \tilde{v}(c)}{v(a) - v(c)} = \frac{\tilde{v}(b) - \tilde{v}(c)}{v(b) - v(c)}$$

whenever $v(a) \neq v(c)$ and $v(b) \neq v(c)$. So, for each fixed $y' \in Y$, there must exist a constant ρ such that

$$\frac{\tilde{v}(y) - \tilde{v}(y')}{v(y) - v(y')} = \rho$$

for all $y \in Y$ with $v(y) \neq v(y')$. Moreover, because the expected values of v and $\tilde{v}$ represent identical preferences, it follows that $v(y) > v(y') \iff \tilde{v}(y) > \tilde{v}(y')$ and so $\rho > 0$. Therefore

$$\tilde{v}(y) = \tilde{v}(y') + \rho\,[v(y) - v(y')] = \delta + \rho v(y)$$

for all $y \in Y$, where $\delta := \tilde{v}(y') - \rho\, v(y')$. This confirms (7) for the case when the domain Y is rich enough to include at least two non-indifferent consequences. But (7) is trivially valid when all consequences in Y are indifferent.

3 Necessary Conditions

3.1 The Ordering Condition

The EU hypothesis implies three obvious but important properties. The first of these concerns the preference relation $\succsim$ defined by

$$\lambda \succsim \mu \iff \sum\nolimits_{y \in Y} \lambda(y)v(y) \geq \sum\nolimits_{y \in Y} \mu(y)v(y) \tag{9}$$

Then, for every finite $F \subset \Delta(Y)$, it follows from (3) that the choice set is given by $C(F) = \{\, \lambda \in F \mid \mu \in F \Longrightarrow \lambda \succsim \mu \,\}$. In particular, of course, $\lambda \succsim \mu$ iff $\lambda \in C(\{\, \lambda, \mu \,\})$.

Evidently, then, the EU hypothesis implies the *ordering condition* (O), requiring the existence of a complete and transitive binary *preference ordering* $\succsim$ on $\Delta(Y)$ satisfying (9). As usual, we write $\lambda \succ \mu$ and say that λ is *strictly preferred* to μ when $\lambda \succsim \mu$ but not $\mu \succsim \lambda$; we write $\lambda \sim \mu$ and say that λ is *indifferent* to μ when $\lambda \succsim \mu$ and also $\mu \succsim \lambda$. Also, $\lambda \precsim \mu$ is equivalent to $\mu \succsim \lambda$, and $\lambda \prec \mu$ is equivalent to $\mu \succ \lambda$. Because $\succsim$ is complete, $\lambda \succsim \mu$ is equivalent to $\lambda \not\prec \mu$.

For general consequence spaces, such preference orderings, and their representation by utility functions, are discussed in the preceding chapter by Mehta. This chapter concentrates entirely on spaces of consequence lotteries. For preference orderings that violate the EU hypothesis, some of the possible "non-expected" utility functions are discussed in other chapters of this *Handbook*. By contrast, this chapter also concentrates on behaviour patterns corresponding to preference orderings with the special structure that the EU hypothesis entails.

3.2 Independence Conditions

An important second implication of the EU hypothesis is *independence*. Indeed, this is the crucial property that distinguishes preferences satisfying the EU hypothesis from non-expected utility theories.

As Marschak (1950), Samuelson (1952) and Malinvaud (1952) pointed out, von Neumann and Morgenstern had originally left one important axiom implicit. In Samuelson (1983, p. 511), the *independence axiom* is expressed by means of the following condition (I′): Whenever $\lambda, \mu, \nu \in \Delta(Y)$ and $0 < \alpha < 1$, then

$$\lambda \succsim \mu \Longrightarrow \alpha\lambda + (1-\alpha)\nu \succsim \alpha\mu + (1-\alpha)\nu \tag{10}$$

Marschak and Malinvaud stated only the weaker condition (I^0), requiring that

$$\lambda \sim \mu \Longrightarrow \alpha\lambda + (1-\alpha)\nu \sim \alpha\mu + (1-\alpha)\nu \tag{11}$$

under the same hypotheses as for condition (I′). Later Herstein and Milnor (1953), following Samuelson's (1952) idea discussed below in connection with condition (I*), required (11) to be true only when $\alpha = \frac{1}{2}$. This is one of Herstein and Milnor's three axioms whose implications are also analysed at length in Fishburn (1982, pp. 13–17).

This chapter, however, will rely on two quite different versions of the independence axiom. Given the other axioms, it will be sufficient to use the following *independence condition* (I), apparently due to Jensen (1967). This requires that, whenever $\lambda, \mu, \nu \in \Delta(Y)$ and $0 < \alpha < 1$, then

$$\lambda \succ \mu \Longrightarrow \alpha\lambda + (1-\alpha)\nu \succ \alpha\mu + (1-\alpha)\nu$$

Combining conditions (I) and (I′) into one gives a single condition that, as shown in Lemma 3.1 below, is equivalent to the *finite dominance axiom* (FD*). This states that, whenever $\lambda_i, \mu_i \in \Delta^*(Y)$ and $\alpha_i > 0$ with $\lambda_i \succsim \mu_i$ $(i = 1, 2, \ldots m)$ and $\sum_{i=1}^m \alpha_i = 1$, then $\sum_{i=1}^m \alpha_i \lambda_i \succsim \sum_{i=1}^m \alpha_i \mu_i$ with strict preference if $\lambda_k \succ \mu_k$ for any k. Condition (FD*) is a simplified version of the infinite dominance condition (D*) due to Blackwell and Girshick (1954, p. 105, H_1) that is discussed in Section 8.4.

Apparently stronger than conditions (I), (I^0) or (I′) is the *strong independence condition* (I*), requiring the logical equivalence

$$\lambda \succsim \mu \iff \alpha\lambda + (1-\alpha)\nu \succsim \alpha\mu + (1-\alpha)\nu \tag{12}$$

to hold whenever $0 < \alpha < 1$. For the case when $\alpha = \frac{1}{2}$, this is equivalent to Samuelson's (1952, p. 672) original "strong independence" axiom. Condition (I*) is not really new, however, because of:

LEMMA 3.1: Provided that condition (O) is satisfied, then condition (FD*), condition (I) combined with (I′), and condition (I*) are all equivalent.

PROOF: (i) Suppose condition (FD*) is true. Obviously condition (I′) holds and so therefore does (10). Conversely, if $\alpha\lambda + (1-\alpha)\nu \succsim \alpha\mu + (1-\alpha)\nu$ and $0 < \alpha < 1$, then $\alpha\mu + (1-\alpha)\nu \not\succ \alpha\lambda + (1-\alpha)\nu$ by condition (O). So $\mu \not\succ \lambda$ by condition (I), from which it follows that $\lambda \succsim \mu$ by condition (O). Therefore (12) holds and condition (I*) is satisfied.

(ii) Suppose condition (I*) is true. Obviously (12) implies (10), so condition (I′) is true. Also, if $\lambda \succ \mu$, then by condition (O), $\mu \not\succsim \lambda$. Therefore, whenever $0 < \alpha < 1$, condition (I*) implies that $\alpha\mu + (1-\alpha)\nu \not\succsim \alpha\lambda + (1-\alpha)\nu$. Then condition (O) implies that $\alpha\lambda + (1-\alpha)\nu \succ \alpha\mu + (1-\alpha)\nu$, which is condition (I).

(iii) Finally, it is easy to show that conditions (I) and (I′) jointly imply (FD*) when $m = 2$. A routine induction proof establishes the same implication for larger values of m.

Thus, it has been proved that (FD*) $\Longrightarrow$ (I*) $\Longrightarrow$ [(I) and (I′)] $\Longrightarrow$ (FD*). ∎

Finally, when (9) is true and $0 < \alpha < 1$, then

$$\begin{aligned} & \alpha\lambda + (1-\alpha)\nu \succsim \alpha\mu + (1-\alpha)\nu \\ \iff & \sum_{y \in Y} \{[\alpha\lambda(y) + (1-\alpha)\nu(y)] - [\alpha\mu(y) + (1-\alpha)\nu(y)]\} v(y) \geq 0 \\ \iff & \sum_{y \in Y} \lambda(y)\, v(y) \geq \sum_{y \in Y} \mu(y)\, v(y) \iff \lambda \succsim \mu \end{aligned}$$

Thus, (9) implies (12). So the strongest independence condition (I*) is an implication of the EU hypothesis.

3.3 Lexicographic Preferences

Though conditions (O) and (I*) are implied by the EU hypothesis, they do not characterize it. There are preference orderings which violate the EU hypothesis even though they satisfy (I*) on a domain $\Delta(Y)$.

As an example, consider the consequence domain $Y = \{ a, b, c \}$. Furthermore, suppose that $v : Y \to \mathbb{R}$ is an NMUF whose expected value $U(\lambda) = \mathbb{E}_\lambda v$ represents a preference ordering $\succsim$ on $\Delta(Y)$ satisfying $1_a \succ 1_b \succ 1_c$. The corresponding indifference map must be as shown in the Marschak triangle of Figure 1 in Section 2.3. Consider the alternative "lexicographic" preference relation $\succsim^*$ on $\Delta(Y)$ defined by

$$\lambda \succsim^* \mu \iff U(\lambda) > U(\mu) \text{ or } [U(\lambda) = U(\mu) \text{ and } \lambda(a) \geq \mu(a)]$$

The preference relation $\succsim^*$ is evidently complete and transitive. It also satisfies condition (I*), as is easily checked. But unlike $\succsim$, the relation $\succsim^*$ is a *total ordering* in the sense that either $\lambda \succ^* \mu$ or $\mu \succ^* \lambda$ whenever $\lambda \neq \mu$. Thus all indifference sets in the Marschak triangle are isolated single points, which is incompatible with the existence of any NMUF whose expected value represents the preference ordering $\succsim^*$. So the EU hypothesis is violated.

3.4 Continuity Conditions

The third and last implication of the EU hypothesis is the requirement that preferences satisfy the following *continuity condition* (C), due to Blackwell and Girshick (1954, p. 106, H_2).[6] Whenever $\lambda, \mu, \nu \in \Delta(Y)$ with $\lambda \succ \mu$ and $\mu \succ \nu$,

[6] This is sometimes called the *Archimedean axiom* — see, for instance, Karni and Schmeidler (1991, p. 1769).

this requires that there exist $\alpha', \alpha'' \in (0,1)$ satisfying

$$\alpha'\lambda + (1-\alpha')\nu \succ \mu \quad \text{and} \quad \mu \succ \alpha''\lambda + (1-\alpha'')\nu \tag{13}$$

Next, define the two preference sets

$$\begin{aligned} A &:= \{\, \alpha \in [0,1] \mid \alpha\lambda + (1-\alpha)\nu \succsim \mu \,\} \\ B &:= \{\, \alpha \in [0,1] \mid \mu \succsim \alpha\lambda + (1-\alpha)\nu \,\} \end{aligned} \tag{14}$$

and let $\underline{\alpha} := \inf A$ and $\overline{\alpha} := \sup B$. An alternative *mixture continuity* condition (C*) introduced by Herstein and Milnor (1953) requires that, whenever $\lambda, \mu, \nu \in \Delta(Y)$ with $\lambda \succ \mu$ and $\mu \succ \nu$, both A and B must be closed. Note that, when combined with condition (O), condition (C*) is a strengthening of (C) because of part (i) of the following Lemma 3.2.

Note that the stronger continuity condition (C*) is entailed by the EU hypothesis (3) because, for instance, (4), (5) and (9) together imply that

$$\begin{aligned} A &= \{\, \alpha \in [0,1] \mid \alpha U(\lambda) + (1-\alpha)U(\nu) \geq U(\mu) \,\} \\ &= \{\, \alpha \in [0,1] \mid \alpha \geq \frac{U(\mu) - U(\nu)}{U(\lambda) - U(\nu)} \,\} \end{aligned}$$

given that $U(\lambda) > U(\nu)$. Similarly,

$$B = \{\alpha \in [0,1] \mid \alpha \leq \frac{U(\mu) - U(\nu)}{U(\lambda) - U(\nu)} \,\}$$

In fact, there must be a unique number $\alpha^* \in (0,1)$ satisfying the condition that $\alpha^*\lambda + (1-\alpha^*)\nu \sim \mu$ and also:

$$\alpha\lambda + (1-\alpha)\nu \succ \mu \iff \alpha > \alpha^*; \quad \alpha\lambda + (1-\alpha)\nu \prec \mu \iff \alpha < \alpha^*$$

On the other hand, note that the lexicographic preferences described in Section 3.3 are discontinuous. For suppose that $\lambda, \mu, \nu \in \Delta(Y)$ with $W(\lambda) > W(\mu) > W(\nu)$. Define $\alpha^* \in (0,1)$ as the critical value

$$\alpha^* := \frac{W(\mu) - W(\nu)}{W(\lambda) - W(\nu)}$$

Then the two sets defined in (14) take the form

$$\begin{aligned} A &= \{\, \alpha \in [0,1] \mid \alpha > \alpha^* \text{ or } [\alpha = \alpha^* \text{ and } \alpha\lambda(y_1) + (1-\alpha)\nu(y_1) \geq \mu(y_1)] \,\} \\ B &= \{\, \alpha \in [0,1] \mid \alpha < \alpha^* \text{ or } [\alpha = \alpha^* \text{ and } \alpha\lambda(y_1) + (1-\alpha)\nu(y_1) \leq \mu(y_1)] \,\} \end{aligned}$$

With lexicographic preferences, the common boundary point α^* of these two sets generally belongs to only one of them, and so either A or B is not closed.

The only exception occurs in the special case when $\alpha^* \lambda(y_1) + (1-\alpha^*)\nu(y_1) = \mu(y_1)$. Then $\alpha^* \in A \cap B$, so both A and B are closed. But for almost all $\lambda, \mu, \nu \in \Delta(Y)$, such lexicographic preferences exhibit a discontinuity.

The following implication of the preceding definitions plays an important role in Section 4.

Lemma 3.2: Suppose that condition (O) is satisfied. Then: (i) condition (C*) implies (C); (ii) on the other hand, condition (C) implies that $\underline{\alpha} \in B$ and $\overline{\alpha} \in A$. Also, under the additional hypothesis $\underline{\alpha} = \overline{\alpha} =: \alpha^*$, it follows that $\alpha^* \lambda + (1-\alpha^*)\nu \sim \mu$.

Proof: Suppose that $\lambda, \mu, \nu \in \Delta(Y)$ with $\lambda \succ \mu$ and $\mu \succ \nu$. The definitions of A and B and condition (O) together imply that $0 \notin A$, $1 \in A$, $0 \in B$, $1 \notin B$.

(i) If condition (C*) is satisfied, then $\underline{\alpha} = \min A$ and $\overline{\alpha} = \max B$, from which it follows that $\underline{\alpha} > 0$ and $\overline{\alpha} < 1$. Choosing any $\alpha', \alpha'' \in (0,1)$ to satisfy $1 > \alpha' > \overline{\alpha}$ and $\underline{\alpha} > \alpha'' > 0$ implies that $\alpha' \notin B$ and $\alpha'' \notin A$. Now (13) follows immediately from definition (14) and condition (O).

(ii) Whenever $\alpha \in (0,1] \setminus B$, condition (O) implies that $\alpha\lambda + (1-\alpha)\nu \succ \mu$. Because $\mu \succ \nu$, condition (C) implies the existence of $\alpha' \in (0,1)$ such that

$$\alpha'[\alpha\lambda + (1-\alpha)\nu] + (1-\alpha')\nu \succ \mu$$

So $\alpha'\alpha \in A$. But $\alpha > \alpha'\alpha$. Because $\underline{\alpha} = \inf A$, it follows that $\alpha > \underline{\alpha}$ whenever $\alpha \in (0,1] \setminus B$. Therefore $\underline{\alpha} \notin (0,1] \setminus B$ and so $\underline{\alpha} \in B \cup \{0\}$. But $0 \in B$ and so $\underline{\alpha} \in B$ even if $\underline{\alpha} = 0$.

Similarly, whenever $\alpha \in [0,1) \setminus A$, condition (O) implies that $\mu \succ \alpha\lambda + (1-\alpha)\nu$. Because $\lambda \succ \mu$, condition (C) implies the existence of $\alpha'' \in (0,1)$ such that

$$\mu \succ \alpha''\lambda + (1-\alpha'')[\alpha\lambda + (1-\alpha)\nu]$$

So $\alpha'' + (1-\alpha'')\alpha = \alpha + (1-\alpha)\alpha'' \in B$. But $\alpha < \alpha + (1-\alpha)\alpha''$. Because $\overline{\alpha} = \sup B$, it follows that $\alpha < \overline{\alpha}$ whenever $\alpha \in [0,1) \setminus A$. Therefore $\overline{\alpha} \notin [0,1) \setminus A$ and so $\overline{\alpha} \in A \cup \{1\}$. But $1 \in A$ and so $\overline{\alpha} \in A$ even if $\overline{\alpha} = 1$.

Under the additional hypothesis $\underline{\alpha} = \overline{\alpha} =: \alpha^*$, it follows that $\alpha^* \in A \cap B$. Obviously, definition (14) implies that $\alpha^* \lambda + (1-\alpha^*)\nu \sim \mu$ in this case. ∎

The three properties (O), (I), and (C) are important because, as shown by Jensen (1967), they are not only necessary, but also sufficient conditions for the EU hypothesis to hold. This is discussed in the next Section. Here it has been shown that the stronger properties (I*) and (C*) are also necessary.

4 Sufficient Conditions

Condition (O) requiring that there be a preference ordering $\succsim$ is, of course, standard in utility theory. The independence condition (I) imposes strong restrictions on the possible form of $\succsim$. And, in combination with the continuity condition (C), it implies EU maximization. This section will be devoted to proving this important result, due to Jensen (1967). Note that, compared to Herstein and Milnor (1953), Jensen uses the stronger independence condition (I) instead of only (I^0) for $\alpha = \frac{1}{2}$. On the other hand, Jensen uses the weaker continuity condition (C) instead of (C*).

It is worth noting that the proof given here becomes rather easier if one assumes conditions (I*) and (C*) instead of (I) and (C). Indeed, there would be no need to prove Lemmas 4.2 or 4.4, since their only role is to show that (I*) is satisfied.

4.1 Ordinality and Independence

In the following, conditions (O) and (I) will be assumed throughout. Also, though the succeeding Lemmas 4.1–4.6 are stated only for the space $\Delta(Y)$, they are actually true in any convex set or mixture space. This important fact will be used later in Sections 8 and 9, as well as in Chapter 3 on subjectively expected utility.

Notice first that if $\lambda \sim \mu$ for all $\lambda, \mu \in \Delta(Y)$, then the EU hypothesis is trivially satisfied: there must exist a constant $\overline{v} \in \mathbb{R}$ such that $U(\lambda) = \overline{v} = v(y)$ for all $\lambda \in \Delta(Y)$ and all $y \in Y$. So, it will be supposed throughout this section that there exist $\lambda, \mu \in \Delta(Y)$ with $\lambda \succ \mu$.

LEMMA 4.1: For any pair of lotteries $\lambda, \mu \in \Delta(Y)$ with $\lambda \succ \mu$, one has:

(a) (Strict Betweenness) $\lambda \succ \alpha\lambda + (1-\alpha)\mu \succ \mu$ whenever $0 < \alpha < 1$.

(b) (Stochastic Monotonicity) $\lambda \succ \alpha\lambda + (1-\alpha)\mu \succ \alpha'\lambda + (1-\alpha')\mu \succ \mu$ whenever $0 < \alpha' < \alpha < 1$.

(c) (Weak Stochastic Monotonicity) if $\alpha, \alpha' \in [0,1]$, then

$$\alpha\lambda + (1-\alpha)\mu \succsim \alpha'\lambda + (1-\alpha')\mu \iff \alpha \geq \alpha'$$

PROOF: (a) Whenever $0 < \alpha < 1$, condition (I) implies that

$$\lambda = \alpha\lambda + (1-\alpha)\lambda \succ \alpha\lambda + (1-\alpha)\mu \succ \alpha\mu + (1-\alpha)\mu = \mu$$

(b) If $0 < \alpha' < \alpha < 1$, then there exists $\delta \in (0,1)$ such that $\alpha' = \delta\alpha$. So

$$\delta[\alpha\lambda + (1-\alpha)\mu] + (1-\delta)\mu = \delta\alpha\lambda + (1-\delta\alpha)\mu = \alpha'\lambda + (1-\alpha')\mu$$

By part (a), $\lambda \succ \alpha\lambda + (1-\alpha)\mu \succ \mu$. Next, applying part (a) a second time to $\delta[\alpha\lambda + (1-\alpha)\mu] + (1-\delta)\mu$ gives

$$\alpha\lambda + (1-\alpha)\mu \succ \delta[\alpha\lambda + (1-\alpha)\mu] + (1-\delta)\mu \succ \delta\mu + (1-\delta)\mu = \mu$$

Together these statements clearly imply part (b).

(c) Immediate from part (b), given that $\succsim$ is a complete ordering. ∎

In particular, if $\lambda \succ \mu$, then a compound lottery with a higher probability of λ and a lower probability of μ is preferred to one with a lower probability of λ and a higher probability of μ.

LEMMA 4.2: Suppose that $0 < \alpha < 1$ and $\lambda, \mu, \nu \in \Delta(Y)$. Then:

(a) $\alpha\lambda + (1-\alpha)\nu \sim \alpha\mu + (1-\alpha)\nu$ implies $\lambda \sim \mu$;

(b) if $\lambda \succ \alpha\lambda + (1-\alpha)\mu$ or $\alpha\lambda + (1-\alpha)\mu \succ \mu$, then $\lambda \succsim \alpha\lambda + (1-\alpha)\mu \succsim \mu$ and so $\lambda \succ \mu$;

(c) (Betweenness) $\lambda \sim \mu$ implies $\lambda \sim \alpha\lambda + (1-\alpha)\mu \sim \mu$;

(d) $\lambda \sim \mu \sim \nu$ implies $\alpha\lambda + (1-\alpha)\nu \sim \alpha\mu + (1-\alpha)\nu$.

PROOF: (a) By condition (I), if $\lambda \not\sim \mu$, then $\alpha\lambda + (1-\alpha)\nu \not\sim \alpha\mu + (1-\alpha)\nu$.

(b) If $\lambda \succ \alpha\lambda + (1-\alpha)\mu$ then condition (I) implies that

$$\alpha\lambda + (1-\alpha)\mu \succ \alpha\,[\alpha\lambda + (1-\alpha)\mu] + (1-\alpha)\,\mu \tag{15}$$

But

$$\alpha\,[\alpha\lambda + (1-\alpha)\mu] + (1-\alpha)\,[\alpha\lambda + (1-\alpha)\mu] = \alpha\lambda + (1-\alpha)\mu$$

Hence, (15) is compatible with condition (I) only if $\mu \not\succ \alpha\lambda + (1-\alpha)\mu$, and so $\alpha\lambda + (1-\alpha)\mu \succsim \mu$, by condition (O). Similarly, if $\alpha\lambda + (1-\alpha)\mu \succ \mu$ then condition (I) implies that

$$\begin{aligned} \alpha\lambda + (1-\alpha)\,[\alpha\lambda + (1-\alpha)\mu] &\succ \alpha\,[\alpha\lambda + (1-\alpha)\mu] + (1-\alpha)\,[\alpha\lambda + (1-\alpha)\mu] \\ &= \alpha\lambda + (1-\alpha)\mu \end{aligned}$$

This is compatible with condition (I) only if $\alpha\lambda + (1-\alpha)\mu \not\succ \alpha$, which implies that $\alpha \succsim \alpha\lambda+(1-\alpha)\mu$, because of condition (O). Therefore, if $\lambda \succ \alpha\lambda+(1-\alpha)\mu$ or $\alpha\lambda + (1-\alpha)\mu \succ \mu$, then $\lambda \succsim \alpha\lambda + (1-\alpha)\mu \succsim \mu$. In either case, transitivity of $\succsim$ implies that $\lambda \succ \mu$.

(c) If $\mu \succsim \lambda$, condition (O) and the contrapositive of part (b) together imply that $\mu \succsim \alpha\lambda + (1-\alpha)\mu \succsim \lambda$. Similarly, with λ and μ interchanged, as well as α and $1-\alpha$, it must be true that $\lambda \succsim \mu$ implies $\lambda \succsim \alpha\lambda + (1-\alpha)\mu \succsim \mu$. Therefore, $\lambda \sim \mu$ must imply that $\lambda \sim \alpha\lambda + (1-\alpha)\mu \sim \mu$.

(d) Suppose that $\lambda \sim \mu \sim \nu$. Applying part (c) first to the pair λ, ν and then to the pair μ, ν implies that $\alpha\lambda + (1-\alpha)\nu \sim \nu \sim \alpha\mu + (1-\alpha)\nu$. ∎

4.2 Continuity

Note that both Lemmas 4.1 and 4.2 rely only on conditions (O) and (I). From now on, assume throughout the rest of this section that the continuity condition (C) is also satisfied. Then:

LEMMA 4.3: Suppose that $\lambda, \mu, \nu \in \Delta(Y)$ with $\lambda \succ \mu$ and $\mu \succ \nu$. Then there exists a unique $\alpha^* \in (0,1)$ such that $\alpha^*\lambda + (1-\alpha^*)\nu \sim \mu$.

PROOF: Consider the two sets A and B as defined in (14) of Section 3.4. Define $\alpha^* := \inf A$. Now whenever $\alpha > \alpha^*$, there exists $\alpha' \in A$ such that $\alpha > \alpha' \geq \alpha^*$. From part (b) of Lemma 4.1 and the definition of A, it follows that

$$\alpha\lambda + (1-\alpha)\nu \succ \alpha'\lambda + (1-\alpha')\nu \succsim \mu$$

By condition (O), $\alpha \notin B$. This is true whenever $\alpha > \alpha^*$. Therefore $\alpha \in B$ implies $\alpha \leq \alpha^*$. So α^* is an upper bound for B, implying that $\alpha^* \geq \sup B$. But completeness of the preference ordering $\succsim$ excludes the possibility that $\inf A > \sup B$. Hence $\alpha^* = \inf A = \sup B$. By Lemma 3.2, condition (C) then implies that

$$\alpha^*\lambda + (1-\alpha^*)\nu \sim \mu \tag{16}$$

Now apply part (b) of Lemma 4.1 to the pair λ, ν. Whenever the inequalities $1 \geq \alpha' > \alpha^* > \alpha'' \geq 0$ are satisfied, it follows that

$$\alpha'\lambda + (1-\alpha')\nu \succ \alpha^*\lambda + (1-\alpha^*)\nu \succ \alpha''\lambda + (1-\alpha'')\nu \tag{17}$$

Because of (16) and (17), condition (O) implies that

$$\alpha'\lambda + (1-\alpha')\nu \succ \mu \succ \alpha''\lambda + (1-\alpha'')\nu$$

This shows that α^* is the unique member of $A \cap B$. ∎

COROLLARY: Suppose that $\lambda, \mu, \nu \in \Delta(Y)$ with $\lambda \succ \nu$ and $\lambda \succsim \mu \succsim \nu$. Then there exists a unique $\alpha^* \in [0,1]$ such that $\alpha^*\lambda + (1-\alpha^*)\nu \sim \mu$.

PROOF: Lemma 4.3 already treats the case when $\lambda \succ \mu \succ \nu$. Alternatively, if $\lambda \sim \mu$, one can take $\alpha^* = 1$, whereas if $\mu \sim \nu$, one can take $\alpha^* = 0$. ∎

LEMMA 4.4: Suppose that $0 < \alpha < 1$ and that $\lambda, \mu, \nu \in \Delta(Y)$. Then:

(a) $\alpha\lambda + (1-\alpha)\nu \succ \alpha\mu + (1-\alpha)\nu$ implies $\lambda \succ \mu$;

(b) Condition (I*) is satisfied.

PROOF: (a) *Case 1:* $\lambda \succsim \nu \succsim \mu$. Here, unless $\lambda \succ \mu$, it must be true that $\lambda \sim \mu \sim \nu$. But then Lemma 4.2(d) would imply $\alpha\lambda+(1-\alpha)\nu \sim \alpha\mu+(1-\alpha)\nu$, a contradiction.

Case 2: $\nu \succ \lambda$. Here, by Lemma 4.1(a), $\nu \succ \alpha\lambda + (1-\alpha)\nu$. So, whenever $\alpha\lambda+(1-\alpha)\nu \succ \alpha\mu+(1-\alpha)\nu$, Lemma 4.3 implies that there exists $\alpha^* \in (0,1)$ satisfying

$$\alpha^* [\alpha\mu + (1-\alpha)\nu] + (1-\alpha^*)\,\nu \sim \alpha\lambda + (1-\alpha)\nu$$

From this it follows that

$$\alpha\,[\alpha^*\mu + (1-\alpha^*)\nu] + (1-\alpha)\,\nu = \alpha\,\alpha^*\mu + (1-\alpha\,\alpha^*)\nu \sim \alpha\lambda + (1-\alpha)\nu$$

Applying Lemma 4.2(a) and condition (I) gives

$$\lambda \sim \alpha^*\mu + (1-\alpha^*)\nu \succ \alpha^*\mu + (1-\alpha^*)\lambda$$

Because $\succsim$ is transitive, Lemma 4.2(b) then yields $\lambda \succ \mu$.

Case 3: $\mu \succ \nu$. Here, by Lemma 4.1(a), $\alpha\mu + (1-\alpha)\nu \succ \nu$. So, whenever $\alpha\lambda+(1-\alpha)\nu \succ \alpha\mu+(1-\alpha)\nu$, Lemma 4.3 implies that there exists $\alpha^* \in (0,1)$ satisfying

$$\alpha^* [\alpha\lambda + (1-\alpha)\nu] + (1-\alpha^*)\,\nu \sim \alpha\mu + (1-\alpha)\nu$$

From this it follows that

$$\alpha\,[\alpha^*\lambda + (1-\alpha^*)\nu] + (1-\alpha)\,\nu = \alpha\,\alpha^*\lambda + (1-\alpha\,\alpha^*)\nu \sim \alpha\mu + (1-\alpha)\nu$$

Applying condition (I) and Lemma 4.2(a) gives

$$\alpha^*\lambda + (1-\alpha^*)\mu \succ \alpha^*\lambda + (1-\alpha^*)\nu \sim \mu$$

Because $\succsim$ is transitive, Lemma 4.2(b) then yields $\lambda \succ \mu$.

(b) Because condition (O) implies that $\succsim$ is complete, the contrapositive of part (a) implies (10) in Section 3.2, which is condition (I′). If condition (I) is also true, the result follows from Lemma 3.1. ∎

4.3 Construction of a Utility Function

We continue to assume that conditions (O), (I), and (C) are satisfied on $\Delta(Y)$. Now that Lemma 4.3 and condition (I*) have been established, one can show how to construct a utility function. In order to do so, first let $\overline{\lambda}, \underline{\lambda} \in \Delta(Y)$ be any two lotteries with $\overline{\lambda} \succ \underline{\lambda}$. Define the associated *order interval*

$$\Lambda := \{ \lambda \in \Delta(Y) \mid \overline{\lambda} \succsim \lambda \succsim \underline{\lambda} \} \tag{18}$$

LEMMA 4.5: (a) There exists a unique utility function $U : \Lambda \to \mathbb{R}$ which: (i) represents the preference relation $\succsim$ on Λ; (ii) takes the values $U(\overline{\lambda}) = 1$ and $U(\underline{\lambda}) = 0$ at $\overline{\lambda}$ and $\underline{\lambda}$ respectively; and (iii) satisfies the mixture preservation property (MP) for all $\lambda, \mu \in \Lambda$ and all $\alpha \in [0, 1]$.

(b) Suppose $V : \Lambda \to \mathbb{R}$ is any other utility function representing $\succsim$ restricted to Λ which satisfies (MP). Then there exist real constants $\rho > 0$ and δ such that $V(\lambda) = \delta + \rho\, U(\lambda)$ for all $\lambda \in \Lambda$.

PROOF: (a) For each $\lambda \in \Lambda$, Lemma 4.3 and its Corollary imply the existence of a unique $U(\lambda) \in [0, 1]$ such that

$$U(\lambda)\,\overline{\lambda} + [1 - U(\lambda)]\,\underline{\lambda} \sim \lambda \tag{19}$$

Furthermore, $\lambda \sim \overline{\lambda}$ implies $U(\lambda) = 1$, and $\lambda \sim \underline{\lambda}$ implies $U(\lambda) = 0$. Also, if $\lambda, \mu \in \Lambda$, then (19) and transitivity of $\succsim$, when combined with Lemma 4.1(c), imply that

$$\lambda \succsim \mu \iff U(\lambda)\,\overline{\lambda} + [1 - U(\lambda)]\,\underline{\lambda} \succsim U(\mu)\,\overline{\lambda} + [1 - U(\mu)]\,\underline{\lambda} \iff U(\lambda) \geq U(\mu)$$

So $U : \Lambda \to \mathbb{R}$ represents $\succsim$ on Λ. Furthermore, for all $\lambda, \mu \in \Lambda$ and $\alpha \in [0, 1]$, (19) and condition (I*) imply that

$$\begin{aligned}
\alpha\lambda + (1-\alpha)\mu &\sim \alpha\{\, U(\lambda)\,\overline{\lambda} + [1 - U(\lambda)]\,\underline{\lambda}\,\} + (1-\alpha)\mu \\
&\sim \alpha\{\, U(\lambda)\,\overline{\lambda} + [1 - U(\lambda)]\underline{\lambda}\,\} \\
&\qquad\qquad + (1-\alpha)\{\, U(\mu)\,\overline{\lambda} + [1 - U(\mu)]\,\underline{\lambda}\,\} \\
&= [\alpha U(\lambda) + (1-\alpha)U(\mu)]\overline{\lambda} + [1 - \alpha U(\lambda) - (1-\alpha)U(\mu)]\underline{\lambda}
\end{aligned}$$

Therefore, definition (19) implies that

$$U(\lambda\alpha + (1-\alpha)\mu) = \alpha U(\lambda) + (1-\alpha)U(\mu)$$

This confirms property (MP).

(b) Given $\overline{\lambda}, \underline{\lambda} \in \Delta(Y)$ and the alternative utility function $V : \Lambda \to \mathbb{R}$ representing $\succsim$, define $\delta := V(\underline{\lambda})$ and $\rho := V(\overline{\lambda}) - V(\underline{\lambda}) > 0$. If V also satisfies (MP), then definition (19) implies that

$$V(\lambda) = V\left(U(\lambda)\,\overline{\lambda} + [1-U(\lambda)]\,\underline{\lambda}\right) = U(\lambda)\,V(\overline{\lambda}) + [1-U(\lambda)]\,V(\underline{\lambda}) = \delta + \rho\,U(\lambda)$$

for all $\lambda \in \Lambda$. ∎

Lemma 4.6: There exists a utility function $U : \Delta(Y) \to \mathbb{R}$ which represents $\succsim$ and satisfies (MP). Also, if $V : \Delta(Y) \to \mathbb{R}$ is any other utility function with the same properties, then there exist real constants $\rho > 0$ and δ such that $V(\lambda) = \delta + \rho\,U(\lambda)$ for all $\lambda \in \Delta(Y)$.

Proof: The first simple case occurs when $\overline{\lambda}$ and $\underline{\lambda}$ can be chosen so that, for all $\lambda \in \Delta(Y)$, one has $\overline{\lambda} \succsim \lambda \succsim \underline{\lambda}$. In this case the order interval Λ defined in (18) is the whole space $\Delta(Y)$. There is nothing more to prove.[7]

It remains to prove the lemma even when the lotteries $\overline{\lambda}, \underline{\lambda} \in \Delta(Y)$ cannot be chosen to satisfy $\overline{\lambda} \succsim \lambda \succsim \underline{\lambda}$ for all $\lambda \in \Delta(Y)$. In this case, one can still construct the order interval Λ as in (18) and the utility function $U : \Lambda \to \mathbb{R}$ to represent $\succsim$ on Λ while satisfying $U(\overline{\lambda}) = 1$, $U(\underline{\lambda}) = 0$, and (MP). Moreover, consider any four lotteries $\mu_1, \mu_2, \nu_1, \nu_2 \in \Delta(Y)$ such that $\mu_i \succsim \overline{\lambda}$ and $\nu_i \precsim \underline{\lambda}$ for $i = 1, 2$, and let Λ_i denote the corresponding order interval $\{\, \lambda \in \Delta(Y) \mid \mu_i \succsim \lambda \succsim \nu_i \,\}$. Now, for $i = 1, 2$, Lemma 4.5(a) implies that there exists a utility function $U_i : \Lambda_i \to \mathbb{R}$ which represents $\succsim$ on Λ_i, while also satisfying $U_i(\mu_i) = 1$, $U_i(\nu_i) = 0$, and (MP).

As discussed in Section 2.4, for $i = 1, 2$, whenever $U_i^*(\lambda) \equiv \delta_i + \rho_i\,U_i(\lambda)$ for some constants $\rho_i > 0$ and δ_i, the alternative utility function $U_i^* : \Lambda_i \to \mathbb{R}$ will also represent $\succsim$ on Λ_i while satisfying (MP). Furthermore, one can make $U_i^*(\overline{\lambda}) = 1$ and $U_i^*(\underline{\lambda}) = 0$ by choosing ρ_i and δ_i to satisfy

$$\delta_i + \rho_i\,U_i(\overline{\lambda}) = 1; \quad \delta_i + \rho_i\,U_i(\underline{\lambda}) = 0$$

These equations will be satisfied provided that one chooses

$$\rho_i = 1/[U_i(\overline{\lambda}) - U_i(\underline{\lambda})] > 0; \quad \delta_i = -U_i(\underline{\lambda})/[U_i(\overline{\lambda}) - U_i(\underline{\lambda})]$$

[7] Of course, many textbooks consider only this easy case.

Then, for all $\lambda \in \Lambda_i$, one has

$$U_i^*(\lambda) = \frac{U_i(\lambda) - U_i(\underline{\lambda})}{U_i(\overline{\lambda}) - U_i(\underline{\lambda})}$$

Now, both utility functions U_1^* and U_2^* represent the same ordering $\succsim$ and also satisfy (MP) on $\Lambda_1 \cap \Lambda_2$. By Lemma 4.5(b), there must exist constants $\rho > 0$ and δ such that

$$U_2^*(\lambda) = \delta + \rho U_1^*(\lambda)$$

for all λ in the set $\Lambda_1 \cap \Lambda_2$, which obviously has Λ as a subset. But $U_1^*(\underline{\lambda}) = U_2^*(\underline{\lambda}) = 0$, implying that $\delta = 0$. Also $U_1^*(\overline{\lambda}) = U_2^*(\overline{\lambda}) = 1$, implying that $1 = \delta + \rho$ and so $\rho = 1$. This proves that $U_1^* \equiv U_2^*$ on $\Lambda_1 \cap \Lambda_2$.

So, given any $\lambda \in \Delta(Y)$, there is a *unique* $U^*(\lambda)$ which can be found by constructing U_0 to represent $\succsim$ and satisfy (MP) on any order interval Λ_0 large enough to include all three lotteries λ, $\overline{\lambda}$, and $\underline{\lambda}$, then re-normalizing by choosing constants $\rho > 0$ and δ so that the transformed utility function U^* satisfies $U^*(\lambda) \equiv \delta + \rho\, U_0(\lambda)$ on Λ_0, as well as $U^*(\overline{\lambda}) = 1$, $U^*(\underline{\lambda}) = 0$. Because U_0 satisfies (MP), so does U^*. Evidently $\lambda \succsim \mu$ iff $U^*(\lambda) \geq U^*(\mu)$ because U^* represents $\succsim$ on any order interval large enough to include all four lotteries λ, μ, $\overline{\lambda}$, and $\underline{\lambda}$.

Next, let U be any alternative utility function representing $\succsim$ on the whole of $\Delta(Y)$. Then U also represents it on any non-trivial order interval Λ_0 that includes Λ. So there exist real constants $\rho > 0$ and δ such that $U(\lambda) = \delta + \rho\, U^*(\lambda)$ throughout Λ_0, including for $\lambda = \overline{\lambda}$ and $\lambda = \underline{\lambda}$. Because $U^*(\overline{\lambda}) = 1$ and $U^*(\underline{\lambda}) = 0$, it follows that $U(\underline{\lambda}) = \delta$ and $U(\overline{\lambda}) = \delta + \rho$. Hence $\rho = U(\overline{\lambda}) - U(\underline{\lambda})$. This implies that $U(\lambda) = U(\underline{\lambda}) + [U(\overline{\lambda}) - U(\underline{\lambda})]\, U^*(\lambda)$ throughout Λ_0 and so, because Λ_0 is arbitrarily large, throughout $\Delta(Y)$. So U^* is unique up to cardinal transformations. ∎

Finally:

THEOREM 4: Suppose that conditions (O), (I), and (C) are satisfied on $\Delta(Y)$. Then there exists a unique cardinal equivalence class of NMUFs $v : Y \to \mathbb{R}$ such that

$$\lambda \succsim \mu \iff \sum\nolimits_{y \in Y} \lambda(y)\, v(y) \geq \sum\nolimits_{y \in Y} \mu(y)\, v(y)$$

PROOF: The theorem is a direct implication of Lemma 4.6, provided that $v(y)$ is defined, for each $y \in Y$, as $U(1_y)$. This is because (2) and repeated application of (MP) together imply that

$$U(\lambda) = \sum\nolimits_{y \in Y} \lambda(y) U(1_y) = \sum\nolimits_{y \in Y} \lambda(y)\, v(y)$$

Also, uniqueness of U up to cardinal transformations obviously implies that v has the same property. ∎

5 Consequentialist Foundations: Ordinality

5.1 The Consequentialist Principle

There has been frequent criticism of the three axioms (O), (I), and (C) which, as Sections 3 and 4 showed, characterize EU maximization. It is clear that (O) and (I) in particular describe actual behaviour rather poorly — see, for example, the other chapters in this *Handbook* on non-expected utility and on experiments. Utility theorists, however, should be no less interested in whether these axioms are an acceptable basis of normative behaviour. This section and the next will show that at least (O) and (I) — indeed, even condition (I*) — can be derived from another, more fundamental, normative principle of behaviour. This is the "consequentialist principle", requiring behaviour to be entirely explicable by its consequences.

As in Section 2, let Y denote a *consequence domain*, and $\Delta(Y)$ the set of all simple lotteries on Y. It will be convenient to regard the members of $\Delta(Y)$ as *random consequences*, to be distinguished from *elementary consequences* $y \in Y$.

The *consequentialist principle* is that, given behaviour β and any non-empty finite *feasible set* $F \subset \Delta(Y)$, the set of all possible random consequences of behaviour, or the *behaviour set*, should be the non-empty "revealed" or *implicit choice set* $C_\beta(F) \subset F$ that depends only on F. In other words, changes in the structure of the decision problem should have no bearing on the possible consequences $C_\beta(F)$ of behaviour β, unless they change the feasible set F. At first, this restriction seems very weak, putting no restrictions at all on the *consequentialist choice function* C_β mapping feasible sets F into choice subsets $C_\beta(F) \subset F$.

This consequentialist principle has a long and controversial history, especially in moral philosophy. Aristotle can be read as suggesting that the moral worth of an act depended on its results or consequences. St. Thomas Aquinas explicitly attacked this doctrine, suggesting that good acts could remain good even if they had evil consequences, and that evil acts would remain evil even if they happened to have good consequences. Later philosophical writers, especially John Stuart Mill and G.E. Moore, then enunciated clearly a doctrine or principle that flew in the face of Aquinas' assertion, claiming that acts should be judged by their consequences. This doctrine was given the name "consequentialism" by Elisabeth Anscombe (1958), in a critical article. To this day it remains controversial in moral philosophy.[8]

In decision theory, Arrow (1951) stated the principle that acts should be valued by their consequences. Savage (1954) defined an act as a mapping from states of the world to consequences, thus espousing consequentialism implicitly.

5.2 Simple Finite Decision Trees

Consequentialism acquires force only in combination with other axioms, or else when applied in a rather obvious way to decision problems that can be described by decision trees in a sufficiently unrestricted domain. This is my next topic.

A *simple finite decision tree* is a list

$$T = \langle\, N, N^*, X, n_0, N_{+1}(\cdot), \gamma(\cdot)\,\rangle \tag{20}$$

in which:

(i) N is the finite set of *nodes*;

(ii) N^* is the finite subset of *decision nodes*;

(iii) X is the finite subset of *terminal nodes*;

(iv) n_0 is the unique *initial node*;

(v) $N_{+1}(\cdot) : N \twoheadrightarrow N$ is the *immediate successor correspondence*;

(vi) $\gamma : X \to \Delta(Y)$ is the *consequence mapping* from terminal nodes to their lottery consequences.[9]

[8] In Hammond (1986, 1996) I have discussed the origins of consequentialism at somewhat greater length, and provided references for the writings cited above, as well as some others. The same issues also receive attention in the contribution by d'Aspremont and Mongin to this *Handbook*.

[9] One could well argue that, strictly speaking, lotteries can only arise in decision trees having chance nodes. Thus, in what I am calling a "simple" decision tree, each $x \in X$ is not

Obviously, N is partitioned into the two disjoint sets N^* and X. The set X must always have at least one member for every finite decision tree. For non-trivial decision trees, both N^* and X are non-empty. But in case $n_0 \in X$, then $X = \{n_0\}$ and $N^* = \emptyset$. Generally, of course, $n \in X \iff N_{+1}(n) = \emptyset$.

In order that N should have a tree structure, there must be a partition of N into a finite collection of pairwise disjoint non-empty subsets N_k $(k = 0, 1, \ldots K)$ such that each N_k consists of all the nodes that can be reached after exactly k steps from n_0 through the decision tree. Thus

$$N_0 = \{n_0\} \quad \text{and} \quad N_k = \bigcup_{n \in N_{k-1}} N_{+1}(n) \quad (k = 1, 2, \ldots K)$$

Evidently the last set N_K is a subset of X, but X can also include some members of the sets N_k with $k < K$.

Similarly, starting with any node $n \in N_k$ $(k = 1, 2, \ldots K)$, define

$$N_k(n) := \{n\} \quad \text{and} \quad N_r(n) = \bigcup_{n' \in N_{r-1}(n)} N_{+1}(n') \quad (r = k+1, k+2, \ldots, K(n))$$

where $K(n) + 1$ is the maximum number of nodes, including the initial node, on any path in T that passes through n. Then $N(n) := \cup_{r=k}^{K(n)} N_r(n)$ is the set of nodes that *succeed* n in the tree T.

Given the consequence domain Y, let $\mathcal{T}(Y)$ denote the collection of all simple finite decision trees given by (20).

5.3 Behaviour and Unrestricted Domain

In any simple finite decision tree $T \in \mathcal{T}(Y)$, at any decision node $n \in N^*$, the agent's *behaviour* or a possible course of action is described by the set $\beta(T, n)$ satisfying

$$\emptyset \neq \beta(T, n) \subset N_{+1}(n) \tag{21}$$

Thus, any decision in $\beta(T, n)$ takes the form of a move to a node $n' \in N_{+1}(n)$ that immediately succeeds n. When $\beta(T, n)$ is multi-valued, this captures the

really a terminal node, but a chance node where a lottery determines which terminal node and which elementary consequence $y \in Y$ will result from earlier decisions. However, later proofs are facilitated by allowing terminal nodes to have random consequences, which is why I have chosen this formulation.

idea that there is no good reason to choose one move rather than another among the different members of $\beta(T, n)$.

The *unrestricted domain* assumption is that $\beta(T, n)$ is defined, and satisfies (21), whenever T is a simple finite decision tree in $\mathcal{T}(Y)$ with n as one of its decision nodes. Note that $\beta(T, n)$ must be defined even at nodes that cannot be reached given earlier behaviour in T.

5.4 Continuation Subtrees and Dynamic Consistency

Let T be any simple finite decision tree, as in (20), and $n \in N$ any node of T. Then there exists a *continuation subtree*

$$T(n) = \langle\, N(n), N^*(n), X(n), n, N_{+1}(\cdot), \gamma(\cdot)\,\rangle$$

with initial node n, and with $N(n)$ consisting of all nodes that succeed n in T. Moreover, $N^*(n) = N(n) \cap N^*$ is the set of decision nodes in $T(n)$, whereas $X(n) = N(n) \cap X$ is the set of terminal nodes. Also, $N_{+1}(\cdot)$ and $\gamma(\cdot)$ in the subtree $T(n)$ are the restrictions to $N(n)$ and $X(n)$ respectively of the same correspondence and mapping in the tree T. In fact, if $n \in N_{k(n)}$, then $N(n) = \cup_{k=0}^{K(n)} N_k(n)$ where

$$\begin{aligned} N_0(n) &= \{n\} = N(n) \cap N_{k(n)} \\ \text{and} \quad N_k(n) &= \bigcup_{n' \in N_{k-1}(n)} N_{+1}(n') = N(n) \cap N_{k(n)+k} \quad (k = 1 \text{ to } K(n)) \end{aligned}$$

Let $n' \in N^*(n)$ be any decision node in the subtree $T(n)$. In this case $\beta(T(n), n')$ describes behaviour at node n', but so does $\beta(T, n')$. It should make no difference whether n' is regarded as belonging to the whole tree T, or only to the subtree $T(n)$. In fact, because $T(n')$ is the relevant decision tree at node n', behaviour there will ultimately be determined at the very last minute by $\beta(T(n'), n')$. This motivates the *dynamic consistency* assumption that

$$\beta(T, n') = \beta(T(n), n') \text{ whenever } n' \in N^*(n)$$

Note that this is a behavioural dynamic consistency condition, requiring consistency between behaviour at the same decision node of a subtree and of the whole tree. It is quite different from dynamic consistency of planned behaviour, of preferences, or of choice. It also differs from consistency between planned and actual behaviour. These differences have been the source of some misunderstanding which is further discussed at the end of Section 5.5.

5.5 Consequentialism in Simple Decision Trees

Applying the consequentialist principle to simple decision trees requires the feasible set of consequences $F(T)$ to be determined, for each simple finite decision tree T, as well as the *behaviour set* $\Phi_\beta(T)$ of possible consequences of behaviour. Both these sets can be found by backward recursion, starting with terminal nodes and then proceeding in reverse to the initial node. Thus, we can calculate successive sets $F(T,n)$ and $\Phi_\beta(T,n)$ for all $n \in N$ by starting from each $n \in X$ and working backwards, until in the end we are able to define $F(T)$ as $F(T,n_0)$ and $\Phi_\beta(T)$ as $\Phi_\beta(T,n_0)$.

Indeed, at any terminal node $x \in X$, there is no alternative to the uniquely specified consequence $\gamma(x) \in \Delta(Y)$. Because $F(T)$ and $\Phi_\beta(T)$ will be *sets* of consequence lotteries, however, rather than single consequence lotteries, it is natural to construct

$$F(T,x) := \Phi_\beta(T,x) := \{\gamma(x)\} \text{ (all } x \in X) \tag{22}$$

Consider any decision node $n \in N^*$. Suppose that for all $n' \in N_{+1}(n)$ the two sets $F(T,n')$ and $\Phi_\beta(T,n')$ have already been constructed and, as the induction hypothesis, that they satisfy

$$\emptyset \neq \Phi_\beta(T,n') \subset F(T,n') \tag{23}$$

After reaching n, by moving to an appropriate node $n' \in N_{+1}(n)$, any consequence in $F(T,n')$ can be made feasible. Hence, $F(T,n)$ is the union of all such sets. But behaviour β allows only moves to nodes $n' \in \beta(T,n)$. Hence, only consequences in $\Phi_\beta(T,n')$ for some $n' \in \beta(T,n)$ can result from behaviour β. Therefore we construct

$$F(T,n) := \bigcup_{n' \in N_{+1}(n)} F(T,n'); \quad \Phi_\beta(T,n) := \bigcup_{n' \in \beta(T,n)} \Phi_\beta(T,n') \tag{24}$$

This construction is essentially the process which LaValle and Wapman (1986) call "rolling back" the decision tree. Now, it is obvious from (23) that

$$\emptyset \neq \Phi_\beta(T,n) \subset F(T,n) \tag{25}$$

This confirms that the induction hypothesis also holds at n. So, by backward induction, (25) holds at any node $n \in N$.

Recalling that $F(T) = F(T,n_0)$ and also $\Phi_\beta(T) = \Phi_\beta(T,n_0)$ in any simple decision tree T, (25) implies in particular

$$\emptyset \neq \Phi_\beta(T) \subset F(T) \tag{26}$$

The *consequentialist axiom* then requires the existence of an implicit choice function C_β such that, for all simple finite decision trees T, one has

$$\Phi_\beta(T) = C_\beta(F(T)) \tag{27}$$

In particular, (27) is satisfied if and only if, for any pair of decision trees T, T' with $F(T) = F(T')$, it is true that $\Phi_\beta(T) = \Phi_\beta(T')$. That is, for trees with identical feasible sets of consequences, there must also be identical behaviour sets of consequences.

At this stage it may be useful to note the distinction between (general) behaviour and consequentialist choice. The former is simply a description of what moves might be made at each decision node of a decision tree. It need not involve, even implicitly, any conscious or even unconscious process of choice. By contrast, consequentialism restricts behaviour to yield the same (random) consequences as if there were a conscious choice mechanism making selections from the relevant feasible set.

Now, given any non-empty finite set F, it is easy to construct a decision tree T such that $F(T) = F$. Indeed, it suffices to construct a simple one-stage or "reduced" decision tree T with components as in (20) specified by

$$N^* = \{n_0\};\ X = N_{+1}(n_0) = \{\, x_\lambda \mid \lambda \in F \subset \Delta(Y) \,\};\ \gamma(x_\lambda) = \lambda \ (\text{all } \lambda \in F) \tag{28}$$

Then, because of (26) and (27), consequentialism implies that

$$\emptyset \neq C_\beta(F) \subset F \tag{29}$$

At this stage, some interesting criticisms due to Machina (1989), McClennen (1990) and others should be noted. Recall that dynamic consistency of behaviour requires the behaviour sets $\beta(T, n')$ and $\beta(T(n), n')$ to be the same at all decision nodes $n' \in N^*(n)$, regardless of whether the subtree $T(n)$ is regarded as part of the whole tree T or not. Then, an obvious implication of consequentialism is that the consequences $\Phi_\beta(T, n) = \Phi_\beta(T(n))$ of behaviour in the subtree $T(n)$ must depend only on the set $F(T, n) = F(T(n))$ of consequences that are feasible in the subtree. Machina and McClennen call this property "separability." They dispute the property, however, and claim that it makes an important difference whether a decision node $n' \in N^*(n)$ is treated as a part of T or of $T(n)$. Their argument is that consequences that occur in T but not in $T(n)$ may after all be relevant to behaviour in $T(n)$, in which case both consequentialism and this separability property are violated.[10] But

[10] Some writers seem to suggest that it is dynamic consistency which is violated. Indeed, unless this dependence of preferences on consequences in $T \setminus T(n)$ is foreseen, there is a

if consequences outside the subtree $T(n)$ are relevant to decisions within the tree, it seems clear that consequences have not been adequately described, and that the consequence domain Y needs to be extended. As Munier (1996) in particular has pointed out, an important issue which then arises is whether in practice the consequence domain can be enriched in a way that allows all behaviour to be explained without at the same time making the unrestricted domain assumption implausible.

5.6 Consequentialism Implies Ordinality

Given behaviour β, define the implicit *weak preference* relation R_β on $\Delta(Y)$ by

$$\lambda \ R_\beta \ \mu \iff \lambda \in C_\beta(\{\,\lambda, \mu\,\}) \tag{30}$$

for all pairs $\lambda, \mu \in \Delta(Y)$. Impose also the reflexivity condition that $\lambda \ R_\beta \ \lambda$ for all $\lambda \in \Delta(Y)$. Thus, R_β is the reflexive binary preference relation revealed by behaviour in decision trees with only a pair of feasible consequence lotteries. Let I_β be the associated *indifference relation* defined by

$$\lambda \ I_\beta \ \mu \iff [\lambda \ R_\beta \ \mu \ \& \ \mu \ R_\beta \ \lambda] \iff C_\beta(\{\,\lambda, \mu\,\}) = \{\,\lambda, \mu\,\}$$

and let P_β be the associated *strict preference* relation defined by

$$\lambda \ P_\beta \ \mu \iff [\lambda \ R_\beta \ \mu \ \& \text{ not } \mu \ R_\beta \ \lambda] \iff C_\beta(\{\,\lambda, \mu\,\}) = \{\lambda\}$$

The following is a striking implication of three axioms set out in Sections 5.3, 5.4 and 5.5 respectively:

Theorem 5 (Consequentialism Implies Ordinality): Let behaviour β be defined for an unrestricted domain of simple finite decision trees, and satisfy both dynamic consistency and consequentialism. Then the implicit preference relation R_β is a (complete and transitive) preference ordering on $\Delta(Y)$. Moreover, the implicit choice function is *ordinal* — i.e., for every non-empty finite set $F \subset \Delta(Y)$, one has

$$C_\beta(F) = \{\,\lambda \in F \mid \mu \in F \Longrightarrow \lambda \ R_\beta \ \mu\,\} \tag{31}$$

---

sense in which both preferences and choice may have to be dynamically inconsistent — cf. Hammond (1976, 1988c). But I prefer to consider dynamic consistency of *behaviour*, which is satisfied virtually automatically. So the criticisms apply to the consequentialist hypothesis and the implied property of separability, rather than to the behavioural dynamic consistency condition.

For a general choice space, this result was first proved in Hammond (1977) by using Arrow's (1959) characterization of an ordinal choice function. The direct proof provided here seems preferable. It proceeds by way of four lemmas:

LEMMA 5.1: For any finite subset $F \subset \Delta(Y)$, if $\lambda \in C_\beta(F)$ and $\mu \in F$, then $\lambda \; R_\beta \; \mu$.

LEMMA 5.2: For any finite subset $F \subset \Delta(Y)$, if $\lambda \in C_\beta(F)$ and $\mu \in F$ with $\mu \; R_\beta \; \lambda$, then $\mu \in C_\beta(F)$.

LEMMA 5.3: Equation (31) is true for any non-empty finite $F \subset \Delta(Y)$.

LEMMA 5.4: The binary relation defined by (30) is a (complete and transitive) preference ordering.

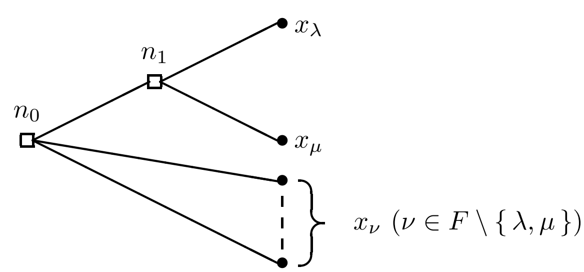

Figure 2 Decision Tree Illustrating Ordinality

Lemmas 5.1 and 5.2 are both proved by considering a particular decision tree T, as illustrated in Figure 2. The components specified in (20) are given by:

$$\begin{aligned} N^* &= \{ n_0, n_1 \}; \quad X \; = \; \{ x_\nu \mid \nu \in F \}; \\ N_{+1}(n_0) &= \{ n_1 \} \cup \{ x_\nu \mid \nu \in F \setminus \{ \lambda, \mu \} \}; \quad N_{+1}(n_1) \; = \; \{ x_\lambda, x_\mu \}; \\ \gamma(x_\nu) &= \nu \text{ (all } \nu \in F) \end{aligned} \tag{32}$$

In fact, for any non-empty finite subset F of $\Delta(Y)$ with $\lambda, \mu \in F$, the finite tree T can be constructed to satisfy (32). By the unrestricted domain hypothesis, $\emptyset \neq \beta(T, n) \subset N_{+1}(n)$ for $n \in \{ n_0, n_1 \}$. Also, $F(T, x_\nu) = \Phi_\beta(T, x_\nu) = \{\nu\}$ for all $\nu \in F$, by (22). Then the construction (24) gives

$$\begin{aligned} F(T(n_1)) &= F(T, n_1) \; = \; F(T, x_\lambda) \cup F(T, x_\mu) \; = \; \{ \lambda, \mu \}; \\ F(T) &= F(T, n_0) \; = \; F(T, n_1) \cup \left[\cup_{\nu \in F \setminus \{ \lambda, \mu \}} F(T, x_\nu) \right] \\ &= \{ \lambda, \mu \} \cup [F \setminus \{ \lambda, \mu \}] \; = \; F \end{aligned}$$

Now, $\lambda \in C_\beta(F)$ implies $\lambda \in \Phi_\beta(T)$. But construction (24) implies that

$$\begin{aligned} \Phi_\beta(T(n_1)) &= \Phi_\beta(T, n_1) = \bigcup_{n' \in \beta(T,n_1)} \Phi_\beta(T, n') = \{\, \nu \mid x_\nu \in \beta(T, n_1)\,\}; \\ \Phi_\beta(T) &= \Phi_\beta(T, n_0) = \bigcup_{n' \in \beta(T,n_0)} \Phi_\beta(T, n') \end{aligned} \tag{33}$$

Because $\lambda \notin \Phi_\beta(T, n')$ for all $n' \in N_{+1}(n_0) \setminus \{n_1\}$, it follows from $\lambda \in C_\beta(F)$ that $n_1 \in \beta(T, n_0)$ and also that $\lambda \in \Phi_\beta(T, n_1)$. The latter implies that $x_\lambda \in \beta(T, n_1)$.

Proof of Lemma 5.1: Because $x_\lambda \in \beta(T, n_1)$, it follows from dynamic consistency that $x_\lambda \in \beta(T(n_1), n_1)$. But $F(T(n_1)) = \{\, \lambda, \mu\,\}$, whereas $\Phi_\beta(T(n_1)) = \Phi_\beta(T, n_1) \ni \lambda$. From this one has $\lambda \in C_\beta(F(T(n_1))) = C_\beta(\{\,\lambda, \mu\,\})$. So $\lambda \; R_\beta \; \mu$, by definition (30) of R_β. ∎

Proof of Lemma 5.2: Suppose also that $\mu \; R_\beta \; \lambda$. Then $\mu \in C_\beta(\{\,\lambda, \mu\,\}) = \Phi_\beta(T(n_1)) = \Phi_\beta(T, n_1)$. Because it has already been proved that $n_1 \in \beta(T, n_0)$, one has

$$C_\beta(F) = \Phi_\beta(T) = \Phi_\beta(T, n_0) = \bigcup_{n' \in \beta(T,n_0)} \Phi_\beta(T, n') \supset \Phi_\beta(T, n_1) \ni \mu$$

as an implication of (33). ∎

Proof of Lemma 5.3: By Lemma 5.1,

$$C_\beta(F) \subset \{\, \lambda \in F \mid \mu \in F \Longrightarrow \lambda \; R_\beta \; \mu\,\} \tag{34}$$

Conversely, by (29) there must exist $\nu \in C_\beta(F)$. Suppose now that $\lambda \in F$ and $\lambda \; R_\beta \; \mu$ for all $\mu \in F$. Then $\lambda \; R_\beta \; \nu$ in particular. Therefore, by Lemma 5.2 with ν and λ replacing λ and μ respectively, it follows that $\lambda \in C_\beta(F)$. This proves that

$$\{\, \lambda \in F \mid \mu \in F \Longrightarrow \lambda \; R_\beta \; \mu\,\} \subset C_\beta(F)$$

Together with (34), this confirms (31). ∎

Proof of Lemma 5.4 — Completeness of R_β: Given any pair $\lambda, \mu \in \Delta(Y)$, the choice set $C_\beta(\{\,\lambda, \mu\,\})$ cannot be empty because of (29). Hence, either $\lambda \in C_\beta(\{\,\lambda, \mu\,\})$, in which case definition (30) implies that $\lambda \; R_\beta \; \mu$, or alternatively $\mu \in C_\beta(\{\,\lambda, \mu\,\})$, in which case (30) implies $\mu \; R_\beta \; \lambda$.

— *Transitivity of R_β:* Suppose $\lambda, \mu, \nu \in \Delta(Y)$ and $\lambda \; R_\beta \; \mu$, $\mu \; R_\beta \; \nu$. Let $F := \{\lambda, \mu, \nu\,\}$. By (29), $C_\beta(F)$ is non-empty. This leaves three possible cases (which need not be mutually exclusive, however):

(i) If $\lambda \in C_\beta(F)$, then Lemma 5.1 implies that $\lambda \; R_\beta \; \nu$.

(ii) If $\mu \in C_\beta(F)$, then $\lambda \; R_\beta \; \mu$ and Lemma 5.2 imply that $\lambda \in C_\beta(F)$, so case (i) applies.

(iii) If $\nu \in C_\beta(F)$, then $\mu \; R_\beta \; \nu$ and Lemma 5.2 imply that $\mu \in C_\beta(F)$, so case (ii) applies.

Therefore, case (i) always applies. So $\lambda \; R_\beta \; \nu$, as required. ∎

6 Consequentialist Foundations: Independence

6.1 Finite Decision Trees with Chance Nodes

Section 5 restricted attention to "simple" finite decision trees, whose non-terminal nodes are all decision nodes. Here the domain of decision trees will be expanded to include trees with a set N^0 of *chance nodes*. Thus, the set N can be partitioned into the three pairwise disjoint subsets N^*, N^0 and X, where as before N^* denotes the set of decision nodes and X the set of terminal nodes. The other new feature will be that (objective) *transition probabilities* $\pi(n'|n)$ are specified for every chance node $n \in N^0$ and for every immediate successor $n' \in N_{+1}(n)$. In particular, the non-negative real numbers $\pi(n'|n)$ $(n' \in N_{+1}(n))$ should satisfy $\sum_{n' \in N_{+1}(n)} \pi(n'|n) = 1$ and so represent a (simple) probability distribution in $\Delta(N_{+1}(n))$.

As before, behaviour β will be described by the sets $\beta(T, n)$ for each finite decision tree T and each decision node $n \in N^*$ of T. The *unrestricted domain* assumption will be that $\beta(T, n)$ is defined as a non-empty subset of $N_{+1}(n)$ for all such pairs (T, n). But there will also be reason to invoke the *almost unrestricted domain* assumption. This requires $\beta(T, n)$ to be specified only for decision trees T in which $\pi(n'|n) > 0$ for all $n' \in N_{+1}(n)$ at any chance node $n \in N^0$. To some extent, this assumption can be justified by requiring all parts of a decision tree that can be reached with only zero probability to be "pruned" off. Such pruning makes good sense in single-person decision theory. But, as discussed in Hammond (1994, 1997), it has unacceptable implications in multi-person game theory.

6.2 Consequentialism in Finite Decision Trees

In this extended domain of finite decision trees that may include chance nodes, most of the analysis set out in Section 5 remains valid. However, rules (22) and (24) in Section 5.5 for calculating the sets $F(T,n)$ and $\Phi_\beta(T,n)$ by backward recursion need supplementing, in order to deal with chance nodes.

First, given subsets $S_i \subset \Delta(Y)$ and non-negative numbers α_i $(i = 1, 2, \ldots, k)$ with $\sum_{i=1}^k \alpha_i = 1$, define

$$\sum_{i=1}^k \alpha_i \, S_i = \{ \, \lambda \in \Delta(Y) \mid \exists \lambda_i \in S_i \ (i = 1, 2, \ldots, k) : \lambda = \sum_{i=1}^k \alpha_i \, \lambda_i \, \}$$

as the corresponding set of convex combinations or probability mixtures. Then, for all $n \in N^0$, construct the two sets

$$F(T,n) \quad := \quad \sum_{n' \in N_{+1}(n)} \pi(n'|n) \, F(T,n') \qquad (35)$$

$$\Phi_\beta(T,n) \quad := \quad \sum_{n' \in N_{+1}(n)} \pi(n'|n) \, \Phi_\beta(T,n') \qquad (36)$$

This construction can be explained as follows. Suppose that $\lambda(n') \in F(T,n')$ $(n' \in N_{+1}(n))$ is any collection of lotteries each of which will be feasible after reaching n, provided that chance selects the appropriate node n'. So the *compound lottery* in which first n' is selected with probabilities $\pi(n'|n)$ $(n' \in N_{+1}(n))$ and then a consequence y is selected with probabilities $\lambda(n')(y)$ $(y \in Y)$ must also be feasible. Now we invoke what Luce and Raiffa (1957, p. 26) call the *reduction of compound lotteries* assumption, originally due to von Neumann and Morgenstern (1953, Axiom (3:C:b), p. 26).[11] This states that the above compound lottery reduces to the single lottery $\lambda \in \Delta(Y)$ with $\lambda = \sum_{n' \in N_{+1}(n)} \pi(n'|n)\lambda(n')$. Then (35) says that this λ belongs to the feasible set $F(T,n)$ at node n. Conversely, (35) requires that, whenever $\lambda \in F(T,n)$, this can only be because there exists a collection of lotteries $\lambda(n') \in F(T,n')$ $(n' \in N_{+1}(n))$ such that $\lambda = \sum_{n' \in N_{+1}(n)} \pi(n'|n)\lambda(n')$. Similarly for $\Phi_\beta(T,n)$, the set of possible consequences of behaviour. See (37) below for a specific example.

[11] Actually, von Neumann and Morgenstern (p. 28) link this axiom to the absence of "utility for gambling", and write (p. 632) that the "axiom expresses the combination rule for multiple chance alternatives, and it is plausible, that a specific utility or disutility for gambling can only exist if this simple combination rule is abandoned." Luce and Raiffa (p. 26) concur with this interpretation, and also write of there being no "pleasure of suspense". Consequentialism requires that, if it is normatively appropriate for feelings of suspense or the excitement of gambling to affect decision-making, then such psychological variables should be included as part of each possible consequence.

Consider the restricted domain of simple decision trees without any chance nodes. Arguing as in Section 5, there must exist an implicit preference ordering R_β on the domain $\Delta(Y)$ of consequence lotteries such that, for all simple decision trees T, one has

$$C_\beta(F(T)) = \{\, \lambda \in F(T) \mid \mu \in F(T) \Longrightarrow \lambda \; R_\beta \; \mu \,\}$$

6.3 Consequentialism Implies Independence

Once chance nodes are allowed into decision trees, however, ordinality is not the only implication of the three axioms set out in Sections 5.3, 5.4 and 5.5 respectively. The strong independence condition (I*) is as well. To see this, consider the particular decision tree T illustrated in Figure 3, with:

$$\begin{aligned} N^0 &= \{\, n_0\};\; N^* \;=\; \{n_1\};\; X \;=\; \{\, x_\lambda, x_\mu, x_\nu\}; \\ N_{+1}(n_0) &= \{\, n_1, x_\nu \,\};\; N_{+1}(n_1) \;=\; \{\, x_\lambda, x_\mu \,\}; \\ \pi(n_1|n_0) &= \alpha;\; \pi(x_\nu|n_0) \;=\; 1-\alpha; \\ \gamma(x_\lambda) &= \lambda;\; \gamma(x_\mu) \;=\; \mu;\; \gamma(x_\nu) \;=\; \nu \end{aligned}$$

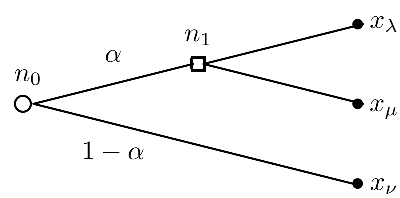

Figure 3 Decision Tree Illustrating Independence

Constructions (22), (24), (35) and (36) imply that:

$$F(T, x_\rho) = \Phi_\beta(T, x_\rho) = \{\rho\} \quad (\rho \in \{\, \lambda, \mu, \nu \,\}); \qquad F(T, n_1) = \{\, \lambda, \mu \,\}$$

and also that

$$\begin{aligned} F(T) &= F(T, n_0) \;=\; \alpha F(T, n_1) + (1-\alpha) F(T, x_\nu) \\ &= \alpha\{\, \lambda, \mu \,\} + (1-\alpha)\{\nu\} \\ &= \{\, \alpha\lambda + (1-\alpha)\nu,\; \alpha\mu + (1-\alpha)\nu \,\}; \qquad (37) \\ \Phi_\beta(T) &= \Phi_\beta(T, n_0) \;=\; \alpha\Phi_\beta(T, n_1) + (1-\alpha)\Phi_\beta(T, x_\nu) \\ &= \alpha\Phi_\beta(T, n_1) + (1-\alpha)\{\nu\} \end{aligned}$$

Therefore

$$\begin{aligned}
\lambda\ R_\beta\ \mu \quad &\Longleftrightarrow \quad \lambda \in C_\beta(\{\,\lambda,\mu\,\}) = C_\beta(F(T,n_1)) = \Phi_\beta(T,n_1) \\
&\Longleftrightarrow \quad \alpha\lambda + (1-\alpha)\nu \in \alpha\Phi_\beta(T,n_1) + (1-\alpha)\{\nu\} = \Phi_\beta(T,n_0) \\
&\Longleftrightarrow \quad \alpha\lambda + (1-\alpha)\nu \in C_\beta(F(T,n_0)) \qquad (38) \\
&\Longleftrightarrow \quad \alpha\lambda + (1-\alpha)\nu\ R_\beta\ \alpha\mu + (1-\alpha)\nu
\end{aligned}$$

This is the strong independence condition (I*), but with the crucial difference that (38) is valid even when $\alpha = 0$.

Indeed, if $\alpha = 0$ really were allowed in the decision tree T illustrated in Figure 3, then (38) would imply that

$$\lambda\ R_\beta\ \mu \iff \nu\ R_\beta\ \nu$$

But Theorem 5 says that R_β is a preference ordering, so $\nu\ R_\beta\ \nu$ is always true because R_β is reflexive. From this it follows that $\lambda\ R_\beta\ \mu$ (and also $\mu\ R_\beta\ \lambda$) for all $\lambda, \mu \in \Delta(Y)$. The implication is that *all* lotteries in $\Delta(Y)$ must be indifferent, and that $\Phi_\beta(T) = F(T)$ in any finite decision tree T. Behaviour must be entirely insensitive.

This explains why it seems natural to invoke only the assumption of an *almost unrestricted domain*, and so to exclude decision trees having any zero transition probability at a chance node. Then (38) applies only when $0 < \alpha < 1$, or trivially when $\alpha = 1$, and so (38) becomes exactly the same as the strong independence condition (I*).

Finally, it should be noted that essentially the same argument for the independence axiom was advanced by LaValle and Wapman (1986). A related argument can also be found in Samuelson (1988). But all these authors *postulate* the existence of a preference ordering, whereas the consequentialist axioms presented here *imply* both the existence of a preference ordering and the independence axiom.

6.4 Ordinality and Strong Independence Characterize Consequentialism

So far, it has been shown that consequentialist and dynamically consistent behaviour on an almost unrestricted domain of finite decision trees must maximize an implicit preference ordering satisfying the strong independence condition (I*). A converse of this result is also true. Given any preference ordering

$\succsim$ on $\Delta(Y)$ satisfying condition (I*), there exists consequentialist and dynamically consistent behaviour β on an almost unrestricted domain of finite decision trees such that the implicit preference ordering R_β is identical to the given ordering $\succsim$.

To show this, for any given finite decision tree T, first construct the feasible sets $F(T, n)$ by backward recursion as in (22), (24) and (35). Second, for each $n \in N$, construct the set of preference maximizing lotteries

$$\Psi(T, n) := \{\, \lambda \in F(T, n) \mid \mu \in F(T, n) \Longrightarrow \lambda \succsim \mu \,\} \tag{39}$$

This set is non-empty because $F(T, n)$ is finite and $\succsim$ is an ordering. Next, for any decision node $n \in N^*$, let $\beta(T, n)$ be any subset of $N_{+1}(n)$ with the property that

$$\bigcup_{n' \in \beta(T,n)} \Psi(T, n') = \Psi(T, n) \tag{40}$$

Such a set always exists, because one can obviously put

$$\beta(T, n) = \beta^*(T, n) := \{\, n' \in N_{+1}(n) \mid \exists \lambda \in F(T, n') : \lambda \in \Psi(T, n) \,\} \tag{41}$$

In fact, $\beta^*(T, n)$ is the largest set satisfying (40). But $\beta(T, n)$ can be any non-empty set such that $\cup_{n' \in \beta(T,n)} \Psi(T, n') = \cup_{n' \in \beta^*(T,n)} \Psi(T, n')$. However, in order for $n' \in \beta^*(T, n) \setminus \beta(T, n)$ to be possible, it must be true that $\Psi(T, n') \subset \cup_{\tilde{n} \in \beta^*(T,n) \setminus \{n'\}} \Psi(T, \tilde{n})$. Thus, some consequences as good as those in $\Psi(T, n')$ must still be available even if the decision-maker refuses to move to n'.

Now, constructing the behaviour sets $\Phi_\beta(T, n)$ by backward recursion as in (22), (24) and (36) yields the following:

LEMMA 6.1: At any decision node $n \in N$ of any finite decision tree T, one has $\Phi_\beta(T, n) = \Psi(T, n)$, so that β is consequentialist and $R_\beta = \succsim$.

PROOF: The proof will proceed by backward induction. First, at any terminal node $x \in X$, (22) implies that $\Psi(T, x) = \Phi_\beta(T, x) = F(T, x) = \{\gamma(x)\}$.

Consider any non-terminal node $n \in N \setminus X$. As the induction hypothesis, suppose that $\Phi_\beta(T, n') = \Psi(T, n')$ is true at all nodes $n' \in N_{+1}(n)$. Then two different cases must be considered:

Case 1: $n \in N^*$. Here n is a decision node. Because of (24) and the induction hypothesis, it follows from (40) that

$$\Phi_\beta(T, n) = \bigcup_{n' \in \beta(T,n)} \Phi_\beta(T, n') = \bigcup_{n' \in \beta(T,n)} \Psi(T, n') = \Psi(T, n)$$

Case 2: $n \in N^0$. Here n is a chance node. So, by (36) and the induction hypothesis,

$$\Phi_\beta(T,n) = \sum\nolimits_{n' \in N_{+1}(n)} \pi(n'|n)\, \Phi_\beta(T,n') = \sum\nolimits_{n' \in N_{+1}(n)} \pi(n'|n)\, \Psi(T,n') \tag{42}$$

Now suppose that $\lambda \in \Phi_\beta(T,n)$. Consider any $\mu \in F(T,n)$. Then, for all $n' \in N_{+1}(n)$, (35) and (36) imply that there exist $\lambda(n') \in \Phi_\beta(T,n')$ and $\mu(n') \in F(T,n')$ satisfying

$$\lambda = \sum\nolimits_{n' \in N_{+1}(n)} \pi(n'|n)\, \lambda(n'), \quad \mu = \sum\nolimits_{n' \in N_{+1}(n)} \pi(n'|n)\, \mu(n') \tag{43}$$

By the induction hypothesis, for all $n' \in N_{+1}(n)$ one has $\lambda(n') \in \Psi(T,n')$, so $\lambda(n') \succsim \mu(n')$. Because of the hypothesis that $\succsim$ satisfies conditions (O) and (I*), Lemma 3.1 implies that it satisfies (FD*) as well. Therefore, because $\lambda(n') \succsim \mu(n')$ for all $n' \in N_{+1}(n)$, it follows that $\lambda \succsim \mu$. Because this is true for all $\mu \in F(T,n)$, one must have $\lambda \in \Psi(T,n)$.

Conversely, suppose that $\lambda \in \Psi(T,n)$. Because $\lambda \in F(T,n)$, for all $n' \in N_{+1}(n)$ there exists $\lambda(n') \in F(T,n')$ such that $\lambda = \sum_{n' \in N_{+1}(n)} \pi(n'|n)\, \lambda(n')$. Because $\lambda \in \Psi(T,n)$, for any $n' \in N_{+1}(n)$ and any $\mu(n') \in F(T,n')$ it must be true that

$$\begin{aligned} \lambda &= \pi(n'|n)\, \lambda(n') + \sum\nolimits_{n'' \in N_{+1}(n) \setminus \{n'\}} \pi(n''|n)\, \lambda(n'') \\ &\succsim \pi(n'|n)\, \mu(n') + \sum\nolimits_{n'' \in N_{+1}(n) \setminus \{n'\}} \pi(n''|n)\, \lambda(n'') \end{aligned}$$

Because $\pi(n'|n) > 0$, condition (I*) then implies that $\lambda(n') \succsim \mu(n')$. This is true for all $\mu(n') \in F(T,n')$, so $\lambda(n') \in \Psi(T,n')$. This is true for all $n' \in N_{+1}(n)$. Then (42) implies that $\lambda \in \Phi_\beta(T,n)$.

In each of these two cases, it has been proved that $\Phi_\beta(T,n) = \Psi(T,n)$. This completes the backward induction argument. ∎

6.5 Summary

The results of this section can be summarized in:

THEOREM 6:
(1) Suppose behaviour β is consequentialist and dynamically consistent for the almost unrestricted domain of finite decision trees with only positive probabilities at all chance nodes. Then β reveals a preference ordering R_β on $\Delta(Y)$ satisfying the strong independence condition (I*).

(2) Conversely, given any ordering $\succsim$ on $\Delta(Y)$ satisfying the strong independence condition (I*), there exists consequentialist behaviour β which is dynamically consistent on the almost unrestricted domain of finite decision trees with only positive probabilities such that the implicit preference ordering R_β on $\Delta(Y)$ is identical to $\succsim$.

7 Continuous Behaviour and Expected Utility

7.1 Continuous Behaviour

As shown in Section 3.3, the ordering and strong independence conditions (O) and (I*) by themselves do not imply EU maximization. So, to arrive at the EU hypothesis, the consequentialist axioms of Sections 5 and 6 need to be supplemented by a continuity condition. To state this condition, consider any infinite sequence of decision trees T^m $(m = 1, 2, \ldots)$ which are all identical except for the positive probabilities $\pi^m(n'|n)$ at each chance node $n \in N^0$. In fact, suppose that at any chance node $n \in N^0$, as $m \to \infty$ one has $\pi^m(n'|n) \to \bar{\pi}(n'|n)$ for all $n' \in N_{+1}(n)$. Let $\bar{T}$ denote the "limit tree" with the probabilities $\bar{\pi}(n'|n)$ at each chance node.

It would be usual to assume behaviour has the *closed graph property* requiring that, at any decision node n^* belonging to each of the trees T^m $(m = 1, 2, \ldots)$, whenever it is true that $n \in \beta(T^m, n^*)$ for all large m, then $n \in \beta(\bar{T}, n^*)$ in the limit tree. Because each set $N_{+1}(n)$ is finite, implying that $\Delta(N_{+1}(n))$ is compact at every chance node n, this is equivalent to upper hemi-continuity of the correspondence from probabilities to behaviour in each decision tree — the condition that is fundamental in, for instance, proving existence of Nash equilibrium in an n-person game where mixed strategies are allowed. However, there is a difficulty here because, if $\bar{\pi}(n'|n) = 0$ for some $n' \in N_{+1}(n)$, then the node n' and all its successors in the set $N(n')$ should be excluded from the tree $\bar{T}$.

Thus, the closed graph property should be weakened as follows. The limit tree $\bar{T}$ should have exactly the same structure and consequence mapping as each tree T^m in the sequence, except that any subtree following a node n' for which $\bar{\pi}(n'|n) = 0$ should be "pruned off" from the limit tree $\bar{T}$. Let $\bar{N}^*$ denote the set of decision nodes in $\bar{T}$ that remain after any necessary pruning. Now, the *continuous behaviour* condition (CB) requires that the closed graph property should hold at each remaining decision node $n^* \in \bar{N}^*$ when the limit tree $\bar{T}$ is constructed in this special way. This condition has the important implication that implicit preferences must be continuous:

LEMMA 7.1: The continuous behaviour condition (CB) implies condition (C*).

PROOF: Suppose that $\lambda, \mu, \nu \in \Delta(Y)$ with $\lambda \; P_\beta \; \mu$ and $\mu \; P_\beta \; \nu$. Recall that the stochastic monotonicity Lemma 4.1(b) is true under conditions (O) and (I) alone. So it applies to the ordering R_β and to the pair $\lambda, \nu \in \Delta(Y)$ satisfying $\lambda \; P_\beta \; \nu$. In particular, it implies that

$$\alpha'\lambda + (1-\alpha')\nu \; P_\beta \; \alpha\lambda + (1-\alpha)\nu \iff \alpha' > \alpha \tag{44}$$

Consider next the two sets

$$\begin{aligned} \overline{A} &:= \{\, \alpha \in [0,1] \mid \alpha\lambda + (1-\alpha)\nu \; R_\beta \; \mu \,\} \\ \underline{A} &:= \{\, \alpha \in [0,1] \mid \mu \; R_\beta \; \alpha\lambda + (1-\alpha)\nu \,\} \end{aligned}$$

Because of (44) and because R_β is transitive, there must exist a unique common boundary point $\alpha^* := \sup \underline{A} = \inf \overline{A}$ such that $\alpha \in \underline{A}$ whenever $0 \leq \alpha < \alpha^*$, whereas $\alpha' \in \overline{A}$ whenever $\alpha^* < \alpha' \leq 1$. Now, $0 \notin \overline{A}$ and $1 \notin \underline{A}$, so $0 < \alpha^* < 1$. Furthermore:

$$\alpha \in [0, \alpha^*) \Longrightarrow \mu \; P_\beta \; \alpha\lambda + (1-\alpha)\nu; \quad \alpha \in (\alpha^*, 1] \Longrightarrow \alpha\lambda + (1-\alpha)\nu \; P_\beta \; \mu \tag{45}$$

Consider any two convergent sequences $\underline{\alpha}^m, \overline{\alpha}^m$ $(m = 1, 2, \ldots)$ of probabilities which have the common limit α^*, while also satisfying $0 < \underline{\alpha}^m < \alpha^* < \overline{\alpha}^m < 1$ for all m. Let $\underline{T}^m, \overline{T}^m$ $(m = 1, 2, \ldots)$ be the two corresponding sequences of decision trees of the form illustrated in Figure 4, where $\alpha = \underline{\alpha}^m$ in each tree $\underline{T}^m$, but $\alpha = \overline{\alpha}^m$ in each tree $\overline{T}^m$.

The constant tree structure and consequence mapping for each of these trees are given by:

$$\begin{aligned} N^* &= \{n_0\}; \quad N^0 \; = \; \{n_1\}; \quad X \; = \; \{\, x_\lambda, x_\mu, x_\nu \,\}; \\ N_{+1}(n_0) &= \{n_1, x_\mu\,\}; \quad N_{+1}(n_1) \; = \; \{\, x_\lambda, x_\nu \,\}; \\ \gamma(x_\lambda) &= \lambda; \quad \gamma(x_\mu) \; = \; \mu; \quad \gamma(x_\nu) \; = \; \nu \end{aligned}$$

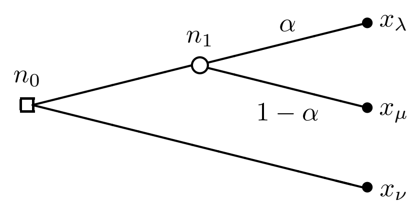

Figure 4 Decision Tree Illustrating Continuity

For each α in the set

$$A^* := \{ \underline{\alpha}^m \mid m = 1, 2, \ldots \} \cup \{ \overline{\alpha}^m \mid m = 1, 2, \ldots \}$$

the transition probabilities at the chance node n_1 are given by:

$$\pi^\alpha(x_\lambda | n_1) = \alpha; \quad \pi^\alpha(x_\nu | n_1) = 1 - \alpha$$

Because $A^* \subset (0, 1)$, the common tree structure is valid for all $\alpha \in A^*$, without any need to prune off either of the terminal nodes x_λ or x_ν. Note that for each tree T in the set

$$\{ \underline{T}^m \mid m = 1, 2, \ldots \} \cup \{ \overline{T}^m \mid m = 1, 2, \ldots \}$$

one has $F(T, x_\rho) = \{\rho\}$ for all $\rho \in \{ \lambda, \mu, \nu \}$. It follows that

$$\begin{aligned} F(T, n_1) &= \alpha\{\lambda\} + (1-\alpha)\{\nu\} = \{ \alpha\lambda + (1-\alpha)\nu \} \\ F(T) &= F(T, n_0) = F(T, n_1) \cup \{\mu\} = \{ \alpha\lambda + (1-\alpha)\nu, \mu \} \end{aligned}$$

Together with (45), the construction of the two sequences $\underline{\alpha}^m, \overline{\alpha}^m$ obviously implies that

$$\beta(\underline{T}^m, n_0) = \{x_\mu\}; \quad \beta(\overline{T}^m, n_0) = \{n_1\}$$

Evidently, applying condition (CB) to the particular tree T^* with limiting probability $\alpha^* \in (0, 1)$ gives $\beta(T^*, n_0) = \{ x_\mu, n_1 \}$. Therefore $\alpha^* \in \underline{A} \cap \overline{A}$. It follows that $\underline{A} = [0, \alpha^*]$ and that $\overline{A} = [\alpha^*, 1]$. In particular, the two sets $\underline{A}$ and $\overline{A}$ are both closed, as condition (C*) requires. ∎

7.2 Dynamic Programming and Continuous Behaviour

Let $v : Y \to \mathbb{R}$ be any NMUF. For each $\lambda \in \Delta(Y)$, let $U(\lambda) := \sum_{y \in Y} \lambda(y)\, v(y)$ denote the expected value of v. Let T be any finite decision tree. Then the *node valuation function* $w(T, \cdot) : N \to \mathbb{R}$ is calculated by backward recursion, starting with terminal nodes $x \in X$ where $w(T, x) := U(\gamma(x))$. When $n \in N^*$ is a decision node, define

$$w(T,n) := \max_{n'} \{ \, w(T,n') \mid n' \in N_{+1}(n) \, \}$$

and when $n \in N^0$ is a chance node, define

$$w(T,n) := \sum\nolimits_{n' \in N_{+1}(n)} \pi(n'|n) \, w(T,n')$$

Evidently, given the continuation subtree $T(n)$ with initial node $n \in N$, the construction by backward recursion ensures that $w(T(n), n') = w(T, n')$ whenever n' is a node of $T(n)$.

An obvious implication of the constructions (41) and of Lemma 6.1 is that, if β is behaviour which maximizes expected utility $U(\lambda)$, then

$$\emptyset \neq \beta(T,n) \subset \arg\max_{n'} \{ \, w(T,n') \mid n' \in N_{+1}(n) \, \} \tag{46}$$

at any decision node $n \in N^*$. Also, at any node $n \in N$, one has

$$\left. \begin{array}{rcl} \Phi_\beta(T,n) & = & \arg \\ w(T,n) & = & \end{array} \right\} \max_{\lambda} \{ \, U(\lambda) \mid \lambda \in F(T,n) \, \} \tag{47}$$

Both these statements can easily be verified by backward induction.

Note that (46) and (47) together constitute the *principle of optimality* in dynamic programming, stating that an optimal policy at each decision node is to choose an immediately succeeding node in order to maximize the node valuation function. The only difference from the standard theory is that the word "node" has replaced "state".

Now one can prove:

LEMMA 7.2: Suppose that $v : Y \to \mathbb{R}$ is any NMUF. Then there exists behaviour β that maximizes the expected value of v in each finite decision tree T of the almost unrestricted domain $\mathcal{T}(Y)$, and also satisfies condition (CB).

PROOF: Let T^m $(m = 1, 2, \ldots)$ be any infinite sequence of finite decision trees in $\mathcal{T}(Y)$ that are all identical except for the positive probabilities $\pi^m(n'|n)$ at each chance node $n \in N^0$. Suppose that $\pi^m(n'|n) \to \bar{\pi}(n'|n)$ as $m \to \infty$ for all $n' \in N_{+1}(n)$ at any chance node $n \in N^0$. Let $\bar{T}$ denote the limit tree, as defined in Section 7.1, with set of nodes $\bar{N}$ after eliminating those that can be reached only with zero probability. Also, let $\bar{T}(n)$ denote the corresponding limit of the subtrees $T^m(n)$, rather than the continuation subtree of $\bar{T}$ that starts from initial node n. The difference is that the limit $\bar{T}(n)$ is well defined even at those nodes $n \in N \setminus \bar{N}$ that can only be reached with zero probability in $\bar{T}$. When $n \in N^0$ is any chance node, note that

$$\bar{N}_{+1}(n) = \{ n' \in N_{+1}(n) \mid \bar{\pi}(n'|n) > 0 \}$$

First it will be confirmed by backward induction that:

INDUCTION HYPOTHESIS (H): At all nodes $n \in N$ of each tree T^m, one has:

(a) (Upper Hemi-continuity) Every sequence satisfying $\lambda^m \in F(T^m, n)$ ($m = 1, 2, \ldots$) has a convergent subsequence, and the limit λ of any convergent subsequence satisfies $\lambda \in F(\bar{T}(n), n)$.

(b) (Lower Hemi-continuity) Any $\lambda \in F(\bar{T}(n), n)$ is the limit of a sequence satisfying $\lambda^m \in F(T^m, n)$ ($m = 1, 2, \ldots$).

These two properties are analogous to those required for the "maximum theorem" of Berge (1963, p. 116) or Hildenbrand (1974, p. 29). Note first that (H) is trivially true at any terminal node $x \in X$, where $F(T^m, x) = F(\bar{T}(x), x) = \{\gamma(x)\}$. At any $n \in N \setminus X$, assume that (H) is true at all $n' \in N_{+1}(n)$.

PROOF WHEN $n \in N^*$ IS A DECISION NODE: (a) Suppose that $\lambda^m \in F(T^m, n)$ ($m = 1, 2, \ldots$). Then there must exist a corresponding sequence of nodes $n^m \in N_{+1}(n)$ such that $\lambda^m \in F(T^m, n^m)$ ($m = 1, 2, \ldots$). Because $N_{+1}(n)$ is a finite set, there must exist a subsequence n^{m_k} and a limit point $\tilde{n} \in N_{+1}(n)$ such that $n^{m_k} = \tilde{n}$ for all large k. But then $\lambda^{m_k} \in F(T^{m_k}, \tilde{n})$ for all large k. Because (H) holds at $\tilde{n} \in N_{+1}(n)$, the sequence λ^{m_k} must have a convergent subsequence $\lambda^{m'_k}$, which is the required convergent subsequence of λ^m.

It follows that if λ^m converges to λ, then λ is the limit of a subsequence which, for some $\tilde{n} \in N_{+1}(n)$, satisfies $\lambda^{m_k} \in F(T^{m_k}, \tilde{n})$ ($k = 1, 2, \ldots$). Because (H) holds at $\tilde{n} \in N_{+1}(n)$, the limit $\lambda \in F(\bar{T}(\tilde{n}), \tilde{n})$. Because n is a decision node, $\tilde{n}$ is included in $\bar{T}(n)$, implying that $F(\bar{T}(\tilde{n}), \tilde{n}) = F(\bar{T}(n), \tilde{n}) \subset F(\bar{T}(n), n)$. So $\lambda \in F(\bar{T}(n), n)$.

(b) Suppose that $\lambda \in F(\bar{T}(n), n)$. There must exist $n' \in N_{+1}(n)$ such that $\lambda \in F(\bar{T}(n), n') = F(\bar{T}(n'), n')$. By the induction hypothesis, there exists a sequence $\lambda^m \in F(T^m(n'), n') = F(T^m, n') \subset F(T^m, n)$ ($m = 1, 2, \ldots$) such that $\lambda^m \to \lambda$ as $m \to \infty$, as required.

PROOF WHEN $n \in N^0$ IS A CHANCE NODE: (a) Suppose that $\lambda^m \in F(T^m, n)$ $(m = 1, 2, \ldots)$. Then

$$\lambda^m = \sum\nolimits_{n' \in N_{+1}(n)} \pi^m(n'|n)\, \lambda^m(n')$$

where $\lambda^m(n') \in F(T^m, n')$ for each $n' \in N_{+1}(n)$. Because (H) holds at each $n' \in N_{+1}(n)$, the finite list of sequences $\langle \lambda^m(n') \rangle_{n' \in N_{+1}(n)}$ has a convergent subsequence $\langle \lambda^{m_k}(n') \rangle_{n' \in N_{+1}(n)}$ $(k = 1, 2, \ldots)$. But then the corresponding subsequence

$$\lambda^{m_k} = \sum\nolimits_{n' \in N_{+1}(n)} \pi^{m_k}(n'|n)\, \lambda^{m_k}(n') \;\; (k = 1, 2, \ldots) \tag{48}$$

also converges. Denote the limit of $\langle \lambda^{m_k}(n') \rangle_{n' \in N_{+1}(n)}$ by $\langle \lambda(n') \rangle_{n' \in N_{+1}(n)}$. Because (H) holds at each $n' \in N_{+1}(n)$, each limit $\lambda(n') \in F(\bar{T}(n'), n')$. Now, if λ^m converges to λ, of course $\lambda^{m_k} \to \lambda$ as $k \to \infty$. Then, taking the limit of (48) as $k \to \infty$ gives

$$\lambda = \sum\nolimits_{n' \in N_{+1}(n)} \bar{\pi}(n'|n)\, \lambda(n')$$

Furthermore, whenever $\bar{\pi}(n'|n) > 0$, then n' is included in $\bar{T}(n)$, implying that $n' \in \bar{N}_{+1}(n)$. Hence,

$$\lambda = \sum\nolimits_{n' \in \bar{N}_{+1}(n)} \bar{\pi}(n'|n)\, \lambda(n')$$

which confirms that $\lambda \in F(\bar{T}(n), n)$.

(b) Suppose that $\lambda \in F(\bar{T}(n), n)$. Then

$$\lambda = \sum\nolimits_{n' \in \bar{N}_{+1}(n)} \bar{\pi}(n'|n)\, \lambda(n')$$

where $\lambda(n') \in F(\bar{T}(n), n')$ for each $n' \in \bar{N}_{+1}(n)$. Because (H) holds at each $n' \in \bar{N}_{+1}(n)$, there exists a sequence $\lambda^m(n') \in F(T^m(n'), n')$ $(m = 1, 2, \ldots)$ such that $\lambda^m \to \lambda$ as $m \to \infty$. For each $n' \in N_{+1}(n) \setminus \bar{N}_{+1}(n)$, choose $\lambda^m(n') \in F(T^m(n'), n')$ arbitrarily. Then

$$\lambda^m = \sum\nolimits_{n' \in N_{+1}(n)} \pi^m(n'|n)\, \lambda^m(n') \in F(T^m, n) \;\; (m = 1, 2, \ldots)$$

Also, because $\bar{\pi}(n'|n) = 0$ for all $n' \in N_{+1}(n) \setminus \bar{N}_{+1}(n)$, one has

$$\lambda^m \to \sum\nolimits_{n' \in N_{+1}(n)} \bar{\pi}(n'|n)\, \lambda(n') = \sum\nolimits_{n' \in \bar{N}_{+1}(n)} \bar{\pi}(n'|n)\, \lambda(n') = \lambda \text{ as } m \to \infty$$

This completes the proof of (H) by induction. ∎

Next, consider the behaviour β^* defined by (41) for all finite decision trees $T \in \mathcal{T}(Y)$ and all decision nodes $n \in N^*$. This behaviour is evidently consequentialist and dynamically consistent. Suppose that n is a decision node of $\bar{T}$, and also that $n^m \in \beta^*(T^m, n)$ $(m = 1, 2, \ldots)$. Because $N_{+1}(n)$ is finite, there exists a subsequence $n^{m_k} \in \beta^*(T^{m_k}, n)$ and $\tilde{n} \in N_{+1}(n)$ such that $n^{m_k} = \tilde{n}$ $(k = 1, 2, \ldots)$. By definition (41), there exists a corresponding sequence satisfying $\lambda^k \in F(T^{m_k}, n) \cap \Psi(T^{m_k}, n)$, where $\Psi(T^{m_k}, n)$ is defined by (39). Because (H) is true at $\tilde{n}$, there is a convergent subsequence λ^{k_r} $(r = 1, 2, \ldots)$ with limit $\lambda \in F(\bar{T}(n), \tilde{n})$. Let μ be any other lottery in $F(\bar{T}(n), \tilde{n})$. Because (H) is true at $\tilde{n}$, there is a sequence $\mu^m \in F(T^m, n)$ $(m = 1, 2, \ldots)$ such that $\mu^m \to \mu$ as $m \to \infty$. Then $\lambda^m \in \Phi_\beta(T^m, n)$ implies that $U(\lambda^m) \geq U(\mu^m)$. Because $U(\lambda)$ is a continuous function of the probabilities $\lambda(y)$ $(y \in Y)$, taking limits as $m \to \infty$ gives $U(\lambda) \geq U(\mu)$. Because μ is any lottery in $F(\bar{T}(n), n)$, it follows that $\lambda \in \Psi(\bar{T}(n), \tilde{n})$. So $\lambda \in F(\bar{T}(n), \tilde{n}) \cap \Psi(\bar{T}(n), \tilde{n})$, implying that $\tilde{n} \in \beta^*(\bar{T}(n), n)$. This verifies that β^* satisfies condition (CB), and so completes the proof of Lemma 7.2. ∎

7.3 Main Theorem

Gathering together the results of Sections 3 and 4 with Theorem 6 and Lemmas 7.1–7.2 gives the following theorem, which constitutes the main result of this chapter for the case of finite lotteries generated by finite decision trees.

THEOREM 7:

(1) Let β be consequentialist behaviour satisfying dynamic consistency and continuity condition (CB) for the almost unrestricted domain $\mathcal{T}(Y)$ of finite decision trees with only positive probabilities at all chance nodes. Then there exists a unique cardinal equivalence class of NMUFs $v : Y \to \mathbb{R}$ such that β maximizes expected utility.

(2) Conversely, let $v : Y \to \mathbb{R}$ be any NMUF. Then there exists consequentialist behaviour β satisfying dynamic consistency and condition (CB) on the domain $\mathcal{T}(Y)$ with the property that the implicit preference ordering R_β is represented by the expected value of v.

8 Discrete Lotteries, Boundedness, and Dominance

8.1 Discrete Lotteries

Previous sections have considered only the space $\Delta(Y)$ of simple lotteries on the set Y of possible consequences. Such lotteries have a finite support of consequences having positive probability. On the other hand, a *countable lottery* on Y is a mapping $\lambda : Y \to [0, 1]$ with the properties that:

(i) there is a countably infinite *support* $K = \{ y_1, y_2, y_3, \ldots \} \subset Y$ of λ such that $\lambda(y) > 0$ for all $y \in K$ and $\lambda(y) = 0$ for all $y \in Y \setminus K$;

(ii) $\sum_{y \in K} \lambda(y) = \sum_{y \in Y} \lambda(y) = 1$.

A *discrete lottery* or probability distribution on Y is either simple or countable; in other words, it satisfies (i) and (ii) above for a support K which is either finite or countably infinite. Let $\Delta^*(Y)$ denote the set of all such discrete lotteries. Like $\Delta(Y)$, it is convex and so a mixture space. Of course, if Y is a finite set, then $\Delta(Y) = \Delta^*(Y)$; to avoid this uninteresting case, the rest of this chapter will assume that Y is infinite.

Many of the earlier results for the space $\Delta(Y)$ of simple lotteries extend in a straightforward way to the space $\Delta^*(Y)$ of discrete lotteries. This section therefore concentrates on the differences between the results for the two spaces.

8.2 Unbounded Utility

Let $v : Y \to \mathbb{R}$ be any NMUF defined on sure consequences. Unless v is bounded, its expected value may not even be defined for some possible lotteries in $\Delta^*(Y)$. Indeed, suppose that v happens to be unbounded both above and below. Then there exists an infinite sequence of consequences y_k $(k = 1, 2, \ldots)$ such that:

$$\begin{aligned} v(y_{2k-1}) &< -4^{2k-1} - \sum_{r=1}^{2k-2} 2^{2k-1-r} \, |v(y_r)|; \\ v(y_{2k}) &> 4^{2k} + \sum_{r=1}^{2k-1} 2^{2k-r} \, |v(y_r)| \end{aligned}$$

Consider now the countable lottery $\lambda = \sum_{r=1}^{\infty} 2^{-r} \, 1_{y_r} \in \Delta^*(Y)$. Its expected value, if it existed, would be the infinite sum $\sum_{r=1}^{\infty} 2^{-r} \, v(y_r)$. But for $k =$

$1, 2, \ldots$ the above pair of inequalities imply that:

$$\sum_{r=1}^{2k-1} 2^{-r}\, v(y_r) \;\leq\; \sum_{r=1}^{2k-2} 2^{-r}\, |v(y_r)| + 2^{1-2k}\, v(y_{2k-1}) \;<\; -2^{2k-1};$$
$$\sum_{r=1}^{2k} 2^{-r}\, v(y_r) \;\geq\; \sum_{r=1}^{2k-1} 2^{-r}\, |v(y_r)| + 2^{-2k}\, v(y_{2k}) \;>\; 2^{2k}$$

It follows that $\lim_{n\to\infty} \sum_{r=1}^{n} 2^{-r}\, v(y_r)$ is undefined, and so therefore is the expected value of v w.r.t. λ.

Even if v is bounded below, but unbounded above, there is still a problem in applying the EU hypothesis to the whole of $\Delta^*(Y)$. This is the point of Menger's (1934) generalization of the St. Petersburg paradox which was briefly reviewed in Section 1. Indeed, there will be an infinite sequence of consequences $y_1', y_2', y_3', \ldots$ such that $v(y_k') > 4^k$ for $k = 1, 2, \ldots$. In this case, the expected utility of the lottery $\lambda' = \sum_{r=1}^{\infty} 2^{-r}\, y_r' \in \Delta^*(Y)$ is $+\infty$. Moreover, for k large enough, one certainly has $v(y_k') > v(y_1')$. Now, if the independence axiom (I) were satisfied, then

$$\lambda'' = \tfrac{1}{2} y_k' + \sum_{r=2}^{\infty} 2^{-r}\, y_r' \succ \lambda' = \tfrac{1}{2} y_1' + \sum_{r=2}^{\infty} 2^{-r}\, y_r'$$

Yet the expected utility of λ'' is also $+\infty$, of course. So the EU criterion fails to distinguish lotteries which, according to condition (I), should not be regarded as indifferent. Similarly if v is bounded above but unbounded below.

Consider next the upper preference set

$$A := \{\, \alpha \in [0,1] \mid \alpha\, \lambda' + (1-\alpha) y_1' \;\succsim\; y_k' \,\}$$

If the EU hypothesis is satisfied, this set must be the half-open interval $(0, 1]$, because the expected utility of $\alpha\, \lambda' + (1-\alpha) y_1'$ is $+\infty$ unless $\alpha = 0$. This contradicts the continuity condition (C) because, even though $\lambda' \succ y_k' \succ y_1'$, there is no $\alpha'' \in (0,1)$ such that $y_k' \succ \alpha''\, \lambda' + (1-\alpha'') y_1'$. A similar contradiction arises if v is bounded above but unbounded below.

In order to avoid such difficulties in applying the EU hypothesis to $\Delta^*(Y)$, it seems natural to assume that $v : Y \to \mathbb{R}$ is bounded both above and below — i.e., that there exist $\underline{v}$ and $\overline{v}$ such that $\underline{v} \leq v(y) \leq \overline{v}$ for all $y \in Y$. Call this the *boundedness* condition (B). Amongst other implications, this will rule out the St. Petersburg paradox — either as originally stated, or in the generalized form due to Menger (1934). Of course, if any one NMUF is bounded, so is any cardinally equivalent NMUF.

However, unlike the three earlier implications of the EU hypothesis — namely conditions (O), (I), (C) and their variants — condition (B) has not been directly expressed in terms of preferences. Yet it can be. Indeed, in combination with conditions (O), (I), and (C), condition (B) is evidently satisfied iff, for all fixed consequences $b, c \in Y$ with $b \succ c$, the constant MRS defined in (6) in Section 2.3 and given by the ratio $[v(a) - v(c)]/[v(b) - v(c)]$ of utility differences is bounded both above and below as a function of $a \in Y$. This is the *bounded preferences* condition. In the following, however, condition (B) will be assumed directly.

Now recall how Section 3 showed that, applied to the space $\Delta(Y)$, the EU hypothesis implies the three conditions (O), (I*) and (C*). Of course, the same three properties should be implications of the EU hypothesis applied to $\Delta^*(Y)$. But, as has just been shown, there are some difficulties with these properties unless utility is bounded (or unless the EU hypothesis is restricted to lotteries in $\Delta^*(Y)$ whose expected utility is finite). It will turn out too that boundedness entails an additional dominance condition. In this sense, the EU hypothesis can have other implications when applied to the whole of $\Delta^*(Y)$.

Conversely, Section 4 showed that the three conditions (O), (I) and (C) imply the EU hypothesis for the whole of $\Delta(Y)$. But for $\Delta^*(Y)$, even the stronger conditions (O), (I*) and (C*), without any extra assumptions, imply only a weakened version of the EU hypothesis. One possibility is that the preference ordering $\succsim$ is represented by a utility function U defined on the whole of $\Delta^*(Y)$. But then any $\lambda \in \Delta^*(Y) \setminus \Delta(Y)$ is an infinite probability mixture, so the (MP) property cannot be applied to the infinite sum $\lambda = \sum_{y \in Y} \lambda(y)\, 1_y$ in order to argue that $U(\lambda) = \sum_{y \in Y} \lambda(y)\, v(y)$. In other words, the EU hypothesis may not extend to the whole of $\Delta^*(Y)$. A rather complicated example to show this possibility, relying on Zorn's lemma or the equivalent axiom of choice, is provided by Fishburn (1970, pp. 141–2); it seems that no simple example exists.

Alternatively, a second possibility is that only on a convex subset of $\Delta^*(Y)$ which includes $\Delta(Y)$ can $\succsim$ be represented by a utility function U satisfying the mixture preservation property (MP). But then U may not be defined on the whole of $\Delta^*(Y)$. For example, if the NMUF $v : Y \to \mathbb{R}$ is unbounded, one could restrict the definition of U to lotteries in $\Delta^*(Y)$ whose expectation is finite. This excludes some lotteries in $\Delta^*(Y) \setminus \Delta(Y)$. For an exploration of what is then possible, see especially Wakker (1993).[12]

Instead of these two possibilities with an unbounded NMUF, I shall follow the standard literature and concentrate on the third case. This occurs when $\succsim$ is

[12] See also Wakker (1989, ch. V) for more on unbounded expected utility when there are uncertain states of the world.

represented on the whole of $\Delta^*(Y)$ by an expected utility function satisfying $U(\lambda) = \sum_{y \in Y} \lambda(y)\, v(y)$ for all $\lambda = \sum_{y \in Y} \lambda(y)\, 1_y$ in $\Delta^*(Y)$, where $v(y) = U(1_y)$ for all $y \in Y$. Moreover, assume that condition (I) is satisfied throughout $\Delta^*(Y)$. Then the arguments above establish that the NMUF $v : Y \to \mathbb{R}$ must be bounded.

8.3 Bounded Expected Utility

The utility function $U : \Delta^*(Y) \to \mathbb{R}$ is said to satisfy the *countable mixture preservation* property (MP*) if, whenever $\lambda_i \in \Delta^*(Y)$ and $\alpha_i \geq 0$ $(i = 1, 2, \ldots)$ with $\sum_{i=1}^{\infty} \alpha_i = 1$, then

$$U\left(\sum_{i=1}^{\infty} \alpha_i\, \lambda_i\right) = \sum_{i=1}^{\infty} \alpha_i\, U(\lambda_i) \tag{49}$$

LEMMA 8.1: Suppose that the function $v : Y \to \mathbb{R}$ is bounded. Let $U : \Delta^*(Y) \to \mathbb{R}$ be the utility function satisfying (MP) whose value at any $\lambda \in \Delta(Y)$ is the expectation $U(\lambda) := \sum_{y \in Y} \lambda(y)\, v(y)$ of v w.r.t. λ. Then U is well defined and bounded on the whole of $\Delta^*(Y)$, and (MP*) is also satisfied.

PROOF: Suppose that $\overline{v}$ and $\underline{v}$ are respectively upper and lower bounds for v on Y. Now, given any $\lambda = \sum_{r=1}^{\infty} \lambda(y_r)\, 1_{y_r} \in \Delta^*(Y)$, it must be true that $\sum_{r=k}^{\infty} \lambda(y_r) \to 0$ as $k \to \infty$. Evidently

$$\sum_{r=k}^{\infty} \lambda(y_r)\, \underline{v} \leq \sum_{r=k}^{\infty} \lambda(y_r)\, v(y_r) \leq \sum_{r=k}^{\infty} \lambda(y_r)\, \overline{v}$$

and so $\sum_{r=k}^{\infty} \lambda(y_r)\, v(y_r) \to 0$ as $k \to \infty$. From this it follows that the utility function

$$U(\lambda) := \sum_{r=1}^{\infty} \lambda(y_r)\, v(y_r) = \sum_{y \in Y} \lambda(y)\, v(y)$$

is also well defined for any $\lambda \in \Delta^*(Y)$. Of course, it must also satisfy the inequalities $\underline{v} \leq U(\lambda) \leq \overline{v}$ for all $\lambda \in \Delta^*(Y)$.

Next, suppose that $\lambda_i \in \Delta^*(Y)$ and $\alpha_i \geq 0$ $(i = 1, 2, \ldots)$ with $\sum_{i=1}^{\infty} \alpha_i = 1$. For every finite k, one can write

$$\sum_{i=1}^{\infty} \alpha_i\, \lambda_i = \sum_{i=1}^{k} \alpha_i\, \lambda_i + \left(\sum_{i=k+1}^{\infty} \alpha_i\right) \mu_k$$

where $\mu_k := \sum_{i=k+1}^{\infty} \delta_i\, \lambda_i$ and $\delta_i := \alpha_i / \sum_{j=k+1}^{\infty} \alpha_j$ $(i = k+1, k+2, \ldots)$. Then

$$U\left(\sum_{i=1}^{\infty} \alpha_i\, \lambda_i\right) = \sum_{i=1}^{k} \alpha_i\, U(\lambda_i) + \sum_{i=k+1}^{\infty} \alpha_i\, U(\mu_k)$$

But $\sum_{i=1}^{\infty} \alpha_i = 1$ implies $\sum_{i=k+1}^{\infty} \alpha_i \to 0$ as $k \to \infty$. Because U is bounded on $\Delta^*(Y)$, one has $\sum_{i=k+1}^{\infty} \alpha_i\, U(\mu_k) \to 0$, and so (49) follows. ∎

Now, by repeating the arguments of Section 3, it is easy to show that when condition (B) is satisfied, then the EU hypothesis really does imply conditions (O), (I*), and (C*) on the whole of $\Delta^*(Y)$.

8.4 Dominance

Another important implication of the EU hypothesis, applied to the whole of $\Delta^*(Y)$, is the following *dominance condition* (D). This states that, whenever $\lambda_i, \mu_i \in \Delta^*(Y)$ and $\alpha_i > 0$ with $\lambda_i \succsim \mu_i$ $(i = 1, 2, \ldots)$ and $\sum_{i=1}^{\infty} \alpha_i = 1$, then $\sum_{i=1}^{\infty} \alpha_i \, \lambda_i \succsim \sum_{i=1}^{\infty} \alpha_i \, \mu_i$. The stronger condition (D*) requires strict preference if in addition $\lambda_i \succ \mu_i$ for some i. Condition (D*) was originally formulated by Blackwell and Girshick (1954, p. 105, H_1). It is a natural extension of the finite dominance condition (FD*) discussed in Section 3.2. The following result shows when (D*) is a necessary condition.

Lemma 8.2: Suppose that the EU hypothesis is satisfied on $\Delta^*(Y)$, and that any NMUF $v : Y \to \mathbb{R}$ is bounded. Then the dominance condition (D*) is satisfied.

Proof: Because of Lemma 8.1, the hypotheses imply that (MP*) is satisfied. So if $U(\lambda_i) \geq U(\mu_i)$ and $\alpha_i > 0$ for $i = 1, 2, \ldots$, then

$$U\left(\sum_{i=1}^{\infty} \alpha_i \, \lambda_i\right) - U\left(\sum_{i=1}^{\infty} \alpha_i \, \mu_i\right) = \sum_{i=1}^{\infty} \alpha_i \, [U(\lambda_i) - U(\mu_i)] \geq 0$$

with strict inequality if $U(\lambda_i) > U(\mu_i)$ for some i. ∎

Condition (D) evidently implies the following property. Suppose that $\lambda, \mu \in \Delta^*(Y)$ satisfy $\mu(\{ y \in Y \mid 1_y \succsim \lambda \}) = 1$. Then $\mu = \sum_{i=1}^{\infty} \mu(y_i) \, 1_{y_i}$ where the support of the distribution μ is the set $\{ y_1, y_2, \ldots \}$, and where $1_{y_i} \succsim \lambda$ for $i = 1, 2, \ldots$. Because $\lambda = \sum_{i=1}^{\infty} \mu(y_i) \, \lambda$, condition (D) implies that $\mu \succsim \lambda$. Similarly, if $\mu(\{ y \in Y \mid 1_y \precsim \lambda \}) = 1$, then $\mu \precsim \lambda$. A related property for probability measures is the probability dominance condition (PD) that will be used in Section 9.2.

8.5 Sufficient Conditions for the EU Hypothesis

It can now be proved that conditions (O), (I), (C) and (D) are sufficient for the EU hypothesis to hold on $\Delta^*(Y)$. In fact, it will turn out that conditions (B) and (D) are logically equivalent in the presence of the other three. As remarked in Section 8.1, however, the EU hypothesis on the whole of $\Delta^*(Y)$ does not follow from conditions (O), (I) and (C) alone.

Lemma 8.3: Suppose that conditions (O), (I), and (C) apply to the whole of the mixture space $\Delta^*(Y)$. Then there exists a unique cardinal equivalence class of utility functions $U : \Delta^*(Y) \to \mathbb{R}$ which represent $\succsim$ and satisfy (MP).

Proof: Note that Lemma 4.6 is true of the space $\Delta^*(Y)$. ∎

Lemma 8.4: Conditions (O), (I), (C), and (B) together imply the EU hypothesis on the whole of $\Delta^*(Y)$.

Proof: Apply Lemma 8.3 and let $U : \Delta^*(Y) \to \mathbb{R}$ be any utility function which represents $\succsim$ and satisfies (MP) on $\Delta^*(Y)$. Define $v : Y \to \mathbb{R}$ by $v(y) := U(1_y)$ for all $y \in Y$. The EU hypothesis is then satisfied on $\Delta(Y)$ and by condition (B), the function v is bounded. So by Lemma 8.1, the function U satisfies (MP*). Hence, $U(\lambda) = \sum_{y \in Y} \lambda(y)v(y)$ for all $\lambda \in \Delta^*(Y)$. ∎

Lemma 8.5: Conditions (O), (I), (C), and (D) together imply condition (B).

Proof: By Lemma 8.3, there exists a utility function $U : \Delta^*(Y) \to \mathbb{R}$ which represents $\succsim$ and satisfies (MP).

Suppose that U were unbounded above. Then there would exist a sequence of lotteries $\langle \lambda_i \rangle_{i=0}^{\infty}$ in $\Delta^*(Y)$ satisfying

$$U(\lambda_0) \geq 1 \quad \text{and} \quad U(\lambda_i) \geq \max\{\, U(\lambda_{i-1}), 2^i \,\} \quad (i = 1, 2, \ldots)$$

Now define $\pi_k := \sum_{i=1}^{\infty} 2^{-i} \lambda_{k+i} \in \Delta^*(Y)$ $(k = 0, 1, 2, \ldots)$. Then, because $U(\lambda_{k+i}) \geq U(\lambda_k)$ and so $\lambda_{k+i} \succsim \lambda_k$ for $i = 1, 2, \ldots$, condition (D) implies that $\pi_k \succsim \lambda_k$. Therefore $U(\pi_k) \geq U(\lambda_k) \geq 2^k$ $(k = 0, 1, 2, \ldots)$. But $\pi_0 = \sum_{i=1}^{k} 2^{-i} \lambda_i + 2^{-k} \pi_k$ and so $U(\pi_0) = \sum_{i=1}^{k} 2^{-i} U(\lambda_i) + 2^{-k} U(\pi_k)$ because of (MP). Hence

$$U(\pi_0) \geq \sum_{i=1}^{k} 2^{-i}\, 2^i + 2^{-k}\, 2^k = k + 1$$

Since this is true for every integer k, this contradicts the requirement that U should be real-valued on the whole of $\Delta^*(Y)$.

The proof that U is bounded below is similar. ∎

8.6 Continuity

In order to state a continuity condition for preferences on $\Delta^*(Y)$, this space must first be given a topology. To do so, define the metric d on $\Delta^*(Y)$ by

$$d(\lambda, \mu) := \sum\nolimits_{y \in Y} |\lambda(y) - \mu(y)|$$

Note that, for all $\lambda, \mu \in \Delta^*(Y)$, one has

$$d(\lambda, \mu) \leq \sum\nolimits_{y \in Y} [|\lambda(y)| + |\mu(y)|] = \sum\nolimits_{y \in Y} [\lambda(y) + \mu(y)] = 2$$

and that $d(\lambda, \mu) = 2$ iff $\lambda(y)\, \mu(y) = 0$ for all $y \in Y$. Furthermore,

$$d(\lambda, \mu) = \max_f \{ |\mathbb{E}_\lambda f - \mathbb{E}_\mu f| \ \mid \forall y \in Y : |f(y)| \leq 1 \}$$

because the maximum is attained by choosing $f(y) =$ sign $(\lambda(y) - \mu(y))$ for all $y \in Y$. It follows that the infinite sequence $\langle \lambda^n \rangle_{n=1}^\infty$ in $\Delta^*(Y)$ converges to λ iff $\mathbb{E}_{\lambda^n} f = \sum_y \lambda^n(y)\, f(y)$ converges to $\mathbb{E}_\lambda f$ for every bounded function $f : Y \to \mathbb{R}$. This implies that the expectation $\mathbb{E}_\lambda\, f$ of any bounded function f is continuous w.r.t. λ on $\Delta^*(Y)$. Hence, if the EU hypothesis is satisfied and the NMUF v is bounded, then the following *continuous preference* condition (CP) is satisfied: for all $\bar{\lambda} \in \Delta^*(Y)$, the two *preference sets*

$$\{ \lambda \in \Delta^*(Y) \mid \lambda \succsim \bar{\lambda} \} \quad \text{and} \quad \{ \lambda \in \Delta^*(Y) \mid \lambda \precsim \bar{\lambda} \}$$

must both be closed. In particular, because of Lemma 8.5, the four conditions (O), (I), (C) and (D) together imply (CP).

A converse result includes (CP) among the set of sufficient conditions for the EU hypothesis to be satisfied.

Lemma 8.6: Condition (CP) implies condition (C*).

Proof: Consider any pair $\lambda, \mu \in \Delta^*(Y)$. Define the convex hull

$$\text{co}\{ \lambda, \mu \} = \{ \nu \in \Delta^*(Y) \mid \exists \alpha \in [0, 1] : \nu = \alpha\, \lambda + (1 - \alpha)\mu \}$$

which is evidently a closed set. Hence, the two sets defined in (14) of Section 3.3 must be closed, because each is the intersection of a closed preference set in $\Delta^*(Y)$ with $\text{co}\{ \lambda, \mu \}$. This verifies condition (C*). ∎

Lemma 8.7: Conditions (O), (I*) and (CP) jointly imply conditions (D) and (D*).

PROOF: Suppose that $\lambda^* = \sum_{i=1}^\infty \alpha_i \lambda_i$ and $\mu^* = \sum_{i=1}^\infty \alpha_i \mu_i$, where $\lambda_i, \mu_i \in \Delta^*(Y)$ and $\alpha_i > 0$ with $\lambda_i \succsim \mu_i$ $(i = 1, 2, \ldots)$ and $\sum_{i=1}^\infty \alpha_i = 1$. Define the two sequences

$$\begin{aligned} \lambda^k &:= \sum_{i=1}^k \alpha_i \lambda_i + \sum_{i=k+1}^\infty \alpha_i \mu_i \in \Delta^*(Y) \\ \nu^k &:= (1-\alpha_k)^{-1} \left(\sum_{i=1}^{k-1} \alpha_i \lambda_i + \sum_{i=k+1}^\infty \alpha_i \mu_i \right) \in \Delta^*(Y) \end{aligned}$$

for $k = 1, 2, \ldots$, with $\lambda^0 := \mu^*$. Note that $\lambda^k = \alpha_k \lambda_k + (1-\alpha_k)\nu^k$ whereas $\lambda^{k-1} = \alpha_k \mu_k + (1-\alpha_k)\nu^k$. Hence, condition (I*) implies that $\lambda^k \succsim \lambda^{k-1}$ $(k = 1, 2, \ldots)$ and so in particular $\lambda^k \succsim \mu^*$ because $\succsim$ is transitive and $\lambda^0 := \mu^*$.

Note too that $\lambda^k \to \lambda^*$ as $k \to \infty$ because $d(\mu_i, \lambda^*) \le 2$ and $\sum_{i=1}^\infty \alpha_i = 1$, so

$$\begin{aligned} \sum_{y\in Y} |\lambda^k(y) - \lambda^*(y)| &= \sum_{y\in Y} \left| \sum_{i=k+1}^\infty \alpha_i [\mu_i(y) - \lambda^*(y)] \right| \\ &\le \sum_{i=k+1}^\infty \alpha_i \sum_{y\in Y} |\mu_i(y) - \lambda^*(y)| \\ &= \sum_{i=k+1}^\infty \alpha_i \, d(\mu_i, \lambda^*) \le 2 \sum_{i=k+1}^\infty \alpha_i \to 0 \end{aligned}$$

Therefore condition (CP) implies that $\lambda^* \succsim \mu^*$. A similar proof can be used to verify condition (D*). ∎

The following summarizes the previous results in this section:

THEOREM 8:

(1) All six conditions (O), (I*), (C*), (B), (D*) and (CP) are necessary if the EU hypothesis is to hold on the whole of $\Delta^*(Y)$, without any violations of either condition (I) or (C).

(2) The four conditions (O), (I), (C) and (D) are sufficient for the EU hypothesis to hold on all of $\Delta^*(Y)$, with a bounded NMUF and a utility function that is continuous. Condition (D) can be replaced by (B) in this list of sufficient conditions. Also, provided that (I) is strengthened to (I*), condition (CP) can replace both (C) and (D).

8.7 Consequentialist Motivation for Dominance

Like its close relatives the strong independence axiom (I*) and the finite dominance axiom (FD*) of Section 3.2, the important dominance condition (D) can also be given a consequentialist justification. Indeed, suppose that $\lambda_i, \mu_i \in \Delta^*(Y)$ and $\alpha_i > 0$ with $\lambda_i \succsim \mu_i$ $(i = 1, 2, \ldots)$ and $\sum_{i=1}^{\infty} \alpha_i = 1$. Consider the (countably infinite) decision tree T with initial chance node n_0 at which $N_{+1}(n_0) = \{\, n_i \mid i = 1, 2, \ldots \}$. Moreover, each transition probability at n_0 should satisfy $\pi(n_i|n_0) = \alpha_i$. Then each node n_i is a decision node at which the agent chooses between two terminal nodes $x_{i\lambda}$ and $x_{i\mu}$. The ensuing lottery consequences in $\Delta^*(Y)$ are given by $\gamma(x_{i\lambda}) = \lambda_i$ and $\gamma(x_{i\mu}) = \mu_i$ $(i = 1, 2, \ldots)$ respectively. A typical branch of this tree is represented in Figure 5.

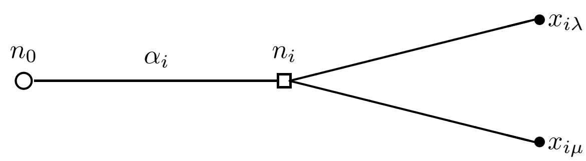

Figure 5 One Typical Tree Branch

Because $\lambda_i \succsim \mu_i$, it must be true that $\lambda_i \in \Phi_\beta(T, n_i)$ and so $x_{i\lambda} \in \beta(T, n_i)$ at each decision node n_i $(i = 1, 2, \ldots)$ of the tree T. But then

$$\lambda := \sum\nolimits_{i=1}^{\infty} \alpha_i \, \lambda_i \in \Phi_\beta(T) = \Phi_\beta(T, n_0) = \sum\nolimits_{i=1}^{\infty} \alpha_i \, \Phi_\beta(T, n_i)$$

Let $\mathcal{N}$ denote the set of natural numbers $\{\, 1, 2, \ldots \}$. Given any subset $J \subset \mathcal{N}$, define the lottery

$$\nu_J := \sum\nolimits_{i \in J}^{\infty} \alpha_i \, \lambda_i + \sum\nolimits_{i \in \mathcal{N} \setminus J}^{\infty} \alpha_i \, \mu_i \in \Delta^*(Y)$$

Note then that

$$F(T) = F(T, n_0) = \sum\nolimits_{i=1}^{\infty} \alpha_i \, F(T, n_i) = \sum\nolimits_{i=1}^{\infty} \alpha_i \, \{\, \lambda_i, \mu_i \,\} = \{\, \nu_J \mid J \subset \mathcal{N} \,\}$$

Consider also an alternative decision tree T' with initial decision node n'_0 at which the set $N_{+1}(n'_0)$ consists of a second decision node n'_1 together with terminal nodes x'_J for every proper subset $J \subset \mathcal{N}$ — i.e., for every non-empty subset J which is not equal to $\mathcal{N}$. Suppose that $\gamma(x'_J) = \nu_J$ for all such J. Suppose too that $N_{+1}(n'_1) = \{\, x'_\lambda, x'_\mu \,\}$ where $\gamma(x'_\lambda) = \lambda$ and $\gamma(x'_\mu) = \mu$. This tree is represented in Figure 6.

Clearly $F(T') = F(T)$, so extending the consequentialist axiom in an obvious way to the countably infinite decision trees T and T' entails $\Phi_\beta(T') = \Phi_\beta(T)$; in particular, because $\lambda \in \Phi_\beta(T)$ it must be true that $\lambda \in \Phi_\beta(T')$. But this is only possible if $x'_\lambda \in \beta(T', n'_1)$. Dynamic consistency then implies that

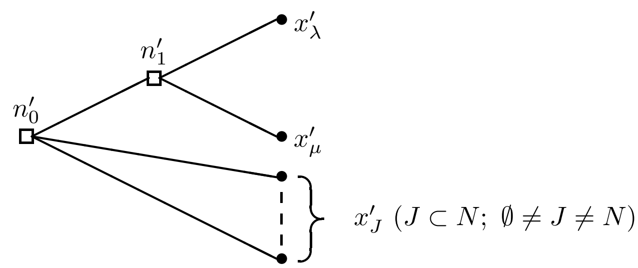

Figure 6 Decision Tree Illustrating Dominance

$x'_\lambda \in \beta(T'(n'_1), n'_1)$, and so $\lambda \in \Phi_\beta(T'(n'_1), n'_1)$. But $F(T'(n'_1), n'_1) = \{\lambda, \mu\}$. Hence, the implicit preference relation must satisfy $\lambda \succsim \mu$. So condition (D) has been given a consequentialist justification, as promised.

Finally, it is evident that when combined with condition (I*), which also has a consequentialist justification, condition (D) implies condition (D*). So (D*) has a consequentialist justification as well.

9 Probability Measures

9.1 Probability Measures and Expectations

This introductory subsection is designed to remind the reader of a few essential but unavoidably technical concepts from the theory of measure and probability that are needed in order to discuss lotteries in which the range of possible outcomes is uncountable. The presentation is not intended as a substitute for a more thorough treatment of the topic such as Halmos (1950), Royden (1988), Billingsley (1995) or — perhaps more suitable for economists — Kirman (1981).

To avoid trivialities, the consequence domain Y should be an infinite set. Then a *σ-field* on Y is a family $\mathcal{F} \subset 2^Y$ of subsets of Y such that: (i) $Y \in \mathcal{F}$; (ii) $B \in \mathcal{F}$ implies $Y \setminus B \in \mathcal{F}$; (iii) if B_i $(i = 1, 2, \ldots)$ is a countable collection in $\mathcal{F}$, then $\cup_{i=1}^\infty B_i \in \mathcal{F}$.[13] The pair $(Y, \mathcal{F})$ constitutes a *measurable space*. Obviously, (i) and (ii) imply that $\emptyset \in \mathcal{F}$.

[13] Royden does not impose the restriction that $Y \in \mathcal{F}$, but this is useful in probability theory. Also, note that the term *σ-algebra* is often used insted of σ-field. Halmos (1950) distinguishes between a *σ-ring*, which need not satisfy $Y \in \mathcal{F}$, and a *σ-algebra*, which must satisfy this condition.

A *probability measure* on $(Y, \mathcal{F})$ is a mapping $\pi : \mathcal{F} \to [0,1]$ satisfying $\pi(\emptyset) = 0$, $\pi(Y) = 1$, and also the *σ-additivity property* that $\pi(\cup_{i=1}^\infty B_i) = \sum_{i=1}^\infty \pi(B_i)$ whenever B_i $(i = 1, 2, \ldots)$ is a countable collection of *pairwise disjoint* sets in $\mathcal{F}$.

A function $v : Y \to \mathbb{R}$ is said to be *$\mathcal{F}$-measurable* if, given any interval $J \subset \mathbb{R}$, the set $v^{-1}(J) := \{ y \in Y \mid v(y) \in J \}$ belongs to $\mathcal{F}$.

An $\mathcal{F}$-measurable *step function* $v : Y \to \mathbb{R}$ has the property that, for some finite partition $\cup_{i=1}^k Y_i$ of Y into pairwise disjoint sets $Y_i \in \mathcal{F}$, and some corresponding collection of real constants $v_i \in \mathbb{R}$ $(i = 1, 2, \ldots, k)$, one has $v(y) \equiv v_i$ throughout each Y_i. It follows that $v = \sum_{i=1}^k v_i \, \chi_{Y_i}$, in effect, where χ_{Y_i} is the particular *characteristic* step function defined by

$$\chi_{Y_i}(y) = \begin{cases} 1 & \text{if } y \in Y_i \\ 0 & \text{otherwise} \end{cases}$$

Let V_0 denote the set of all such step functions. The *expectation* $\mathbb{E}_\pi v$ of the $\mathcal{F}$-measurable step function $v = \sum_{i=1}^k v_i \, \chi_{Y_i}$ w.r.t. any probability measure π on $(Y, \mathcal{F})$ is defined as the integral

$$\mathbb{E}_\pi v := \int_Y v(y) \, \pi(dy) = \sum_{i=1}^k v_i \, \pi(Y_i)$$

A fundamental result in the theory of Lebesgue integration is that the expectation $\mathbb{E}_\pi v$ can also be well defined for *any* bounded measurable $v : Y \to \mathbb{R}$, not just for step functions. In fact, given such a general measurable function, define the "greater" and "lesser" sets of step functions as

$$\begin{aligned} V_0^+(v) &:= \{ v_0 \in V_0 \mid \forall y \in Y : v_0(y) \geq v(y) \} \\ V_0^-(v) &:= \{ v_0 \in V_0 \mid \forall y \in Y : v_0(y) \leq v(y) \} \end{aligned}$$

Then one can define

$$\mathbb{E}_\pi v := \int_Y v(y) \pi(dy) := \inf_{v_0} \{ \mathbb{E}_\pi v_0 \mid v_0 \in V_0^+(v) \} = \sup_{v_0} \{ \mathbb{E}_\pi v_0 \mid v_0 \in V_0^-(v) \} \tag{50}$$

For a bounded function v, this definition is unambiguous if and only if there is a measurable function $\hat{v} : Y \to \mathbb{R}$ such that $\pi(\{ y \in Y \mid v(y) = \hat{v}(y) \}) = 1$. See Royden (1988, Ch. 4, Prop. 3 and proof) while recalling that Royden uses a more inclusive definition of measurability.

9.2 Necessary Conditions for EU Maximization

As in Sections 3 and 8, I shall begin by considering some key implications of the EU hypothesis. Indeed, suppose that there is a domain D of probability distributions over Y and a preference ordering $\precsim$ on D that can be represented by the expected value of some NMUF defined on Y. What properties must the space D and the ordering $\precsim$ have for such a representation to be possible?

The answer to this question hinges on the definition of "expected value". For simple probability distributions $\pi \in \Delta(Y)$ and discrete probability distributions $\pi \in \Delta^*(Y)$, the answer was unambiguous — in fact, $\mathbb{E}_\pi v = \sum_{y \in Y} \pi(y)\, v(y)$ for any utility function $v : Y \to \mathbb{R}$. More generally, it is natural to assume that, for some σ-field $\mathcal{F}$ on Y, the domain D is the set $\Delta(Y, \mathcal{F})$ of $\mathcal{F}$-measurable probability distributions on Y, and that $\mathbb{E}_\pi v = \int_Y v(y)\, \pi(dy)$ for all $\pi \in \Delta(Y, \mathcal{F})$. For this integral to be defined, however, the function v has to be approximable by $\mathcal{F}$-measurable step functions, which requires v itself to be $\mathcal{F}$-measurable.

In considering sufficient conditions for such a representation, however, one should not simply assume that v is $\mathcal{F}$-measurable, because v itself represents preferences. For this reason, it is better to adopt a slightly different approach. Conditions for an ordering $\precsim^*$ on $\Delta^*(Y)$ to have an EU representation have already been given. So the most pertinent question is this: When can $\precsim^*$ on $\Delta^*(Y)$ be extended to an ordering $\precsim$ that also has an EU representation on some domain D of probability measures on Y?

For this extension to be possible, it has already been seen that D should be the set of probability measures on a σ-field $\mathcal{F}$ with respect to which v is measurable. But for the NMUF $v : Y \to \mathbb{R}$ to be $\mathcal{F}$-measurable, the set $v^{-1}(I)$ must be measurable for every interval I of the real line. This last condition evidently implies that the sets $v^{-1}((-\infty, \alpha))$ and $v^{-1}((\alpha, \infty))$ should be measurable for every real α. In particular, this must be true whenever $\alpha = \mathbb{E}_\pi v$ for some discrete probability distribution $\pi \in \Delta^*(Y)$.

What this implies is that the σ-field $\mathcal{F}$ has to include all upper and lower preference sets of the ordering $\precsim^*$ on $\Delta^*(Y)$. In other words, it must satisfy the *preference measurability* condition (PM) stating that, for every $\bar{\pi} \in \Delta^*(Y)$, the two *preference sets*

$$\{\, y \in Y \mid 1_y \ \precsim\ \bar{\pi} \,\}, \quad \{\, y \in Y \mid \bar{\pi} \ \precsim\ 1_y \,\} \tag{51}$$

are both members of $\mathcal{F}$.

Another desirable property of the σ-field $\mathcal{F}$ will be that $\Delta^*(Y) \subset \Delta(Y, \mathcal{F})$. Moreover, given any lottery $\lambda \in \Delta^*(Y)$, the probabilities $\lambda(y)$ are well-defined. So, given any $\pi \in \Delta(Y, \mathcal{F})$, it is desirable that the probabilities $\pi(\{y\})$ should be well-defined. Accordingly, the stronger *measurability* condition (M) will be imposed, requiring not only condition (PM), but also that $\{y\} \in \mathcal{F}$ for all $y \in Y$. Because $\mathcal{F}$ is a σ-field, this implies that $K \in \mathcal{F}$ for every finite or countably infinite set $K \subset Y$. So, it will be true that $\Delta^*(Y) \subset \Delta(Y, \mathcal{F})$. The implication of condition (M) is that $\mathcal{F}$ contains the smallest σ-field generated by both the singleton sets $\{y\}$ (all $y \in Y$) and the preference sets defined by (51).

Because $\Delta(Y, \mathcal{F})$ extends $\Delta^*(Y)$, the arguments of Section 8.2 apply *a fortiori*, so it will be assumed that condition (B) holds, otherwise the EU hypothesis is inconsistent with condition (I) and with condition (C).

Finally, another obvious necessary condition for $\succsim$ to have an EU representation is the following *probability dominance* condition (PD). This states that for every $\pi \in \Delta(Y, \mathcal{F})$ and $\lambda \in \Delta(Y)$, one has

$$\begin{aligned} \pi(\{\, y \in Y \mid 1_y \succsim \lambda \,\}) = 1 \quad &\Longrightarrow \quad \pi \succsim \lambda \\ \pi(\{\, y \in Y \mid 1_y \precsim \lambda \,\}) = 1 \quad &\Longrightarrow \quad \pi \precsim \lambda \end{aligned}$$

Section 8.4 discusses the relationship between condition (PD) applied to $\Delta^*(Y)$ and dominance condition (D).

9.3 Sufficient Conditions for the EU Hypothesis

The main result of this Section is:

Theorem 9: The six conditions (O), (I), (C), (M), (D) and (PD) are sufficient for the EU hypothesis to apply to the whole of $\Delta(Y, \mathcal{F})$.

Proof: Because $\Delta(Y, \mathcal{F})$ is a convex mixture space and conditions (O), (I), and (C) are all satisfied, Lemma 4.6 is applicable. So there exists a utility function $U : \Delta(Y, \mathcal{F}) \to \mathbb{R}$ which represents $\succsim$ and satisfies (MP). Because of (M), the set $\{y\}$ is measurable for all $y \in Y$, so $1_y \in \Delta(Y, \mathcal{F})$. Hence, one can define the NMUF $v : Y \to \mathbb{R}$ so that $v(y) := U(1_y)$ for all $y \in Y$. Because of condition (D), Theorem 8 implies that the EU hypothesis is satisfied on $\Delta^*(Y)$, and also that condition (B) is satisfied. So the function v is bounded both above and below. Also, Lemma 8.1 implies that U satisfies the extended mixture preservation property (MP*).

Consider next the two sets $\{ y \in Y \mid v(y) \leq r \}$ and $\{ y \in Y \mid v(y) \geq r \}$ for any real r. It will be shown that these two sets are $\mathcal{F}$-measurable. First, there are the two trivial cases where: either (a) $v(y) > r$ for all $y \in Y$; or (b) $v(y) < r$ for all $y \in Y$. Then $\{ y \in Y \mid v(y) \leq r \}$ and $\{ y \in Y \mid v(y) \geq r \}$ are respectively the empty set and the whole of Y in case (a), and *vice versa* in case (b). In either case, both are certainly measurable.

Alternatively, there exist $y_1, y_2 \in Y$ such that $v(y_1) \geq r \geq v(y_2)$. Then, because of (MP), there exists a mixture $\alpha 1_{y_1} + (1-\alpha) 1_{y_2} \in \Delta(Y)$ with $0 \leq \alpha \leq 1$ such that

$$U(\alpha 1_{y_1} + (1-\alpha) 1_{y_2}) = \alpha v(y_1) + (1-\alpha) v(y_2) = r$$

This implies that

$$\begin{aligned} \{ y \in Y \mid v(y) \leq r \} &= \{ y \in Y \mid y \precsim \alpha 1_{y_1} + (1-\alpha) 1_{y_2} \} \\ \text{and} \quad \{ y \in Y \mid v(y) \geq r \} &= \{ y \in Y \mid y \succsim \alpha 1_{y_1} + (1-\alpha) 1_{y_2} \} \end{aligned}$$

By condition (M), both these sets are measurable. Therefore the function $v : Y \to \mathbb{R}$ is $\mathcal{F}$-measurable. It follows that the integral $\mathbb{E}_\pi v$ is well defined for every $\pi \in \Delta(Y, \mathcal{F})$. So it remains only to show that $U(\pi) \equiv \mathbb{E}_\pi v$ on $\Delta(Y, \mathcal{F})$.

As in Section 9.1, let V_0 be the set of all $\mathcal{F}$-measurable step functions — i.e., functions that can be expressed in the form $v_0 \equiv \sum_{i=1}^k v_i \, \chi_{Y_i}$ for some finite partition $\cup_{i=1}^k Y_i$ of Y into pairwise disjoint sets $Y_i \in \mathcal{F}$, and some corresponding collection of real constants $v_i \in \mathbb{R}$ $(i = 1, 2, \ldots, k)$. Also, let $V_0^+(v)$ and $V_0^-(v)$ be the set of all $\mathcal{F}$-measurable step functions v_0 satisfying respectively $v_0(y) \geq v(y)$ and $v_0(y) \leq v(y)$ for all $y \in Y$.

Suppose that $v_0 \equiv \sum_{i=1}^k v_i \, \chi_{Y_i} \in V_0^+(v)$ is any step function which, for $i = 1$ to k, satisfies $v_0(y) = v_i \geq v(y)$ for all $y \in Y_i$, where $\cup_{i=1}^k Y_i$ is a partition of Y. Let y_j $(j = 1, 2, \ldots)$ be any infinite sequence of consequences such that $v(y_j)$ increases strictly with j, and $v(y_j) \to \bar{v} := \sup \{ v(y) \mid y \in Y \}$ as $j \to \infty$. Define the sequence of measurable sets

$$\begin{aligned} Z_1 \quad &:= \quad \{ y \in Y \mid v(y) \leq v(y_1) \} \\ \text{and} \quad Z_j \quad &:= \quad \{ y \in Y \mid v(y_{j-1}) < v(y) \leq v(y_j) \} \quad (j = 2, 3, \ldots) \end{aligned}$$

Then the function defined by $v^+ := \sum_{j=1}^\infty v(y_j) \, \chi_{Z_j}$ satisfies $v^+(y) \geq v(y)$ for all $y \in Y$. So the function defined by $v^* := \min\{ v_0, v^+ \}$ also satisfies $v^*(y) \geq v(y)$ for all $y \in Y$. Moreover,

$$\begin{aligned} v^* \quad &\equiv \quad \sum\nolimits_{i=1}^k \sum\nolimits_{j=1}^\infty v_{ij}^* \, \chi_{W_{ij}} \quad \text{where} \quad v_{ij}^* := \min\{ v_i, v(y_j) \} \\ \text{and} \quad W_{ij} \quad &:= \quad Y_i \cap Z_j \quad (i = 1, 2, \ldots, k;\ j = 1, 2, \ldots) \end{aligned} \tag{52}$$

Of course, whenever $W_{ij} = \emptyset$, it is understood that $\chi_{Y_i \cap Z_j}(y) = \chi_\emptyset(y) = 0$ for all $y \in Y$. Now, either $v^*_{ij} = v(y_j)$ or else $v(y) \leq v^*_{ij} = v_i < v(y_j)$ for all $y \in Y_i$. In either case, there exist $y_{ij} \in Y$ and $\alpha_{ij} \in [0, 1]$ such that $v^*_{ij} = \alpha_{ij} v(y_{ij}) + (1 - \alpha_{ij}) v(y_j)$. Then $v^*_{ij} = U(\lambda_{ij})$ where $\lambda_{ij} = \alpha_{ij} 1_{y_{ij}} + (1 - \alpha_{ij}) 1_{y_j} \in \Delta(Y)$. Therefore, for $i = 1$ to k and for $j = 1, 2, \ldots$, one has

$$v^*_{ij} = U(\lambda_{ij}) \geq v(y) \text{ for all } y \in W_{ij} \tag{53}$$

Now, for each variable probability measure $\pi \in \Delta(Y, \mathcal{F})$, as well as some arbitrary fixed measure $\bar{\pi} \in \Delta(Y, \mathcal{F})$, define the associated conditional probability measures $\pi_{ij} = \pi(\cdot | W_{ij}) \in \Delta(Y, \mathcal{F})$ ($i = 1$ to k and $j = 1, 2, \ldots$) by

$$\pi_{ij}(S) := \begin{cases} \pi(S \cap W_{ij}) / \pi(W_{ij}) & \text{if } \pi(W_{ij}) > 0 \\ \bar{\pi}(S) & \text{if } \pi(W_{ij}) = 0 \end{cases} \tag{54}$$

for every measurable set $S \in \mathcal{F}$. Because $\cup_{i=1}^k \cup_{j=1}^\infty W_{ij}$ is a partition of Y, it follows that $\cup_{i=1}^k \cup_{j=1}^\infty (S \cap W_{ij})$ is a partition of S. Therefore σ-additivity of π implies that

$$\pi(S) = \sum_{i=1}^k \sum_{j=1}^\infty \pi(S \cap W_{ij}) = \sum_{i=1}^k \sum_{j=1}^\infty \pi(W_{ij})\, \pi_{ij}(S) \tag{55}$$

Next, because of (53) and (54), for each pair i and j such that $\pi(W_{ij}) > 0$, it follows that $\pi_{ij}(\{\, y \in Y \mid y \precsim \lambda_{ij} \,\}) = \pi_{ij}(W_{ij}) = 1$. Therefore the probability dominance condition (PD) implies that, whenever $\pi(W_{ij}) > 0$, then $\pi_{ij} \precsim \lambda_{ij}$, and so, because of (53), that

$$U(\pi_{ij}) \leq U(\lambda_{ij}) = v^*_{ij} \tag{56}$$

Also, $\cup_{j=1}^\infty W_{ij}$ is a partition of Y_i ($i = 1, 2, \ldots, k$), implying that

$$\pi(Y_i) = \sum_{j=1}^\infty \pi(W_{ij}) \tag{57}$$

Because (55) implies that π is the countable mixture $\sum_{i=1}^k \sum_{j=1}^\infty \pi(W_{ij})\, \pi_{ij}$ of the measures π_{ij}, and because U satisfies (MP*), it follows from (56), (52), and (57) that

$$\begin{aligned} U(\pi) &= \sum_{i=1}^k \sum_{j=1}^\infty \pi(W_{ij})\, U(\pi_{ij}) \leq \sum_{i=1}^k \sum_{j=1}^\infty \pi(W_{ij})\, v^*_{ij} \\ &\leq \sum_{i=1}^k \sum_{j=1}^\infty \pi(W_{ij})\, v_i = \sum_{i=1}^k \pi(Y_i)\, v_i = \mathbb{E}_\pi v_0 \end{aligned}$$

by definition of v_0. This is true even when $\pi(W_{ij}) = 0$ for some i and j. Moreover $U(\pi) \leq \mathbb{E}_\pi v_0$ holds for all $v_0 \in V_0^+(v)$, so definition (50) of the Lebesgue integral implies that $\mathbb{E}_\pi v \geq U(\pi)$.

Similarly, replacing U by $-U$ and v by $-v$ throughout this demonstration shows that $\mathbb{E}_\pi(-v) \geq -U(\pi)$ or $\mathbb{E}_\pi v \leq U(\pi)$. This completes the proof that $U(\pi) = \mathbb{E}_\pi v$. ∎

9.4 Continuity of Expected Utility

Here, conditions for an NMUF $v : Y \to \mathbb{R}$ to be continuous will be investigated. So that continuity of v can have meaning, suppose that the consequence domain Y is a *metric space*. That is, there must be a *metric* $d : Y \times Y \to \mathbb{R}_+$ satisfying the three conditions that

(i) $d(x, y) = 0 \iff x = y$; (ii) $d(x, y) = d(y, x)$; (iii) $d(x, y) + d(y, z) \leq d(x, z)$

for all $x, y, z \in Y$. Moreover, assume that Y is *separable* in the sense that there exists a countable *dense* subset $S = \{ y_1, y_2, \ldots \}$ whose closure in the metric topology is the whole space Y. That is, for any $y \in Y$, there must exist a sequence $\langle x_k \rangle_{k=1}^\infty$ of points in S such that $d(x_k, y) \to 0$ as $k \to \infty$. As will explained further below, the separability of Y plays an important role in ensuring that any probability measure $\pi \in \Delta(Y, \mathcal{F})$ can be approximated arbitrarily closely by a simple probability measure in $\Delta(Y)$.

Let $\mathcal{B}$ denote the *Borel* σ-field — i.e., the smallest σ-field that includes all the open sets of Y. Then $\mathcal{B}$ also includes all closed subsets of Y, and many but not all subsets that are neither open nor closed. Condition (M) of Section 9.2 will now be strengthened to condition (M*), requiring that the σ-field $\mathcal{F}$ is equal to $\mathcal{B}$. In particular, the preference sets specified by (14) in Section 3.4 must belong to the Borel σ-field. Of course, when they are closed sets, this condition is automatically satisfied.

In Section 8.6, for discrete lotteries in the space $\Delta^*(Y)$, it was shown that the continuity condition (C) and dominance condition (D) could be replaced by the single continuous preference condition (CP). Here, for the space of probability measures $\Delta(Y, \mathcal{F})$, it will be shown that (C), (D) and the probability dominance condition (PD) can all be replaced by a single continuous preference condition (CP*) which strengthens (CP).

To discuss continuity of expected utility, the space $\Delta(Y, \mathcal{F})$ must also be given a topology. Following Grandmont (1972) and Hildenbrand (1974), it is customary to use the *topology of weak convergence of measures*. This extends the similar topology on $\Delta^*(Y)$ that was used in Section 8.6. Specifically, say that the sequence $\pi^n \in \Delta(Y, \mathcal{F})$ $(n = 1, 2, \ldots)$ *converges weakly* to $\pi \in \Delta(Y, \mathcal{F})$ iff, for every bounded continuous function $f : Y \to \mathbb{R}$, one has $\mathbb{E}_{\pi^n} f \to \mathbb{E}_\pi f$. In this case, write $\pi^n \xrightarrow{w} \pi$. Unlike for the corresponding topology on $\Delta^*(Y)$, here the function $f : Y \to \mathbb{R}$ is required to be continuous. Properties of this topology are fully discussed in Billingsley (1968) — see also Parthasarathy (1967), Billingsley (1971), Huber (1981, ch. 2) and Kirman (1981, pp. 196–8). It turns out that $\pi^n \xrightarrow{w} \pi$ iff $\pi^n(E) \to \pi(E)$ for every set $E \in \mathcal{F}$ whose boundary $\text{bd}\, E$ satisfies $\pi(\text{bd}\, E) = 0$. Also, that $d(y^n, y) \to 0$ in Y iff $1_{y^n} \xrightarrow{w} 1_y$ in the weak topology on $\Delta(Y, \mathcal{F})$. This implies in turn that the set $\{ 1_y \in \Delta(Y, \mathcal{F}) \mid y \in Y \}$ of degenerate lotteries is closed in the weak topology.

Suppose that the EU hypothesis is satisfied on $\Delta(Y, \mathcal{F})$ for an NMUF $v : Y \to \mathbb{R}$ which is both bounded and continuous. Then it is true by definition that the utility function $U(\pi) = \mathbb{E}_\pi v$ is continuous when $\Delta(Y, \mathcal{F})$ is given the topology of weak convergence. In particular, the induced preference ordering $\succsim$ satisfies the *continuous preference* condition (CP*) requiring that both preference sets

$$\{ \pi \in \Delta(Y, \mathcal{F}) \mid \pi \succsim \bar{\pi} \} \quad \text{and} \quad \{ \pi \in \Delta(Y, \mathcal{F}) \mid \pi \precsim \bar{\pi} \}$$

must be closed, for all $\bar{\pi} \in \Delta(Y, \mathcal{F})$.

More interesting is the converse, which includes (CP*) among a set of sufficient conditions for the EU hypothesis to be true with a continuous NMUF.

LEMMA 9.1: Condition (CP*) implies the corresponding condition (CP) for preferences restricted to $\Delta^*(Y)$.

PROOF: Suppose that $\lambda^n \in \Delta^*(Y)$ $(n = 1, 2, \ldots)$ and that, as $n \to \infty$, so $\lambda^n \to \lambda$ in the topology of $\Delta^*(Y)$. Then $\mathbb{E}_{\lambda^n} f \to \mathbb{E}_\lambda f$ for all bounded functions, and so for all bounded continuous functions $f : Y \to \mathbb{R}$. Therefore $\lambda^n \xrightarrow{w} \lambda$ as $n \to \infty$. So, if $\lambda^n \succsim \bar{\lambda}$ $(n = 1, 2, \ldots)$, then condition (CP*) implies that $\lambda \succsim \bar{\lambda}$. Similarly, if $\lambda^n \precsim \bar{\lambda}$ $(n = 1, 2, \ldots)$, then $\lambda \precsim \bar{\lambda}$. This confirms condition (CP). ∎

LEMMA 9.2: Conditions (O), (I*) and (CP*) together imply conditions (C*), (D) and (D*).

PROOF: Immediate from Lemmas 9.1, 8.6 and 8.7. ∎

LEMMA 9.3: Conditions (O), (I*), (M*) and (CP*) together imply condition (PD).

PROOF: Given any $\lambda \in \Delta(Y)$, define the two sets

$$Y^+(\lambda) := \{ y \in Y \mid 1_y \succsim \lambda \} \quad \text{and} \quad Y^-(\lambda) := \{ y \in Y \mid 1_y \precsim \lambda \}$$

By condition (CP*), the upper weak preference set $U := \{ \mu \in \Delta(Y, \mathcal{F}) \mid \mu \succsim \lambda \}$ is closed. So therefore is $Y^+(\lambda)$, as the intersection of U with the closed set $\{ 1_y \in \Delta(Y, \mathcal{F}) \mid y \in Y \}$. A similar argument shows that $Y^-(\lambda)$ is also closed.

Let $\Delta(Y^+(\lambda), \mathcal{F})$ denote the set of probability measures in $\Delta(Y, \mathcal{F})$ satisfying $\pi(Y^+(\lambda)) = 1$, and let $\Delta(Y^+(\lambda))$ be the subset of simple probability distributions in $\Delta(Y)$.

Suppose that $\pi \in \Delta(Y^+(\lambda), \mathcal{F})$. By Parthasarathy (1967, Theorem 6.3, p. 44), the set $\Delta(Y^+(\lambda))$ is dense in $\Delta(Y^+(\lambda), \mathcal{F})$ when $\Delta(Y, \mathcal{F})$ is given the topology of weak convergence. So π is the weak limit of a sequence of simple distributions $\pi^n \in \Delta(Y^+(\lambda))$ $(n = 1, 2, \ldots)$.[14] By Lemma 3.1, condition (I*) implies (FD*), so $\pi^n \succsim \lambda$ for $n = 1, 2, \ldots$. Therefore condition (CP*) implies that $\pi \succsim \lambda$.

A similar proof shows that, if $\pi(Y^-(\lambda)) = 1$, then $\pi \precsim \lambda$. Hence, condition (PD) is satisfied. ∎

LEMMA 9.4: Conditions (O), (I*), (M*) and (CP*) together imply the EU hypothesis, with a bounded continuous NMUF.

PROOF: By Theorem 9 together with Lemmas 9.2 and 9.3, the four conditions (O), (I*), (M) and (CP*) together imply that there exists a utility function $U(\pi)$ on $\Delta(Y, \mathcal{F})$ which represents $\succsim$ while also satisfying (MP). Suppose that $\pi^n \in \Delta(Y, \mathcal{F})$ $(n = 1, 2, \ldots)$ and that $\pi^n \xrightarrow{w} \bar{\pi} \in \Delta(Y, \mathcal{F})$ as $n \to \infty$.

Suppose first that there exists $\pi_+ \in \Delta(Y, \mathcal{F})$ for which $\pi_+ \succ \bar{\pi}$. Then for every $\epsilon > 0$ the set

$$\begin{aligned} & \{ \pi \in \Delta(Y, \mathcal{F}) \mid \epsilon\, \pi_+ + (1 - \epsilon)\, \bar{\pi} \succ \pi \succ \bar{\pi} \} \\ = \; & \{ \pi \in \Delta(Y, \mathcal{F}) \mid 0 < U(\pi) - U(\bar{\pi}) < \epsilon\, [U(\pi_+) - U(\bar{\pi})] \} \end{aligned}$$

is open. So there exists an integer $n_+(\epsilon)$ for which, if $U(\pi^n) > U(\bar{\pi})$ for any $n > n_+(\epsilon)$, then $U(\pi^n) - U(\bar{\pi}) < \epsilon\, [U(\pi_+) - U(\bar{\pi})]$.

[14] In fact, by a trivial application of the Glivenko–Cantelli Theorem (Parthasarathy, 1967, Theorem 7.1, p. 53), for any infinite sequence $\langle y_k \rangle_{k=1}^{\infty}$ of independently and identically distributed random draws from the probability distribution π on Y, one can approximate π by the associated sequence $\pi^n = n^{-1} \sum_{k=1}^{n} 1_{y_k} \in \Delta(Y^+(\lambda))$ $(n = 1, 2, \ldots)$ of empirical distributions.

Alternatively, suppose that there exists $\pi_- \in \Delta(Y, \mathcal{F})$ for which $\pi_- \prec \bar{\pi}$. In this case, reversing the preferences and inequalities in the argument of the previous paragraph shows that there must exist an integer $n_-(\epsilon)$ for which, if $U(\pi^n) < U(\bar{\pi})$ for any $n > n_-(\epsilon)$, then $U(\pi^n) - U(\bar{\pi}) > \epsilon\,[U(\pi_-) - U(\bar{\pi})]$.

The last two paragraphs together imply that $U(\pi^n) \to U(\bar{\pi})$ as $n \to \infty$. Therefore U is continuous when $\Delta(Y, \mathcal{F})$ is given the topology of weak convergence. Also, $y_n \to y$ as $n \to \infty$ implies that $1_{y_n} \xrightarrow{w} 1_y$. It follows that $v(y_n) = U(1_{y_n}) \to U(1_y) = v(y)$. This shows that the NMUF $v : Y \to \mathbb{R}$ is continuous. ∎

Thus, provided that conditions (I) and (M) are strengthened to (I*) and (M*) respectively, the one condition (CP*) can replace all three conditions (C), (D) and (PD) in the set of sufficient conditions listed in Theorem 9. Moreover, the resulting four sufficient conditions (O), (I*), (M*) and (CP*) imply the stronger conclusion that not only is the EU hypothesis satisfied for a bounded NMUF, but in fact any possible NMUF is continuous.

9.5 Consequentialism and Probability Dominance

The probability dominance condition (PD) can be given a consequentialist justification, just as condition (D) was in Section 8.7. Indeed, suppose that $\pi, \bar{\pi} \in \Delta(Y, \mathcal{F})$. Let $H \subset Y$ be any set large enough to satisfy $\pi(H) = 1$; when Y is a topological space and $\mathcal{F}$ coincides with its Borel σ-algebra, it is natural to take H as the *support* of π — i.e., the smallest closed set satisfying $\pi(H) = 1$.

Consider the infinite decision tree T with initial chance node n_0 whose immediate successors constitute the set $N_{+1}(n_0) = \{\, n_y \mid y \in H \,\}$. Corresponding to the σ-field $\mathcal{F}$ on Y and the set $H \subset Y$, define the collection

$$\mathcal{F}_N(H) := \{\, K_N \subset N_{+1}(n_0) \mid \exists K_Y \in \mathcal{F} : K_Y \subset H,\ K_N = \{\, n_y \mid y \in K_Y \,\}\,\}$$

It is easy to verify that $\mathcal{F}_N(H)$ is a σ-field on $N_{+1}(n_0)$. Moreover, suppose that the transition probabilities at n_0 satisfy

$$\pi(K_N|n_0) = \pi(\{\, y \in Y \mid n_y \in K_N \,\})$$

for every $K_N \in \mathcal{F}_N(H)$. Also, for each $y \in Y$, suppose that n_y is a decision node with $N_{+1}(n_y) = \{\, x_y, \bar{x}_y \,\}$, where x_y and $\bar{x}_y$ are terminal nodes whose lottery consequences are 1_y and $\bar{\pi}$ respectively. This tree is very similar to the one whose typical branch was illustrated in Figure 5 of Section 8.7.

Consider also a second decision tree T' with initial decision node n'_0 at which the set $N'_{+1}(n'_0)$ consists of a second decision node n'_1, together with a unique terminal node x'_K corresponding to each measurable set $K \in \mathcal{F}$ with $K \subset H$ and $0 < \pi(K) < 1$. Let $N'_{+1}(n'_1)$ consist of the two terminal nodes x'_π and $x'_{\bar{\pi}}$ whose lottery consequences are $\gamma'(x'_\pi) = \pi$ and $\gamma'(x'_{\bar{\pi}}) = \bar{\pi}$ respectively. Thus $F(T', n'_1) = \{\, \pi, \bar{\pi} \,\}$. Suppose too that, for each $K \in \mathcal{F}$ which satisfies both $K \subset H$ and $0 < \pi(K) < 1$, the lottery consequence $\gamma'(x'_K)$ at the corresponding terminal node x'_K is $\pi(K)\, \pi_{|K} + [1 - \pi(K)]\, \bar{\pi}$, where $\pi_{|K}$ denotes the conditional probability measure that is derived from π given the event K. This tree is very similar to the one represented in Figure 6 of Section 8.7.

Following the reasoning behind equation (35) of Section 6.2, the obvious feasible set $F(T)$ consists of all possible integrals of integrable selections from the correspondence $y \mapsto\!\!\!\to F(T, n_y) = \{\, 1_y, \bar{\pi} \,\}$, defined on the domain H as in Hildenbrand (1974, p. 53). That is

$$F(T) = \int_H F(T, n_y)\, \pi(dy) = \int_H \{\, 1_y, \bar{\pi} \,\}\, \pi(dy)$$

Evidently both $F(T)$ and $F(T')$ are equal to the set

$$\{\, \pi, \bar{\pi} \,\} \cup \{\, \pi(K)\, \pi_{|K} + [1 - \pi(K)]\, \bar{\pi} \mid K \in \mathcal{F},\ K \subset H,\ 0 < \pi(K) < 1 \,\} \quad (58)$$

Suppose that $\pi(\{\, y \in Y \mid y \succsim \bar{\pi} \,\}) = 1$. Then $\pi(\{\, y \in H \mid x_y \in \beta(T, n_y) \,\}) = 1$, and so $\pi = \int_H 1_y\, \pi(dy) \in \Phi_\beta(T)$. From this and (58), an obvious extension of the consequentialist hypothesis to the "measurable" trees T and T' implies that $\pi \in \Phi_\beta(T')$. But, unless $\bar{\pi} = \pi$, this is only possible if $\pi \in \Phi_\beta(T', n'_1)$. Because $\bar{\pi} \in F(T', n'_1)$, it follows that the implicit preference ordering must satisfy $\pi \succsim \bar{\pi}$.

On the other hand, if $\pi(\{\, y \in Y \mid y \precsim \bar{\pi} \,\}) = 1$, then one has $\pi(\{\, y \in H \mid \bar{x}_y \in \beta(T, n_y) \,\}) = 1$, and so $\bar{\pi} = \int_H \bar{\pi}\, \pi(dy) \in \Phi_\beta(T)$. The rest of the proof is as in the previous paragraph, but with π and $\bar{\pi}$ interchanged.

In both cases, condition (PD) is satisfied. As promised, it has been given a consequentialist justification.

10 Summary and Concluding Remarks

In Section 2, the EU hypothesis was stated and ratios of utility differences were interpreted as marginal rates of substitution between corresponding probability shifts. Utility is then determined only up to a unique cardinal equivalence class. Thereafter, the chapter has concentrated on necessary and sufficient conditions for the EU hypothesis to be valid for lotteries on a given consequence domain Y, as well as the "consequentialist" axioms that can be used to justify some of these conditions.

For the space $\Delta(Y)$ of simple lotteries with finite support, Sections 3 and 4 showed that necessary and sufficient conditions are ordinality (O), independence (I), and continuity (C). Actually, stronger versions (I*) and (C*) of conditions (I) and (C) were shown to be necessary. These classical results are well known.

The space $\Delta^*(Y)$ of discrete lotteries that can have countably infinite support was considered in Section 8. For this space, a somewhat less well known dominance condition (D) due to Blackwell and Girshick (1954) enters the set of necessary and sufficient conditions. Furthermore, utility must be bounded. Provided that (I*) is satisfied, a continuous preference condition (CP) can replace both conditions (C) and (D).

Finally, Section 9 considered the space $\Delta(Y, \mathcal{F})$ of probability measures on the σ-field $\mathcal{F}$ of measurable sets generated by the singleton and preference subsets of Y. Here two extra conditions enter the list — the obvious measurability condition (M), and a probability dominance condition (PD) that is different from condition (D). When the consequence domain Y is a separable metric space, then provided that conditions (I) and (M) are strengthened somewhat to (I*) and (M*), it is possible to replace conditions (C), (D) and (PD) with a single continuous preference condition (CP*). Moreover, then each utility function in the unique cardinal equivalence class must be continuous.

The heart of the chapter (Sections 5, 6 and 7) considered the implications of assuming that behaviour in an (almost) unrestricted domain of decision trees could be explained by its consequences while satisfying dynamic consistency in subtrees. This assumption was expressed through three "consequentialist" axioms. These three axioms were shown to imply conditions (O) and (I). By allowing a richer domain of trees, conditions (D) and (PD) could also be given a consequentialist justification. Conditions (C) (or (CB)) and (M), however, remain as supplementary hypotheses, without a consequentialist justification. All six conditions play an important role in the succeeding separate chapter on subjectively expected utility. So do the three consequentialist axioms.

The three consequentialist axioms appear natural when applied to decision trees. Nevertheless, McLennen (1990) and Cubitt (1996) have both offered interesting decompositions of these axioms into a larger set of individually weaker axioms. Moreover, these decompositions invoke the notion of a plan which could differ from actual behaviour.

At least one open problem remains. Theorem 7 of Section 7.3 offered a complete characterization of consequentialist behaviour satisfying dynamic consistency and continuity condition (CB) for an almost unrestricted domain of finite decision trees. Still lacking is a similar result for a richer domain of infinite decision trees giving rise to random consequences in the space $\Delta^*(Y)$ or $\Delta(Y, \mathcal{F})$. Indeed, beyond some results in classical decision analysis due to LaValle (1978), there appears to have been little systematic analysis of general infinite decision trees.

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