# Lecture Notes 1: Matrix Algebra
# Part D: Similar Matrices and Diagonalization

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# Outline

# Definitions in the Real Case

### Definition

Consider any $n \times n$ matrix $\mathbf{A}$.

The scalar $\lambda \in \mathbb{R}$ is an eigenvalue of $\mathbf{A}$,
just in case the equation $\mathbf{Ax} = \lambda \mathbf{x}$ has a non-zero solution.

In this case the solution $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ is an eigenvector,
and the pair $(\lambda, \mathbf{x})$ is an eigenpair.

The spectrum of the matrix $\mathbf{A}$ is the set $S_{\mathbf{A}}$ of its eigenvalues.

Let $S_{\mathbf{A}}^{\mathbb{R}}$ denote the subset of its real eigenvalues.

Let $S_{\mathbf{A}}^{\mathbb{C}}$ denote the subset of its complex eigenvalues,
which satisfies $S_{\mathbf{A}}^{\mathbb{C}} = S_{\mathbf{A}} \setminus S_{\mathbf{A}}^{\mathbb{R}}$.

# Summary of Main Properties

We will be demonstrating the following properties:

1. $S^{\mathbb{R}}_{\mathbf{A}} \subseteq S_{\mathbf{A}}$ and $\# S_{\mathbf{A}} \leq n$
2. The number $\# S^{\mathbb{C}}_{\mathbf{A}}$ of complex eigenvalues is even, and the members of $S^{\mathbb{C}}_{\mathbf{A}}$ are complex conjugate pairs $\lambda \pm \mu i$.
3. $S^{\mathbb{R}}_{\mathbf{A}} = \emptyset$ is possible in case $n$ is even, but not if $n$ is odd.
4. In case $\mathbf{A}$ is symmetric, one has $S^{\mathbb{C}}_{\mathbf{A}} = \emptyset$ and $S^{\mathbb{R}}_{\mathbf{A}} = S_{\mathbf{A}}$.

# The Eigenspace

Given any eigenvalue $\lambda$, let $E_\lambda := \{\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\} \mid \mathbf{A}\mathbf{x} = \lambda\mathbf{x}\}$ denote the associated set of eigenvectors.

Given any two eigenvectors $\mathbf{x}, \mathbf{y} \in E_\lambda$
and any two scalars $\alpha, \beta \in \mathbb{R}$, note that

$$\mathbf{A}(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha\mathbf{A}\mathbf{x} + \beta\mathbf{A}\mathbf{y} = \alpha\lambda\mathbf{x} + \beta\lambda\mathbf{y} = \lambda(\alpha\mathbf{x} + \beta\mathbf{y})$$

Hence the linear combination $\alpha\mathbf{x} + \beta\mathbf{y}$,
unless it is $\mathbf{0}$, is also an eigenvector in $E_\lambda$.

It follows that the set $E_\lambda \cup \{\mathbf{0}\}$ is a linear subspace of $\mathbb{R}^n$
which we call the eigenspace associated with the eigenvalue $\lambda$.

# Powers of a Matrix

## Theorem

*Suppose that* $(\lambda, \mathbf{x})$ *is an eigenpair of the* $n \times n$ *matrix* $\mathbf{A}$*.*

*Then* $\mathbf{A}^m \mathbf{x} = \lambda^m \mathbf{x}$ *for all* $m \in \mathbb{N}$*.*

## Proof.

By definition, $\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$.

Premultiplying each side of this equation by the matrix $\mathbf{A}$ gives

$$\mathbf{A}^2 \mathbf{x} = \mathbf{A}(\mathbf{A}\mathbf{x}) = \mathbf{A}(\lambda \mathbf{x}) = \lambda(\mathbf{A}\mathbf{x}) = \lambda(\lambda \mathbf{x}) = \lambda^2 \mathbf{x}$$

As the induction hypothesis,
suppose that $\mathbf{A}^{m-1}\mathbf{x} = \lambda^{m-1}\mathbf{x}$ for any $m = 2, 3, \ldots$

Premultiplying each side of this last equation by the matrix $\mathbf{A}$ gives

$$\mathbf{A}^m \mathbf{x} = \mathbf{A}(\mathbf{A}^{m-1}\mathbf{x}) = \mathbf{A}(\lambda^{m-1}\mathbf{x}) = \lambda^{m-1}(\mathbf{A}\mathbf{x}) = \lambda^{m-1}(\lambda \mathbf{x}) = \lambda^m \mathbf{x}$$

This completes the proof by induction on $m$. □

# Characteristic Equation

The equation $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ holds for $\mathbf{x} \neq \mathbf{0}$
if and only if $\mathbf{x} \neq \mathbf{0}$ solves $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$.

This holds iff the matrix $\mathbf{A} - \lambda\mathbf{I}$ is singular,
which holds iff $\lambda$ is a characteristic root
— i.e., it solves the characteristic equation $|\mathbf{A} - \lambda\mathbf{I}| = 0$.

Equivalently, $\lambda$ is a zero of the polynomial $|\mathbf{A} - \lambda\mathbf{I}|$ of degree $n$.

Suppose $|\mathbf{A} - \lambda\mathbf{I}| = 0$ has $k$ distinct real roots $\lambda_1, \lambda_2, \ldots, \lambda_k$
whose multiplicities are respectively $m_1, m_2, \ldots, m_k$.

This means that

$$\begin{aligned} |\mathbf{A} - \lambda\mathbf{I}| &= (-1)^n(\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k} \\ &= (-1)^n \prod_{j=1}^{k}(\lambda - \lambda_j)^{m_j} \end{aligned}$$

The polynomial has degree $m_1 + m_2 + \ldots + m_k$, which equals $n$.

This implies that $k \leq n$,
so there can be at most $n$ distinct real eigenvalues.

# Eigenvalues of a $2 \times 2$ matrix

Consider the $2 \times 2$ matrix $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

The characteristic equation for its eigenvalues is

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0$$

Evaluating the determinant gives the equation

$$\begin{aligned} 0 &= (a_{11} - \lambda)(a_{22} - \lambda) - a_{12} a_{21} \\ &= \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11} a_{22} - a_{12} a_{21}) \\ &= \lambda^2 - (\operatorname{tr} \mathbf{A})\lambda + |\mathbf{A}| \ = \ (\lambda - \lambda_1)(\lambda - \lambda_2) \end{aligned}$$

where the two roots $\lambda_1$ and $\lambda_2$ of the quadratic equation have:

- a sum $\lambda_1 + \lambda_2$ equal to the trace $\operatorname{tr} \mathbf{A}$ of $\mathbf{A}$ (the sum of its diagonal elements);
- a product $\lambda_1 \cdot \lambda_2$ equal to the determinant of $\mathbf{A}$.

Let $\mathbf{\Lambda}$ denote the diagonal matrix $\mathbf{diag}(\lambda_1, \lambda_2)$ whose diagonal elements are the eigenvalues.

Note that $\operatorname{tr} \mathbf{A} = \operatorname{tr} \mathbf{\Lambda}$ and $|\mathbf{A}| = |\mathbf{\Lambda}|$.

# The Case of a Diagonal Matrix, I

For the diagonal matrix $\mathbf{D} = \mathbf{diag}(d_1, d_2, \ldots, d_n)$,
the characteristic equation $|\mathbf{D} - \lambda \mathbf{I}| = 0$
takes the degenerate form $\prod_{k=1}^n (d_k - \lambda) = 0$.

So the spectrum $S_{\mathbf{A}}$ equals $\{d_1, d_2, \ldots, d_n\}$,
the set of diagonal elements.

$\# S_{\mathbf{A}}$ could be any number beteen 1 and $n$.

The $i$th component of the vector equation $\mathbf{D}\mathbf{x} = d_k \mathbf{x}$
takes the form $d_i x_i = d_k x_i$,
which has a non-trivial solution if and only if $d_i = d_k$.

The $k$th vector $\mathbf{e}^k = (\delta_{jk})_{j=1}^n$
of the canonical orthonormal basis of $\mathbb{R}^n$
always solves the equation $\mathbf{D}\mathbf{x} = d_k \mathbf{x}$,
and so is an eigenvector associated with the eigenvalue $d_k$.

# The Case of a Diagonal Matrix, II

Apart from non-zero multiples of $\mathbf{e}^k$,
there are other eigenvectors associated with $d_k$
only if a different element $d_i$ of the diagonal also equals $d_k$.

In fact, the eigenspace associated with each eigenvalue $d_k$
equals the space spanned by the set $\{\mathbf{e}^i \mid d_i = d_k\}$
of canonical basis vectors.

## Example

In case $\mathbf{D} = \mathbf{diag}(1, 1, 0)$ the spectrum is $\{0, 1\}$ with:

- the one-dimensional eigenspace

$$E_0 = \{x_3\,(0, 0, 1)^\top \mid x_3 \in \mathbb{R}\}$$

- the two-dimensional eigenspace

$$E_1 = \{x_1\,(1, 0, 0)^\top + x_2\,(0, 1, 0)^\top \mid (x_1, x_2) \in \mathbb{R}^2\}$$

# Characterizing $2 \times 2$ Orthogonal Matrices

By definition, an orthogonal matrix $\mathbf{P}$ satisfies $\mathbf{P}^\top \mathbf{P} = \mathbf{P}\mathbf{P}^\top = \mathbf{I}$.

In the $2 \times 2$ case when $\mathbf{P} = (p_{ij})_{2\times 2}$ , the matrix $\mathbf{P}\mathbf{P}^\top$ equals

$$\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} p_{11} & p_{21} \\ p_{12} & p_{22} \end{pmatrix} = \begin{pmatrix} (p_{11})^2 + (p_{12})^2 & p_{11}p_{21} + p_{21}p_{22} \\ p_{21}p_{11} + p_{22}p_{12} & (p_{21})^2 + (p_{22})^2 \end{pmatrix}$$

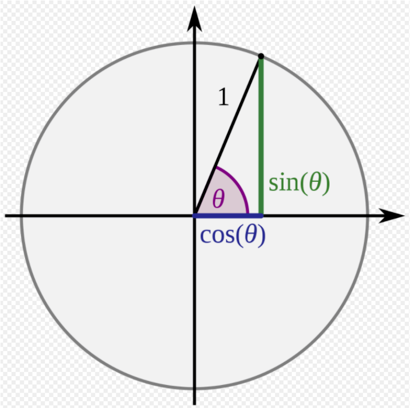
Here we use trigonometric identities
and put $p_{11} = \cos\theta, \quad p_{22} = \cos\eta, \quad p_{12} = -\sin\theta, \quad p_{21} = \sin\eta.$
This makes $\mathbf{P}\mathbf{P}^\top$ equal

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\eta & \cos\eta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\eta \\ -\sin\theta & \cos\eta \end{pmatrix} = \begin{pmatrix} 1 & \sin(\eta - \theta) \\ \sin(\eta - \theta) & 1 \end{pmatrix}$$

This equals the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

if and only if $\mathbf{P}$ equals the rotation matrix $\mathbf{R}_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

# Rotations Illustrated




Illustrating $P_\theta = \mathbf{R}_\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$.

Also $P_{\theta+\frac{1}{2}\pi} = \mathbf{R}_\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$.

# Rotations in Polar Coordinates

Recall that a 2-dimensional *rotation matrix* takes the form

$$\mathbf{R}_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

for $\theta \in \mathbb{R}$, which is the *angle of rotation* measured in radians.
The rotation $\mathbf{R}_\theta$ transforms any vector $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ to

$$\mathbf{R}_\theta \mathbf{x} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \cos\theta - x_2 \sin\theta \\ x_1 \sin\theta + x_2 \cos\theta \end{pmatrix}$$

Introduce polar coordinates $(r, \eta)$,
where $\mathbf{x} = (x_1, x_2) = r(\cos\eta, \sin\eta)$. Then

$$\mathbf{R}_\theta \mathbf{x} = r \begin{pmatrix} \cos\eta\cos\theta - \sin\eta\sin\theta \\ \cos\eta\sin\theta + \sin\eta\cos\theta \end{pmatrix} = r \begin{pmatrix} \cos(\eta+\theta) \\ \sin(\eta+\theta) \end{pmatrix}$$

This makes it easy to verify that $\mathbf{R}_{\theta+2k\pi} = \mathbf{R}_\theta$ for all $\theta \in \mathbb{R}$
and $k \in \mathbb{Z}$, and that $\mathbf{R}_\theta \mathbf{R}_\eta = \mathbf{R}_\eta \mathbf{R}_\theta = \mathbf{R}_{\theta+\eta}$ for all $\theta, \eta \in \mathbb{R}$.

# Does a Rotation Matrix Have Real Eigenvalues?

The characteristic equation $|\mathbf{R}_\theta - \lambda \mathbf{I}| = 0$ takes the form

$$0 = \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = (\cos\theta - \lambda)^2 + \sin^2\theta = 1 - 2\lambda\cos\theta + \lambda^2$$

1. A degenerate case occurs when $\theta = 2k\pi$ for some $k \in \mathbb{Z}$ so $\cos\theta = 1$ and $\sin\theta = 0$.

   Then $\mathbf{R}_\theta$ reduces to the identity matrix $\mathbf{I}_2$.
2. Otherwise, the real matrix $\mathbf{R}_\theta$ has no real eigenvalues.

   Indeed, if $\cos\theta < 1$, the characteristic equation has two roots $\lambda = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$.

   Because $\sin\theta = \sqrt{1 - \cos^2\theta} \neq 0$, there are two distinct complex conjugate eigenvalues.

   The associated eigenspaces must both consist of complex eigenvectors.

# Outline

# Complex Eigenvalues

To consider complex eigenvalues properly,
we need to leave $\mathbb{R}^n$ and consider instead the linear space $\mathbb{C}^n$
whose elements are *n*-vectors with complex coordinates.

That is, we consider a linear space whose field of scalars
is the plane $\mathbb{C}$ of complex numbers,
rather than the line $\mathbb{R}$ of real numbers.

Suppose $\mathbf{A}$ is any $n \times n$ matrix
whose elements may be real or complex.

The complex scalar $\lambda \in \mathbb{C}$ is an eigenvalue
just in case the equation $\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$ has a non-zero solution,
in which case that solution $\mathbf{x} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ is an eigenvector.

# Fundamental Theorem of Algebra

## Theorem

*Let* $P(\lambda) = \lambda^n + \sum_{k=0}^{n-1} p_k \lambda^k$
*be a polynomial function of* $\lambda$ *of degree* $n$ *in the complex plane* $\mathbb{C}$.

*Then there exists at least one root* $\hat{\lambda} \in \mathbb{C}$ *such that* $P(\hat{\lambda}) = 0$.

## Corollary

*The polynomial* $P(\lambda)$ *can be factorized*
*as the product* $P_n(\lambda) \equiv \prod_{r=1}^{n}(\lambda - \lambda_r)$ *of exactly* $n$ *linear terms.*

## Proof.

The proof will be by induction on $n$.

When $n = 1$ one has $P_1(\lambda) = \lambda + p_0$, whose only root is $\lambda = -p_0$.

Suppose the result is true when $n = m - 1$.

By the fundamental theorem of algebra,
there exists $\hat{\lambda} \in \mathbb{C}$ such that $P_m(\hat{\lambda}) = 0$.

Polynomial division gives $P_m(\lambda) \equiv P_{m-1}(\lambda)(\lambda - \hat{\lambda})$, etc. □

# Characteristic Roots as Eigenvalues

### Theorem

*Every $n \times n$ matrix $\mathbf{A} \in \mathbb{C}^{n \times n}$ with complex elements has exactly $n$ eigenvalues (real or complex) corresponding to the roots, counting multiple roots, of the characteristic equation $|\mathbf{A} - \lambda \mathbf{I}| = 0$.*

### Proof.

The characteristic equation can be written in the form $P_n(\lambda) = 0$ where $P_n(\lambda) \equiv |\lambda \mathbf{I} - \mathbf{A}|$ is a polynomial of degree $n$.

By the fundamental theorem of algebra, together with its corollary, the polynomial $|\lambda \mathbf{I} - \mathbf{A}|$ equals the product $\prod_{r=1}^{n} (\lambda - \lambda_r)$ of $n$ linear terms.

For any of these roots $\lambda_r$ the matrix $\mathbf{A} - \lambda_r \mathbf{I}$ is singular.

So there exists $\mathbf{x} \neq \mathbf{0}$ such that $(\mathbf{A} - \lambda_r \mathbf{I})\mathbf{x} = \mathbf{0}$ or $\mathbf{A}\mathbf{x} = \lambda_r \mathbf{x}$, implying that $\lambda_r$ is an eigenvalue. □

# Outline

## Eigenvalues and Eigenvectors

### Real Case
### The Complex Case
### Linear Independence of Eigenvectors

## Diagonalizing a General Matrix

### Similar Matrices

## Properties of Adjoint and Symmetric Matrices

### A Self-Adjoint Matrix has only Real Eigenvalues

## Diagonalizing a Symmetric Matrix

### Orthogonal Matrices
### Orthogonal Projections
### Rayleigh Quotient
### The Spectral Theorem

## Quadratic Forms and Their Definiteness

### Quadratic Forms
### The Eigenvalue Test of Definiteness
### Sylvester's Criterion for Definiteness

# Linear Independence of Eigenvectors

The following theorem tells us
that eigenvectors associated with distinct eigenvalues
must be linearly independent.

### Theorem

*Let* $\{\lambda_k\}_{k=1}^m = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$
*be any collection of* $m \leq n$ *distinct eigenvalues.*

*Then any corresponding set* $\{\mathbf{x}_k\}_{k=1}^m$ *of associated eigenvectors must be linearly independent.*

The proof will be by induction on $m$.

Because $\mathbf{x}_1 \neq \mathbf{0}$, the set $\{\mathbf{x}_1\}$ is linearly independent.

So the result is evidently true when $m = 1$.

As the induction hypothesis, suppose the result holds for $m - 1$.

# Completing the Proof by Induction, I

Suppose that one solution of the equation $\mathbf{A}\mathbf{x} = \lambda_m \mathbf{x}$, which may be zero, is the linear combination $\mathbf{x} = \sum_{k=1}^{m-1} \alpha_k \mathbf{x}_k$ of the preceding $m-1$ eigenvectors. Hence

$$\mathbf{A}\mathbf{x} = \lambda_m \mathbf{x} = \sum_{k=1}^{m-1} \alpha_k \lambda_m \mathbf{x}_k$$

Then the hypothesis that $\{(\lambda_k, \mathbf{x}_k)\}_{k=1}^{m-1}$ is a collection of eigenpairs implies that this $\mathbf{x}$ satisfies

$$\mathbf{A}\mathbf{x} = \sum_{k=1}^{m-1} \alpha_k \mathbf{A}\mathbf{x}_k = \sum_{k=1}^{m-1} \alpha_k \lambda_k \mathbf{x}_k$$

Subtracting this equation from the prior equation gives

$$\mathbf{0} = \sum_{k=1}^{m-1} \alpha_k (\lambda_m - \lambda_k) \mathbf{x}_k$$

# Completing the Proof by Induction, II

So we have

$$\mathbf{0} = \sum_{k=1}^{m-1} \alpha_k (\lambda_m - \lambda_k) \mathbf{x}_k$$

The induction hypothesis is that the set $\{\mathbf{x}_k\}_{k=1}^{m-1}$
of distinct eigenvectors is linearly independent, implying that

$$\alpha_k (\lambda_m - \lambda_k) \mathbf{x}_k = 0 \quad \text{for } k = 1, \ldots, m-1$$

But we are assuming that $\lambda_m \notin \{\lambda_k\}_{k=1}^{m-1}$,
so $\lambda_m - \lambda_k \neq 0$ for $k = 1, \ldots, m-1$.

It follows that $\alpha_k = 0$ for $k = 1, \ldots, m-1$.

We have proved that if $\mathbf{x} = \sum_{k=1}^{m-1} \alpha_k \mathbf{x}_k$ solves $\mathbf{A}\mathbf{x} = \lambda_m \mathbf{x}$,
then $\mathbf{x} = \mathbf{0}$, so $\mathbf{x}$ is not an eigenvector.

This completes the proof by induction that no eigenvector $\mathbf{x} \in E_{\lambda_m}$
can be a linear combination
of the eigenvectors $\mathbf{x}_k \in E_{\lambda_k}$ $(k = 1, \ldots, m-1)$. $\square$

# Outline

# Similar Matrices

**Definition**

The two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ are similar
just in case there exists an invertible $n \times n$ matrix $\mathbf{S}$
such that the following three equivalent statements all hold

$$\mathbf{B} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S} \iff \mathbf{S}\mathbf{B} = \mathbf{A}\mathbf{S} \iff \mathbf{A} = \mathbf{S}\mathbf{B}\mathbf{S}^{-1}$$

in which case we write $\mathbf{A} \sim \mathbf{B}$.

# Similarity is an Equivalence Relation

## Theorem

*The similarity relation is an equivalence relation — i.e., $\sim$ is:*

- *reflexive* $\mathbf{A} \sim \mathbf{A}$;
- *symmetric* $\mathbf{A} \sim \mathbf{B} \iff \mathbf{B} \sim \mathbf{A}$;
- *transitive* $\mathbf{A} \sim \mathbf{B}$ & $\mathbf{B} \sim \mathbf{C} \Longrightarrow \mathbf{A} \sim \mathbf{C}$

## Proof.

The proofs that $\sim$ is reflexive and symmetric are elementary.

Suppose that $\mathbf{A} \sim \mathbf{B}$ and $\mathbf{B} \sim \mathbf{C}$.

By definition, there exist invertible matrices $\mathbf{S}$ and $\mathbf{T}$ such that $\mathbf{B} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ and $\mathbf{C} = \mathbf{T}^{-1}\mathbf{B}\mathbf{T}$.

Define $\mathbf{U} := \mathbf{S}\mathbf{T}$, which is invertible with $\mathbf{U}^{-1} = \mathbf{T}^{-1}\mathbf{S}^{-1}$.

Then $\mathbf{C} = \mathbf{T}^{-1}(\mathbf{S}^{-1}\mathbf{A}\mathbf{S})\mathbf{T} = (\mathbf{T}^{-1}\mathbf{S}^{-1})\mathbf{A}(\mathbf{S}\mathbf{T}) = \mathbf{U}^{-1}\mathbf{A}\mathbf{U}$.

So $\mathbf{A} \sim \mathbf{C}$. □

# Similar Matrices Have Identical Spectra

### Theorem

*If* $\mathbf{A} \sim \mathbf{B}$ *then* $\mathcal{S}_{\mathbf{A}} = \mathcal{S}_{\mathbf{B}}$.

### Proof.

Suppose that $\mathbf{A} = \mathbf{SBS}^{-1}$ and that $(\lambda, \mathbf{x})$ is an eigenpair of $\mathbf{A}$.

Then $\mathbf{x} \neq \mathbf{0}$ solves $\mathbf{Ax} = \mathbf{SBS}^{-1}\mathbf{x} = \lambda\mathbf{x}$.

Premultiplying each side of the equation $\mathbf{SBS}^{-1}\mathbf{x} = \lambda\mathbf{x}$ by $\mathbf{S}^{-1}$, it follows that $\mathbf{y} := \mathbf{S}^{-1}\mathbf{x}$ solves $\mathbf{By} = \lambda\mathbf{y}$.

Moreover, because $\mathbf{S}^{-1}$ has the inverse $\mathbf{S}$, the equation $\mathbf{S}^{-1}\mathbf{x} = \mathbf{y}$ would have only the trivial solution $\mathbf{x} = \mathbf{Sy} = \mathbf{0}$ in case $\mathbf{y} = \mathbf{0}$.

Hence $\mathbf{y} \neq \mathbf{0}$, implying that $(\lambda, \mathbf{y})$ is an eigenpair of $\mathbf{B}$.

A symmetric argument shows
that if $(\lambda, \mathbf{y})$ is an eigenpair of $\mathbf{B} = \mathbf{S}^{-1}\mathbf{SA}$,
then $(\lambda, \mathbf{Sy})$ is an eigenpair of $\mathbf{A}$. □

# Diagonalization

## Definition

An $n \times n$ matrix $\mathbf{A}$ matrix is diagonalizable just in case it is similar to a diagonal matrix $\mathbf{\Lambda} = \mathbf{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

## Theorem

*Given any diagonalizable $n \times n$ matrix $\mathbf{A}$:*

1. *The columns of any matrix $\mathbf{S}$ that diagonalizes $\mathbf{A}$ must consist of $n$ linearly independent eigenvectors of $\mathbf{A}$.*
2. *The matrix $\mathbf{A}$ is diagonalizable if and only if it has a set of $n$ linearly independent eigenvectors.*
3. *The matrix $\mathbf{A}$ and its diagonalization $\mathbf{\Lambda} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ have the same set of eigenvalues.*

# Proof of Part 1

Suppose that $\mathbf{AS} = \mathbf{S\Lambda}$ where $\mathbf{A} = (a_{ij})^{n \times n}$, $\mathbf{S} = (s_{ij})^{n \times n}$, and $\mathbf{\Lambda} = \mathbf{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Then for each $i, k \in \{1, 2, \ldots, n\}$,
equating the elements in row $i$ and column $k$
of the equal matrices $\mathbf{AS}$ and $\mathbf{S\Lambda}$ implies that

$$\sum\nolimits_{j=1}^{n} a_{ij} s_{jk} = \sum\nolimits_{j=1}^{n} s_{ij} \delta_{jk} \lambda_k = s_{ik} \lambda_k$$

It follows that $\mathbf{A}\mathbf{s}^k = \lambda_k \mathbf{s}^k$
where $\mathbf{s}^k = (s_{ik})_{i=1}^{n}$ denotes the $k$th column of the matrix $\mathbf{S}$.

Because $\mathbf{S}$ must be invertible:

- each column $\mathbf{s}^k$ must be non-zero, so an eigenvector of $\mathbf{A}$;
- the set of all these $n$ columns must be linearly independent.

□

# Proofs of Parts 2 and 3

**Proof of Part 2:** By part 1, if the diagonalizing matrix $\mathbf{S}$ exists, its columns must form a set of $n$ linearly independent eigenvectors for the matrix $\mathbf{A}$.

Conversely, suppose that $\mathbf{A}$ does have a set $\{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^n\}$ of $n$ linearly independent eigenvectors, with $\mathbf{A}\mathbf{x}^k = \lambda_k \mathbf{x}^k$ for $k = 1, 2, \ldots, n$.

Now define $\mathbf{S}$ as the $n \times n$ matrix whose $k$th column is the eigenvector $\mathbf{x}^k$, for each $k = 1, 2, \ldots, n$.

Then it is easy to check that $\mathbf{AS} = \mathbf{S\Lambda}$ where $\mathbf{\Lambda} = \mathbf{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. $\square$

**Proof of Part 3:** This follows from the general property that similar matrices have the same spectrum of eigenvalues.

# Outline

# Complex Conjugates and Adjoint Matrices

Recall that any complex number $c \in \mathbb{C}$ can be expressed as $a + ib$ with $a \in \mathbb{R}$ as the real part and $b \in \mathbb{R}$ as the imaginary part.

The complex conjugate of $c$ is $\bar{c} = a - ib$.

Note that $c\bar{c} = \bar{c}c = (a + ib)(a - ib) = a^2 + b^2 = |c|^2$, where $|c|$ is the modulus of $c$.

Any $m \times n$ complex matrix $\mathbf{C} = (c_{ij})_{m \times n} \in \mathbb{C}^{m \times n}$ can be written as $\mathbf{A} + i\mathbf{B}$, where $\mathbf{A}$ and $\mathbf{B}$ are real $m \times n$ matrices.

The adjoint of the $m \times n$ complex matrix $\mathbf{C} = \mathbf{A} + i\mathbf{B}$, is the $n \times m$ complex matrix $\mathbf{C}^* := (\mathbf{A} - i\mathbf{B})^\top = \mathbf{A}^\top - i\mathbf{B}^\top$.

This is the transpose of the matrix $\mathbf{A} - i\mathbf{B}$ whose elements are the complex conjugates $\bar{c}_{jk}$ of the corresponding elements of $\mathbf{C}$.

That is, each element of $\mathbf{C}^*$ is given by $c^*_{jk} = a_{kj} - ib_{kj}$.

In the case of a real matrix $\mathbf{A}$, whose imaginary part is $\mathbf{0}$, its adjoint is simply the transpose $\mathbf{A}^\top$.

# Self-Adjoint and Symmetric Matrices

An $n \times n$ complex matrix $\mathbf{C} = \mathbf{A} + i\mathbf{B}$ is self-adjoint just in case $\mathbf{C}^* = \mathbf{C}$, which holds if and only if $\mathbf{A}^\top - i\mathbf{B}^\top = \mathbf{A} + i\mathbf{B}$, and so if and only if:

- the real part $\mathbf{A}$ is symmetric;
- the imaginary part $\mathbf{B}$ is anti-symmetric in the sense that $\mathbf{B}^\top = -\mathbf{B}$.

Of course, a real matrix is self-adjoint if and only if it is symmetric.

**Theorem**

*Any eigenvalue of a self-adjoint complex matrix is a real scalar.*

**Corollary**

*Any eigenvalue of a symmetric real matrix is a real scalar.*

# Proof that Eigenvalues of a Self-Adjoint Matrix are Real

Suppose that the scalar $\lambda \in \mathbb{C}$ and vector $\mathbf{x} \in \mathbb{C}^n$ together satisfy the eigenvalue equation $\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$ for any $\mathbf{A} \in \mathbb{C}^{n \times n}$.

Taking complex conjugates throughout, one has $\bar{\lambda}\mathbf{x}^* = \mathbf{x}^*\mathbf{A}^*$.

By the associative law of complex matrix multiplication,
one has $\mathbf{x}^*\mathbf{A}\mathbf{x} = \mathbf{x}^*(\mathbf{A}\mathbf{x}) = \mathbf{x}^*(\lambda\mathbf{x}) = \lambda(\mathbf{x}^*\mathbf{x})$
as well as $\mathbf{x}^*\mathbf{A}^*\mathbf{x} = (\mathbf{x}^*\mathbf{A}^*)\mathbf{x} = (\bar{\lambda}\mathbf{x}^*)\mathbf{x} = \bar{\lambda}(\mathbf{x}^*\mathbf{x})$.

In case $\mathbf{A}$ is self-adjoint and so $\mathbf{A}^* = \mathbf{A}$,
subtracting the second equation from the first gives

$$\mathbf{x}^*\mathbf{A}\mathbf{x} - \mathbf{x}^*\mathbf{A}^*\mathbf{x} = \mathbf{x}^*(\mathbf{A} - \mathbf{A}^*)\mathbf{x} = 0 = (\lambda - \bar{\lambda})(\mathbf{x}^*\mathbf{x})$$

But in case $\mathbf{x}$ is an eigenvector, one has $\mathbf{x} = (x_i)_{i=1}^n \in \mathbb{C}^n \setminus \{\mathbf{0}\}$
and so $\mathbf{x}^*\mathbf{x} = \sum_{i=1}^n |x_i|^2 > 0$.

Because $0 = (\lambda - \bar{\lambda})\mathbf{x}^*\mathbf{x}$, it follows that the eigenvalue $\lambda$
satisfies $\lambda - \bar{\lambda} = 0$, implying that $\lambda$ is real. □

# Outline

# Orthogonal and Orthonormal Sets of Vectors

Recall our earlier definition:

### Definition

A set of $k$ vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\} \subset \mathbb{R}^n$ is said to be:

- pairwise orthogonal just in case $\mathbf{x}_i \cdot \mathbf{x}_j = 0$ whenever $j \neq i$;
- orthonormal just in case, in addition, each $\|\mathbf{x}_i\| = 1$
  — i.e., all $k$ elements of the set are vectors of unit length.

The set of $k$ vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\} \subset \mathbb{R}^n$ is orthonormal just in case $\mathbf{x}_i \cdot \mathbf{x}_j = \delta_{ij}$ for all pairs $i, j \in \{1, 2, \ldots, k\}$.

# Orthogonal Matrices: Recall

Definition
Any $n \times n$ matrix is orthogonal
just in case its $n$ columns (or rows) form an orthonormal set.

Theorem
*Given any $n \times n$ matrix $\mathbf{P}$, the following are equivalent:*

1. $\mathbf{P}$ *is orthogonal;*
2. $\mathbf{P}\mathbf{P}^\top = \mathbf{P}^\top \mathbf{P} = \mathbf{I}$;
3. $\mathbf{P}^{-1} = \mathbf{P}^\top$;
4. $\mathbf{P}^\top$ *is orthogonal.*

# The Complex Case: Self-Adjoint and Unitary Matrices

We briefly consider matrices with complex elements.

Recall that the adjoint $\mathbf{A}^*$ of an $m \times n$ matrix $\mathbf{A}$
is the matrix formed from the transpose $\mathbf{A}^\top$
by taking the complex conjugate of each element.

The appropriate extension to complex numbers of:

- a symmetric matrix satisfying $\mathbf{A}^\top = \mathbf{A}$
  is a self-adjoint matrix satisfying $\mathbf{A}^* = \mathbf{A}$;
- an orthogonal matrix satisfying $\mathbf{P}^{-1} = \mathbf{P}^\top$
  is a unitary matrix satisfying $\mathbf{U}^{-1} = \mathbf{U}^*$.

# Outline

# Orthogonal Projection Matrices

### Definition

An $n \times n$ matrix $\mathbf{P}$ is an orthogonal projection if $\mathbf{P}^2 = \mathbf{P}$ and $\mathbf{u}^\top \mathbf{v} = 0$ whenever $\mathbf{P}\mathbf{v} = \mathbf{0}$ and $\mathbf{u} = \mathbf{P}\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$.

### Theorem

*Suppose that the $n \times m$ matrix $\mathbf{X}$ has full rank $m < n$.*

*Let $L \subset \mathbb{R}^n$ be the linear subspace spanned by $m$ linearly independent columns of $\mathbf{X}$.*

*Define the $n \times n$ matrix $\mathbf{P} := \mathbf{X}(\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top$. Then:*

1. *The matrix $\mathbf{P}$ is a symmetric orthogonal projection onto $L$.*
2. *The matrix $\mathbf{I} - \mathbf{P}$ is a symmetric orthogonal projection onto the orthogonal complement $L^\perp$ of $L$.*
3. *For each vector $\mathbf{y} \in \mathbb{R}^n$, its orthogonal projection onto $L$ is the unique vector $\mathbf{v} = \mathbf{P}\mathbf{y} \in L$ that minimizes the distance $\|\mathbf{y} - \mathbf{v}\|$ between $\mathbf{y}$ and $L$ — i.e., $\|\mathbf{y} - \mathbf{v}\| \leq \|\mathbf{y} - \mathbf{u}\|$ for all $\mathbf{u} \in L$.*

# Orthogonal Complements

### Definition

A subset $L \subseteq \mathbb{R}^n$ is a linear subspace just in case $\lambda \mathbf{x} + \mu \mathbf{y} \in L$ for every pair of vectors $\mathbf{x}, \mathbf{y}$ in $L$ and every pair of scalars $\lambda, \mu$ in $\mathbb{R}$.

### Definition

Given any linear subspace $L$ of $\mathbb{R}^n$, its orthogonal complement $L^\perp$ is the set of all vectors $\mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{x} \cdot \mathbf{y} = 0$ for all $\mathbf{x} \in L$.

# Two Examples

**Example**

Suppose that $L$ is the space spanned
by any finite subset $\{\mathbf{e}^i \mid i \in I\}$
of the canonical basis $\{\mathbf{e}^i \mid i = 1, 2, \ldots, n\}$ of $\mathbb{R}^n$.

Then $L^\perp$ is the space spanned
by the complementary set $\{\mathbf{e}^i \mid i \notin I\}$ of canonical basis vectors.

**Example**

Any $\mathbf{c} \neq \mathbf{0}$ in $\mathbb{R}^n$ generates the straight line $L(\mathbf{c}) = \{\lambda\mathbf{c} \mid \lambda \in \mathbb{R}\}$,
which is a one-dimensional linear subspace in $\mathbb{R}^n$.

Its orthogonal complement $L(\mathbf{c})^\perp = \{\mathbf{x} \mid \mathbf{c} \cdot \mathbf{x} = 0\}$
consists of an $n-1$-dimensional subspace
which is the unique hyperplane in $\mathbb{R}^n$ that has $\mathbf{c}$ as a normal.

# Proof of Part 1

First note that if $\mathbf{X}$ is an $n \times m$ matrix, then $\mathbf{X}^\top \mathbf{X}$ is $m \times m$.

Then, provided that $(\mathbf{X}^\top \mathbf{X})^{-1}$ exists,
so does the $n \times n$ matrix $\mathbf{P} = \mathbf{X}(\mathbf{X}^\top \mathbf{X})^{-1}\mathbf{X}^\top$.

Because of the rules for the transposes of products and inverses, the definition $\mathbf{P} := \mathbf{X}(\mathbf{X}^\top \mathbf{X})^{-1}\mathbf{X}^\top$ implies that $\mathbf{P}^\top = \mathbf{P}$ and also

$$\mathbf{P}^2 = \mathbf{X}(\mathbf{X}^\top \mathbf{X})^{-1}\mathbf{X}^\top \mathbf{X}(\mathbf{X}^\top \mathbf{X})^{-1}\mathbf{X}^\top = \mathbf{X}(\mathbf{X}^\top \mathbf{X})^{-1}\mathbf{X}^\top = \mathbf{P}$$

Moreover, if $\mathbf{P}\mathbf{v} = \mathbf{0}$ and $\mathbf{u} = \mathbf{P}\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$, then

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^\top \mathbf{v} = \mathbf{x}^\top \mathbf{P}^\top \mathbf{v} = \mathbf{x}^\top \mathbf{P}\mathbf{v} = 0$$

Finally, for every $\mathbf{y} \in \mathbb{R}^n$, the vector $\mathbf{P}\mathbf{y}$ equals $\mathbf{X}\mathbf{b}$, where

$$\mathbf{b} = (\mathbf{X}^\top \mathbf{X})^{-1}\mathbf{X}^\top \mathbf{y}$$

Hence $\mathbf{P}\mathbf{y} \in L$. □

# Proof of Part 2

Evidently $(\mathbf{I} - \mathbf{P})^\top = \mathbf{I} - \mathbf{P}^\top = \mathbf{I} - \mathbf{P}$, and

$$(\mathbf{I} - \mathbf{P})^2 = \mathbf{I} - 2\mathbf{P} + \mathbf{P}^2 = \mathbf{I} - 2\mathbf{P} + \mathbf{P} = \mathbf{I} - \mathbf{P}$$

Hence $\mathbf{I} - \mathbf{P}$ is a projection.

This projection is also orthogonal because if $(\mathbf{I} - \mathbf{P})\mathbf{v} = \mathbf{0}$ and $\mathbf{u} = (\mathbf{I} - \mathbf{P})\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$, then

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^\top \mathbf{v} = \mathbf{x}^\top (\mathbf{I} - \mathbf{P})^\top \mathbf{v} = \mathbf{x}^\top (\mathbf{I} - \mathbf{P})\mathbf{v} = 0$$

Next, suppose that $\mathbf{v} = \mathbf{X}\mathbf{b} \in L$ and that $\mathbf{y} = (\mathbf{I} - \mathbf{P})\mathbf{x}$ belongs to the range of $(\mathbf{I} - \mathbf{P})$. Then

$$\mathbf{y} \cdot \mathbf{v} = \mathbf{y}^\top \mathbf{v} = \mathbf{x}^\top (\mathbf{I} - \mathbf{P})^\top \mathbf{X}\mathbf{b} = \mathbf{x}^\top \mathbf{X}\mathbf{b} - \mathbf{x}^\top \mathbf{X}\mathbf{b} = 0$$

Hence $\mathbf{y} \in L^\perp$. □

# Proof of Part 3

For any vector $\mathbf{v} = \mathbf{X}\mathbf{b} \in L$ and any $\mathbf{y} \in \mathbb{R}^n$, because $\mathbf{y}^\top \mathbf{X}\mathbf{b}$ and $\mathbf{b}^\top \mathbf{X}^\top \mathbf{y}$ are equal scalars, one has

$$\|\mathbf{y} - \mathbf{v}\|^2 = (\mathbf{y} - \mathbf{X}\mathbf{b})^\top (\mathbf{y} - \mathbf{X}\mathbf{b}) = \mathbf{y}^\top \mathbf{y} - 2\mathbf{y}^\top \mathbf{X}\mathbf{b} + \mathbf{b}^\top \mathbf{X}^\top \mathbf{X}\mathbf{b}$$

Now define $\hat{\mathbf{b}} := (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{y}$ (which is the OLS estimator of $\mathbf{b}$ in the linear regression equation $\mathbf{y} = \mathbf{X}\mathbf{b} + \mathbf{e}$) and $\hat{\mathbf{v}} := \mathbf{X}\hat{\mathbf{b}} = \mathbf{P}\mathbf{y}$.

Because $\mathbf{P}^\top \mathbf{P} = \mathbf{P}^\top = \mathbf{P} = \mathbf{P}^2$, one has

$$\begin{aligned}
\|\mathbf{y} - \mathbf{v}\|^2 &= \mathbf{y}^\top \mathbf{y} - 2\mathbf{y}^\top \mathbf{X}\mathbf{b} + \mathbf{b}^\top \mathbf{X}^\top \mathbf{X}\mathbf{b} \\
&= (\mathbf{b} - \hat{\mathbf{b}})^\top \mathbf{X}^\top \mathbf{X}(\mathbf{b} - \hat{\mathbf{b}}) + \mathbf{y}^\top \mathbf{y} - \hat{\mathbf{b}}^\top \mathbf{X}^\top \mathbf{X}\hat{\mathbf{b}} \\
&= \|\mathbf{v} - \hat{\mathbf{v}}\|^2 + \mathbf{y}^\top \mathbf{y} - \mathbf{y}^\top \mathbf{P}^\top \mathbf{P}\mathbf{y} = \|\mathbf{v} - \hat{\mathbf{v}}\|^2 + \mathbf{y}^\top \mathbf{y} - \mathbf{y}^\top \mathbf{P}\mathbf{y}
\end{aligned}$$

On the other hand, given that $\hat{\mathbf{v}} = \mathbf{P}\mathbf{y}$, one also has

$$\begin{aligned}
\|\mathbf{y} - \hat{\mathbf{v}}\|^2 &= \mathbf{y}^\top \mathbf{y} - 2\mathbf{y}^\top \hat{\mathbf{v}} + \hat{\mathbf{v}}^\top \hat{\mathbf{v}} \\
&= \mathbf{y}^\top \mathbf{y} - 2\mathbf{y}^\top \mathbf{P}\mathbf{y} + \mathbf{y}^\top \mathbf{P}^\top \mathbf{P}\mathbf{y} = \mathbf{y}^\top \mathbf{y} - \mathbf{y}^\top \mathbf{P}\mathbf{y}
\end{aligned}$$

So $\|\mathbf{y} - \mathbf{v}\|^2 - \|\mathbf{y} - \hat{\mathbf{v}}\|^2 = \|\mathbf{v} - \hat{\mathbf{v}}\|^2 \geq 0$ with $=$ iff $\mathbf{v} = \hat{\mathbf{v}}$. $\square$

# Outline

# A Trick Function for Generating Eigenvalues

For all $\mathbf{x} \neq \mathbf{0}$, define the Rayleigh quotient function

$$\mathbb{R}^n \setminus \{\mathbf{0}\} \ni \mathbf{x} \mapsto f(\mathbf{x}) := \frac{\mathbf{x}^\top \mathbf{A} \mathbf{x}}{\mathbf{x}^\top \mathbf{x}} = \frac{\sum_{i=1}^n \sum_{j=1}^n x_i a_{ij} x_j}{\sum_{i=1}^n x_i^2}$$

It is homogeneous of degree zero, and left undefined at $\mathbf{x} = \mathbf{0}$.

Its partial derivative w.r.t. any component $x_h$ of the vector $\mathbf{x}$ is

$$\frac{\partial f}{\partial x_h} = \frac{2}{(\mathbf{x}^\top \mathbf{x})^2} \left[ \sum_{j=1}^n a_{hj} x_j (\mathbf{x}^\top \mathbf{x}) - (\mathbf{x}^\top \mathbf{A} \mathbf{x}) x_h \right]$$

At any critical point $\hat{\mathbf{x}} \neq \mathbf{0}$ where $\partial f / \partial x_h = 0$ for all $h$,
one therefore has $(\hat{\mathbf{x}}^\top \hat{\mathbf{x}}) \mathbf{A} \hat{\mathbf{x}} = (\hat{\mathbf{x}}^\top \mathbf{A} \hat{\mathbf{x}}) \hat{\mathbf{x}}$.
Hence $\mathbf{A} \hat{\mathbf{x}} = \lambda \hat{\mathbf{x}}$ where $\lambda = (\hat{\mathbf{x}}^\top \mathbf{A} \hat{\mathbf{x}}) / (\hat{\mathbf{x}}^\top \hat{\mathbf{x}}) = f(\hat{\mathbf{x}})$.

That is, a stationary point $\hat{\mathbf{x}} \neq \mathbf{0}$ must be an eigenvector,
with the corresponding function value $f(\hat{\mathbf{x}})$
as the associated eigenvalue.

# More Properties of the Rayleigh Quotient

Using the Rayleigh quotient

$$\mathbb{R}^n \setminus \{\mathbf{0}\} \ni \mathbf{x} \mapsto f(\mathbf{x}) := \frac{\mathbf{x}^\top \mathbf{A}\mathbf{x}}{\mathbf{x}^\top \mathbf{x}} = \frac{\sum_{i=1}^n \sum_{j=1}^n x_i a_{ij} x_j}{\sum_{i=1}^n x_i^2}$$

one can state and prove the following lemma.

## Lemma

*Every $n \times n$ symmetric square matrix $\mathbf{A}$:*

1. *has a maximum eigenvalue $\lambda^*$ with $\lambda^*$ real, and any associated eigenvector $\mathbf{x}^*$ as a maximum point of $f$;*
2. *has a minimum eigenvalue $\lambda_*$ with $\lambda_*$ real, and any associated eigenvector $\mathbf{x}_*$ as a minimum point of $f$;*
3. *satisfies $\mathbf{A} = \lambda \mathbf{I}$ if and only if $\lambda^* = \lambda_* = \lambda$.*

# Proof of Parts 1 and 2

The unit sphere $S^{n-1}$ is a closed and bounded subset of $\mathbb{R}^n$.

Moreover, the Rayleigh quotient function $f$ is continuous when restricted to $S^{n-1}$.

By the extreme value theorem, $f$ restricted to $S^{n-1}$ must have:

- a maximum value $\lambda^*$ attained at some point $\mathbf{x}^*$;
- a minimum value $\lambda_*$ attained at some point $\mathbf{x}_*$.

Because $f$ is homogeneous of degree zero,
these are the maximum and minimum values of $f$
over the whole domain $\mathbb{R}^n \setminus \{\mathbf{0}\}$.

In particular, $f$ must be critical at any maximum point $\mathbf{x}^*$,
as well as at any minimum point $\mathbf{x}_*$.

But critical points must be eigenvectors.

This proves parts 1 and 2 of the lemma.

Part 3 is left as an exercise. □

# Outline

# A Minor Lemma

### Lemma

*Let* $\mathbf{A}$ *be a symmetric* $n \times n$ *matrix.*

*Suppose that* $\lambda$ *and* $\mu$ *are distinct eigenvalues, with corresponding eigenvectors* $\mathbf{x}$ *and* $\mathbf{y}$.

*Then* $\mathbf{x}$ *and* $\mathbf{y}$ *are orthogonal — that is,* $\mathbf{x} \cdot \mathbf{y} = 0$.

### Proof.

Suppose that the non-zero vectors $\mathbf{x}$ and $\mathbf{y}$ satisfy $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ and $\mathbf{A}\mathbf{y} = \mu\mathbf{y}$.

Because $\mathbf{A}$ is symmetric, one has

$$\lambda\mathbf{x}^\top\mathbf{y} = (\mathbf{A}\mathbf{x})^\top\mathbf{y} = \mathbf{x}^\top\mathbf{A}^\top\mathbf{y} = \mathbf{x}^\top\mathbf{A}\mathbf{y} = \mu\mathbf{x}^\top\mathbf{y}$$

In case $\lambda \neq \mu$, it follows that $0 = \mathbf{x}^\top\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$. □

# A Useful Lemma

### Lemma

*Let* $\mathbf{A}$ *be a symmetric* $n \times n$ *matrix.*

*Suppose that there are* $m < n$ *eigenvectors* $\{\mathbf{u}_k\}_{k=1}^m$ *which form an orthonormal set of column* $n$*-vectors, as well as the columns of an* $n \times m$ *matrix* $\mathbf{U}$.

*Then there is at least one more eigenvector* $\mathbf{x} \neq \mathbf{0}$ *that satisfies* $\mathbf{U}^\top \mathbf{x} = 0$
*— i.e., it is orthogonal to each of the* $m$ *eigenvectors* $\mathbf{u}_k$.

# Constructive Proof, Part 1

For each eigenvector $\mathbf{u}_k$, let $\lambda_k$ be the associated eigenvalue, so that $\mathbf{A}\mathbf{u}_k = \lambda_k \mathbf{u}_k$ for $k = 1, 2, \ldots, m$.

Then the $n \times m$ matrix $\mathbf{U}$ satisfies $\mathbf{AU} = \mathbf{U}\boldsymbol{\Lambda}$ where $\boldsymbol{\Lambda}$ is the $m \times m$ matrix $\mathbf{diag}(\lambda_k)_{k=1}^m$.

Also, because the eigenvectors $\{\mathbf{u}_k\}_{k=1}^m$ form an orthonormal set, one has $\mathbf{U}^\top \mathbf{U} = \mathbf{I}_m$.

Hence $\mathbf{U}^\top \mathbf{A} \mathbf{U} = \mathbf{U}^\top \mathbf{U} \boldsymbol{\Lambda} = \boldsymbol{\Lambda}$.

Also, transposing $\mathbf{AU} = \mathbf{U}\boldsymbol{\Lambda}$ gives $\mathbf{U}^\top \mathbf{A} = \boldsymbol{\Lambda} \mathbf{U}^\top$.

# Constructive Proof, Part 2

Consider now the $n \times n$ matrix $\hat{\mathbf{A}} := (\mathbf{I}_n - \mathbf{U}\mathbf{U}^\top)\mathbf{A}(\mathbf{I}_n - \mathbf{U}\mathbf{U}^\top)$.

Then $\hat{\mathbf{A}}$ is symmetric because both $\mathbf{A}$ and $\mathbf{U}\mathbf{U}^\top$ are symmetric.

Note that, because $\mathbf{A}\mathbf{U} = \mathbf{U}\mathbf{\Lambda}$ and so $\mathbf{U}^\top\mathbf{A} = \mathbf{\Lambda}\mathbf{U}^\top$, one has

$$\begin{aligned}\hat{\mathbf{A}} &= \mathbf{A} - \mathbf{U}\mathbf{U}^\top\mathbf{A} - \mathbf{A}\mathbf{U}\mathbf{U}^\top + \mathbf{U}\mathbf{U}^\top\mathbf{A}\mathbf{U}\mathbf{U}^\top \\ &= \mathbf{A} - \mathbf{U}\mathbf{\Lambda}\mathbf{U}^\top - \mathbf{U}\mathbf{\Lambda}\mathbf{U}^\top + \mathbf{U}\mathbf{\Lambda}\mathbf{U}^\top = \mathbf{A} - \mathbf{U}\mathbf{\Lambda}\mathbf{U}^\top\end{aligned}$$

The symmetric matrix $\hat{\mathbf{A}}$ has at least one real eigenvalue $\lambda$.

Let $\mathbf{x} \neq \mathbf{0}$ be an associated real eigenvector, which must satisfy

$$\hat{\mathbf{A}}\mathbf{x} = (\mathbf{I} - \mathbf{U}\mathbf{U}^\top)\mathbf{A}(\mathbf{I} - \mathbf{U}\mathbf{U}^\top)\mathbf{x} = (\mathbf{A} - \mathbf{U}\mathbf{\Lambda}\mathbf{U}^\top)\mathbf{x} = \lambda\mathbf{x}$$

Pre-multiplying each side of the last equality
by the $m \times n$ matrix $\mathbf{U}^\top$ shows that

$$\lambda\mathbf{U}^\top\mathbf{x} = \mathbf{U}^\top\mathbf{A}\mathbf{x} - \mathbf{U}^\top\mathbf{U}\mathbf{\Lambda}\mathbf{U}^\top\mathbf{x} = \mathbf{\Lambda}\mathbf{U}^\top\mathbf{x} - \mathbf{\Lambda}\mathbf{U}^\top\mathbf{x} = \mathbf{0}_m$$

# Constructive Proof, Part 3

There are now two cases.

Consider first the generic case
when $\hat{\mathbf{A}}$ has at least one eigenvalue $\lambda \neq 0$.

Then there is a corresponding eigenvector $\mathbf{x} \neq \mathbf{0}$ of $\hat{\mathbf{A}}$
that satisfies $\lambda \mathbf{U}^\top \mathbf{x} = \mathbf{0}_m$ and so $\mathbf{U}^\top \mathbf{x} = \mathbf{0}_m^\top$.

But then the earlier equation

$$\hat{\mathbf{A}}\mathbf{x} = (\mathbf{I} - \mathbf{U}\mathbf{U}^\top)\mathbf{A}(\mathbf{I} - \mathbf{U}\mathbf{U}^\top)\mathbf{x} = (\mathbf{A} - \mathbf{U}\mathbf{\Lambda}\mathbf{U}^\top)\mathbf{x} = \lambda\mathbf{x}$$

implies that

$$\mathbf{A}\mathbf{x} = (\hat{\mathbf{A}} + \mathbf{U}\mathbf{\Lambda}\mathbf{U}^\top)\mathbf{x} = \hat{\mathbf{A}}\mathbf{x} = \lambda\mathbf{x}$$

Hence $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ as well as of $\hat{\mathbf{A}}$.

# Constructive Proof, Part 4

The remaining exceptional case occurs
when the only eigenvalue of the symmetric matrix $\hat{\mathbf{A}}$ is $\lambda = 0$.

Given the properties of the Rayleigh quotient function,
this implies that $\hat{\mathbf{A}} = \mathbf{0}$ and so $\mathbf{A} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^\top$.

Then any vector $\mathbf{x} \neq \mathbf{0}$ satisfying $\mathbf{U}^\top \mathbf{x} = \mathbf{0}$
must satisfy $\mathbf{A}\mathbf{x} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^\top \mathbf{x} = \mathbf{0}$.

This implies that $\mathbf{x}$ is an eigenvector of $\mathbf{A}$
associated with the eigenvalue $\lambda = 0$.

In both cases there is an eigenvector $\mathbf{x}$ of $\mathbf{A}$
associated with the common eigenvalue $\lambda$ of both $\hat{\mathbf{A}}$ and $\mathbf{A}$
that satisfies $\mathbf{U}^\top \mathbf{x} = \mathbf{0}_m^\top$.

This completes the proof. □

# Spectral Theorem

### Theorem

*Given any symmetric $n \times n$ matrix $\mathbf{A}$:*

1. *the set of all its eigenvectors spans the whole of $\mathbb{R}^n$;*
2. *there exists an orthogonal matrix $\mathbf{P}$ that diagonalizes $\mathbf{A}$ in the sense that $\mathbf{P}^\top \mathbf{A} \mathbf{P} = \mathbf{P}^{-1} \mathbf{A} \mathbf{P}$ is a diagonal matrix $\mathbf{\Lambda}$, whose elements are the eigenvalues of $\mathbf{A}$, all real.*

When $\mathbf{A}$ is required to be self-adjoint rather than symmetric, and the eigenvectors may be complex, the corresponding result is:

Given any self-adjoint $n \times n$ matrix $\mathbf{A}$:

1. the set of all its eigenvectors spans the whole of $\mathbb{C}^n$;
2. there exists a unitary matrix $\mathbf{U}$ that diagonalizes $\mathbf{A}$ in the sense that $\mathbf{U}^* \mathbf{A} \mathbf{U} = \mathbf{U}^{-1} \mathbf{A} \mathbf{U}$ is a diagonal matrix $\mathbf{\Lambda}$ whose elements are the eigenvalues of $\mathbf{A}$, all real.

We give a proof for the case when $\mathbf{A}$ is symmetric.

# Proof of Spectral Theorem, Part 1

The symmetric matrix $\mathbf{A}$ has at least one eigenvalue $\lambda$, which must be real.

The associated eigenvector $\mathbf{x}$, normalized to satisfy $\mathbf{x}^\top \mathbf{x} = 1$, forms an orthonormal set $\{\mathbf{u}_1\}$ satisfying $\mathbf{u}_j^\top \mathbf{u}_k = \delta_{jk}$ for all $j, k = 1, 2, \ldots, m$.

As the induction hypothesis, suppose that there are $m < n$ eigenvectors $\{\mathbf{u}_k\}_{k=1}^m$ which form an orthonormal set of vectors.

We have just proved that this hypothesis holds for $m = 1$.

The "useful lemma" shows that, if the hypothesis holds for any $m = 1, 2, \ldots, n-1$, then it holds for $m + 1$.

So the result follows for $m = n$ by induction.

In particular, when $m = n$, there exists an orthonormal set of $n$ eigenvectors, which must then span the whole of $\mathbb{R}^n$.

# Proof of Spectral Theorem, Part 2

Also, by the previous result,
we can take $\mathbf{P}$ as an orthogonal matrix
whose columns are an orthonormal set of $n$ eigenvectors.

Then $\mathbf{A}\mathbf{P} = \mathbf{P}\boldsymbol{\Lambda}$.

So premultiplying by $\mathbf{P}^\top = \mathbf{P}^{-1}$ gives $\mathbf{P}^\top\mathbf{A}\mathbf{P} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \boldsymbol{\Lambda}$. $\square$

# Outline

# Definition of Quadratic Form

### Definition

A quadratic form on the $n$-dimensional Euclidean space $R^n$ is a mapping

$$R^n \ni \mathbf{x} \mapsto q(\mathbf{x}) = \mathbf{x}^\top \mathbf{Q} \mathbf{x} = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i q_{ij} x_j \in \mathbb{R}$$

where $\mathbf{Q}$ is a symmetric $n \times n$ matrix.

The quadratic form $\mathbf{x}^\top \mathbf{Q} \mathbf{x}$ is diagonal just in case the matrix $\mathbf{Q}$ is diagonal, with $\mathbf{Q} = \mathbf{\Lambda} = \mathbf{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

In this case $\mathbf{x}^\top \mathbf{Q} \mathbf{x}$ reduces to $\mathbf{x}^\top \mathbf{\Lambda} \mathbf{x} = \sum_{i=1}^{n} \lambda_i (x_i)^2$.

# The Hessian Matrix of a Quadratic Form

## Example

1. Given the quadratic form $q(x,y) = (x,y)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$,

   show that, even if $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is not symmetric,
   its Hessian matrix of second-order partial derivatives
   is the symmetric matrix $\begin{pmatrix} q''_{xx} & q''_{xy} \\ q''_{yx} & q''_{yy} \end{pmatrix} = \begin{pmatrix} 2a & b+c \\ b+c & 2d \end{pmatrix}$.
2. Given the quadratic form defined for $\mathbf{x} \in \mathbb{R}^n$ by $q(\mathbf{x}) = \mathbf{x}^\top \mathbf{A}\mathbf{x}$,
   show that, even if $\mathbf{A}$ is not symmetric,
   its Hessian matrix $\left(\partial^2 q / \partial x_i \partial x_j\right)_{n \times n}$
   of second-order partial derivatives
   is the symmetric matrix $\mathbf{A} + \mathbf{A}^\top$.

# Symmetry Loses No Generality

Requiring $\mathbf{Q}$ in $\mathbf{x}^\top \mathbf{Q} \mathbf{x}$ to be symmetric loses no generality.

This is because, given a general non-symmetric $n \times n$ matrix $\mathbf{A}$, repeated transposition implies that

$$\mathbf{x}^\top \mathbf{A} \mathbf{x} = (\mathbf{x}^\top \mathbf{A} \mathbf{x})^\top = \tfrac{1}{2}[\mathbf{x}^\top \mathbf{A} \mathbf{x} + (\mathbf{x}^\top \mathbf{A} \mathbf{x})^\top] = \tfrac{1}{2} \mathbf{x}^\top (\mathbf{A} + \mathbf{A}^\top) \mathbf{x}$$

Hence $\mathbf{x}^\top \mathbf{A} \mathbf{x} = \mathbf{x}^\top \mathbf{A}^\top \mathbf{x} = \mathbf{x}^\top \mathbf{Q} \mathbf{x}$
where $\mathbf{Q}$ is the symmetrized matrix $\frac{1}{2}(\mathbf{A} + \mathbf{A}^\top)$.

Note that $\mathbf{Q}$ is indeed symmetric because

$$\mathbf{Q}^\top = \tfrac{1}{2}(\mathbf{A} + \mathbf{A}^\top)^\top = \tfrac{1}{2}[\mathbf{A}^\top + (\mathbf{A}^\top)^\top] = \tfrac{1}{2}(\mathbf{A}^\top + \mathbf{A}) = \mathbf{Q}$$

# Definiteness of a Quadratic Form

When $\mathbf{x} = \mathbf{0}$, then $\mathbf{x}^\top \mathbf{Q} \mathbf{x} = 0$. Otherwise:

**Definition**

The quadratic form $\mathbb{R}^n \ni \mathbf{x} \mapsto \mathbf{x}^\top \mathbf{Q} \mathbf{x} \in \mathbb{R}$ is:

positive definite just in case $\mathbf{x}^\top \mathbf{Q} \mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$;

negative definite just in case $\mathbf{x}^\top \mathbf{Q} \mathbf{x} < 0$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$;

positive semi-definite just in case $\mathbf{x}^\top \mathbf{Q} \mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$;

negative semi-definite just in case $\mathbf{x}^\top \mathbf{Q} \mathbf{x} \leq 0$ for all $\mathbf{x} \in \mathbb{R}^n$;

indefinite just in case there exist $\mathbf{x}^+$ and $\mathbf{x}^-$ in $\mathbb{R}^n$ such that $(\mathbf{x}^+)^\top \mathbf{Q} \mathbf{x} > 0$ and $(\mathbf{x}^-)^\top \mathbf{Q} \mathbf{x} < 0$.

# Definiteness of a Diagonal Quadratic Form

**Theorem**

*The diagonal quadratic form* $\sum_{i=1}^{n} \lambda_i (x_i)^2 \in \mathbb{R}$ *is:*

*positive definite if and only if* $\lambda_i > 0$ *for* $i = 1, 2, \ldots, n$*;*

*negative definite if and only if* $\lambda_i < 0$ *for* $i = 1, 2, \ldots, n$*;*

*positive semi-definite if and only if* $\lambda_i \geq 0$ *for* $i = 1, 2, \ldots, n$*;*

*negative semi-definite if and only if* $\lambda_i \leq 0$ *for* $i = 1, 2, \ldots, n$*;*

*indefinite if and only if there exist* $i, j \in \{1, 2, \ldots, n\}$ *such that* $\lambda_i > 0$ *and* $\lambda_j < 0$*.*

**Proof.**

The proof is left as an exercise.

The result is obvious if $n = 1$, and straightforward if $n = 2$.

Working out these two cases first suggests the proof for $n > 2$. □

# Outline

# Diagonalizing Quadratic Forms

Consider a quadratic form $\mathbb{R}^n \ni \mathbf{x} \mapsto \mathbf{x}^\top \mathbf{Q} \mathbf{x} \in \mathbb{R}$ where,
without losing generality,
we assume that the $n \times n$ matrix $\mathbf{Q}$ is symmetric.

By the spectral theorem for symmetric matrices,
there exists a matrix $\mathbf{P}$ that diagonalizes $\mathbf{Q}$,
meaning that $\mathbf{P}^{-1}\mathbf{Q}\mathbf{P}$ is a diagonal matrix that we denote by $\boldsymbol{\Lambda}$.

Moreover $\mathbf{P}$ can be made orthogonal, meaning that $\mathbf{P}^{-1} = \mathbf{P}^\top$.

Given any $\mathbf{x} \neq \mathbf{0}$, because $\mathbf{P}^{-1}$ exists, we can define $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$.

This implies that $\mathbf{x} = \mathbf{P}\mathbf{y}$, where $\mathbf{y} \neq \mathbf{0}$ because $(\mathbf{P}^{-1})^{-1} = \mathbf{P}$.

Then $\mathbf{x}^\top \mathbf{Q} \mathbf{x} = \mathbf{y}^\top \mathbf{P}^\top \mathbf{Q} \mathbf{P} \mathbf{y} = \mathbf{y}^\top \boldsymbol{\Lambda} \mathbf{y}$,
so the diagonalization leads to a diagonal quadratic form.

# The Eigenvalue Test of Definiteness

A standard result says that $\mathbf{Q}$
and its diagonalization $\mathbf{P}^{-1}\mathbf{Q}\mathbf{P}$ have the same set of eigenvalues.

From the theorem on the definiteness of a diagonal quadratic form,
it follows that:

### Theorem

*The quadratic form* $\mathbf{x}^\top \mathbf{Q}\mathbf{x}$ *is:*

*positive definite if and only if all its eigenvalues are positive;*

*negative definite if and only if all its eigenvalues are negative;*

*positive semi-definite if and only if
all its eigenvalues are non-negative;*

*negative semi-definite if and only if
all its eigenvalues are non-positive;*

*indefinite if and only if
it has both positive and negative eigenvalues.*

# Concavity or Convexity of a Quadratic Form

### Theorem

*As a function of* $\mathbf{x}$, *the quadratic form* $\mathbf{x}^\top \mathbf{Q} \mathbf{x}$ *is:*

*strictly convex if and only if all it is positive definite;*

*strictly concave if and only if all it is negative definite;*

*convex if and only if it is positive semi-definite;*

*concave if and only if it is negative semi-definite.*

*Otherwise* $\mathbf{x}^\top \mathbf{Q} \mathbf{x}$ *is neither concave nor convex if and only if it is indefinite.*

# Outline

# The Case of a Quadratic Form in Two Variables

The general quadratic form in 2 variables is

$$(x, y) \begin{pmatrix} a & h \\ h & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + 2hxy + by^2$$

If it is positive definite, it is positive whenever $x \neq 0$ and $y = 0$, implying that $a > 0$.

If $a > 0$, completing the square implies that
$ax^2 + 2hxy + by^2 = a(x + hy/a)^2 + (b - h^2/a)y^2$.

Given that $a > 0$, this is positive definite if and only if $b > h^2/a$, or if and only if $ab - h^2 = \begin{vmatrix} a & h \\ h & b \end{vmatrix} > 0$.

For the case of 2 variables, this proves Sylvester's Criterion:
The real-symmetric matrix $\mathbf{A}$ is positive definite
if and only if all the leading principal minors of $\mathbf{A}$ are positive.

# The Case of a Diagonal Quadratic Form

The general diagonal quadratic form in $n$ variables is $\mathbf{x}^\top \Lambda \mathbf{x}$
where $\mathbf{x}$ is an $n$-vector
and $\Lambda$ is an $n \times n$ diagonal matrix $\mathbf{diag}(\lambda_1, \ldots, \lambda_n)$.

Then the quadratic form $\mathbf{x}^\top \Lambda \mathbf{x} = \sum_{i=1}^n \lambda_i (x_i)^2$:

1. is positive definite if and only if $\lambda_i > 0$ for $i = 1, 2, \ldots, n$.

   This is true if and only if the $k$-fold product $\prod_{i=1}^k \lambda_i$
   is positive for $k = 1, 2, \ldots, n$.

   But $\prod_{i=1}^k \lambda_i = |\,\mathbf{diag}(\lambda_1, \ldots, \lambda_k)|$
   is the leading principal minor of order $k$
   for the diagonal matrix $\Lambda = \mathbf{diag}(\lambda_1, \ldots, \lambda_n)$.

2. is positive semi-definite if and only if $\lambda_i \geq 0$ for $i = 1, 2, \ldots, n$.

   This is true if and only if the product $\prod_{i \in K} \lambda_i$ is nonnegative
   for every nonempty $K \subseteq \mathbb{N}_n = \{1, 2, \ldots, n\}$.

   But each product $\prod_{i \in K} \lambda_i$ is a principal minor
   for the diagonal matrix $\Lambda = \mathbf{diag}(\lambda_1, \ldots, \lambda_n)$.

# Sylvester's Criterion: General Statement

### Theorem (Sylvester's criterion)

*Given the symmetric matrix* $\mathbf{A}$*, for each* $k = 1, \ldots, n$*,*
*and for each non-empty subset* $K \subseteq \mathbb{N}_n$ *with* $k = \#K$*:*
*let* $D_k$ *denote the leading principal minor of order* $k$*;*
*let* $\Delta_k^K$ *denote an arbitrary principal minor of order* $k$*.*
*Then the quadratic form* $\mathbf{x}^\top \mathbf{A} \mathbf{x}$ *is:*

$$\begin{aligned}
\textit{positive definite} &\iff D_k > 0 \;\; \textit{for all } k = 1, \ldots, n \\
\textit{positive semidefinite} &\iff \Delta_k^K \geq 0 \;\; \textit{for all } \Delta_k^K \textit{ of any order } k \\
\textit{negative definite} &\iff (-1)^k D_k > 0 \;\; \textit{for all } k = 1, \ldots, n \\
\textit{negative semidefinite} &\iff (-1)^k \Delta_k^K \geq 0 \;\; \textit{for all } \Delta_k^K \textit{ of any order } k
\end{aligned}$$

Note that the necessary and sufficient conditions
for $\mathbf{A}$ to be negative (semi-) definite
are exactly those for $-\mathbf{A}$ to be positive (semi-) definite,

# Necessity of Sylvester's Criterion

## Necessity for a Positive (Semi-) Definite Matrix.

Given any non-empty subset $K \subseteq \mathbb{N}_n$ with with $k = \#K$,
let $\mathbf{A}^K$ denote the $k \times k$ matrix $(a_{ij})_{(i,j) \in K \times K}$.

Then, given any $n$-vector $\mathbf{x}$,
let $\mathbf{x}^K$ denote $(x_j)_{j \in K}$ and let $\mathbf{x}^{-K}$ denote $(x_j)_{j \notin K}$.

In case $\mathbf{A}$ is positive definite, whenever $\mathbf{x}^K \neq \mathbf{0}$ and $\mathbf{x}^{-K} = \mathbf{0}$,
then $\mathbf{x}^\top \mathbf{A} \mathbf{x} = (\mathbf{x}^K)^\top \mathbf{A}^K \mathbf{x}^K > 0$, so $\mathbf{A}^K$ is positive definite.

Because the determinant of a symmetric matrix
is the product of its eigenvalues,it follows that $|\mathbf{A}^K| > 0$.

This holds in particular when $K = \mathbb{N}_k$, and so $|\mathbf{A}^K| = D_k$.

In case $\mathbf{A}$ is positive semi-definite,
the same argument shows that $\mathbf{A}^K$ is positive semi-definite,
and so $\Delta_k^K = |\mathbf{A}^K| \geq 0$. □

# Sufficiency of Sylvester's Criterion

Apart from the $2 \times 2$ and diagonal cases,
the proof of sufficiency is much more challenging!

See you, at least online, on Tuesday September 29th.

Enjoy the next five days of lectures with my colleague Pablo Beker!