1 Introduction and Outline

The expected utility (EU) hypothesis was originally formulated to be used with specified or "objective" probabilities. Objectively expected utility is the subject of Chapter 5. Not all uncertainty, however, can be described by a specified or objective probability distribution. The pioneering work of Frank Ramsey (1926) and Bruno de Finetti (1937, 1949) demonstrated how, under certain assumptions, "subjective" probabilities could still be inferred from behaviour in the face of such uncertainty.[1] The task of this chapter is to set out some recent developments in this decision-theoretic approach to subjective probability, and in the closely associated theory of subjectively expected utility. As in Chapter 5, an explicitly "consequentialist" perspective will be maintained. A much more thorough survey of earlier developments can be found in Fishburn (1981) — see also Karni and Schmeidler (1991) and Fishburn (1994).

An important body of earlier work, especially that most directly inspired by Savage (1954), will not be covered here. One reason for this omission is that extensive reviews of Savage's particular approach can be found elsewhere, including Arrow (1965, 1971, 1972) and Fishburn (1970, ch. 14). A second reason is the continuing ready and affordable accessibility of Savage's original work. A third reason is that more recent writers such as Harsanyi (1977, 1983), Myerson (1979) and Fishburn (1982) have chosen, as I do here, to derive subjective probabilities from preferences for state dependent lotteries involving objective probabilities. Of course, this latter approach was first set out in full by Anscombe and Aumann (1963), though earlier related ideas can be found in Rubin (1949), Arrow (1951), Chernoff (1954), Suppes (1956), and Ellsberg (1961).[2]

A rather better reason suggests itself, however. Following Keynes (1921), de Finetti (1937) and many others, Savage (1954, pp. 30–32) used a crucial "ordering of events" assumption in order to derive "qualitative" probabilities. This axiom is described in Section 2.4. Later these qualitative probabilities are made quantitative as a result of additional assumptions. For example, Villegas (1964) and DeGroot (1970) simply assume that, for every $p \in [0, 1]$, there exists an event whose objective probability is p. Then the ordering of events assumption implies that every event E has subjective probability equal to the objective probability of an equally likely event. For surveys of qualitative probability

[1] De Finetti, who founds subjective probabilities on the willingness of a risk-neutral agent to take bets, cites the earlier work by Bertrand (1889) using a similar idea. But this concept surely pre-dates Bertrand — see, for example, the original works by Hobbes (1650, IV, 10) and Bayes (1763), as well as the recent commentaries by Hacking (1975) and Lindley (1987).

[2] Indeed, Ellsberg's "paradox" concerns violations of an extended sure thing principle that had not even been properly formulated before Anscombe and Aumann's pioneering work.

theory, see Narens (1980) and Fishburn (1986). However, in Section 3 it will be shown that without more structure such as objective probabilities, the ordering of events condition cannot be derived from the consequentialist axioms. For this reason, the consequentialist approach followed both in Chapter 5 and here does not lend support to Savage's axiom system.

Instead of objective probabilities, another approach which was pioneered by Ramsey uses a continuum of deterministic consequences and at least one even chance event — see Gul (1992) for a more modern exposition and references to other work. Here, an "even chance event" E has the property that, for all consequences y, y', the prospect of y if E occurs combined with y' if E does not occur is indifferent to the prospect of y' if E occurs combined with y if E does not occur. Thus, the events E and not E are regarded as equally likely, implying that E has a subjective probability of $\frac{1}{2}$, in effect. This condition turns out to be a significant weakening of the ordering of events assumption. In a similar way de Finetti (1937), followed by Pratt, Raiffa and Schlaifer (1964), proposed the *uniformity* axiom requiring that, for $m = 2, 3, \ldots$, there should be m equally likely events. Savage (1954, p. 33) also discusses this approach. Not surprisingly, one can then avoid postulating a continuum of deterministic consequences. Another implication of the examples in Section 3, however, is that neither do such assumptions of equally likely events have any consequentialist justification. Nevertheless, this chapter uses Anscombe and Aumann's assumption requiring objective probabilities to exist, which is in some sense a stronger form of uniformity. The view taken here is that it is more straightforward to assume directly that objective probabilities can appear in decision problems.

Finally, Wakker (1989) has a connected topological space of consequences but no even chance events. Instead, an assumption of "noncontradictory tradeoffs on consequences" is used to ensure an appropriate additively separable utility function. Once again, however, the examples in Section 3 show that this assumption lacks a consequentialist justification.

Following this introduction and outline, Section 2 of the chapter presents four key conditions which are necessary for behaviour to correspond to the maximization of subjective expected utility (or SEU) — i.e., expected utility with subjective probabilities attached to events. Actually it will be convenient to distinguish two different versions of the SEU hypothesis. Whereas maximizing SEU in general allows null events whose subjective probabilities must be zero, a stronger version of the hypothesis, called SEU*, excludes null events. Thus, SEU* requires all subjective probabilities to be positive.

Of the four necessary conditions for SEU maximization, the first states that, for each non-null event, there must be a contingent preference ordering over the possible consequences of acts, in Savage's form of "state contingent consequence functions" (or CCFs). Second, these orderings must satisfy Savage's sure thing principle (STP), which is the counterpart for subjectively expected utility theory of the equally controversial independence axiom in objectively expected utility theory. In fact, when SEU* is satisfied, a stronger version of (STP), called (STP*), must hold. Third, when either the state of the world is known, or the consequence is independent of the state of the world, there must be a state independent contingent preference ordering over certain consequences. Fourth and last, preferences over CCFs must induce an ordering of events according to relative likelihood.

Section 3 turns to consequentialist foundations. So that different states of the world may occur, decision trees are allowed to include "natural nodes", at which nature refines the set of possible states. The axioms that were set out in Chapter 5 for finite decision trees are then adapted in a rather obvious way. They imply the existence of a preference ordering over CCFs satisfying STP*. There need not be an implied ordering of events, however, nor subjective probabilities. Thus, SEU theory cannot be derived by applying consequentialism only to decision trees with natural nodes.

Next, Section 4 turns instead to the Anscombe and Aumann theory of subjective probability and subjectively expected utility. This approach is based on combinations of "horse lotteries", for which probabilities are not specified, with "roulette lotteries", for which they are. In fact, roulette lotteries involve objective probabilities, as considered in Chapter 5, and so implicitly the previously mentioned uniformity axiom must be satisfied. Anyway, a version of Anscombe and Aumann's axioms is set out, and their main theorem proved. This particular proof owes much to Fishburn (1970, ch. 13) and also to Myerson (1991, ch. 1).

In Section 5 the discussion returns to consequentialist analysis in decision trees, and a simplified version of the analysis set out in Hammond (1988). Corresponding to the distinction between horse and roulette lotteries, decision trees will contain "natural" nodes at whic nature moves but probabilities are not specified, as opposed to chance nodes of the kind considered in Chapter 5, where "objective" probabilities are specified. It will follow that decisions give rise to lotteries over CCFs rather than to sure CCFs. The decision tree which the agent faces will induce a strategic game against both chance and nature. The consequentialist axiom is then somewhat re-formulated so that it becomes normal form invariance in this strategic game. This axiom, together with both a form of state independence and also continuity of behaviour as probabilities vary, will justify the axioms used by Anscombe and Aumann. And actually these conditions imply rather more, since it will also turn out that null events having zero subjective probability are excluded. Thus, consequentialism justifies the stronger SEU* hypothesis.

The theories of Savage and of Anscombe and Aumann both rely on the assumption that there are "constant acts" yielding the same consequence in all states of the world. More precisely, they postulate that the domain of consequences is state independent. But there is a whole class of decision problems where this hypothesis makes no sense — for instance, where there is a risk of death or serious injury. See the chapter in this *Handbook* by Drèze and Rustichini as well as Karni (1993a, b). For such cases, Section 6 begins by finding sufficient conditions for behaviour to maximize the expectation of an evaluation function (Wilson, 1968).[3] Then it considers one possible way of deriving subjective probabilities and utilities in this case also. Moreover, the utilities will be state independent in the sense of giving equal value to any consequence that happens to occur in more than one state dependent consequence domain. The key is to consider decision trees having "hypothetical" probabilities attached to states of nature, following the suggestion of Karni, Schmeidler and Vind (1983), and even to allow hypothetical choices of these probabilities, as in Drèze (1961, 1987) and also Karni (1985).

The first part of the chapter will assume throughout that the set S of possible states of the world is finite. Section 7 considers the implications of allowing S to be countably infinite. Extra conditions of bounded utility, event continuity and event dominance are introduced. When combined with the conditions of Section 4, they are necessary and sufficient for the SEU hypothesis to hold. Necessity is obvious, whereas sufficiency follows as a special case of the results in Section 8.

Finally, Section 8 allows S to be a general measurable space, and considers measurable lotteries mapping S into objective probability measures on the consequence domain. For this domain, two more conditions were introduced in Section 9 of Chapter 5 — namely measurability of singletons and of upper and lower preference sets in the consequence domain, together with probability dominance. Not surprisingly, these two join the list of necessary and sufficient conditions for the SEU hypothesis. This gives eleven conditions in all, which are summarized in a table at the end of the brief concluding Section 9. Only this last part of the chapter requires some familiarity with measure theory,

[3]See also Myerson (1979) for a somewhat different treatment of this issue.

and with corresponding results for objective probability measures set out in Chapter 5.

2 Necessary Conditions

2.1 Subjective Expected Utility Maximization

Let Y be a fixed domain of possible *consequences*, and let S be a fixed finite set of possible *states of the world*. No probability distribution over S is specified. An *act*, according to Savage (1954), is a mapping $a : S \to Y$ specifying what consequence results in each possible state of the world. Inspired by the Arrow (1953) and Debreu (1959) device of "contingent" securities or commodities in general equilibrium theory, I shall prefer to speak instead of *contingent consequence functions*, or CCFs for short. Also, each CCF will be considered as a list $y^S = \langle y_s \rangle_{s \in S}$ of contingent consequences in the Cartesian product space $Y^S := \prod_{s \in S} Y_s$, where each set Y_s is a copy of the consequence domain Y.

The *subjective expected utility* (SEU) maximization hypothesis requires that there exist non-negative subjective or personal probabilities p_s of different states $s \in S$ satisfying $\sum_{s \in S} p_s = 1$. Also, there must be a von Neumann–Morgenstern utility function (or NMUF) $v : Y \to \mathbb{R}$, as in the objective expected utility (EU) theory considered in Chapter 5. Moreover, it is hypothesized that the agent will choose a CCF y^S from the relevant feasible set in order to maximize the *subjectively expected utility* function

$$U^S(y^S) := \sum_{s \in S} p_s \, v(y_s) \tag{1}$$

As in Chapter 5, the NMUF v could be replaced by any $\tilde{v}$ that is cardinally equivalent, without affecting EU maximizing behaviour. The difference from the earlier theory arises because the probabilities p_s are not objectively specified, but are revealed by the agent's behaviour. The SEU* hypothesis strengthens SEU by adding the requirement that the subjective probabilities satisfy $p_s > 0$ for all $s \in S$.

Obviously, if behaviour maximizes the utility function (1), there is a corresponding complete and transitive *preference ordering* $\succsim$ over the domain Y^S. This is the *ordering* condition (O). Let $\succ$ and $\sim$ respectively denote the corresponding strict preference and indifference relations. There is an uninteresting trivial case of *universal indifference* when $y^S \sim \tilde{y}^S$ for all $y^S, \tilde{y}^S \in Y^S$; in the spirit of Savage's P5 postulate, it will usually be assumed in this chapter that there exists at least one pair $\overline{y}^S, \underline{y}^S \in Y^S$ such that $\overline{y}^S \succ \underline{y}^S$.

2.2 Contingent Preferences and the Sure Thing Principle

An *event* E is any non-empty subset of S. Its *subjective probability* is defined as $P(E) := \sum_{s\in E} p_s$.

Given the function $U^S : Y^S \to \mathbb{R}$ defined by (1) and any event E, there is a *contingent expected utility* function U^E defined on the Cartesian subproduct $Y^E := \prod_{s\in E} Y_s$ by the partial sum

$$U^E(y^E) := \sum_{s\in E} p_s\, v(y_s) \tag{2}$$

Obviously, U^E induces a *contingent preference ordering* $\succsim^E$ on Y^E satisfying

$$y^E \succsim^E \tilde{y}^E \iff U^E(y^E) \geq U^E(\tilde{y}^E)$$

for all pairs $y^E, \tilde{y}^E \in Y^E$. This is intended to describe the agent's preference or behaviour given that E is the set of possible states, so that consequences $y^{S\setminus E} = \langle y_s \rangle_{s\in S\setminus E}$ are irrelevant. Evidently, then, for all pairs $y^E, \tilde{y}^E \in Y^E$ and all $\bar{y}^{S\setminus E} = \langle y_s \rangle_{s\in S\setminus E} \in Y^{S\setminus E} := \prod_{s\in S\setminus E} Y_s$ one has

$$y^E \succsim^E \tilde{y}^E \iff \sum_{s\in E} p_s\, v(y_s) \geq \sum_{s\in E} p_s\, v(\tilde{y}_s)$$

But the right hand side is true iff

$$\sum_{s\in E} p_s\, v(y_s) + \sum_{s\in S\setminus E} p_s\, v(\bar{y}_s) \geq \sum_{s\in E} p_s\, v(\tilde{y}_s) + \sum_{s\in S\setminus E} p_s\, v(\bar{y}_s)$$

and so

$$y^E \succsim^E \tilde{y}^E \iff (y^E, \bar{y}^{S\setminus E}) \succsim (\tilde{y}^E, \bar{y}^{S\setminus E}) \tag{3}$$

Furthermore, $(y^E, \bar{y}^{S\setminus E}) \precsim (\tilde{y}^E, \bar{y}^{S\setminus E})$ either for all $\bar{y}^{S\setminus E}$ because $y^E \sim^E \tilde{y}^E$, or else for no $\bar{y}^{S\setminus E}$ because $y^E \succ^E \tilde{y}^E$. It follows that $y^E \succsim^E \tilde{y}^E$ iff $y^E \succsim \tilde{y}^E$ *given* E, in the sense specified in Savage (1954, definition D1).

The requirement that (3) be satisfied, with the contingent ordering $\succsim^E$ on Y^E independent of $\bar{y}^{S\setminus E} \in Y^{S\setminus E}$, will be called the *sure thing principle* (or STP). It is a second implication of SEU maximization.[4]

[4] In Savage (1954) there is no formal statement of (STP), though the informal discussion on p. 21 seems to correspond to the "dominance" result stated as Theorem 3 on p. 26. This

When conditions (O) and (STP) are satisfied, say that the event E is *null* if $y^E \sim^E \tilde{y}^E$ for all $y^E, \tilde{y}^E \in Y^E$. Otherwise, if $y^E \succ^E \tilde{y}^E$ for some $y^E, \tilde{y}^E \in Y^E$, say that E is *non-null*. When the SEU hypothesis holds, event E is null iff its subjective probability $P(E) := \sum_{s \in E} p_s$ satisfies $P(E) = 0$, and E is non-null iff $P(E) > 0$. Note that S is null if and only if there is universal indifference; outside this trivial case, S is non-null. Of course, one also says that the state $s \in S$ is *null* iff the event $\{s\}$ is null, otherwise s is non-null. Obviously, under the SEU hypothesis, state $s \in S$ is null iff $p_s = 0$ and non-null iff $p_s > 0$.

The *strong sure thing principle*, or condition (STP*), requires that no state $s \in S$ be null. Thus, the difference between SEU* and SEU is that SEU* excludes null states and null events, whereas SEU allows them. When SEU* holds, so that no event is null, then condition (STP*) must be satisfied.

Suppose that $E \subset E' \subset S$ and that conditions (O) and (STP) are satisfied. Clearly, for all $y^E, \tilde{y}^E \in Y^E$ and all $\bar{y}^{S \setminus E} \in Y^{S \setminus E}$, it will be true that

$$y^E \succsim^E \tilde{y}^E \iff (y^E, \bar{y}^{S \setminus E}) \succsim (\tilde{y}^E, \bar{y}^{S \setminus E}) \iff (y^E, \bar{y}^{E' \setminus E}) \succsim^{E'} (\tilde{y}^E, \bar{y}^{E' \setminus E}) \tag{4}$$

2.3 State Independent Preferences

Given any event E and any consequence $y \in Y$, let $y\,1^E$ denote the *constant* or *sure consequence* with $y_s = y$ for all $s \in E$. It describes a CCF for which, even though there is uncertainty about the state of the world, the consequence y is certain. Note how (2) implies that

$$U^E(y\,1^E) = \sum_{s \in E} p_s\, v(y) = P(E)\, v(y)$$

for all $y \in Y$. Because $P(E) > 0$ whenever E is non-null, there is a *state independent* preference ordering $\succsim^*$ on Y such that

$$y \succsim^* \bar{y} \iff v(y) \geq v(\bar{y}) \iff y\,1^E \succsim^E \bar{y}\,1^E \text{ (all non-null events } E)$$

This condition (SI) is the third implication of SEU maximization.

Under conditions (O), (STP), and (SI), if $y \succ^* \bar{y}$, then $y\,1^E \succ^E \bar{y}\,1^E$ for all non-null events E.

theorem is a logical consequence of his postulates P1, which is condition (O), together with P2, which is close to what I have chosen to call the "sure thing principle", and P3 which, at least when combined with conditions (O) and (STP), is equivalent to the state-independence condition (SI) set out in Section 2.3 below.

2.4 A Likelihood Ordering of Events

Given two events $E_1, E_2 \subset S$, their respective subjective probabilities induce a clear "likelihood" ordering between them, depending on whether $P(E_1) \geq P(E_2)$ or *vice versa*. When the SEU hypothesis holds, this likelihood ordering can be derived from preferences. Indeed, suppose that $\overline{y}$, $\underline{y} \in Y$ with $\overline{y} \succ^* \underline{y}$. Consider then the two CCFs $(\overline{y}\, 1^{E_1}, \underline{y}\, 1^{S \setminus E_1})$ and $(\overline{y}\, 1^{E_2}, \underline{y}\, 1^{S \setminus E_2})$. Suppose that the better consequence $\overline{y}$ is interpreted as "winning", and the worse consequence $\underline{y}$ as "losing". Then the first CCF arises when the agent wins iff E_1 occurs and loses iff E_2 occurs; the second when winning and losing are interchanged. The subjective expected utilities of these two CCFs are

$$\begin{aligned} U^S(\overline{y}\, 1^{E_1}, \underline{y}\, 1^{S \setminus E_1}) &= \sum\nolimits_{s \in E_1} p_s\, v(\overline{y}) + \sum\nolimits_{s \in S \setminus E_1} p_s\, v(\underline{y}) \\ &= P(E_1)\, v(\overline{y}) + [1 - P(E_1)]\, v(\underline{y}) \\ \text{and} \quad U^S(\overline{y}\, 1^{E_2}, \underline{y}\, 1^{S \setminus E_2}) &= P(E_2)\, v(\overline{y}) + [1 - P(E_2)]\, v(\underline{y}) \end{aligned}$$

respectively. Therefore

$$U^S(\overline{y}\, 1^{E_1}, \underline{y}\, 1^{S \setminus E_1}) - U^S(\overline{y}\, 1^{E_2}, \underline{y}\, 1^{S \setminus E_2}) = [P(E_1) - P(E_2)]\, [v(\overline{y}) - v(\underline{y})]$$

implying that

$$(\overline{y}\, 1^{E_1}, \underline{y}\, 1^{S \setminus E_1}) \succsim^E (\overline{y}\, 1^{E_2}, \underline{y}\, 1^{S \setminus E_2}) \iff P(E_1) \geq P(E_2)$$

Moreover, this must be true no matter what the consequences $\overline{y}$, $\underline{y} \in Y$ may be, provided only that $\overline{y} \succ^* \underline{y}$. So the agent weakly prefers winning conditional on E_1 to winning conditional on E_2 iff E_1 is no less likely than E_2. The implication is condition (OE), which requires that a (complete and transitive) likelihood *ordering of events* can be inferred from preferences for winning as against losing conditional on those events. This is the fourth implication of SEU maximization. It is equivalent to Savage's (1954) P4 postulate.

The four conditions (O), (STP), (SI) and (OE) do not by themselves imply the SEU hypothesis. For example, they amount to only the first four of Savage's seven postulates — or the first five if one excludes the trivial case of universal indifference. I shall not present the last two postulates, except peripherally in Sections 7.2 and 7.3. Instead, I shall turn to consequentialist axioms like those set out in Section 5 of Chapter 5. It will turn out that these fail to justify (OE). In my view, this detracts considerably from the normative appeal of Savage's approach, though it remains by far the most important and complete theory of decision-making under uncertainty prior to 1960.

3 Consequentialist Foundations

3.1 Decision Trees with Natural Nodes

Chapter 5 was concerned with objective EU maximization. There the ordering and independence properties were derived from consequentialist axioms applied to decision trees incorporating chance nodes, at which random moves occurred with specified objective probabilities. Here, similar arguments will be applied to decision trees in which no probabilities are specified. The chance nodes that have specified probabilities will be replaced by natural nodes at each of which nature has a move that restricts the remaining set of possible states of nature.

Formally, then, the finite decision trees considered in this section all take the form

$$T = \langle\, N, N^*, N^1, X, n_0, N_{+1}(\cdot), S(\cdot), \gamma(\cdot)\,\rangle \tag{5}$$

As in Chapter 5, N denotes the set of all nodes, N^* the set of decision nodes, X the set of terminal nodes, n_0 the initial node, $N_{+1}(\cdot) : N \twoheadrightarrow N$ the immediate successor correspondence, and $\gamma(\cdot)$ the consequence mapping. The three new features are as follows.

First, N^1 denotes the set of natural nodes, replacing the earlier set N^0 of chance nodes.

Second, $S(\cdot) : N \twoheadrightarrow S$ denotes the *event correspondence*, with $S(n) \subset S$ as the set of states of the world which are still possible after reaching node n. Within the decision tree the agent is assumed to have perfect recall in the sense that $S(n') \subset S(n)$ whenever node n' follows node n in the decision tree. In fact, the event correspondence should have the properties that: (i) whenever $n \in N^*$ and $n' \in N_{+1}(n)$, then $S(n') = S(n)$ because the agent's decision at node n does not restrict the set of possible states; (ii) whenever $n \in N^1$, then $\cup_{n' \in N_{+1}(n)} S(n')$ is a partition of $S(n)$ into pairwise disjoint events because nature's move at n creates an information partition of $S(n)$.

Third, at each terminal node $x \in X$, where $S(x)$ is the set of states that remain possible, the consequence mapping γ determines a CCF $\gamma(x) \in Y^{S(x)}$ specifying, for each state $s \in S(x)$, a state contingent consequence $\gamma_s(x)$ in the fixed domain Y of possible consequences.[5]

[5] It might seem more natural to define each decision tree so that by the time a terminal node $x \in X$ is reached, all uncertainty must be resolved and so $S(x)$ is a singleton $\{s(x)\}$. I have avoided doing this, not just to increase generality, but more important, to allow results

3.2 Feasible and Chosen CCFs

Let T be any decision tree with natural nodes, as defined in (5). Then $F(T)$ will denote the set of feasible CCFs in T, whereas $\Phi_\beta(T)$ will denote the possible CCFs which can result from behaviour β. As in Chapter 5, these two sets are respectively equal to the values at $n = n_0$ of the sets

$$F(T, n) := F(T(n)) \quad \text{and} \quad \Phi_\beta(T, n) := \Phi_\beta(T(n))$$

where $T(n)$ denotes the continuation subtree T that starts with initial node n. Moreover, $F(T, n)$ and $\Phi_\beta(T, n)$ can be constructed by backward recursion, starting at terminal nodes $x \in X$ where

$$F(T, x) := \Phi_\beta(T, x) := \{\gamma(x)\} \subset Y^{S(x)} \tag{6}$$

At a decision node $n \in N^*$, one has $S(n') = S(n)$ (all $n' \in N_{+1}(n)$). Then

$$F(T, n) := \bigcup_{n' \in N_{+1}(n)} F(T, n') \quad \text{and} \quad \Phi_\beta(T, n) := \bigcup_{n' \in \beta(T(n), n)} \Phi_\beta(T, n') \tag{7}$$

where both are subsets of $Y^{S(n)} = Y^{S(n')}$. At a natural node $n \in N^1$, on the other hand, where $S(n)$ is partitioned into the pairwise disjoint sets $S(n')$ ($n' \in N_{+1}(n)$), one has Cartesian product sets of CCFs given by

$$F(T, n) = \prod\nolimits_{n' \in N_{+1}(n)} F(T, n') \quad \text{and} \quad \Phi_\beta(T, n) = \prod\nolimits_{n' \in N_{+1}(n)} \Phi_\beta(T, n') \tag{8}$$

Both of these are subsets of $Y^{S(n)} = \prod_{n' \in N_{+1}(n)} Y^{S(n')}$. It is easy to prove by backward induction that, for all $n \in N$, including $n = n_0$, one has

$$\emptyset \neq \Phi_\beta(T, n) \subset F(T, n) \subset Y^{S(n)} \tag{9}$$

3.3 Consequentialism and Contingent Orderings

As in Section 5.5 of Chapter 5, the consequentialist hypothesis requires that there exist a consequence choice function specifying how the behaviour set (of possible consequences of behaviour) depends upon the feasible set of consequences. Here, however, any such consequence is a CCF $y^S \in Y^S$. In fact, each different event $E \subset S$ gives rise to a different domain Y^E of possible CCFs. Accordingly, for each event $E \subset S$ there must be a corresponding event-contingent

in Section 5 of Chapter 5 to be applied directly, especially Theorem 5 concerning the existence of a preference ordering.

choice function C^E_β defined on the domain of non-empty finite subsets of Y^E, and satisfying

$$\Phi_\beta(T) = C^E_\beta(F(T)) \subset F(T) \subset Y^E \tag{10}$$

for all decision trees T such that $S(n_0) = E$ at the initial node n_0 of T.

Consider now any fixed event E, and the restricted domain of all finite decision trees with no natural nodes which have the property that $S(n) = E$ at all nodes $n \in N$, including at all terminal nodes. On this restricted domain, impose the *dynamic consistency* hypothesis that $\beta(T(n), n') = \beta(T, n')$ at any decision node n' of the continuation tree $T(n)$. Then the arguments in Section 5.6 of Chapter 5 apply immediately and yield the result that C^E_β must correspond to a preference ordering R^E_β on the set Y^E. Thus, the family of event-contingent choice functions $\{\, C^E_\beta \mid \emptyset \neq E \subset S \,\}$ gives rise to a corresponding family of *contingent preference orderings* $\{\, R^E_\beta \mid \emptyset \neq E \subset S \,\}$ such that, whenever Z is a non-empty finite subset of Y^E, then

$$C^E_\beta(Z) = \{\, y^E \mid \tilde{y}^E \in Z \Longrightarrow y^E \; R^E_\beta \; \tilde{y}^E \,\} \tag{11}$$

3.4 Consequentialism and the Sure Thing Principle

In Section 6 of Chapter 5 it was argued that, for decision trees with chance nodes, consequentialism implied the independence axiom. Here, for decision trees with natural nodes, a very similar argument establishes that consequentialism implies the sure thing principle (STP) described in equation (3) of Section 2.2. Indeed, suppose that E_1 and E_2 are disjoint events in S, whereas $E = E_1 \cup E_2$, and $a^{E_1}, b^{E_1} \in Y^{E_1}, c^{E_2} \in Y^{E_2}$. Then consider the tree T as in (5), with

$$\begin{aligned}
N^* = \{n_1\}; \quad N^1 = \{n_0\}; \quad X = \{\, x_a, x_b, x_c \,\}; \\
N_{+1}(n_0) = \{\, n_1, x_c \,\}; \quad N_{+1}(n_1) = \{\, x_a, x_b \,\}; \\
S(n_0) = E; \quad S(n_1) = S(x_a) = S(x_b) = E_1; \quad S(x_c) = E_2; \\
\gamma(x_a) = a^{E_1}; \quad \gamma(x_b) = b^{E_1}; \quad \gamma(x_c) = c^{E_2}.
\end{aligned}$$

This tree is illustrated in Figure 1. Notice how (6) and (7) imply that

$$F(T, n_1) \;=\; F(T, x_a) \cup F(T, x_b) = \{a^{E_1}\} \cup \{b^{E_1}\} \;=\; \{\, a^{E_1}, b^{E_1} \,\}$$

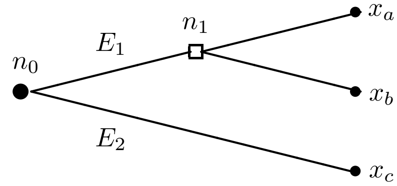

Figure 1 Decision Tree Illustrating the Sure Thing Principle

Then (8) implies that

$$\begin{aligned} F(T) = F(T, n_0) &= F(T, n_1) \times F(T, x_c) \\ &= F(T, n_1) \times \{c^{E_2}\} = \{\, (a^{E_1}, c^{E_2}),\, (b^{E_1}, c^{E_2})\, \} \end{aligned}$$

Also (11), (10) and (8) together imply that

$$\begin{aligned} & a^{E_1}\ R_\beta^{E_1}\ b^{E_1} \quad \iff \quad a^{E_1} \in C_\beta(F(T, n_1)) = \Phi_\beta(T, n_1) \\ \iff \quad & (a^{E_1}, c^{E_2}) \in \Phi_\beta(T, n_1) \times \{c^{E_2}\} = \Phi_\beta(T, n_1) \times \Phi_\beta(T, x_c) \\ & \qquad = \Phi_\beta(T, n_0) = C_\beta(F(T, n_0)) \\ \iff \quad & (a^{E_1}, c^{E_2})\ R_\beta^E\ (b^{E_1}, c^{E_2}) \end{aligned}$$

This is exactly the sure thing principle (STP), as expressed by (4) in Section 2.2, but applied to the contingent orderings $R_\beta^{E_1}$ and R_β^E instead of to $\succsim^E$ and $\succsim^{E'}$ respectively.

3.5 Consequentialism Characterized

In fact, consequentialist behaviour satisfying dynamic consistency on an unrestricted domain of finite decision trees is possible whenever there exist, for all events $E \subset S$, contingent orderings R_β^E that satisfy (STP). To see this, consider *any* family of contingent orderings R^E ($\emptyset \neq E \subset S$) satisfying (STP). Essentially the same arguments as in Section 6.4 of Chapter 5 can then be used to construct behaviour β satisfying the consequentialist axioms whose family of contingent revealed preference orderings satisfies $R_\beta^E = R^E$ whenever $\emptyset \neq E \subset S$. The proof of Lemma 6.1 in that chapter does need extending to deal with one new case, which is when $n \in N^1$ is a natural node. However, this is a routine modification of the proof for when $n \in N^0$ is a chance node, with condition (STP) replacing condition (I). Also, see the subsequent Section 5.5 in this chapter for a proof which applies in a more general setting.

3.6 Unordered Events

It has just been shown that consequentialist behaviour is completely characterized by any family of contingent orderings satisfying (STP). Therefore, no further restrictions on behaviour can be inferred from consequentialism, unless these restrictions are implications of there being a family of orderings satisfying (STP). In particular, the crucial ordering of events condition (OE) is generally violated, implying that consequentialist behaviour does *not* maximize SEU.

To confirm this, it is enough to exhibit a family of contingent preference orderings that fails to induce an ordering of events despite satisfying (STP*). So, let $S = \{ s_1, s_2 \}$ and $Y = \{ a, b, c \}$. Then define the state independent utility function $v : Y \to \mathbb{R}$ so that:

$$v(a) = 1; \quad v(b) = 0; \quad v(c) = -1. \tag{12}$$

Now consider the preference ordering on Y^S induced by the specific additive utility function

$$U^S(y^S) = \phi_1(v(y_{s_1})) + \phi_2(v(y_{s_2})) \tag{13}$$

where ϕ_1 and ϕ_2 are increasing functions satisfying

$$\begin{array}{rclrclrcl} \phi_1(1) & = & 2, & \phi_1(0) & = & 0, & \phi_1(-1) & = & -1 \\ \phi_2(1) & = & 1, & \phi_2(0) & = & 0, & \phi_2(-1) & = & -2 \end{array} \tag{14}$$

Suppose that the two contingent orderings on Y_{s_1} and Y_{s_2} are represented by the utility functions $\phi_1(v(y_{s_1}))$ and $\phi_2(v(y_{s_2}))$ respectively. Neither state is null, because neither $\phi_1(v(y))$ nor $\phi_2(v(y))$ are constant functions independent of y. Because (13) has an additive form, (STP*) is evidently satisfied. Moreover, the preferences on Y_{s_1}, Y_{s_2} are even state independent, as are those on the set $Y\,1^S := \{ (y_{s_1}, y_{s_2}) \in Y_{s_1} \times Y_{s_2} \mid y_{s_1} = y_{s_2} \}$, since all are represented by the same utility function $v(y)$. Nevertheless

$$\begin{array}{rclrcl} U^S(a,b) & = & 2, & U^S(b,a) & = & 1 \\ U^S(b,c) & = & -2, & U^S(c,b) & = & -1 \end{array}$$

So the agent's behaviour reveals a preference for winning a in state s_1 to winning it in state s_2, when the alternative losing outcome is b. On the other hand, it also reveals a preference for winning b in state s_2 to winning it in state s_1, when the alternative losing outcome is c. Hence, there is no induced ordering of the events $\{s_1\}$ and $\{s_2\}$.

Savage, of course, introduced other postulates whose effect is to ensure a rather rich set of states — see the later discussion in Section 7.2. Adding such postulates, however, in general will not induce an ordering of events. To see this, suppose that S is the entire interval $[0,1]$ of the real line instead of just the doubleton $\{ s_1, s_2 \}$. Instead of the additive utility function (13), consider the integral

$$\bar{U}^S(y^S) = \int_0^{1/2} \phi_1(v(y(s)))ds + \int_{1/2}^1 \phi_2(v(y(s)))ds$$

with v given by (12) and ϕ_1, ϕ_2 by (14). Also, so that the integrals are well defined, y^S should be a measurable function from S to Y, in the sense that the set $\{ s \in S \mid y(s) = y \}$ is measurable for all $y \in Y$. Then the particular CCF $y^S = \left(a\, 1_{[0,\frac{1}{2}]}, b\, 1_{(\frac{1}{2},1]}\right)$ with

$$y(s) = \begin{cases} a & \text{if} \quad s \in [0, \frac{1}{2}] \\ b & \text{if} \quad s \in (\frac{1}{2}, 1] \end{cases}$$

is preferred to the lottery represented by $\left(b\, 1_{[0,\frac{1}{2}]}, a\, 1_{(\frac{1}{2},1]}\right)$ in the same notation. But $\left(c\, 1_{[0,\frac{1}{2}]}, b\, 1_{(\frac{1}{2},1]}\right)$ is preferred to $\left(b\, 1_{[0,\frac{1}{2}]}, c\, 1_{(\frac{1}{2},1]}\right)$. So there is no induced likelihood ordering of the two events $[0, \frac{1}{2}]$ and $(\frac{1}{2}, 1]$. In fact, it is easy to confirm that this example satisfies Savage's six postulates P1–P3 and P5–P7; only the ordering of events postulate P4 is violated. See Wakker and Zank (1996) for a systematic study of the rather rich extra possibilities which arise when all but P4 of Savage's seven postulates are satisfied.

4 Anscombe and Aumann's Axioms

4.1 Horse Lotteries versus Roulette Lotteries

Anscombe and Aumann's (1963) article is the definitive statement of a different approach to the derivation of subjective probabilities. They allowed subjective probabilities for the outcomes of "horse lotteries" or CCFs to be inferred from expected utility representations of preferences over compounds of horse and "roulette lotteries". Formally, the framework of Section 2 is extended to allow preferences over, not only CCFs of the form $y^E \in Y^E$ for some non-empty $E \subset S$, but also (finitely supported) simple roulette lotteries in the space $\Delta(Y)$, as considered in Chapter 5. And in fact general compound lotteries are allowed, taking the form of simple lotteries $\lambda^E \in \Delta(Y^E)$ attaching the objective probabilities $\lambda^E(y^E)$ to CCFs $y^E \in Y^E$ instead of to consequences $y \in Y$. Then the finite collection of random variables y_s $(s \in E)$ has a multivariate distribution with probabilities $\lambda^E(y^E)$.

Within this extended framework, the SEU hypothesis extends that of Section 2 by postulating that there is a preference ordering $\succsim$ over the domain $\Delta(Y^S)$ of mixed lotteries, and that $\succsim$ is represented by the objective expected utility function

$$U^S(\lambda^S) = \sum\nolimits_{y^S \in Y^S} \lambda^S(y^S)\, v^S(y^S) \tag{15}$$

where $v^S(y^S)$ is the subjective expected utility function defined by

$$v^S(y^S) := \sum\nolimits_{s \in S} p_s\, v(y_s) \tag{16}$$

Thus $U^S(\lambda^S)$ involves the double expectation w.r.t. both the objective probabilities $\lambda^S(y^S)$ of different CCFs $y^S \in Y^S$ and the subjective probabilities p_s of different states $s \in S$. The SEU* hypothesis implies in addition that $p_s > 0$ for all $s \in S$.

4.2 Ratios of Utility Differences

Let a^S, b^S, c^S be any three CCFs in Y^S with $v^S(b^S) \neq v^S(c^S)$. As in Section 2.3 of Chapter 5, the ratio $[v^S(a^S) - v^S(c^S)]/[v^S(b^S) - v^S(c^S)]$ of utility differences is equal to the constant marginal rate of substitution (MRS) between, on the one hand, an increase in the probability of a^S that is compensated by an equal decrease in the probability of c^S, and on the other hand, an increase in the probability of b^S that is also compensated by an equal decrease in the probability of c^S. Furthermore, because only these ratios of utility differences are uniquely determined, each NMUF v^S is unique only up to a cardinal equivalence class. Also, because of (16), one has $v^S(y\,1^S) = v(y)$ for all $y \in Y$ and

$$v^S(a, b\,1^{S\setminus\{s\}}) = p_s\, v(a) + (1 - p_s)\, v(b) = v(b) + p_s\,[v(a) - v(b)]$$

for all $a, b \in Y$ and all $s \in S$. Provided that $v(a) \neq v(b)$, it follows that the subjective probability of each state $s \in S$ is given by

$$p_s = \frac{v^S(a, b\,1^{S\setminus\{s\}}) - v(b)}{v(a) - v(b)} = \frac{v^S(a, b\,1^{S\setminus\{s\}}) - v^S(b\,1^S)}{v^S(a\,1^S) - v^S(b\,1^S)} \tag{17}$$

It is therefore the constant MRS between an increase in the probability of the CCF $(a, b\,1^{S\setminus\{s\}})$ that is compensated by an equal decrease in the probability of $b\,1^S$, and an increase in the probability of $a\,1^S$ that is also compensated by an equal decrease in the probability of $b\,1^S$. Note that this MRS must be independent of the two consequences a, b. Of course, the uniqueness of each such MRS implies that each subjective probability p_s ($s \in S$) is unique, even though the utility function is unique only up to a cardinal equivalence class.

One particular advantage of Anscombe and Aumann's version of the SEU hypothesis is that subjective probabilities can be interpreted in this way. No interpretation quite as simple emerges from Savage's version of the theory.

4.3 Ordinality, Independence and Continuity

As obvious notation, let $1_{y^S} \in \Delta(Y^S)$ denote the degenerate lottery which attaches probability 1 to the CCF y^S. Next, define $v^S(y^S) := U^S(1_{y^S})$ for every CCF $y^S \in Y^S$. Note how (15) implies that $v^S(y^S) = \sum_{s \in S} p_s\, v(y_s)$, which is exactly the SEU expression that was introduced in (1) of Section 2.1. Equation (15) also implies that

$$U^S(\lambda^S) = \sum\nolimits_{y^S \in Y^S} \lambda^S(y^S)\, v^S(y^S) = \mathbb{E}_{\lambda^S}\, v^S$$

Hence, the preference ordering $\succsim$ is represented by the objectively expected value of the NMUF $v^S : Y^S \to \mathbb{R}$. So, as discussed in Chapter 5, the following three conditions must be satisfied:

(O) *Ordering.* There exists a preference ordering $\succsim$ on $\Delta(Y^S)$.

(I*) *Strong Independence Axiom.* For any $\lambda^S, \mu^S, \nu^S \in \Delta(Y^S)$ and $0 < \alpha \leq 1$, it must be true that

$$\lambda^S \succsim \mu^S \iff \alpha\lambda^S + (1-\alpha)\nu^S \succsim \alpha\mu^S + (1-\alpha)\nu^S$$

(C*) *Strong Continuity as Probabilities Vary.* For each $\lambda^S, \mu^S, \nu^S \in \Delta(Y^S)$ with $\lambda^S \succ \mu^S \succ \nu^S$, the two sets

$$A := \{\alpha \in [0,1] \mid \alpha\lambda^S + (1-\alpha)\nu^S \succsim \mu^S\}$$
$$B := \{\alpha \in [0,1] \mid \alpha\lambda^S + (1-\alpha)\nu^S \precsim \mu^S\}$$

must both be closed in $[0,1]$.

Indeed, these three properties are precisely those that were used to characterize objective EU maximization in Chapter 5. More precisely, conditions (O), (I*) and (C*) were shown to be necessary; for sufficiency, conditions (I*) and (C*) could be replaced by the following two weaker conditions, which both apply for each $\lambda^S, \mu^S, \nu^S \in \Delta(Y^S)$:

(I) *Independence.* Whenever $0 < \alpha \leq 1$, then

$$\lambda^S \succ \mu^S \Longrightarrow \alpha\lambda^S + (1-\alpha)\nu^S \succ \alpha\mu^S + (1-\alpha)\nu^S$$

(C) *Continuity.* Whenever $\lambda^S \succ \mu^S$ and $\mu^S \succ \nu^S$, there must exist $\alpha', \alpha'' \in (0,1)$ such that

$$\alpha'\lambda^S + (1-\alpha')\nu^S \succ \mu^S \quad \text{and} \quad \mu^S \succ \alpha''\lambda^S + (1-\alpha'')\nu^S$$

4.4 Reversal of Order

For each $y \in Y$ and $s \in S$, define $Y^S_s(y) := \{\, y^S \in Y^S \mid y_s = y \,\}$ as the set of CCFs yielding the particular consequence y in state s. Then, given any $\lambda^S \in \Delta(Y^S)$ and any $s \in S$, define

$$\lambda_s(y) := \sum\nolimits_{y^S \in Y^S_s(y)} \lambda^S(y^S) \tag{18}$$

Note that $\lambda_s(y) \geq 0$ and that

$$\sum\nolimits_{y \in Y} \lambda_s(y) = \sum\nolimits_{y^S \in Y^S} \lambda^S(y^S) = 1$$

Therefore λ_s is itself a simple probability distribution in $\Delta(Y)$, called the *marginal distribution* of the consequence y_s occurring in state s. Moreover, (15), (16) and (18) imply that

$$U^S(\lambda^S) = \sum\nolimits_{y^S \in Y^S} \lambda^S(y^S) \sum\nolimits_{s \in S} p_s\, v(y_s) = \sum\nolimits_{s \in S} p_s \sum\nolimits_{y \in Y} \lambda_s(y)\, v(y) \tag{19}$$

thus demonstrating that only the marginal probabilities $\lambda_s(y)$ $(s \in S,\ y \in Y)$ matter in the end. So the SEU hypothesis also implies:

(RO) *Reversal of Order.* Whenever $\lambda^S, \mu^S \in \Delta(Y^S)$ have marginal distributions satisfying $\lambda_s = \mu_s$ for all $s \in S$, then $\lambda^S \sim \mu^S$.

This condition owes its name to the fact that the compound lottery λ^S, in which a roulette lottery determines the random CCF y^S before the horse lottery that resolves which $s \in S$ occurs, is indifferent to the reversed compound lottery in which the horse lottery is resolved first, and its outcome $s \in S$ determines which marginal roulette lottery λ_s generates the ultimate consequence y.

In particular, suppose that $\mu^S = \prod_{s \in E} \lambda_s$ is the *product lottery* defined, for all $y^S = \langle y_s \rangle_{s \in S} \in Y^S$, by $\mu^S(y^S) := \prod_{s \in S} \lambda_s(y_s)$. Thus, the different random consequences y_s ($s \in S$) all have independent distributions. Then condition (RO) requires λ^S to be treated as equivalent to μ^S, whether or not the different consequences y_s ($s \in S$) are correlated random variables when the joint distribution is λ^S. Only marginal distributions matter. So any $\lambda^S \in \Delta(Y^S)$ can be regarded as equivalent to the list $\langle \lambda_s \rangle_{s \in S}$ of corresponding marginal distributions. This has the effect of reducing the space $\Delta(Y^S)$ to the Cartesian product space $\prod_{s \in S} \Delta(Y_s)$, with $Y_s = Y$ for all $s \in S$.

4.5 Sure Thing Principle

For each event $E \subset S$, there is obviously a corresponding *contingent expected utility function*

$$U^E(\lambda^E) = \sum_{s \in E} p_s \sum_{y \in Y} \lambda_s(y)\, v(y) \tag{20}$$

which represents the contingent preference ordering $\succsim^E$ on the set $\Delta(Y^E)$.

Given $\lambda^E, \mu^E \in \Delta(Y^E)$ and $\nu^{S \setminus E} \in \Delta(Y^{S \setminus E})$, let $(\lambda^E, \nu^{S \setminus E})$ denote the combination of the conditional lottery λ^E if E occurs with $\nu^{S \setminus E}$ if $S \setminus E$ occurs. Similarly for $(\mu^E, \nu^{S \setminus E})$. Then the following extension of the sure thing principle (STP) in Section 2.2 can be derived in exactly the same way as (3):

(STP) *Sure Thing Principle.* Given any event $E \subset S$, there exists a contingent preference ordering $\succsim^E$ on $\Delta(Y^E)$ satisfying

$$\lambda^E \succsim^E \mu^E \iff (\lambda^E, \nu^{S \setminus E}) \succsim (\mu^E, \nu^{S \setminus E})$$

for all $\lambda^E, \mu^E \in \Delta(Y^E)$ and all $\nu^{S \setminus E} \in \Delta(Y^{S \setminus E})$.

The following preliminary Lemma 4.1 shows that the four conditions (O), (I*), (RO) and (STP) are not logically independent, In fact, as Raiffa (1961) implicitly suggests in his discussion of the Ellsberg paradox, condition (STP) is an implication of the three conditions (O), (I*) and (RO) — see also Blume, Brandenburger and Dekel (1991).

LEMMA 4.1: Suppose that the three axioms (O), (I*), and (RO) are satisfied on $\Delta(Y^S)$. Then so is (STP).

PROOF: Consider any event $E \subset S$ and also any lotteries $\lambda^E, \mu^E \in \Delta(Y^E)$, $\bar{\nu}^{S \setminus E} \in \Delta(Y^{S \setminus E})$ satisfying $(\lambda^E, \bar{\nu}^{S \setminus E}) \succsim (\mu^E, \bar{\nu}^{S \setminus E})$. For any other lottery $\nu^{S \setminus E} \in \Delta(Y^{S \setminus E})$, axioms (I*) and (RO) respectively imply that

$$\tfrac{1}{2}(\lambda^E, \nu^{S \setminus E}) + \tfrac{1}{2}(\lambda^E, \bar{\nu}^{S \setminus E}) \quad \succsim \quad \tfrac{1}{2}(\lambda^E, \nu^{S \setminus E}) + \tfrac{1}{2}(\mu^E, \bar{\nu}^{S \setminus E})$$

$$\sim \quad \tfrac{1}{2}(\mu^E, \nu^{S\setminus E}) + \tfrac{1}{2}(\lambda^E, \bar{\nu}^{S\setminus E})$$

But then transitivity of $\succsim$ and axiom (I*) imply that $(\lambda^E, \nu^{S\setminus E}) \succsim (\mu^E, \nu^{S\setminus E})$. This confirms condition (STP). ∎

4.6 State Independence

For each non-null state $s \in S$ there is an associated event $\{s\} \subset S$. Because $p_s > 0$, according to (20) the corresponding contingent preference ordering $\succsim^{\{s\}}$ on the set $\Delta(Y)$ is represented by the conditional objectively expected utility function $\sum_{y \in Y} \lambda_s(y)\, v(y)$. This makes the following condition (SI) an obvious implication of the fact that the NMUF v is independent of s:

(SI) *State Independence.* Given any non-null state $s \in S$, the contingent preference ordering $\succsim^{\{s\}}$ over $\Delta(Y^{\{s\}}) = \Delta(Y)$ is independent of s; let $\succsim^*$ denote this state independent preference ordering, which must satisfy conditions (O), (I*) and (C*), of course.

To summarize, the SEU hypothesis implies the six conditions (O), (I), (C), (RO), (STP), and (SI). Of course, the stronger SEU* hypothesis has the same implications, except that there can be no null events.

4.7 Sufficient Conditions for the SEU and SEU* Hypotheses

The principal contribution of Anscombe and Aumann (1963) was to demonstrate how, in effect, these six conditions (O), (I), (C), (RO), (STP), and (SI) are sufficient for the SEU hypothesis to hold.[6] In particular, unlike Savage, there was no explicit need for the ordering of events assumption. Nor for the assumption that events can be refined indefinitely. Moreover, they were able to give a much simpler proof, based on the corresponding result for the objective version of the EU hypothesis. Their proof, however, requires that there exist both best and worst consequences in the domain Y. The proof given here relaxes this unnecessary requirement. It proceeds by way of several intermediate lemmas. The key Lemma 4.4 is proved by means of an elegant argument that apparently originated with Fishburn (1970, p. 176).

[6] In fact Anscombe and Aumann assumed the EU hypothesis directly when the state of the world is known. This has the effect of merging conditions (O), (I) and (C) into one.

Of course, the six conditions (O), (I), (C), (RO), (STP), and (SI) are assumed throughout.

LEMMA 4.2: (a) Suppose that $E \subset S$ is an event and that $\lambda^E, \mu^E \in \Delta(Y^E)$ satisfy $\lambda_s \succsim^* \mu_s$ for all $s \in E$. Then $\lambda^E \succsim^E \mu^E$. (b) If in addition $\lambda^E \sim^E \mu^E$, then $\lambda_s \sim^* \mu_s$ for every non-null state $s \in E$.

PROOF: The proof is by induction on m, the number of states in E. For $m = 1$, the result is trivial. Suppose that $m > 1$. As the induction hypothesis, suppose that the result is true for any event with $m - 1$ states.

Let s' be any state in E, and let $E' := E \setminus \{s'\}$. If $\lambda_s \succsim^* \mu_s$ for all $s \in E$, then the same is true for all $s \in E' \subset E$, so the induction hypothesis implies that $\lambda^{E'} \succsim^{E'} \mu^{E'}$. But $\lambda_{s'} \succsim^* \mu_{s'}$ also, and so for every $\nu^{S \setminus E'} \in \Delta(Y^{S \setminus E'})$, applying (STP) twice yields

$$
\begin{aligned}
(\lambda^E, \nu^{S \setminus E}) = (\lambda^{E'}, \lambda_{s'}, \nu^{S \setminus E}) \; & \succsim \; (\mu^{E'}, \lambda_{s'}, \nu^{S \setminus E}) \\
& \succsim \; (\mu^{E'}, \mu_{s'}, \nu^{S \setminus E}) = (\mu^E, \nu^{S \setminus E})
\end{aligned}
\tag{21}
$$

Then (STP) implies $\lambda^E \succsim^E \mu^E$, which confirms that (a) holds for E.

To prove (b), suppose in addition that $\lambda^E \sim \mu^E$. Then (STP) implies that $(\lambda^E, \nu^{S \setminus E}) \sim (\mu^E, \nu^{S \setminus E})$. From (21) and transitivity of $\succsim$, it follows that

$$
(\lambda^{E'}, \lambda_{s'}, \nu^{S \setminus E}) \sim (\mu^{E'}, \lambda_{s'}, \nu^{S \setminus E}) \sim (\mu^{E'}, \mu_{s'}, \nu^{S \setminus E}) \tag{22}
$$

But then (STP) implies $\lambda^{E'} \sim^{E'} \mu^{E'}$, so the induction hypothesis implies that $\lambda_s \sim^* \mu_s$ for all non-null $s \in E'$. However, (22) and (STP) also imply that $\lambda_{s'} \sim^{s'} \mu_{s'}$ and so, unless s' is null, that $\lambda_{s'} \sim^* \mu_{s'}$. Therefore $\lambda_s \sim^* \mu_s$ for all non-null $s \in E$.

The proof by induction is complete. ∎

Suppose it were true that $\lambda \sim^* \mu$ for all pure roulette lotteries $\lambda, \mu \in \Delta(Y)$. Because S is finite, Lemma 4.2 would then imply that $\lambda^S \sim \mu^S$ for all $\lambda^S, \mu^S \in \Delta(Y^S)$. However, the ordering $\succsim$ could then be represented by the trivial subjective expected utility function $\sum_{s \in E} p_s \, U^*(\lambda_s)$ for arbitrary subjective probabilities p_s and any constant utility function satisfying $U^*(\lambda) = c$ for all $\lambda \in \Delta(Y)$. So from now on, exclude the trivial case of universal indifference by assuming there exist two pure roulette lotteries $\overline{\lambda}, \underline{\lambda} \in \Delta(Y)$ with $\overline{\lambda} \succ^* \underline{\lambda}$. Equivalently, assume that the set S is not itself a null event.

The key idea of the following proof involves adapting the previous construction of an NMUF for the objective version of the EU hypothesis in Chapter 5. Because $\succsim^*$ satisfies conditions (O), (I) and (C), Lemma 4.5 of Chapter 5 can be applied. It implies that $\succsim^*$ can be represented by a normalized expected utility function $U^* : \Delta(Y) \to \mathbb{R}$ which satisfies

$$U^*(\underline{\lambda}) = 0 \quad \text{and} \quad U^*(\overline{\lambda}) = 1 \tag{23}$$

and also the *mixture preservation* property (MP) that, whenever $\lambda, \mu \in \Delta(Y)$ and $0 \leq \alpha \leq 1$, then

$$U^*(\alpha\,\lambda + (1-\alpha)\,\mu) = \alpha\,U^*(\lambda) + (1-\alpha)\,U^*(\mu) \tag{24}$$

As an obvious extension of the notation introduced in Section 2.3, given any event $E \subset S$ and any lottery $\lambda \in \Delta(Y)$, let $\lambda\,1^E$ denote the lottery in $\Delta(Y^E)$ whose marginal distribution in each state $s \in E$ is $\lambda_s = \lambda$, independent of s.

Lemma 4.3: Given any $\lambda, \mu \in \Delta(Y)$, one has

$$\lambda \succsim^* \mu \iff \lambda\,1^S \succsim \mu\,1^S$$

Proof: Lemma 4.2 immediately implies that $\lambda \succsim^* \mu \Longrightarrow \lambda\,1^S \succsim \mu\,1^S$. On the other hand, because the event S cannot be null, Lemma 4.2 also implies that $\mu \succ^* \lambda \Longrightarrow \mu\,1^S \succ \lambda\,1^S$. But $\succsim^*$ and $\succsim$ are complete orderings, so $\lambda\,1^S \succsim \mu\,1^S \Longrightarrow \mu\,1^S \not\succ \lambda\,1^S \Longrightarrow \mu \not\succ^* \lambda \Longrightarrow \lambda \succsim^* \mu$. ∎

By Lemma 4.3, $\overline{\lambda}\,1^S \succ \underline{\lambda}\,1^S$. Because the ordering $\succsim$ on $\Delta(Y^S)$ satisfies conditions (O), (I) and (C), Lemma 4.5 of Chapter 5 shows that $\succsim$ can also be represented by a normalized expected utility function $U^S : \Delta(Y^S) \to \mathbb{R}$ which satisfies

$$U^S(\underline{\lambda}\,1^S) = 0 \quad \text{and} \quad U^S(\overline{\lambda}\,1^S) = 1 \tag{25}$$

and also (MP). Then Lemma 4.6 of Chapter 5 and Lemma 4.3 above imply that $U^S(\lambda\,1^S)$ and $U^*(\lambda)$ must be cardinally equivalent functions of λ on the domain $\Delta(Y)$. Because of the two normalizations (23) and (25), for all $\lambda \in \Delta(Y)$ one has

$$U^*(\lambda) = U^S(\lambda\,1^S) \tag{26}$$

Next, define the functions $g_s : \Delta(Y) \to \mathbb{R}$ and constants p_s (all $s \in S$) by

$$g_s(\lambda) := U^S(\underline{\lambda}\,1^{S\setminus\{s\}}, \lambda) \quad \text{and} \quad p_s := g_s(\overline{\lambda}) \tag{27}$$

Evidently, because of the normalization (25), it must be true that

$$g_s(\underline{\lambda}) = 0 \tag{28}$$

LEMMA 4.4: For all $\lambda^S \in \Delta(Y^S)$ one has

$$U^S(\lambda^S) \equiv \sum_{s\in S} p_s\, U^*(\lambda_s) \tag{29}$$

where $p_s = 0$ iff s is null¿ Also $p = \langle p_s \rangle_{s\in S}$ is a probability distribution in $\Delta(S)$ because $p_s \geq 0$ for all $s \in S$ and $\sum_{s\in S} p_s = 1$.

PROOF: Let m be the number of elements in the finite set S. For all $\lambda^S \in \Delta(Y^S)$, the two members

$$\sum_{s\in S} \frac{1}{m}\,(\underline{\lambda}\,1^{S\setminus\{s\}}, \lambda_s) \quad \text{and} \quad \frac{m-1}{m}\,\underline{\lambda}\,1^S + \frac{1}{m}\,\lambda^S \tag{30}$$

of $\Delta(Y^S)$ have the common marginal distribution $(1-\frac{1}{m})\,\underline{\lambda} + \frac{1}{m}\,\lambda_s$ for each $s \in S$. So condition (RO) implies that they are indifferent. Because U^S satisfies (MP), applying U^S to the two indifferent mixtures in (30) gives the equality

$$\sum_{s\in S} \frac{1}{m}\, U^S(\underline{\lambda}\,1^{S\setminus\{s\}}, \lambda_s) = \frac{m-1}{m}\, U^S(\underline{\lambda}\,1^S) + \frac{1}{m}\, U^S(\lambda^S) \tag{31}$$

But $U^S(\underline{\lambda}\,1^S) = 0$ by (25), so (31) and definition (27) imply that

$$U^S(\lambda^S) = \sum_{s\in S} U^S(\underline{\lambda}\,1^{S\setminus\{s\}}, \lambda_s) = \sum_{s\in S} g_s(\lambda_s) \tag{32}$$

Because of (27) and (28), if s is a null state, then $g_s(\lambda) = 0$ for all $\lambda \in \Delta(Y)$. In particular, $p_s = g_s(\overline{\lambda}) = 0$. Otherwise, if s is not null, then (STP) and (27) jointly imply that $p_s := g_s(\overline{\lambda}) > 0$.

Because the function U^S satisfies (MP), equations (26) and (27) evidently imply that the functions U^* and g_s $(s \in S)$ do the same. Also, by (STP), $g_s(\lambda)$ and $U^*(\lambda)$ both represent $\succsim^{\{s\}}$ on $\Delta(Y)$ while satisfying (MP). So by Lemma 4.6 of Chapter 5, they must be cardinally equivalent utility functions. By (23) and (28), $U^*(\underline{\lambda}) = g_s(\underline{\lambda}) = 0$. Hence, there exists $\rho > 0$ for which

$$g_s(\lambda) \equiv \rho\, U^*(\lambda) \tag{33}$$

By (27), putting $\lambda = \overline{\lambda}$ in (33) yields $p_s = g_s(\overline{\lambda}) = \rho\, U^*(\overline{\lambda}) = \rho$, where the last equality is true because $U^*(\overline{\lambda}) = 1$, by (23). Therefore (33) becomes $g_s(\lambda) \equiv p_s\, U^*(\lambda)$. Substituting this into (32) gives (29). Finally, (25), (29) and (23) jointly imply that

$$1 = U^S(\overline{\lambda}\,1^S) = \sum_{s\in S} p_s\, U^*(\overline{\lambda}) = \sum_{s\in S} p_s$$

which completes the proof. ∎

Theorem 4: Under conditions (O), (I), (C), (RO), (STP), and (SI), there exists a unique cardinal equivalence class of NMUFs $v : Y \to \mathbb{R}$ and, unless there is universal indifference, unique subjective probabilities p_s $(s \in S)$ such that the ordering $\succsim$ on $\Delta(Y^S)$ is represented by the expected utility function

$$U^S(\lambda^S) \equiv \sum\nolimits_{s \in S} p_s \sum\nolimits_{y \in Y} \lambda_s(y)\, v(y)$$

Proof: First, define $v^*(y) := U^*(1_y)$ for all $y \in Y$. Because λ_s is the (finite) mixture $\sum_{y \in Y} \lambda_s(y)\, 1_y$ and U^* satisfies (MP), it follows that $U^*(\lambda_s) = \sum_{y \in Y} \lambda_s(y)\, v^*(y)$ for all $s \in S$. But then, by Lemma 4.4, one has

$$U^S(\lambda^S) = \sum\nolimits_{s \in S} p_s\, U^*(\lambda_s) = \sum\nolimits_{s \in S} p_s \sum\nolimits_{y \in Y} \lambda_s(y)\, v^*(y)$$

As in Chapter 5, v^* could be replaced by any cardinally equivalent NMUF $v : Y \to \mathbb{R}$. But, whenever s is non-null, any such replacement leaves the ratio of utility differences on the right hand side of (17) in Section 4.2 unaffected. On the other hand, $p_s = 0$ iff s is null. So the subjective probabilities p_s are unique. ▮

When there is no null event, $p_s > 0$ for all $s \in S$, so SEU* is satisfed instead of SEU.

5 Consequentialist Foundations Reconsidered

5.1 Decision Trees with Both Chance and Natural Nodes

Simple finite decision trees were introduced in Section 5.2 of Chapter 5, and trees with chance nodes in Section 6.1 of that chapter. Section 3.1 of this chapter introduced decision trees with natural nodes but no chance nodes. Now, in order to allow both horse and roulette lotteries to arise as consequences of decisions, it is natural to consider decision trees with both natural and chance nodes, like those in Hammond (1988). These take the form

$$T = \langle\, N, N^*, N^0, N^1, X, n_0, N_{+1}(\cdot), \pi(\cdot|\cdot), S(\cdot), \gamma(\cdot)\,\rangle$$

Compared with (5) of Section 3.1, the tree T now has the set of all nodes N partitioned into four instead of only three sets; the new fourth set is N^0, the set of chance nodes. Also, at each $n \in N^0$ the transition probabilities $\pi(n'|n)$ are specified for all $n' \in N_{+1}(n)$, just as in Section 6.1 of Chapter 5. For reasons explained in that chapter, it is assumed that each $\pi(n'|n) > 0$. In addition, at each terminal node $x \in X$ the consequence takes the form of a lottery $\gamma(x) \in \Delta(Y^{S(x)})$ over CCFs.

The construction of the feasible set $F(T)$ and of the behaviour set $\Phi_\beta(T)$ proceeds by backward recursion, much as it did in Sections 5.5 and 6.2 of Chapter 5 and in Section 3.2 of this chapter. When n is a natural node, however, (8) needs reinterpreting because the sets

$$\prod\nolimits_{n' \in N_{+1}(n)} F(T, n') \quad \text{and} \quad \prod\nolimits_{n' \in N_{+1}(n)} \Phi_\beta(T, n') \tag{34}$$

are no longer Cartesian products. Instead, the set $S(n) \subset S$ is partitioned into the disjoint non-empty sets $S(n')$ ($n' \in N_{+1}(n)$), so that $Y^{S(n)}$ is the Cartesian product $\prod_{n' \in N_{+1}(n)} Y^{S(n')}$. Now, given the finite collection of lotteries $\lambda(n') \in \Delta(Y^{S(n')})$ ($n' \in N_{+1}(n)$), their *probabilistic product* $\prod_{n' \in N_{+1}(n)} \lambda(n')$, like the product lottery defined in Section 4.4, is the lottery $\lambda(n) \in \Delta(Y^{S(n)})$ which satisfies

$$\lambda(n)\,(y^{S(n)}) := \prod\nolimits_{n' \in N_{+1}(n)} \lambda(n')\,(y^{S(n')})$$

for all combinations $y^{S(n)} = \langle\, y^{S(n')}\,\rangle_{n' \in N_{+1}(n)} \in Y^{S(n)}$. That is, $\lambda(n)$ is the multivariate joint distribution of $y^{S(n)}$ which arises when each of its components $y^{S(n')}$ ($n' \in N_{+1}(n)$) is an independent random variable with distribution $\lambda(n')$. Then the sets in (34) are *probabilistic product sets* consisting of all the possible probabilistic products of independent lotteries $\lambda(n')$ which, for each $n' \in N_{+1}(n)$, belong to the sets $F(T, n')$ and $\Phi_\beta(T, n')$ respectively.

Furthermore, because $\gamma(x) \in \Delta(Y^{S(x)})$ at each terminal node $x \in X$, from (6) and (7) it follows that, instead of (9), the constructed sets satisfy

$$\emptyset \neq \Phi_\beta(T, n) \subset F(T, n) \subset \Delta(Y^{S(n)})$$

for all $n \in N$ including $n = n_0$.

5.2 Consequentialism and Contingent Utilities

In this new framework, the consequentialist hypothesis of Chapter 5 and of Section 3.3 requires that, for each event $E \subset S$, there must be a contingent choice function C^E_β defined on the domain of non-empty finite subsets of $\Delta(Y^E)$, and satisfying

$$\Phi_\beta(T) = C_\beta^E(F(T)) \subset F(T) \subset \Delta(Y^E) \tag{35}$$

for all decision trees T such that $S(n_0) = E$ at the initial node n_0 of T. Suppose also that behaviour is dynamically consistent on the almost unrestricted domain of finite decision trees in which all transition probabilities $\pi(n'|n)$ ($n \in N^0$; $n' \in N_{+1}(n)$) are positive.

Consider the restricted domain of decision trees with no natural nodes. Arguing as in Sections 5.6 and 6.3 of Chapter 5, in the first place each C_β^E must correspond to a contingent revealed preference ordering R_β^E on the set $\Delta(Y^E)$; this is condition (O), of course. Second, each contingent ordering R_β^E must satisfy the strong independence condition (I*).

Moreover, the following *sure thing principle for independent lotteries* must be satisfied. Suppose that the two events $E_1, E_2 \subset S$ are disjoint, and that $E = E_1 \cup E_2$. Then, whenever $\lambda^{E_1}, \mu^{E_1} \in \Delta(Y^{E_1})$, and $\nu^{E_2} \in \Delta(Y^{E_2})$, arguing as in Section 3.4 shows that

$$\lambda^{E_1} \; R_\beta^{E_1} \; \mu^{E_1} \iff \left(\lambda^{E_1} \times \nu^{E_2}\right) \; R_\beta^E \; \left(\mu^{E_1} \times \nu^{E_2}\right) \tag{36}$$

where $\lambda^{E_1} \times \nu^{E_2}$ and $\mu^{E_1} \times \nu^{E_2}$ denote probabilistic products. Condition (36) remains weaker than (STP). Nevertheless, the next subsection introduces an extra assumption that implies condition (RO). Then (STP) will follow from (36) because $\lambda^{E_1} \times \nu^{E_2}$ and $\mu^{E_1} \times \nu^{E_2}$ will be indifferent to $(\lambda^{E_1}, \nu^{E_2})$ and (μ^{E_1}, ν^{E_2}) respectively; only the marginal distributions will matter.

Suppose that behaviour also satisfies the continuity hypothesis discussed in Section 7.1 of Chapter 5. Then, for each non-empty $E \subset S$, there is a unique cardinal equivalence class of NMUFs $v^E : Y^E \to \mathbb{R}$ whose expected values represent the revealed preference ordering R_β^E. The complete family of all possible NMUFs v^E ($\emptyset \neq E \subset S$) is characterized in Hammond (1988).

5.3 Consequentialist Normal Form Invariance and Condition (RO)

One of the six conditions discussed in Section 4 was (RO) — reversal of order. An implication of the results in Hammond (1988) is that this condition cannot be deduced from the other consequentialist axioms introduced so far. But one can argue that it is reasonable to impose it anyway as an additional axiom.

Indeed, let λ^S be any lottery in $\Delta(Y^S)$. Then there exist CCFs $y_i^S \in Y^S$ and associated probabilities $q_i \geq 0$ $(i = 1, 2, \ldots, k)$ such that $\sum_{i=1}^k q_i = 1$ and $\lambda^S = \sum_{i=1}^k q_i \, 1_{y_i^S}$. For each $s \in S$ the corresponding marginal distribution is $\lambda_s = \sum_{i=1}^k q_i \, 1_{y_{is}}$. Now consider two decision trees T and T' described as follows.

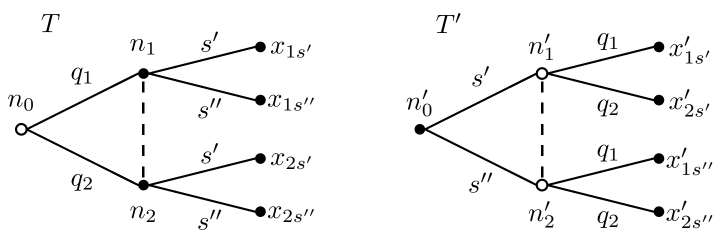

Figure 2 Decision trees T and T' (when $k = 2$ and $E = \{ s', s'' \}$)

Tree T begins with the chance node n_0, which is succeeded by the set of natural nodes $N_{+1}(n_0) = \{ n_i \mid i = 1, 2, \ldots, k \}$. The transition probabilities are $\pi(n_i|n_0) = q_i$ $(i = 1, 2, \ldots, k)$. Each $n_i \in N_{+1}(n_0)$ is succeeded by the set of terminal nodes $N_{+1}(n_i) = \{ x_{is} \mid s \in S \}$. The tree T is illustrated in the left half of Figure 2 for the case when $k = 2$ and $S = \{ s', s'' \}$. The consequences are assumed to be given by $\gamma(x_{is}) = 1_{y_{is}} \in \Delta(Y)$ (all $s \in S$).

On the other hand, tree T' begins with the natural node n'_0, whose successors form the set $N'_{+1}(n'_0) = \{ n'_s \mid s \in S \}$. Then each $n'_s \in N'_{+1}(n'_0)$ is a chance node whose successors form the set $N'_{+1}(n'_s) = \{ x'_{is} \mid i = 1, 2, \ldots, k \}$ of terminal nodes. The transition probabilities are $\pi'(x'_{is}|n'_s) = q_i$ $(i = 1, 2, \ldots, k)$. The tree T' is illustrated in the right half of Figure 2, again for the case when $k = 2$ and $S = \{ s', s'' \}$. The consequences are assumed to be given by $\gamma'(x'_{is}) = 1_{y_{is}} \in \Delta(Y)$ (all $s \in S$).

Both trees represent a three-person extensive game between chance, nature, and the decision maker, who actually has no decision to make. In tree T it is natural to assume that $N_{+1}(n_0)$ is a single information set for nature. Similarly, in tree T' it is natural to assume that $N'_{+1}(n'_0)$ is a single information set for chance. Then the extensive form games represented by the two trees will have identical normal forms, in which the decision maker has only one strategy, whereas chance's strategies are indexed by $i \in \{ 1, 2, \ldots, k \}$ and nature's strategies are indexed by $s \in S$. Furthermore, in either extensive form game, when chance chooses i and nature chooses s, the consequence is $y_{is} \in Y$. In fact, the *consequentialist normal form invariance* condition is that trees like T and T' with identical three-person normal forms should be regarded as giving rise to equivalent feasible sets, and that behaviour should generate equivalent choice sets of consequences in each case. This is a natural extension of the normal form invariance hypothesis which will be discussed in a later chapter on utility in non-cooperative games.

Evidently, in tree T the only feasible consequence is $\lambda^S = \sum_{i=1}^k q_i \, 1_{y_i^S}$, whereas in tree T' it is $\langle \lambda_s \rangle_{s \in S} = \langle \sum_{i=1}^k q_i \, 1_{y_{is}} \rangle_{s \in S}$. Consequentialist normal form invariance requires these consequences to be regarded as equivalent. But this is precisely condition (RO), as stated in Section 4.4.

5.4 State Independent Consequentialism, No Null Events, and Continuity

Let T and T' be two finite decision trees such that $S(n) = \{s\}$ for all $n \in N$ and $S'(n') = \{s'\}$ for all $n' \in N'$, so only a single state of nature is possible in each tree. Thus, there is really no uncertainty about the state. Suppose also that the feasible sets $F(T)$ and $F(T')$ of random consequences are equal when regarded as subsets of the set $\Delta(Y)$. Then a very minor extension of the consequentialist hypothesis (35), to be called *state independent consequentialism*, requires that the behaviour sets $\Phi_\beta(T)$ and $\Phi_\beta(T')$ should be equal subsets of $F(T) = F(T') \subset \Delta(Y)$. Thus, because only consequences matter and the state is certain, that state should not affect the consequences of behaviour. Equivalently, there must be a state independent consequence choice function C^*_β satisfying $C^*_\beta = C^{\{s\}}_\beta$ for all $s \in S$. Obviously, this implies that the preference orderings $R^{\{s\}}_\beta$ are state independent, so condition (SI) of Section 4.6 must be satisfied — i.e., there must exist R^*_β such that $R^{\{s\}}_\beta = R^*_\beta$ for all $s \in S$.

Next, note that $s \in S$ will be a null state iff $\lambda \; I^{\{s\}}_\beta \; \mu$ for all $\lambda, \mu \in \Delta(Y)$. Now, state independent consequentialism implies that C^*_β must equal $C^{\{s\}}_\beta$ for *all* $s \in S$, not only for all non-null $s \in S$. In particular, the indifference relation $I^{\{s\}}_\beta$ must be state independent. It follows that, if any state $s \in S$ is null, then all states are null. Because S is finite, Lemma 4.2 implies that then we must be in the trivial case of universal indifference throughout the domain $\Delta(Y^S)$, with $\Phi_\beta(T) = F(T)$ for every finite decision tree T. This is similar to the result noted in Section 6.3 of Chapter 5 that, if zero probabilities are allowed into decision trees, the consequentialist axioms imply universal indifference.

Outside this trivial case, therefore, state independent consequentialism implies that there can be no null events.

As for the continuity conditions (C) and (C*) of Section 4.3, the stronger condition (C*) follows provided that behaviour satisfies the continuity hypothesis set out in Section 7.1 of Chapter 5. Then all six conditions of Section 4 in this chapter must be satisfied. It has therefore been shown that the usual consequentialist axioms, supplemented by the continuity hypothesis of Chapter 5 and by the above state independence hypothesis, imply that all six conditions of Theorem 4 are met and also that null events are impossible. By Theorem 4, the SEU* hypothesis must be satisfied.

5.5 Sufficient Conditions for Consequentialism

State independent consequentialist behaviour that is dynamically consistent on the almost unrestricted domain of finite decision trees is completely characterized by the existence of a preference ordering $\succsim$ on $\Delta(Y^S)$ satisfying conditions (I*), (RO), (STP), and (SI), as well as the absence of null states. It has been shown that these conditions are necessary. Conversely, if these conditions are all satisfied, then consequentialist and dynamically consistent behaviour β can be defined on an almost unrestricted domain of finite decision trees in order to satisfy the property that, for each event $E \subset S$, the contingent revealed preference ordering R^E_β is equal to the contingent preference ordering $\succsim^E$ on $\Delta(Y^E)$. This can be proved by essentially the same argument as in Chapter 5. Indeed, for each node n of each finite decision tree T, define

$$\Psi(T,n) := \{\, \lambda \in F(T,n) \mid \mu \in F(T,n) \Longrightarrow \lambda \ \succsim^{S(n)} \ \mu \,\} \tag{37}$$

as in Section 6.4 of Chapter 5. Also, for any decision node $n \in N^*$, let $\beta(T(n), n)$ be any subset of $N_{+1}(n)$ with the property that

$$\bigcup_{n' \in \beta(T(n),n)} \Psi(T,n') = \Psi(T,n)$$

As was argued in Chapter 5, $\Psi(T,n) \neq \emptyset$ for all $n \in N$, and $\beta(T,n) \neq \emptyset$ at every decision node n of tree T. Obviously, to make β dynamically consistent, put $\beta(T,n^*) = \beta(T(n^*), n^*)$ whenever n^* is a decision node of tree T.

Now Lemma 6.1 of Chapter 5 can be proved almost as before, showing in particular that $\Phi_\beta(T,n) = \Psi(T,n)$ everywhere. This last equality evidently implies that $R^{S(n)}_\beta = \succsim^{S(n)}$. Apart from minor changes in notation, with

$\succsim^{S(n)}$ replacing $\succsim$ throughout, the only new feature of the proof by backward induction is the need to include an extra case 3, when $n \in N^1$ is a natural node. The following proof is a routine modification of that for when $n \in N^0$ is a chance node.

When $n \in N^1$ is a natural node, the relevant new version of the induction hypothesis used in proving Lemma 6.1 of Chapter 5 is that $\Phi_\beta(T, n') = \Psi(T, n')$ for all $n' \in N_{+1}(n)$. Recall too that

$$\Phi_\beta(T, n) = \prod\nolimits_{n' \in N_{+1}(n)} \Phi_\beta(T, n') \quad \text{and} \quad F(T, n) = \prod\nolimits_{n' \in N_{+1}(n)} F(T, n') \tag{38}$$

where, as in (34) of this chapter, the products are probabilistic.

Suppose that $\lambda \in \Phi_\beta(T, n)$ and $\mu \in F(T, n)$. By (38), for all $n' \in N_{+1}(n)$ there exist $\lambda(n') \in \Phi_\beta(T, n') = \Psi(T, n')$ and $\mu(n') \in F(T, n')$ such that

$$\lambda = \prod\nolimits_{n' \in N_{+1}(n)} \lambda(n'), \quad \mu = \prod\nolimits_{n' \in N_{+1}(n)} \mu(n') \tag{39}$$

For all $n' \in N_{+1}(n)$, because $\lambda(n') \in \Psi(T, n')$, one has $\lambda(n') \succsim^{S(n')} \mu(n')$. By repeatedly applying condition (STP) as in the proof of Lemma 4.2, it follows that $\lambda \succsim^{S(n)} \mu$. This is true for all $\mu \in F(T, n)$, so (37) implies that $\lambda \in \Psi(T, n)$.

Conversely, suppose that $\lambda \in \Psi(T, n)$. For all $n' \in N_{+1}(n)$, there must then exist $\lambda(n') \in F(T, n')$ such that $\lambda = \prod_{n' \in N_{+1}(n)} \lambda(n')$. So for any $n'' \in N_{+1}(n)$ and any $\mu(n'') \in F(T, n'')$ it must be true that

$$\lambda = \prod\nolimits_{n' \in N_{+1}(n)} \lambda(n') \succsim^{S(n)} \mu(n'') \times \prod\nolimits_{n' \in N_{+1}(n) \setminus \{n''\}} \lambda(n') \tag{40}$$

Applying condition (STP) to (40) implies that $\lambda(n'') \succsim^{S(n'')} \mu(n'')$. Since this holds for all $\mu(n'') \in F(T, n'')$, it follows that $\lambda(n'') \in \Psi(T, n'')$. But this is true for all $n'' \in N_{+1}(n)$. Hence, by the induction hypothesis, $\lambda(n') \in \Psi(T, n') = \Phi_\beta(T, n')$ for all $n' \in N_{+1}(n)$. So (38) and (39) together imply that $\lambda \in \Phi_\beta(T, n)$.

Therefore $\Phi_\beta(T, n) = \Psi(T, n)$ also when $n \in N^1$. The proof by induction is complete for this third case as well.

5.6 Dynamic Programming and Continuous Behaviour

Suppose that there are no null states in S. Suppose too that, for each event $E \subset S$, the contingent preference ordering $\succsim^E$ is represented by the *conditional* expected utility function

$$\sum_{s\in E} p_s \left[\sum_{y\in Y} \lambda_s(y)\, v(y)\right] / \sum_{s\in E} p_s$$

This is proportional to $U^E(\lambda^E)$ given by (20) in Section 4.5, but with the strictly positive subjective probabilities p_s replaced by conditional probabilities $p_s/\sum_{s\in E} p_s$. Let T be any finite decision tree. As in Section 7.2 of Chapter 5, a node valuation function $w(T,\cdot) : N \to \mathbb{R}$ can be constructed by backward recursion. The process starts at each terminal node $x \in X$, where $w(T,x)$ is simply the conditional expected utility of the random consequence $\gamma(x) = \langle \gamma_s(x)\rangle_{s\in S(x)} \in \Delta(Y^{S(x)})$. That is,

$$w(T,x) := \sum_{s\in S(x)} p_s \sum_{y\in Y} \gamma_s(x)(y)\, v(y) / \sum_{s\in S(x)} p_s$$

As in Section 7.2 of Chapter 5, at any decision node $n \in N^*$ one has

$$w(T,n) := \max_{n'} \{\, w(T,n') \mid n' \in N_{+1}(n) \,\}$$

while at any chance node $n \in N^0$ one has

$$w(T,n) := \sum_{n'\in N_{+1}(n)} \pi(n'|n)\, w(T,n')$$

Now, at any natural node $n \in N^1$ one constructs in addition

$$w(T,n) := \sum_{n'\in N_{+1}(n)} P(S(n')|S(n))\, w(T,n')$$

where $P(S(n')|S(n)) := \sum_{s\in S(n')} p_s/\sum_{s\in S(n)} p_s$ is the well defined conditional subjective probability that nature will select a state in the set $S(n')$, given that a state in $S(n)$ is already bound to occur after reaching node n. As in Section 7.2 of Chapter 5, the principle of optimality in dynamic programming still holds, with subjective expected utility maximizing behaviour β satisfying

$$\emptyset \neq \beta(T,n) \subset \arg\max_{n'} \{\, w(T,n') \mid n' \in N_{+1}(n) \,\} \tag{41}$$

at any decision node $n \in N^*$. Also, at any node $n \in N$, one has

$$\left.\begin{array}{rcl} \Phi_\beta(T,n) & = & \arg \\ w(T,n) & = & \end{array}\right\} \max_{\lambda} \{\, U^{S(n)}(\lambda) \mid \lambda \in F(T,n) \,\} \tag{42}$$

The proof of Lemma 7.2 in Chapter 5 can then be adapted easily to show that SEU behaviour β must satisfy continuity condition (CB). Indeed, it is only necessary to realize that each natural node $n \in N^1$ can be replaced with an equivalent chance node where nature moves to each node $n' \in N_{+1}(n)$ with positive conditional probability $P(S(n')|S(n))$.

5.7 Main Theorem

Corresponding to the main result in Section 7.3 of Chapter 5 is the following:

THEOREM 5:

(1) Let β be non-trivial state independent consequentialist behaviour which, for the almost unrestricted domain of finite decision trees with only positive probabilities at all chance nodes, satisfies consequentialist normal form invariance, dynamic consistency, and the continuous behaviour condition (CB) stated in Section 7.1 of Chapter 5. Then there exists a unique cardinal equivalence class of NMUFs $v : Y \to \mathbb{R}$ and unique strictly positive subjective probabilities such that β maximizes subjectively expected utility.

(2) Conversely, let $v : Y \to \mathbb{R}$ be any NMUF, and p_s $(s \in S)$ any strictly positive subjective probabilities. Then state independent consequentialist behaviour β satisfying consequentialist normal form invariance, dynamic consistency and condition (CB) can be defined on the almost unrestricted domain of finite decision trees with only positive probabilities at all chance nodes in order that the associated preference ordering R_β revealed by behaviour β should be represented by the subjective expected value of v.

6 State-Dependent Consequence Domains

6.1 Evaluation Functions

Up to this point, it has been assumed throughout that there is a fixed consequence domain Y, independent of the state of the world $s \in S$. Yet, as discussed in the accompanying chapter by Rustichini and Drèze, this is a poor assumption in many practical examples — for example, if some states lead to the death or permanent impairment of the decision maker. Consequences arising in such disastrous states seem quite different from those that can be experienced while the decision maker still enjoys good health. Accordingly, this section considers the implications of allowing there to be a consequence domain Y_s that depends on the state of the world $s \in S$. Also, in contrast to Karni (1993a, b), it will not be assumed that consequences in different state-dependent consequence domains are in any way related through "constant valuation acts" or "state invariance". Evidently, the state independence condition (SI) must be dropped.

Eventually, Lemma 6.1 in this Section will demonstrate what happens when the five axioms of Sections 4.3–4.5 are applied to the domain $\Delta(Y^S)$, where Y^S is now the Cartesian product $\prod_{s\in S} Y_s$ of state dependent consequence domains, and S is once again the finite domain of possible states of the world. In order to state the result, first let $Y_{Ss} := \{s\} \times Y_s$ for each $s \in S$, and then define the corresponding *universal domain* of state–consequence ordered pairs (s, y) as the set

$$Y_S := \cup_{s\in S} Y_{Ss} \tag{43}$$

This is an obvious generalization of the domain of "prize–state lotteries" considered by Karni (1985) — see also the chapter by Rustichini and Drèze. Second, define an *evaluation function* (Wilson, 1968; Myerson, 1979) as a real-valued mapping $w(s, y)$ on the domain Y_S with the property that the preference ordering $\precsim$ on $\Delta(Y^S)$ is represented by the expected total evaluation, which is defined for all $\lambda^S \in \Delta(Y^S)$ by

$$U^S(\lambda^S) = \sum\nolimits_{s\in S} \sum\nolimits_{y_s\in Y_s} \lambda_s(y_s)\, w(s, y_s) \tag{44}$$

We shall see that evaluation functions differ from state-dependent utility functions because the latter are separate from and independent of subjective probabilities, whereas the former conflate utility functions with subjective probabilities. Also, say that two evaluation functions $w(s, y)$ and $\tilde{w}(s, y)$ are *co-cardinally equivalent* if and only if there exist real constants $\rho > 0$, independent of s, and δ_s $(s \in S)$, such that

$$\tilde{w}(s, y) = \delta_s + \rho\, w(s, y) \tag{45}$$

Note then that the transformed expected evaluation satisfies

$$\tilde{U}^S(\lambda^S) = \sum\nolimits_{s\in S} \sum\nolimits_{y_s\in Y_s} \lambda_s(y_s)\, \tilde{w}(s, y_s) = \sum\nolimits_{s\in S} \delta_s + \rho\, U^S(\lambda^S) \tag{46}$$

because $\sum_{y_s\in Y_s} \lambda_s(y_s) = 1$ for each $s \in S$. Hence $\tilde{U}^S$ and U^S are cardinally equivalent, so they both represent the same contingent preference ordering on $\Delta(Y^S)$.

Conversely, suppose that (44) and (46) both represent the same ordering $\precsim$ on $\Delta(Y^S)$. Let s, s' be any pair of states in S, and $a, b \in Y_s$, $c, d \in Y_{s'}$ any four consequences with $w(s, a) \neq w(s, b)$ and $w(s', c) \neq w(s', d)$. As was argued in Section 2.3 of Chapter 5 as well as in connection with (17) of Section 4.2, the common ratio

$$\frac{w(s, a) - w(s, b)}{w(s', c) - w(s', d)} = \frac{\tilde{w}(s, a) - \tilde{w}(s, b)}{\tilde{w}(s', c) - \tilde{w}(s', d)} \tag{47}$$

of evaluation differences is the constant MRS between shifts in probability from consequence b to a in state s and shifts in probability from consequence d to c in state s'. So for all such configurations of s, s', a, b, c, d there must exist a constant ρ such that

$$\frac{\tilde{w}(s,a) - \tilde{w}(s,b)}{w(s,a) - w(s,b)} = \frac{\tilde{w}(s',c) - \tilde{w}(s',d)}{w(s',c) - w(s',d)} = \rho \tag{48}$$

Moreover $\rho > 0$ because the non-zero numerator and denominator of each fraction in (48) must have the same sign. Then (45) follows, so $\tilde{w}(s,y)$ and $w(s,y)$ must be co-cardinally equivalent functions on the domain Y_S.

For the rest of this section, assume that for every state $s \in S$ there exist $\overline{\lambda}_s, \underline{\lambda}_s \in \Delta(Y_s)$ such that the contingent ordering $\succsim^{\{s\}}$ on $\Delta(Y_s)$ satisfies $\overline{\lambda}_s \succ^{\{s\}} \underline{\lambda}_s$. This loses no generality because, by (STP), states without this property are null states which can be omitted from S without affecting preferences.

Lemma 6.1: Under the five conditions (O), (I), (C), (RO) and (STP), there exists a unique co-cardinal equivalence class of evaluation functions $w(s,y)$ such that the expected sum $U^S(\lambda^S)$ defined by (44) represents the corresponding preference ordering $\succsim$ on $\Delta(Y^S)$.

Proof: Because the ordering $\succsim$ satisfies conditions (O), (I) and (C), Theorem 4 of Chapter 5 shows that $\succsim$ can be represented by a unique cardinal equivalence class of expected utility functions $U^S : \Delta(Y^S) \to \mathbb{R}$ which satisfy the mixture property (MP) on $\Delta(Y^S)$ (like (24) in Section 4.7). Then normalize U^S so that

$$U^S(\underline{\lambda}^S) = 0 \quad \text{and} \quad U^S(\overline{\lambda}^S) = 1 \tag{49}$$

Next, for each state $s \in S$ and lottery $\lambda \in \Delta(Y_s)$, define

$$u_s(\lambda) := U^S(\underline{\lambda}^{S \setminus \{s\}}, \lambda) \tag{50}$$

Note that whenever $s \in S$ is a null state, then $(\underline{\lambda}^{S\setminus\{s\}}, \lambda) \sim \underline{\lambda}^S$ for all $\lambda \in \Delta(Y_s)$. In this case, it follows from (50) and (49) that $u_s(\lambda) \equiv 0$ on $\Delta(Y_s)$.

By an argument similar to that used in the proof of Lemma 4.4, if m is the number of elements in the finite set S and $\lambda^S \in \Delta(Y^S)$, the two lotteries

$$\sum\nolimits_{s \in S} \frac{1}{m} \, (\underline{\lambda}^{S\setminus\{s\}}, \lambda_s) \quad \text{and} \quad \frac{m-1}{m} \, \underline{\lambda}^S + \frac{1}{m} \, \lambda^S \tag{51}$$

in $\Delta(Y^S)$ both have the common marginal distribution $(1 - \frac{1}{m}) \, \underline{\lambda}_s + \frac{1}{m} \, \lambda_s$ for each $s \in S$. So condition (RO) implies that they are indifferent. Because U^S

satisfies (MP), applying U^S to the two indifferent mixtures in (51) gives the equality

$$\sum_{s \in S} \frac{1}{m} U^S(\underline{\lambda}^{S \setminus \{s\}}, \lambda_s) = \frac{m-1}{m} U^S(\underline{\lambda}^S) + \frac{1}{m} U^S(\lambda^S) \tag{52}$$

But $U^S(\underline{\lambda}^S) = 0$ by (49), so (52) and (50) together imply that

$$U^S(\lambda^S) = \sum_{s \in S} U^S(\underline{\lambda}^{S \setminus \{s\}}, \lambda_s) = \sum_{s \in S} u_s(\lambda_s) \tag{53}$$

Finally, define $w(s, y) := u_s(1_y)$ for each $s \in S$ and $y \in Y_s$. Because U^S satisfies (MP), (50) implies that so does each function u_s on the corresponding domain $\Delta(Y_s)$. It follows that $u_s(\lambda_s) \equiv \sum_{y \in Y_s} \lambda_s(y)\, w(s, y)$ and so, because of (53), that $U^S(\lambda^S)$ is given by (44).

The fact that there is a unique co-cardinal equivalence class of the functions $w(s, y)$ follows easily from the discussion preceding the lemma. ∎

6.2 Chosen Probabilities and State Dependent Utilities

An extreme case occurs if the state dependent consequence domains Y_s and $Y_{s'}$ are disjoint whenever $s \neq s'$. In this case, there seems no hope of inferring subjective probabilities from behaviour. To see why, suppose that an agent's behaviour is observed to maximize the SEU function

$$U^S(y^S) = \sum_{s \in S} p_s\, v(y_s)$$

where $p_s > 0$ for all $s \in S$. Then the same behaviour will also maximize the equivalent SEU function

$$U^S(y^S) = \sum_{s \in S} \tilde{p}_s\, \tilde{v}(y_s)$$

for *any* positive subjective probabilities $\tilde{p}_s$ satisfying $\sum_{s \in S} \tilde{p}_s = 1$, provided that $\tilde{v}(y) = p_s\, v(y_s)/\tilde{p}_s$ for all $y \in Y_s$. Without further information, there is no way of disentangling subjective probabilities from utilities.

Following a suggestion of Karni, Schmeidler and Vind (1983), such additional information could be inferred from hypothetical behaviour when probabilities p_s ($s \in S$) happen to be specified. The idea is that, though the agent does not know the true probabilities of the different states of the world, nevertheless it should be possible for coherent decisions to emerge if the agent happened to discover what the true probabilities are. In particular, if those true probabilities happen to coincide with the agent's subjective probabilities, the agent's behaviour should be the same whether or not these true probabilities are known.

A somewhat extreme version of this assumption will be used here. Following Karni (1985, Section 1.6), Schervish, Seidenfeld and Kadane (1990), and also Karni and Schmeidler (1991), it will be assumed that the decision-maker can handle problems involving not only hypothetical probabilities, but also hypothetical choices of probabilities. Take, for instance, problems where the states of nature are indeed natural disasters, weather events, etc. It will be assumed that the decision-maker can rank prospects of the following general kind: A probability of 2% each year of a major earthquake? Or 1% each year of a devastating hundred year flood? Or 4% each year of a serious forest fire set off by lightning? More specifically, the assumption is that the decision-maker can resolve such issues within a coherent framework of decision analysis. Certainly, if the SEU hypothesis holds, it can be applied to decide such issues. Drèze's (1961, 1987) theory of "moral hazard" is based on a somewhat related idea. But Drèze assumes that the agent can influence the choice of state, as opposed to the choice of probabilities of different states.

For this reason, it will be assumed that there exists an additional preference ordering $\succsim_S$ on the whole extended lottery domain $\Delta(Y_S)$, where Y_S, defined by (43), is the universal state–consequence domain of pairs (s, y). Thus, $\succsim_S$ satisfies condition (O). Furthermore, assume that $\succsim_S$ satisfies the obvious counterparts of conditions (I) and (C) for the domain $\Delta(Y_S)$. Obviously, these conditions (O) and (I) can be given a consequentialist justification, along the lines of that in Sections 5 and 6 of Chapter 5, by considering a suitably extended domain of decision trees in which natural nodes become replaced by chance nodes, and there are even several copies of natural nodes so that opportunities to affect the probabilities attached to states of nature are incorporated in the tree. Arguing as in that chapter, there must exist a unique cardinal equivalence class of extended NMUFs v_S on the domain Y_S whose expected values all represent the ordering $\succsim_S$ on $\Delta(Y_S)$. As a function $v_S(s, y)$ of both the state $s \in S$ and the consequence $y \in Y_s$, each such function is a state-dependent utility of the kind considered in the accompanying chapter by Rustichini and Drèze.

Note next that each $\Delta(Y_s)$ is effectively the same as the set

$$\Delta(Y_{Ss}) := \{\, \lambda \in \Delta(Y_S) \mid \lambda(\{s\} \times Y_s) = 1 \,\} \tag{54}$$

of lotteries attaching probability one to the state $s \in S$. So, after excluding states in which the contingent preference ordering $\succsim^{\{s\}}$ on $\Delta(Y_s)$ is trivial, as in Section 6.1, it will be assumed that each $\succsim^{\{s\}}$ is identical to the ordering $\succsim_S$ restricted to the corresponding set $\Delta(Y_{Ss})$. But these orderings are represented by the expected values of the two respective NMUFs $w(s, y)$ and $v_S(s, y)$ on the common domain Y_s. So these NMUFs are cardinally equivalent functions of y. Hence, there must exist constants $\rho_s > 0$ and δ_s such that on Y_s one has

$$w(s, y) \equiv \delta_s + \rho_s \, v_S(s, y) \tag{55}$$

Now define $\rho := \sum_{s \in S} \rho_s > 0$ and, for all $s \in S$, the ratios $p_s := \rho_s/\rho$. Clearly each $p_s > 0$ and $\sum_{s \in S} p_s = 1$. Therefore the ratios p_s can be interpreted as subjective probabilities. Furthermore, because $\succsim$ on $\Delta(Y^S)$ is represented by the expected total evaluation (44), it is also represented by the expectation of the cardinally equivalent NMUF $v^S(y^S) := \sum_{s \in S} p_s \, v_S(s, y_s)$. Note that each CCF $y^S \in Y^S$ is subjectively equivalent to the lottery in $\Delta(Y)$ with corresponding objective probabilities $p_s(y_s)$ for all $s \in S$.

Because of (55), one has $w(s, \tilde{y}_s) - w(s, y_s) = \rho_s \, [v_S(s, \tilde{y}_s) - v_S(s, y_s)]$ for any state $s \in S$ and any pair of consequences $y_s, \tilde{y}_s \in Y_s$. Therefore,

$$\frac{p_s}{p_{s'}} = \frac{\rho_s}{\rho_{s'}} = \frac{w(s, \tilde{y}_s) - w(s, y_s)}{w(s', \tilde{y}_{s'}) - w(s', y_{s'})} \cdot \frac{v_S(s', \tilde{y}_{s'}) - v_S(s', y_{s'})}{v_S(s, \tilde{y}_s) - v_S(s, y_s)} \tag{56}$$

This formula enables ratios of subjective probabilities to be inferred uniquely in an obvious way from marginal rates of substitution (MRSs) between shifts in objective probability, expressed in the form of ratios of utility differences. The first term of the product is the MRS between changes in the probabilities of consequences in two different states of the kind considered in (47). The second term is a four-way ratio of utility differences that equals the MRS between shifts in probability from $(s', \tilde{y}_{s'})$ to $(s', y_{s'})$ and shifts in probability from $(s, \tilde{y}_s)$ to (s, y_s).

To summarize the results of the above discussion:

LEMMA 6.2. Suppose that:

1. conditions (O), (I), and (C) apply to the ordering $\succsim_S$ on the domain $\Delta(Y_S)$;
2. conditions (O), (I), (C), (RO) and (STP) apply to the ordering $\succsim$ on $\Delta(Y^S)$;
3. for each $s \in S$, the contingent preference ordering $\succsim^{\{s\}}$ on $\Delta(Y_s)$ is identical to the restriction of the ordering $\succsim_S$ to this set, regarded as equal to $\Delta(Y_{Ss})$ defined by (54);

4. for each $s \in S$, there exist $\bar{\lambda}_s, \underline{\lambda}_s \in \Delta(Y_s)$ such that $\bar{\lambda}_s \succ^{\{s\}} \underline{\lambda}_s$.

Then there exist unique positive subjective probabilities p_s ($s \in S$) and a unique cardinal equivalence class of state independent NMUFs $v_S : Y_S \to \mathbb{R}$ such that the ordering $\succsim$ on $\Delta(Y^S)$ is represented by the expected utility function

$$U^S(\lambda^S) \equiv \sum\nolimits_{s \in S} p_s \sum\nolimits_{y_s \in Y_s} \lambda_s(y_s)\, v_S(s, y_s)$$

6.3 State Independent Utilities

So far, no attention has been paid to the possibility of the same consequence arising in different states of the world. Apart from being unrealistic, this also means that the theory set out in the previous sections of this chapter has not really been generalized. In fact, we have merely gone from one extreme of identical consequence domains in all states to the other extreme of consequence domains in different states being treated as if they were pairwise disjoint. Here the implications of treating the same consequence in a different state of the world as really the same consequence will be explored.

First introduce the notation

$$\hat{Y} = \cup_{s \in S} Y_s \tag{57}$$

for the *union domain* of all consequences that can occur in some state of the world. Then there is a natural embedding $\phi : \Delta(Y_S) \to \Delta(\hat{Y})$ from lotteries λ_S over the universal domain Y_S defined by (43) to lotteries over $\hat{Y}$. After adopting the convention that $\lambda_S(s, y) = 0$ whenever $y \notin Y_s$, this embedding can be defined by

$$\phi(\lambda_S)(y) := \sum\nolimits_{s \in S} \lambda_S(s, y) \tag{58}$$

for all $\lambda_S \in \Delta(Y_S)$ and all $y \in \hat{Y}$. Thus, $\phi(\lambda_S)(y)$ is the total probability of all state–consequence pairs (s, y) in which the particular consequence y occurs. Evidently, for all $\lambda_S, \mu_S \in \Delta(Y_S)$ and all $\alpha \in (0, 1)$, definition (58) implies that

$$\phi(\alpha\, \lambda_S + (1 - \alpha)\, \mu_S) = \alpha\, \phi(\lambda_S) + (1 - \alpha)\, \phi(\mu_S) \tag{59}$$

LEMMA 6.3: The mapping $\phi : \Delta(Y_S) \to \Delta(\hat{Y})$ is onto.

PROOF: Given any $\lambda \in \Delta(\hat{Y})$, let $K_\lambda := \{\, y \in \hat{Y} \mid \lambda(y) > 0 \,\}$ denote the (finite) support of the distribution λ. For each consequence $y \in K_\lambda$, choose any state $s(y) \in S$ with the property that $y \in Y_{s(y)}$; at least one such state always exists. Then define $\lambda_S \in \Delta(Y_S)$ so that $\lambda_S(s(y), y) = \lambda(y)$ for all

$y \in K_\lambda$, but $\lambda_S(s,y) = 0$ unless both $y \in K_\lambda$ and $s = s(y)$. Evidently $\phi(\lambda_S)(y) = \lambda_S(s(y),y) = \lambda(y)$ for all $y \in K_\lambda$, and $\phi(\lambda_S)(y) = \lambda(y) = 0$ for all $y \notin K_\lambda$. This shows that $\phi(\lambda_S) = \lambda$. ∎

The pre-image correspondence $\Phi_S : \Delta(\hat{Y}) \twoheadrightarrow \Delta(Y_S)$ of ϕ can be defined, for all $\lambda \in \Delta(\hat{Y})$, by

$$\Phi_S(\lambda) := \{ \lambda_S \in \Delta(Y_S) \mid \phi(\lambda_S) = \lambda \}$$

Because of Lemma 6.3, $\Phi_S(\lambda)$ is never empty. In this framework, it now seems natural to impose the following *modified state independence* condition (SI*): there exists a "state independent consequence" preference relation $\succsim_Y$ on $\Delta(\hat{Y})$ with the property that, for all pairs $\lambda_S, \mu_S \in \Delta(Y_S)$, one has

$$\lambda_S \succsim_S \mu_S \iff \phi(\lambda_S) \succsim_Y \phi(\mu_S) \tag{60}$$

Thus, in deciding between the pair $\lambda_S, \mu_S \in \Delta(Y_S)$, it is enough to consider the induced consequence lotteries $\phi(\lambda_S), \phi(\mu_S)$; the states in which the various consequences occur are irrelevant.

In the special case of a state independent consequence domain, so that $Y_s = Y$ for all $s \in S$, condition (SI*) evidently implies that $\succsim_S$ reduces to an ordering on $\Delta(Y)$. But condition (SI*) can also hold when the domains Y_s depend on the state; they could even be pairwise disjoint.

LEMMA 6.4: Suppose that conditions (O), (I), (C) and (SI*) apply to the ordering $\succsim_S$ on the domain $\Delta(Y_S)$. Then the relation $\succsim_Y$ satisfies conditions (O), (I), and (C) on $\Delta(\hat{Y})$.

PROOF: Throughout the following proof, given any three lotteries $\lambda, \mu, \nu \in \Delta(\hat{Y})$, let $\lambda_S, \mu_S, \nu_S \in \Delta(Y_S)$ denote arbitrarily chosen members of $\Phi_S(\lambda)$, $\Phi_S(\mu)$ and $\Phi_S(\nu)$ respectively. That is, suppose $\lambda = \phi(\lambda_S)$, $\mu = \phi(\mu_S)$, and $\nu = \phi(\nu_S)$. Because of (59), it follows that

$$\begin{aligned} \phi(\alpha\,\lambda_S + (1-\alpha)\,\nu_S) &= \alpha\,\lambda + (1-\alpha)\,\nu \\ \text{and} \quad \phi(\alpha\,\mu_S + (1-\alpha)\,\nu_S) &= \alpha\,\mu + (1-\alpha)\,\nu \end{aligned} \tag{61}$$

Condition (O). First, for any $\lambda \in \Delta(\hat{Y})$ one has $\lambda_S \succsim_S \lambda_S$ and so $\lambda \succsim_Y \lambda$, thus confirming that $\succsim_Y$ is reflexive. Second, for any $\lambda, \mu \in \Delta(\hat{Y})$ one has $\lambda_S \succsim_S \mu_S$ or $\mu_S \succsim_S \lambda_S$, and so $\lambda \succsim_Y \mu$ or $\mu \succsim_Y \lambda$, thus confirming that $\succsim_Y$ is complete. Finally, if $\lambda, \mu, \nu \in \Delta(\hat{Y})$ satisfy $\lambda \succsim_Y \mu$ and $\mu \succsim_Y \nu$, then $\lambda_S \succsim_S \mu_S$ and $\mu_S \succsim_S \nu_S$. Therefore transitivity of $\succsim_S$ implies $\lambda_S \succsim_S \nu_S$ and so $\lambda \succsim_Y \nu$, thus confirming that $\succsim_Y$ is transitive. So $\succsim_Y$ is a preference ordering.

Condition (I). Suppose that $0 < \alpha < 1$. Because $\succsim_S$ satisfies condition (I), it follows from (60) and (61) that

$$\begin{aligned} \lambda \succ_Y \mu \quad \Longrightarrow \quad \lambda_S \succ_S \mu_S \quad &\Longrightarrow \quad \alpha\,\lambda_S + (1-\alpha)\,\nu_S \succ_S \alpha\,\mu_S + (1-\alpha)\,\nu_S \\ &\Longrightarrow \quad \alpha\,\lambda + (1-\alpha)\,\nu \succ_Y \alpha\,\mu + (1-\alpha)\,\nu \end{aligned}$$

Therefore $\succsim_Y$ also satisfies condition (I).

Condition (C). Suppose that $\lambda \succ_Y \mu$ and $\mu \succ_Y \nu$. Then $\lambda_S \succ_S \mu_S$ and also $\mu_S \succ_S \nu_S$. Because $\succ_S$ satisfies condition (C), it follows that there exist $\alpha', \alpha'' \in (0,1)$ such that $\alpha'\,\lambda_S + (1-\alpha')\,\nu_S \succ_S \mu_S$ and $\mu_S \succ_S \alpha''\,\lambda_S + (1-\alpha'')\,\nu_S$. Then (59) and (61) together imply that $\alpha'\,\lambda + (1-\alpha')\,\nu \succ_Y \mu$, and also that $\mu \succ_Y \alpha''\,\lambda + (1-\alpha'')\,\nu$. Therefore $\succsim_Y$ also satisfies condition (C). ∎

The following is the main result in this chapter for state-dependent consequence domains:

Theorem 6. Suppose that:

1. conditions (O), (I), (C) and (SI*) apply to the ordering $\succsim_S$ on the domain $\Delta(Y_S)$;
2. conditions (O), (I), (C), (RO) and (STP) apply to the ordering $\succsim$ on the domain $\Delta(Y^S)$;
3. for each $s \in S$, the contingent preference ordering $\succsim^{\{s\}}$ on $\Delta(Y_s)$ is identical to the restriction of the ordering $\succsim_S$ to this set, when $\Delta(Y_s)$ is regarded as equal to $\Delta(Y_{Ss})$ defined by (54);
4. for each $s \in S$, there exist consequences $\underline{y}_s, \overline{y}_s \in Y_s$ such that $(s, \overline{y}_s) \succ_S (s, \underline{y}_s)$.

Then there exists a unique cardinal equivalence class of state-independent NMUFs $\hat{v}$ defined on the union consequence domain $\hat{Y}$, as well as unique positive subjective probabilities p_s $(s \in S)$ such that, for every $\hat{v}$ in the equivalence class, the ordering $\succsim$ on $\Delta(Y^S)$ is represented by the expected value of

$$v^S(y^S) \equiv \sum\nolimits_{s \in S} p_s\, \hat{v}(y_s) \tag{62}$$

Proof: By the first hypothesis and Lemma 6.4, the associated ordering $\succsim_Y$ on $\Delta(\hat{Y})$ satisfies conditions (O), (I), and (C). So Theorem 4 which concludes Section 4 of Chapter 5 implies that there exists a unique cardinal equivalence class of NMUFs $\hat{U} : \Delta(\hat{Y}) \to \mathbb{R}$ which represent $\succsim_Y$ while satisfying the mixture property (MP). Define $\hat{v}(y) := \hat{U}(1_y)$ for all $y \in \hat{Y}$. Then $\hat{v}$ is state independent and belongs to a unique cardinal equivalence class. Because $\hat{U}$ satisfies (MP), condition (SI*) implies that $\succsim_S$ on $\Delta(Y_S)$ must be represented by the expected utility function U_S defined by

$$\begin{aligned} U_S(\lambda_S) \quad := \quad \hat{U}(\phi(\lambda_S)) &= \sum\nolimits_{y \in Y} \phi(\lambda_S)(y)\, \hat{v}(y) \\ &= \sum\nolimits_{s \in S} \sum\nolimits_{y \in Y_s} \lambda_S(s,y)\, \hat{v}(y) \end{aligned}$$

where the last equality follows from (58).

Let $s \in S$ be any state. Note that each $\lambda_s \in \Delta(Y^{\{s\}}) = \Delta(Y_s)$ corresponds uniquely to the lottery $\lambda_{Ss} \in \Delta(Y_S)$ satisfying $\lambda_{Ss}(s,y) = \lambda_s(y)$ for all $y \in Y_s$, and so $\lambda_{Ss}(\{s\} \times Y_s) = 1$. It follows that the restriction $\succsim^{\{s\}}$ of the preference ordering to $\Delta(Y^{\{s\}})$ is represented by both the expected utility functions $\sum_{y \in Y_s} \lambda_s(y)\, \hat{v}(y)$ and $\sum_{y \in Y_s} \lambda_s(y)\, w(s,y)$ of λ_s. Hence, the two functions $\hat{v}$ and $w(s,\cdot)$ of y must be cardinally equivalent on $\Delta(Y^{\{s\}})$ and on $\Delta(Y_s)$. This implies that for each state $s \in S$, there exist constants $\rho_s > 0$ and δ_s such that $w(s,y) \equiv \delta_s + \rho_s\, \hat{v}(y)$ on Y_s.

Now let $p_s := \rho_s/\rho$, where $\rho := \sum_{s \in S} \rho_s > 0$. Also $p_s > 0$. Then $\sum_{s \in S} p_s = 1$, so the constants p_s $(s \in S)$ are probabilities. Also, $w(s,y) \equiv \delta_s + \rho\, p_s\, \hat{v}(y)$. Therefore, by Lemma 6.1 in Section 6.1 and (44), the preference ordering $\succsim$ on $\Delta(Y^S)$ is represented by the expected value of

$$v^S(y^S) := U^S(1_{y^S}) = \sum\nolimits_{s \in S} w(s, y_s) = \sum\nolimits_{s \in S} \delta_s + \rho \sum\nolimits_{s \in S} p_s\, \hat{v}(y_s)$$

It follows that $\succsim$ is also represented by the objectively expected value of the NMUF (62).

Finally, the subjective conditional probabilities p_s $(s \in S)$ are unique because each ratio $p_s/p_{s'}$ is given by the unique corresponding ratio (56) of utility differences.

7 Countable Events

7.1 Bounded Preferences

Up to now, the set S of possible states of the world has been finite throughout. Here, this assumption will be relaxed to allow both S and some conditioning events $E \subset S$ to be countably infinite. The Anscombe and Aumann framework of Section 4 will still be used. Then the SEU hypothesis remains unchanged except that the summation $\sum_{s \in S}$ in (16) may be over infinitely many states. Of course, the six conditions that were set out in Section 4 all remain necessary.

Suppose that the set S is countably infinite and that p_s $(s \in S)$ are any subjective probabilities. Then any $\lambda^S \in \Delta(Y^S)$ is subjectively equivalent to the lottery λ with $\lambda(y) = \sum_{s \in S} p_s \, \lambda_s(y)$. Because $\lambda(y)$ can be positive for infinitely many $y \in Y$, it follows that λ is not in general a member of $\Delta(Y)$, the set of simple or finitely supported lotteries on Y. Instead, λ belongs to $\Delta^*(Y)$, the set of all discrete lotteries over Y, whose support can be countably infinite rather than finite. In order to extend the objective EU hypothesis to $\Delta^*(Y)$, Section 8.2 of Chapter 5 introduced condition (B), requiring each NMUF $v : Y \to \mathbb{R}$ to be bounded. Thus, following the argument of that section, if subjective expected utility is to be well defined for all possible $\lambda^S \in \Delta(Y^S)$, even when S is infinite, then condition (B) must hold. But also necessary is the appropriately reformulated dominance condition (D) requiring that, whenever $\lambda_i, \mu_i \in \Delta^*(Y)$ and $\alpha_i > 0$ with $\lambda_i \succsim^* \mu_i$ $(i = 1, 2, \ldots)$ and $\sum_{i=1}^{\infty} \alpha_i = 1$, then $\sum_{i=1}^{\infty} \alpha_i \, \lambda_i \succsim^* \sum_{i=1}^{\infty} \alpha_i \, \mu_i$. Section 8.5 of Chapter 5 shows how (D) can replace (B) in the set of sufficient conditions, and Section 8.7 of that chapter shows how (D) can be given a consequentialist justification. For this reason, condition (D) will be included in the following discussion, but condition (B) will be excluded.

7.2 Event Continuity

However, the seven conditions (O), (I*), (C*), (RO), (STP), (SI) and (D) are not sufficient on their own for the SEU hypothesis to hold. Indeed, there exist utility functions $U^E(\lambda^E)$ $(E \subset S)$ satisfying (MP) such that, whenever E and E' are infinite sets but $E \setminus E'$ and $E' \setminus E$ are finite, then

$$U^{E \cup E'}(\lambda^{E \cup E'}) = U^E(\lambda^E) = U^{E'}(\lambda^{E'}) = U^{E \cap E'}(\lambda^{E \cap E'})$$

whereas whenever E_1 and E_2 are disjoint sets, then

$$U^{E_1 \cup E_2}(\lambda^{E_1 \cup E_2}) = U^{E_1}(\lambda^{E_1}) + U^{E_2}(\lambda^{E_2})$$

In this case, each individual state of the world $s \in S$ and each finite event $E \subset S$ must be null. In particular, any subjective probabilities must satisfy $p_s = 0$ for all $s \in S$. Therefore, except in the trivial case of universal indifference, it cannot be true that $U^S(\lambda^S) = \sum_{s \in S} p_s \, U^*(\lambda_s)$ when S is infinite.

One attempt to exclude such awkward possibilities would be to postulate, following Savage (1954, p. 39, P6), that if E is any infinite subset of S, and if $\lambda^E, \mu^E \in \Delta(Y^E)$ satisfy $\lambda^E \succ^E \mu^E$, then for any $\nu \in \Delta(Y)$ there is a finite partition $\cup_{k=1}^r E_k$ of E into r small enough pairwise disjoint subsets such that, for $k = 1, 2, \ldots, r$, both $(\lambda^{E \setminus E_k}, \nu\, 1^{E_k}) \succ^E \mu^E$ and $\lambda^E \succ^E (\mu^{E \setminus E_k}, \nu\, 1^{E_k})$. However, as shown by Fishburn (1970, ch. 14), this axiom implies that S must be uncountably infinite and also that $p_s = 0$ for all $s \in S$.

So a more suitable alternative seems to be the following condition, suggested by Fishburn (1982, p. 126, axiom F7). Suppose that

$$E_1 \subset E_2 \subset \ldots \subset E_k \subset E_{k+1} \subset \ldots \subset S \quad \text{and} \quad E^* = \cup_{k=1}^\infty E_k \tag{63}$$

Then the *event continuity* condition (EC) requires that, for all events E and all $\lambda, \mu \in \Delta(Y)$ satisfying both $\lambda \succ^* \mu$ and $(\lambda\, 1^{E^*}, \mu\, 1^{S \setminus E^*}) \succ (\lambda\, 1^E, \mu\, 1^{S \setminus E})$, there must exist a finite k such that $(\lambda\, 1^{E_k}, \mu\, 1^{S \setminus E_k}) \succ (\lambda\, 1^E, \mu\, 1^{S \setminus E})$.

To see why this condition is necessary for the SEU hypothesis to hold, note how the hypotheses of condition (EC) imply that $U^*(\lambda) > U^*(\mu)$ and also that

$$P(E^*)\, U^*(\lambda) + [1 - P(E^*)]\, U^*(\mu) > P(E)\, U^*(\lambda) + [1 - P(E)]\, U^*(\mu)$$

Hence $P(E^*) > P(E)$. But $E^* = E_1 \cup [\bigcup_{k=1}^\infty (E_{k+1} \setminus E_k)]$, where E_1 and $E_{k+1} \setminus E_k$ $(k = 1, 2, \ldots)$ are all pairwise disjoint events. So

$$P(E^*) = P(E_1) + \sum_{k=1}^\infty [P(E_{k+1}) - P(E_k)] = \lim_{k \to \infty} P(E_k)$$

From this it follows that, for all large enough k, one has $P(E_k) > P(E)$ and so $(\lambda\, 1^{E_k}, \mu\, 1^{S \setminus E_k}) \succ (\lambda\, 1^E, \mu\, 1^{S \setminus E})$. This confirms condition (EC).

7.3 Event Dominance

One other condition will be needed in order to establish the SEU hypothesis for a countably infinite state space. This is the *event dominance* condition (ED) requiring that, whenever the lotteries $\mu^S \in \Delta^*(Y^S)$ and $\lambda \in \Delta(Y)$ are given, then $\mu_s \succsim^* \lambda$ (all $s \in S$) implies $\mu^S \succsim \lambda\, 1^S$, and $\mu_s \precsim^* \lambda$ (all $s \in S$) implies $\mu^S \precsim \lambda\, 1^S$. Clearly, this condition is closely related to the probability dominance condition (PD) discussed in Section 9 of Chapter 5. It even has a similar consequentialist justification. Moreover, Savage's (1954) postulate P7 is analogous — it requires that whenever $y^S, z^S \in Y^S$, then $y^S \succsim z_s\, 1^S$ (all $s \in S$) implies $y^S \succsim z^S$, and $y^S \precsim z_s\, 1^S$ (all $s \in S$) implies $y^S \precsim z^S$. In fact, a postulate which evidently strengthens both (ED) and Savage's P7 is the *strong event dominance* condition (ED*) requiring that, whenever $\lambda^S, \mu^S \in \Delta^*(Y^S)$ are given, then $\mu_s \succsim^* \lambda_s 1^S$ (all $s \in S$) implies $\mu^S \succsim \lambda^S$, and $\mu_s \precsim \lambda_s 1^S$ (all $s \in S$) implies $\mu^S \precsim \lambda^S$.

Obviously, condition (ED*) is necessary for the SEU hypothesis to hold when S is countably infinite.

7.4 Sufficient Conditions for SEU and SEU*

Whenever S is a countably infinite set, the nine conditions (O), (I), (C), (RO), (STP), (SI), (D), (EC), and (ED) are sufficient for the SEU hypothesis to be valid on the domain $\Delta^*(Y^S)$. This will not be proved here, however, since it is an obvious corollary of a more general result in the next section, and much of the proof would then have to be repeated there.

8 Subjective Probability Measures

8.1 Measurable Expected Utility

In this section, the set S of possible states of the world can be an infinite set, not necessarily even countably infinite. However, S will be equipped with a σ-field $\mathcal{S}$, to use the standard terminology set out in Section 9.1 of Chapter 5. Thus the pair $(S, \mathcal{S})$ constitutes a measurable space. So, in this section, an *event* E will be defined as any member of $\mathcal{S}$, implying that all events are measurable.

As in Section 9.2 of Chapter 5, let $\mathcal{F}$ denote the σ-field generated by the singleton sets $\{y\}$ $(y \in Y)$ together with the upper and lower preference sets that correspond to the state independent preference ordering $\succsim^*$ defined on sure consequences in Y. Indeed, condition (M) of that chapter states precisely that all sets in $\mathcal{F}$ should be measurable. Finally, let $\Delta(Y, \mathcal{F})$ denote the set of (countably additive) probability measures on the σ-field $\mathcal{F}$.

Next, for any event $E \in \mathcal{S}$, let $\mathcal{S}^E$ denote the σ-field consisting of all sets $G \in \mathcal{S}$ such that $G \subset E$. Of course $\mathcal{S}^S = \mathcal{S}$. Then define $\Delta(Y^E, \mathcal{F})$ as the set of mappings $\pi^E : E \to \Delta(Y, \mathcal{F})$ with the property that, for every measurable set $K \in \mathcal{F}$, the real-valued mapping $s \mapsto \pi(s, K)$ on the domain E is measurable when E is given the σ-field $\mathcal{S}^E$ and the real line is given its

Borel σ-field. In other words, given any Borel measurable set $B \subset [0,1]$, the set $\{ s \in E \mid \pi(s,K) \in B \}$ must belong to $\mathcal{S}$. The following result is important later in Section 8.2:

LEMMA 8.1: The set $\Delta(Y^S, \mathcal{F})$ is a convex mixture space.

PROOF (cf. Fishburn, 1982, p. 134): Suppose that $\pi^S, \tilde{\pi}^S \in \Delta(Y^S, \mathcal{F})$ and $0 < \alpha < 1$. Given any $\delta \in [0,1]$ and any $K \in \mathcal{F}$, define the two sets

$$\begin{aligned} A_\delta^+ &:= \{ s \in S \mid \alpha\,\pi(s,K) + (1-\alpha)\,\tilde{\pi}(s,K) > \delta \} \\ A_\delta^- &:= \{ s \in S \mid \alpha\,\pi(s,K) + (1-\alpha)\,\tilde{\pi}(s,K) < \delta \} \end{aligned}$$

Clearly, in order to show that $\alpha\,\pi^S + (1-\alpha)\,\tilde{\pi}^S \in \Delta(Y^S, \mathcal{F})$, it is enough to prove that, for all $\delta \in [0,1]$, both A_δ^+ and A_δ^- are measurable sets in $\mathcal{S}$. In fact, it will be proved that $A_\delta^+ \in \mathcal{S}$; the proof that $A_\delta^- \in \mathcal{S}$ is similar.

Let $Q \subset \mathbb{R}$ denote the set of rational numbers. Given any pair $r, \tilde{r} \in Q$ with $r + \tilde{r} > \delta$, define

$$E(r,\tilde{r}) := \{ s \in S \mid \pi(s,K) > r/\alpha \quad \text{and} \quad \tilde{\pi}(s,K) > \tilde{r}/(1-\alpha) \}$$

Note that $E(r,\tilde{r})$ is measurable as the intersection of two measurable sets in $\mathcal{S}$. Obviously $E(r,\tilde{r}) \subset A_\delta^+$ whenever $r + \tilde{r} > \delta$. Also, for all $s \in A_\delta^+$, there exist rational numbers $r, \tilde{r}$ satisfying $r + \tilde{r} > \delta$ such that $\alpha\,\pi(s,K) > r$ and $(1-\alpha)\,\tilde{\pi}(s,K) > \tilde{r}$. Therefore

$$A_\delta^+ = \bigcup_{r,\tilde{r} \in Q} \{ E(r,\tilde{r}) \mid r + \tilde{r} > \delta \}$$

Because A_δ^+ is the union of a subfamily of the countable family of measurable sets $E(r,\tilde{r})$ with r and $\tilde{r}$ both rational, A_δ^+ must also be measurable. ∎

The SEU hypothesis requires that there exist an NMUF $v : Y \to \mathbb{R}$ and a subjective probability measure $P(\cdot)$ defined on $\mathcal{S}$ such that the preference ordering $\succsim$ on the domain $\Delta(Y^S, \mathcal{F})$ is represented by a subjective expected utility function in the form of the double integral

$$U^S(\pi^S) = \int_Y \left[\int_S \pi(s,dy)\, P(ds) \right] v(y)$$

A standard result in the theory of integration is Fubini's theorem, stating that the order of integration in a double integral is immaterial. A useful extension of this result, due to Halmos (1950, Section 36, exercise 3), here implies that the mapping $s \mapsto U^*(\pi(s)) := \int_Y \pi(s, dy)\, v(y)$ is measurable, and moreover

$$U^S(\pi^S) = \int_S \left[\int_Y \pi(s, dy)\, v(y) \right] P(ds) = \int_S U^*(\pi(s))\, P(ds) \tag{64}$$

Next, for any measurable event $E \in \mathcal{S}$, the contingent ordering $\succsim^E$ on the domain $\Delta(Y^E, \mathcal{F})$ is represented by the double integral

$$U^E(\pi^E) = \int_Y \int_E \pi(s, dy)\, P(ds)\, v(y) = \int_E U^*(\pi(s))\, P(ds)$$

In an obvious extension of previous notation, for any measurable event $E \in \mathcal{S}$ and any $\pi \in \Delta(Y, \mathcal{F})$, let $\pi\, 1^E \in \Delta(Y^E, \mathcal{F})$ denote the state independent measure with $\pi(s, K) = \pi(K)$ for all $s \in E$ and all $K \in \mathcal{F}$. Observe that, for all $\pi \in \Delta(Y, \mathcal{F})$, one has

$$U^E(\pi\, 1^E) = P(E) \int_Y \pi(dy)\, v(y) = P(E)\, U^*(\pi) \tag{65}$$

and in particular, $U^S(\pi\, 1^S) = U^*(\pi)$.

Note that the previous state independence condition (SI) should be changed because when S is uncountably infinite, it is possible for every state to be null without being forced into the trivial case of universal indifference. Instead, a reformulated condition (SI) will require the existence of a state independent preference ordering $\succsim^*$ on $\Delta(Y, \mathcal{F})$ with the property that, for all pairs $\pi, \tilde{\pi} \in \Delta(Y, \mathcal{F})$ and any non-null event $E \in \mathcal{S}$, one has $\pi \succsim^* \tilde{\pi}$ iff $\pi\, 1^E \succsim^E \tilde{\pi}\, 1^E$. Then (65) implies that the expected utility function U^* must represent the preference ordering $\succsim^*$ on $\Delta(Y, \mathcal{F})$. Furthermore, note that condition (PD) of Chapter 5 is satisfied — i.e., whenever $\pi \in \Delta(Y, \mathcal{F})$ and $\lambda \in \Delta(Y)$, then:

$$\pi(\{\, y \in Y \mid 1_y \succsim^* \lambda \,\}) = 1 \Longrightarrow \pi \succsim \lambda; \quad \pi(\{\, y \in Y \mid 1_y \precsim^* \lambda \,\}) = 1 \Longrightarrow \pi \precsim \lambda.$$

The event continuity condition (EC) of Section 7.2 and the event dominance condition (ED) of Section 7.3 will also be slightly modified so that they apply to probability measures. Indeed, the reformulation of condition (EC) in Section 7.2 requires that, whenever $E_1 \subset E_2 \subset \ldots \subset E_k \subset E_{k+1} \subset \ldots \subset S$, $E^* = \cup_{k=1}^{\infty} E_k$ and $\pi, \tilde{\pi} \in \Delta(Y, \mathcal{F})$ satisfy both $\pi \succ^* \tilde{\pi}$ and $(\pi\, 1^{E^*}, \tilde{\pi}\, 1^{S \setminus E^*}) \succ (\pi\, 1^E, \tilde{\pi}\, 1^{S \setminus E})$, then there must exist a finite k such that $(\pi\, 1^{E_k}, \tilde{\pi}\, 1^{S \setminus E_k}) \succ (\pi\, 1^E, \tilde{\pi}\, 1^{S \setminus E})$. On the other hand, suppose that the event $E \subset S$, the measure $\pi^E \in \Delta(Y^E, \mathcal{F})$, and the simple lottery $\lambda \in \Delta(Y)$ are all given. The reformulated condition (ED) will then require that $\pi(s) \succsim^* \lambda$ (all $s \in E$) implies $\pi^E \succsim^E \lambda\, 1^E$, and also that $\pi(s) \precsim^* \lambda$ (all $s \in E$) implies $\pi^E \precsim^E \lambda\, 1^E$.

Finally, arguing as in Sections 4 and 7 above and as in Chapter 5, the eleven conditions (O), (I), (C), (RO), (STP), (SI), (D), (EC), (ED), (M) and (PD), appropriately modified so that they apply to the domain $\Delta(Y^S, \mathcal{F})$, are clearly necessary for the SEU hypothesis to extend to suitable probability measures.

8.2 Sufficient Conditions for SEU and SEU*

The purpose of this section is to prove that the eleven conditions (O), (I), (C), (RO), (STP), (SI), (D), (EC), (ED), (M) and (PD) are together sufficient for the SEU hypothesis to apply to $\Delta(Y^S, \mathcal{F})$. Much of the proof below is adapted from Fishburn (1982, ch. 10).

First, suppose that $\overline{\pi}$, $\underline{\pi} \in \Delta(Y, \mathcal{F})$ satisfy $\overline{\pi} \succ^* \underline{\pi}$. If no such pair existed, there would be universal indifference, in which case any subjective probabilities and any constant NMUF would allow the SEU hypothesis to be satisfied. Now, arguing as in Section 4 and using the result of Lemma 8.1, conditions (O), (I), and (C) imply that there exist a real-valued expected utility function U^S defined on the mixture space $\Delta(Y^S, \mathcal{F})$ which represents $\succsim$ while satisfying the mixture preservation property (MP). Moreover, U^S can be normalized to satisfy

$$U^S(\overline{\pi}\, 1^S) = 1 \quad \text{and} \quad U^S(\underline{\pi}\, 1^S) = 0 \tag{66}$$

Then define the *revealed subjective probability* of each event $E \in \mathcal{S}$ by

$$p(E) := U^S(\overline{\pi}\, 1^E, \underline{\pi}\, 1^{S \setminus E}) \tag{67}$$

Later, Lemma 8.4 will confirm that this definition does yield a countably additive probability measure on the σ-field $\mathcal{S}$.

LEMMA 8.2: Suppose that the events E_k ($k = 1, 2, \ldots$) and E^* in $\mathcal{S}$ satisfy the conditions that $E_1 \subset E_2 \subset \ldots \subset E_k \subset E_{k+1} \subset \ldots \subset S$ and $E^* = \cup_{k=1}^{\infty} E_k$, as in (63) of Section 7.2. Then for any event E satisfying $p(E) < p(E^*)$, one has $p(E) < p(E_k)$ for some finite k.

PROOF: The proof involves applying condition (EC) for probability measures to the pair of lotteries $\overline{\pi}, \underline{\pi} \in \Delta(Y, \mathcal{F})$. Because of (67), for any event E satisfying $p(E) < p(E^*)$ one has $U^S(\overline{\pi}\, 1^{E^*}, \underline{\pi}\, 1^{S \setminus E^*}) > U^S(\overline{\pi}\, 1^E, \underline{\pi}\, 1^{S \setminus E})$. Then condition (EC) implies that for some finite k one has $U^S(\overline{\pi}\, 1^{E_k}, \underline{\pi}\, 1^{S \setminus E_k}) > U^S(\overline{\pi}\, 1^E, \underline{\pi}\, 1^{S \setminus E})$ and so, because of (67), that $p(E_k) > p(E)$. ∎

For the following lemmas, let $U^* : \Delta(Y, \mathcal{F}) \to \mathbb{R}$ be the normalized expected utility function satisfying $U^*(\pi) = U^S(\pi\, 1^S)$ for all $\pi \in \Delta(Y, \mathcal{F})$.

Then U^* satisfies (MP) because U^S does. Next, let $v : Y \to \mathbb{R}$ satisfy $v(y) = U^*(1_y)$ for all $y \in Y$. Then, as in Chapter 5, because conditions (O), (I), (C), (D), (M) and (PD) are satisfied on $\Delta(Y, \mathcal{F})$, it will be true that $U^*(\pi) = \int_Y \pi(dy)\, v(y)$ for all $\pi \in \Delta(Y, \mathcal{F})$. Also, for any $\pi^S \in \Delta(Y^S, \mathcal{F})$, the mapping $s \mapsto U^*(\pi(s)) := \int_Y \pi(s, dy)\, v(y)$ must be measurable, as was argued in Section 8.1 when demonstrating (64).

Lemma 8.3: Let $\{ E_k \mid k = 1, 2, \ldots, r \}$ be a partition of S into a finite collection of pairwise disjoint measurable subsets. Whenever $\pi_k \in \Delta(Y, \mathcal{F})$ $(k = 1, 2, \ldots, r)$ one has

$$U^S \left(\langle \pi_k\, 1^{E_k} \rangle_{k=1}^r \right) = \sum\nolimits_{k=1}^r p(E_k)\, U^*(\pi_k) \tag{68}$$

Proof: Argue as in Lemma 4.4 of Section 4.7, but with the finite set of states $s \in S$ replaced by the finite collection of events E_k $(k = 1, 2, \ldots, r)$, and the lotteries λ_s replaced by the measures π_k. ∎

Lemma 8.4: The function $p(E)$ on the domain $\mathcal{S}$ of measurable subsets of S is a (countably additive) probability measure.

Proof: First, it is obvious from definition (67) and the normalizations in (66) that $p(S) = 1$ and $p(\emptyset) = 0$. So, from (65) and (68), it follows that

$$U^*(\overline{\pi}) = U^S(\overline{\pi}\, 1^S) = 1 \quad \text{and} \quad U^*(\underline{\pi}) = U^S(\underline{\pi}\, 1^S) = 0 \tag{69}$$

Also, given any $E \in \mathcal{S}$, because of (67) and (66), (STP) implies that

$$p(E) = U^S(\overline{\pi}\, 1^E, \underline{\pi}\, 1^{S \setminus E}) \geq U^S(\underline{\pi}\, 1^S) = 0$$

Next, given any disjoint pair E_1, E_2 of measurable events, by (67), (68) and (69) one must have

$$\begin{aligned} p(E_1 \cup E_2) &= U^S(\overline{\pi}\, 1^{E_1 \cup E_2}, \underline{\pi}\, 1^{S \setminus (E_1 \cup E_2)}) = U^S(\overline{\pi}\, 1^{E_1}, \overline{\pi}\, 1^{E_2}, \underline{\pi}\, 1^{S \setminus (E_1 \cup E_2)}) \\ &= [\, p(E_1) + p(E_2)]\, U^*(\overline{\pi}) + p(S \setminus (E_1 \cup E_2))\, U^*(\underline{\pi}) \\ &= p(E_1) + p(E_2) \end{aligned}$$

An easy induction argument now shows that, whenever $r > 2$ and the events E_k $(k = 1, 2, \ldots, r)$ are pairwise disjoint, then $p(\cup_{k=1}^r E_k) = \sum_{k=1}^r p(E_k)$. Therefore $p(\cdot)$ is finitely additive.

Next, suppose that G is the countable union $\cup_{k=1}^\infty G_k$ of pairwise disjoint events $G_k \in \mathcal{S}$. For $r = 1, 2, \ldots$, define $E_r := \cup_{k=1}^r G_k$. Then G and all the sets E_r

are measurable. Also,

$$E_1 \subset E_2 \subset \ldots \subset E_k \subset E_{k+1} \subset \ldots \subset S \quad \text{and} \quad G = \cup_{k=1}^{\infty} E_k$$

Because $p(\cdot)$ is finitely additive, $p(E_r) = \sum_{k=1}^{r} p(G_k)$. Because each $p(G_k) \geq 0$, the sequence $p(E_r)$ $(r = 1, 2, \ldots)$ is non-decreasing. Define p^* as the supremum of the probabilities $p(E_r)$ $(r = 1, 2, \ldots)$. Then

$$p^* = \lim_{r \to \infty} p(E_r) = \sum\nolimits_{k=1}^{\infty} p(G_k) \leq p(G) \tag{70}$$

In particular, $p(G_k) \to 0$ as $k \to \infty$.

Now, one possibility is that, for some finite r, one has $p(G_k) = 0$ for all $k > r$. Then $p(E_r) = \sum_{k=1}^{r} p(G_k) = \sum_{k=1}^{\infty} p(G_k) = p^*$. In particular, $p(E_r) \geq p(E_k)$ for $k = 1, 2, \ldots$. By Lemma 8.2, $p(E_r) < p(G)$ would imply that $p(E_r) < p(E_k)$ for some finite k, a contradiction. Therefore $p^* = p(E_r) \geq p(G)$ in this first case.

Because $p(G_k) \to 0$, the only other possibility is that, for every $\epsilon > 0$, there exists r (which depends on ϵ) such that $0 < p(G_r) < \epsilon$. Because $p(G \setminus G_r) = p(G) - p(G_r)$, it follows that

$$p(G) > p(G \setminus G_r) > p(G) - \epsilon \tag{71}$$

Then Lemma 8.2 implies that, for some finite k, one has

$$p(E_k) > p(G \setminus G_r) \tag{72}$$

Yet the definition of p^* implies that $p^* \geq p(E_k)$. So from (71) and (72) one has

$$p^* \geq p(E_k) > p(G \setminus G_r) > p(G) - \epsilon$$

But this is true for all $\epsilon > 0$, so $p(G) \leq p^*$ in this case as well.

Thus, $p(G) \leq p^*$ in both cases. But then (70) implies that $p(G) = p^* = \sum_{k=1}^{\infty} p(G_k)$, verifying that p is countably additive. ∎

For the next lemma, given any $\pi^S \in \Delta(Y^S, \mathcal{F})$, define the two bounds

$$\underline{U}(\pi^S) := \inf_s \{\, U^*(\pi(s)) \mid s \in S \,\} \text{ and } \overline{U}(\pi^S) := \sup_s \{\, U^*(\pi(s)) \mid s \in S \,\} \tag{73}$$

LEMMA 8.5: For all $\pi^S \in \Delta(Y^S, \mathcal{F})$ one has $\underline{U}(\pi^S) \leq U^S(\pi^S) \leq \overline{U}(\pi^S)$.

Proof: First, by Lemma 8.5 of Chapter 5, the bounded utility condition (B) is also satisfied. Therefore U^* must be bounded both above and below. So (73) implies that $-\infty < \underline{U}(\pi^S) \leq \overline{U}(\pi^S) < \infty$.

The general case occurs when there exists $\mu \in \Delta(Y)$ such that $U^*(\mu) < \overline{U}(\pi^S)$. Then there must exist an infinite sequence of lotteries λ_k $(k = 1, 2, \ldots)$ in $\Delta(Y)$ such that $U^*(\lambda_k)$ is increasing and $U^*(\lambda_k) \to \overline{U}(\pi^S)$ as $k \to \infty$. Clearly, then $U^*(\lambda_k) > U^*(\mu)$ for k large enough. In this case, define

$$\alpha_k := \frac{U^*(\lambda_k) - U^*(\mu)}{\overline{U}(\pi^S) - U^*(\mu)}$$

Then, whenever $0 < \alpha < \alpha_k$, one has $U^*(\lambda_k) > \alpha\, \overline{U}(\pi^S) + (1-\alpha)\, U^*(\mu)$. Because $U^*(\pi(s)) \leq \overline{U}(\pi^S)$ for all $s \in S$, it follows that $\lambda_k \succ^* \alpha\, \pi(s) + (1-\alpha)\, \mu$. By condition (ED), $\lambda_k\, 1^S \succsim \alpha\, \pi^S + (1-\alpha)\, \mu\, 1^S$ and so

$$\alpha\, U^S(\pi^S) + (1-\alpha)\, U^*(\mu) \leq U^*(\lambda_k) \leq \overline{U}(\pi^S)$$

Now, as $k \to \infty$, so $\alpha_k \to 1$, implying that $\alpha\, U^S(\pi^S) + (1-\alpha)\, U^*(\mu) \leq \overline{U}(\pi^S)$ for all $\alpha < 1$. Therefore $U^S(\pi^S) \leq \overline{U}(\pi^S)$.

The other, special, case is when $U^*(\mu) \geq \overline{U}(\pi^S)$ for all $\mu \in \Delta(Y)$. Given any fixed $\bar{s} \in S$, note that $\overline{U}(\pi^S) \geq U^*(\pi(\bar{s})) = \int_Y \pi(\bar{s}, dy)\, v(y)$. So, given any $\epsilon > 0$, there certainly exists a simple lottery $\lambda_\epsilon \in \Delta(Y)$ such that

$$U^*(\lambda_\epsilon) = \sum\nolimits_{y \in Y} \lambda_\epsilon(y)\, v(y) \leq \overline{U}(\pi^S) + \epsilon \tag{74}$$

But $v(y) = U^*(1_y) \geq \overline{U}(\pi^S)$ for all $y \in Y$. Hence $U^*(\lambda_\epsilon) \geq \overline{U}(\pi^S) \geq U^*(\pi(s))$ and so $\lambda_\epsilon \succsim^* \pi(s)$ for all $s \in S$. By condition (ED), it follows that $\lambda_\epsilon\, 1^S \succsim \pi^S$. Because of (74), this implies that $\overline{U}(\pi^S) + \epsilon \geq U^*(\lambda_\epsilon) \geq U^S(\pi^S)$. This is true for all $\epsilon > 0$. Hence, $U^S(\pi^S) \leq \overline{U}(\pi^S)$ in this second case as well.

The proof that $U^S(\pi^S) \geq \underline{U}(\pi^S)$ is similar, with each inequality sign reversed. ∎

Lemma 8.6: For all $\pi^S \in \Delta(Y^S, \mathcal{F})$ one has $U^S(\pi^S) = \int_S U^*(\pi(s))\, p(ds)$.

Proof: Let $\underline{U}(\pi^S)$ and $\overline{U}(\pi^S)$ be as in (73). Then, for $n = 2, 3, \ldots$, define:

$$\begin{aligned} \delta_n &:= \tfrac{1}{n}\, [\overline{U}(\pi^S) - \underline{U}(\pi^S)]; \quad J_{1n} := [\, \underline{U}(\pi^S),\ \underline{U}(\pi^S) + \delta_n\,]; \\ J_{in} &:= (\, \underline{U}(\pi^S) + (i-1)\, \delta_n,\ \underline{U}(\pi^S) + i\, \delta_n\,] \quad (i = 2, 3, \ldots, n); \\ \text{and}\quad E_{in} &:= \{\, s \in S \mid U^*(\pi(s)) \in J_{in}\, \} \quad (i = 1, 2, \ldots, n). \end{aligned}$$

The sets J_{in} ($i = 1, 2, \ldots, n$) are pairwise disjoint intervals of the real line, whose union is the closed interval $[\underline{U}(\pi^S), \overline{U}(\pi^S)]$. Also, as remarked in connection with showing (64) in Section 8.1, the mapping $s \mapsto U^*(\pi(s))$ is measurable. Hence, each set E_{in} is also measurable. But the family E_{in} ($i = 1, 2, \ldots, n$) is a partition of S into n pairwise disjoint events, so Lemma 8.4 implies that

$$\sum_{i=1}^n p(E_{in}) = 1 \tag{75}$$

For each $i = 1, 2, \ldots, n$, let λ_i be any lottery in $\Delta(Y)$ with the property that $U^*(\lambda_i) \in J_{in}$. By Lemma 8.5, it must be true that

$$(i-1)\,\delta_n \leq U^S\left(\pi^{E_{in}}, \lambda_i 1^{S \setminus E_{in}}\right) - \underline{U}(\pi^S) \leq i\,\delta_n \tag{76}$$

Also, condition (RO) implies that the two members of $\Delta(Y^S, \mathcal{F})$ specified by

$$\frac{1}{n}\sum_{i=1}^n \left(\pi^{E_{in}}, \lambda_i 1^{S \setminus E_{in}}\right), \; \frac{1}{n}\pi^S + \frac{n-1}{n}\left\langle \frac{1}{n-1}\left(\sum_{j \neq i} \lambda_j\right) 1^{E_{in}} \right\rangle_{i=1}^n \tag{77}$$

respectively are indifferent because, for each state $s \in S$, the common marginal measure is $(1 - \frac{1}{n})\,\lambda_i + \frac{1}{n}\,\pi(s)$. But then, because U^S satisfies (MP), and because of Lemma 8.3, applying U^S to the two indifferent prospects in (77) gives

$$\begin{aligned} & \frac{1}{n}\sum_{i=1}^n U^S\left(\pi^{E_{in}}, \lambda_i 1^{S \setminus E_{in}}\right) \\ = \; & \frac{1}{n}U^S(\pi^S) + \frac{n-1}{n}\sum_{i=1}^n p(E_{in}) \cdot \frac{1}{n-1}\sum_{j \neq i} U^*(\lambda_j) \end{aligned}$$

Therefore, because of (75),

$$\begin{aligned} U^S(\pi^S) &= \sum_{i=1}^n U^S\left(\pi^{E_{in}}, \lambda_i 1^{S \setminus E_{in}}\right) - \sum_{i=1}^n p(E_{in}) \sum_{j \neq i} U^*(\lambda_j) \\ &= \sum_{i=1}^n U^S\left(\pi^{E_{in}}, \lambda_i 1^{S \setminus E_{in}}\right) - \sum_{i=1}^n [1 - p(E_{in})]\, U^*(\lambda_i) \end{aligned}$$

which implies that

$$\begin{aligned} U^S(\pi^S) - \underline{U}(\pi^S) \;=\; & \sum_{i=1}^n \left[U^S\left(\pi^{E_{in}}, \lambda_i 1^{S \setminus E_{in}}\right) - \underline{U}(\pi^S)\right] \\ & - \sum_{i=1}^n [1 - p(E_{in})]\,[U^*(\lambda_i) - \underline{U}(\pi^S)] \end{aligned} \tag{78}$$

To simplify notation later, define

$$Q_n := \sum_{i=1}^n i\, p(E_{in}) \tag{79}$$

First, consider what happens when, for each $i = 1, 2, \ldots, n$, the lottery $\lambda_i \in \Delta(Y)$ with $U^*(\lambda_i) \in J_{in}$ satisfies the extra restriction

$$U^*(\lambda_i) - \underline{U}(\pi^S) \leq \left(i - 1 + \frac{1}{n-1} \right) \delta_n \tag{80}$$

Then (78), (76), (80), (75) and (79) together imply that

$$\begin{aligned} & U^S(\pi^S) - \underline{U}(\pi^S) \\ \geq\ & \delta_n \sum\nolimits_{i=1}^n (i-1) - \delta_n \sum\nolimits_{i=1}^n \left(i - 1 + \frac{1}{n-1} \right) [1 - p(E_{in})] \\ =\ & -\delta_n \sum\nolimits_{i=1}^n \frac{1}{n-1} [1 - p(E_{in})] + \delta_n \sum\nolimits_{i=1}^n (i-1)\, p(E_{in}) \\ =\ & -2\, \delta_n + \delta_n\, Q_n \end{aligned} \tag{81}$$

Alternatively, consider what happens when, for each $i = 1, 2, \ldots, n$, the lottery $\lambda_i \in \Delta(Y)$ with $U^*(\lambda_i) \in J_{in}$ satisfies the extra restriction

$$\left(i - \frac{1}{n-1} \right) \delta_n \leq U^*(\lambda_i) - \underline{U}(\pi^S) \tag{82}$$

Then (78), (76), (82), (75) and (79) together imply that

$$\begin{aligned} U^S(\pi^S) - \underline{U}(\pi^S) \ &\leq\ \delta_n \sum\nolimits_{i=1}^n i - \delta_n \sum\nolimits_{i=1}^n \left(i - \frac{1}{n-1} \right) [1 - p(E_{in})] \\ &=\ \delta_n \sum\nolimits_{i=1}^n \frac{1}{n-1} [1 - p(E_{in})] + \delta_n \sum\nolimits_{i=1}^n i\, p(E_{in}) \\ &=\ \delta_n + \delta_n\, Q_n \end{aligned} \tag{83}$$

But the definitions of the intervals J_{in} and of the Lebesgue integral imply that

$$\delta_n \sum\nolimits_{i=1}^n (i-1)\, p(E_{in}) \leq \int_S U^*(\pi(s))\, p(ds) - \underline{U}(\pi^S) \leq \delta_n \sum\nolimits_{i=1}^n i\, p(E_{in})$$

So, by (75) and (79), it follows that

$$-\delta_n + \delta_n\, Q_n \leq \int_S U^*(\pi(s))\, p(ds) - \underline{U}(\pi^S) \leq \delta_n\, Q_n \tag{84}$$

Now subtract the second inequality in (84) from (81), and the first from (83). These two operations lead to the double inequality

$$-2\, \delta_n \leq U^S(\pi^S) - \int_S U^*(\pi(s))\, p(ds) \leq 2\, \delta_n \tag{85}$$

Finally, because $\delta_n = \frac{1}{n}[\overline{U}(\pi^S) - \underline{U}(\pi^S)]$, the result follows from taking the limit of (85) as $n \to \infty$ and so $\delta_n \to 0$. ∎

8.3 Eleven Sufficient Conditions

THEOREM 8: Conditions (O), (I), (C), (RO), (STP), (SI), (D), (EC), (ED), (M) and (PD) are sufficient for the SEU hypothesis to apply to $\Delta(Y^E, \mathcal{F})$.

PROOF: Lemma 8.6 shows that the ordering $\succsim$ on $\Delta(Y^S, \mathcal{F})$ is represented by the utility integral $U^S(\pi^S) = \int_S U^*(\pi(s))\, p(ds)$, where $U^*(\pi(s))$ is defined by $\int_Y \pi(s, dy)\, v(y)$. So $U^S(\pi^S)$ takes the form (64), as required. Also, by Lemma 8.4, p is a probability measure on the space $(S, \mathcal{S})$. ∎

The above result used eleven sufficient conditions for the SEU model. With so many conditions, it may help to group them in order to assess the contribution each makes to the overall result. This is done in Table 1.

domain of probability distributions	simple	discrete	measures
domain	$\Delta(Y)$	$\Delta^*(Y)$	$\Delta(Y, \mathcal{F})$
conditions for objective EU	(O), (I), (C) [3 conditions]	+ (D) [4 conditions]	+ (M), (PD) [6 conditions]
domain	$\Delta(Y^S)$	$\Delta^*(Y^S)$	$\Delta(Y^S, \mathcal{F})$
extra conditions for subjective EU	+ (RO), (STP), (SI) [6 conditions]	+ (EC), (ED) [9 conditions]	——— [11 conditions]

(Extra conditions enter as one moves either down or to the right.)

Table 1 Eleven Sufficient Conditions for Expected Utility Maximization

First come the three conditions that were introduced in Chapter 5 as sufficient for objectively expected utility with simple lotteries. These are **ordinality (O), independence (I)**, and **continuity (C)**. Second, the same chapter introduced one extra condition for objectively expected utility with discrete lotteries: **dominance (D)**. Third, the same chapter also introduced two extra conditions for objectively expected utility with probability measures: **measurability (M)** and **probability dominance (PD)**. It was noted that conditions (O), (I), (C) and (D) imply that utility is bounded.

This chapter first introduced three extra conditions (in addition to (O), (I) and (C)) for subjectively expected utility with simple lotteries. These are **reversal of order (RO)**, the **sure thing principle (STP)**, and **state independence (SI)**. Second, it introduced two extra conditions (in addition to (O), (I), (C), (D), (RO), (STP) and (SI)) for subjectively expected utility with discrete lotteries and an infinite set of states of the world: **event continuity (EC)** and **event dominance (ED)**. Finally, adding the two conditions (M) and (PD) that had already been included for objectively expected utility with probability measures gives the entire list of all eleven conditions that are sufficient for subjectively expected utility with probability measures over both states and consequences — no further conditions need be added.

Note that the eight conditions (O), (I), (D), (RO), (STP), (SI), (PD) and (ED) are all justified by consequentialism (or weak extensions). Only the domain condition (M) and two continuity conditions (C) and (EC) lack a consequentialist justification.

9 Summary and Conclusions

In Section 2, the subjective expected utility (or SEU) hypothesis was stated for the case when there are no events with objective probabilities. It was shown to imply the ordering of events condition (OE) in particular. Turning to sufficient conditions like those in Chapter 5 for the EU hypothesis, Section 3 showed how consequentialist axioms justify the existence of a preference ordering satisfying (STP), which is a form of Savage's sure thing principle. Furthermore, the axioms rule out null events, but they fail to justify condition (OE), and so do not imply the SEU hypothesis.

After this essentially false start, Section 4 turned to the framework inspired by Anscombe and Aumann (1963), with roulette as well as horse lotteries. Particular ratios of utility differences can then be interpreted as subjective probabilities. For the space $\Delta(Y^S)$ whose members are simple lotteries with finite support on the space Y^S of contingent consequence functions, Section 4 showed that necessary and sufficient conditions for the SEU hypothesis are ordinality (O), independence (I), continuity (C), reversal of order (RO), the sure-thing principle (STP), and state independence (SI). In fact, as in Chapter 5, stronger versions of conditions (I) and (C) are also necessary.

In order to provide a consequentialist justification for conditions (O), (I), (RO), (STP), and even (SI), Section 5 considered decision trees including moves made by chance which have objective probabilities, as well as moves made by nature which lack objective probabilities. It was also shown that these conditions exhaust all the implications of consequentialism because consequentialist behaviour is always possible whenever conditions (O), (I), (RO), (STP), and (SI) are all satisfied.

Next, Section 6 considered conditions for the SEU model to apply with state dependent consequence domains, but with state independent utility for consequences which arise in more than one state of the world. These conditions involve the hypothetical choice of the objective probabilities which might apply to different states of the world.

The corresponding space $\Delta^*(Y^S)$ of discrete lotteries on Y^S that can have countably infinite support was considered in Section 7. In addition, S was allowed to be an arbitrary infinite set. Apart from the dominance condition (D) that was introduced in Chapter 5, two new conditions of event continuity (EC) and event dominance (ED) enter the set of necessary and sufficient conditions for the SEU hypothesis to hold.

Finally, Section 8 considered the space $\Delta(Y^S, \mathcal{F})$ of measurable mappings from states of the world $s \in S$ to probability measures on the σ-field $\mathcal{F}$ of measurable sets generated by the singleton and preference subsets of Y^S. Here, as in Chapter 5, two extra conditions enter the list — the obvious measurability condition (M), and a probability dominance condition (PD) that is different from condition (D).

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