MULTILATERALLY STRATEGY-PROOF MECHANISMS IN RANDOM AUMANN–HILDENBRAND MACROECONOMIES

PETER J. HAMMOND

Abstract. By definition, multilaterally strategy-proof mechanisms are immune to manipulation not only by individuals misrepresenting their preferences, but also by finite coalitions exchanging tradeable goods on the side. Continuum economies are defined in which both agents' identifiers and their privately known characteristics are joint i.i.d. random variables. For such economies, conditions are given for multilateral strategy-proofness to imply decentralization by a budget constraint with linear prices for tradeable goods and lump-sum transfers independent of individual characteristics. Also, adapting Aumann's [1964a] key proof avoids using Lyapunov's theorem or its corollary, Richter's theorem on integrating a correspondence w.r.t. a non-atomic measure.

1. Introduction and Outline

A macroeconomy is one that is adequately described by statistical aggregates. Usually macroeconomists look at the means of important variables like income, hours worked, etc. But other moments of the relevant distribution are also important. For example, when different income groups have different marginal propensities to spend on various goods, aggregate demand is influenced by the distribution of income between those groups. Accordingly, the obvious step for an economic theorist to take is to consider the whole distribution. This suggests examining economies with a continuum of agents — i.e., one in which the usual finite set of agents is replaced by a non-atomic measure space.

The earliest paper I recall seeing that uses a continuum of economic agents explicitly is by Vickrey [1945], in his discussion of the limits to redistribution. Much later, Mirrlees [1971] used a model like Vickrey's with a one-dimensional continuum of skill levels in order to formulate and solve a problem involving optimal non-linear income taxation. There have been many successors, of course. But the first general equilibrium model of a continuum economy is due to Aumann [1964a, 1966]. Such models can be motivated using the words of Aumann [1964a, p. 41] himself:

> The idea of a continuum of traders ... is no stranger than ... a continuum of "particles" in fluid mechanics. ... [T]he continuum can be considered as an approximation to the "true" situation in which

---

Date: May 1998 revised version, for publication in M. Wooders (ed.) *Topics in Game Theory and Mathematical Economics* (American Mathematical Society).

Key words and phrases. continuum economies, strategy-proofness, core equivalence.

It is a great pleasure to recall the extensive fruitful interaction that I have enjoyed with Bob Aumann during the many summer workshops on economic theory we have both attended at Stanford over the years since 1975, as well as on other occasions elsewhere, especially during my visit to Jerusalem in 1983. It has been a great privilege also to observe how strongly Bob upholds the highest standards of both scientific and personal integrity.

> there is a large but finite number of particles .... The purpose of adopting the continuous approximation is to make available the powerful and elegant methods of the branch of mathematics called "analysis," in a situation where treatment by finite methods would be much more difficult or even hopeless (think of trying to do fluid mechanics by solving n-body problems for large n).
>
> There is perhaps a certain psychological difference between a fluid with a continuum of particles and a market with a continuum of traders. Though we are intellectually convinced that a fluid contains only finitely many particles, to the naked eye it still looks quite continuous. The economic structure of a shopping center, on the other hand, does not look continuous at all. But, *for the economic policy maker in Washington, or for any professional macroeconomist*, there is no such difference. He works with figures that are summarized for geographic regions, different industries, and so on; the individual consumer (or merchant) is as anonymous to him as the individual molecule is to the physicist.
>
> (emphasis added)

Indeed, continuum economies have come to play a central role in much recent work in macroeconomics, as well as in general equilibrium theory. Continuum economies capture very well the intuitive idea that individuals have negligible power to affect prices. They also represent how individuals acting alone cannot control the levels of "widespread externalities" like carbon dioxide in the atmosphere, or those associated with learning by doing in models of "endogenous" growth.[1]

Aumann used his general equilibrium model in order to prove first core equivalence and then later the existence of Walrasian equilibrium. These results were soon extended in several articles by Hildenbrand and others — see especially Hildenbrand [1974]. In addition, following an idea he ascribes to Mertens [1970], Hildenbrand also introduced a metric on the space of preferences. However, Hildenbrand's [1971] earlier work on finite random economies suggests what may be the most satisfactory interpretation of a continuum economy — as the statistical limit of a sequence of large finite economies in which individuals' identifiers, characteristics, and also their corresponding allocations are all sampled independently from an appropriate common joint probability distribution. By an application of the law of large numbers — or, more precisely, of Glivenko–Cantelli's theorem in the form set out by Parthasarathy [1967, Theorem 7.1, p. 53] — the sample distributions converge almost surely to their values in the continuum economy.[2]

In general equilibrium theory, two well known and several less well known results are true under much weaker convexity assumptions about individuals in economies with many agents. These include results on the existence of Walrasian equilibrium

[1] For work on general widespread externalities, see for example Kaneko and Wooders [1986, 1989, 1994], Hammond, Kaneko and Wooders [1989], and Hammond [1993b, 1995]. For some of the many recently published surveys and general discussions of endogenous growth theory, see for example Romer [1991], Stern [1991], Helpman [1992], Lucas [1993], Aghion and Howitt [1995], Barro and Sala-i-Martin [1995], and Hahn [1995], as well as Hammond and Rodríguez-Clare [1993].

[2] Actually, as Walter Trockel has kindly reminded me, the continuum economy model is just one particular way of representing an economy with many participants. One prominent alternative is the non-standard model with a hyperfinite set of agents. This was pioneered by Brown and Robinson [1975a, b] — see also Anderson [1991]. A less well known alternative with finitely additive measures is due to Weiss [1981].

and the characterization of Pareto efficient allocations as Walrasian equilibria with lump-sum redistribution of wealth. Neither of these results requires preferences to be convex. Typically, both use proofs relying on Richter's [1963] theorem which states that integrating w.r.t. a non-atomic measure any measurable correspondence with values in $\Re^n$ produces a convex set.[3] Convexity of individual feasible sets, however, still helps avoid difficulties like Arrow's [1951] "exceptional case" — see Hammond [1993].

Aumann's core equivalence theorem, however, falls into a different class of results that, except for economies with many agents, are simply not true in general. Moreover, unlike the existence and second efficiency theorems mentioned above, Aumann's [1964a] proof of core equivalence might seem at first to avoid Lyapunov's theorem. However, it does rely on the range of a non-atomic measure being an interval of the real line; at this step of essentially the same proof, Hildenbrand [1974, p. 144; 1982, p. 844] explicitly invokes Lyapunov's theorem "in one dimension, the proof of which is much easier".[4] For Hildenbrand [1974], on the other hand, Aumann's proof is left as an exercise; in the main text (p. 133), he uses Richter's theorem. Other results similar to core equivalence include Mas-Colell's [1989] discussion of the bargaining set for a continuum economy, and Aumann and Shapley's [1974] treatment of values in non-atomic cooperative games without transferable utility.

For Aumann, as well as in later work on the core of a continuum economy that immediately succeeded his pioneering contribution, coalitions were always sets of positive measure, even if this measure might be arbitrarity small as in Grodal [1972], Schmeidler [1972] and Vind [1972]. In the 1980s, however, Mamoru Kaneko, Myrna Wooders and I all became interested in what finite coalitions could achieve in continuum economies. Of course, because a continuum economy involves a non-atomic measure, any finite coalition has zero measure — i.e., it is a null set.

In fact, Kaneko and Wooders [1986] is the first paper on what came to be called the "f-core." Using a related idea, Hammond [1987] improved some very preliminary ideas found in Hammond [1979], which had been taken further by Gale [1980, 1982] and Guesnerie [1981, 1995]. These concerned strategy-proof allocation mechanisms in a continuum economy which are also immune to manipulation by finite coalitions exchanging on an unofficial or hidden market the goods allocated to them by the official mechanism. Similar results were applied in different contexts by Jacklin [1987], Blackorby and Donaldson [1988], Haubrich [1988], and Haubrich and King [1990].

In the work on multilateral strategy-proofness reported in Hammond [1987] and also in our joint paper (Hammond, Kaneko and Wooders [1989]) on an equivalence

[3] In symbols, Richter's theorem says that $\int_T F(t)\,d\tau$ is a convex set whenever $F : T \twoheadrightarrow \Re^n$ is a correspondence with a measurable graph and τ is a non-atomic measure. As Hildenbrand's [1974, Theorem 3, p. 62] elegant and concise proof makes clear, Richter's theorem is really a corollary of the well known theorem of Lyapunov [1940] stating that the range of a finite-dimensional vector-valued non-atomic measure is a convex set. Note too that Richter's theorem is different from what Clarke [1983] calls "Aumann's theorem" stating that the integral of the convex hull of an integrably bounded, measurable and non-empty valued correspondence $F : T \twoheadrightarrow \Re^n$ is identical to the integral of the correspondence — or in symbols, that $\int_T F(t)\,d\tau = \int_T \mathrm{co}\,F(t)\,d\tau$. Aumann [1965] proved this using Lyapunov's theorem. See also Artstein [1980]. Clarke [1983] provides a proof of Aumann's result without using Lyapunov's theorem.

[4] Actually, sometimes a measure on a space of one dimension is defined to be non-atomic iff Lyapunov's theorem is true.

theorem for the f-core with widespread externalities, the main results were proved using Richter's theorem. It has since become clear to me that both proofs can be made considerably simpler by following Aumann's [1964a] original approach. Moreover, because only finite coalitions are being considered, one can avoid using Richter's theorem or even Lyapunov's theorem in one dimension. There is a small price to pay, however: the conventional assumption of local non-satiation must be strengthened so that it becomes local non-satiation among suitable vectors with rational coordinates — an assumption that is automatically satisfied if preferences are monotone, for instance.

For the case of the f-core equivalence theorem in an economy with rather general widespread externalities, such a proof is included in Hammond [1998]. The present paper concentrates instead on the multilateral incentive compatibility or strategy-proofness result of Hammond [1987], and extends the key lemma in that paper. In particular, mechanisms that are not anonymous or symmetric can now be considered. Also, a subtle difficulty raised by Guesnerie [1995] is overcome by considering a random continuum economy. Moreover, some of the earlier assumptions are replaced by a dispersion assumption along the lines suggested by Yamazaki [1978, 1981]. Finally, exceptional null sets of agents are dealt with more satisfactorily than in Hammond [1979, 1987].

In what follows, Section 2 sets out the model of a random economy with a continuum of agents whose identifiers and privately known characteristics are pairs of independently and identically distributed random variables with an unknown joint distribution. For such random continuum economies, Section 3 defines a multilaterally strategy-proof mechanism which is immune to manipulation even when finite coalitions can exchange tradeable goods unofficially on the side. Then Sections 4 and 5 discuss the conditions under which such mechanisms are decentralizable by a budget constraint with linear prices for tradeable goods, but non-linear pricing of non-tradeable goods.

2. Random Continuum Economies

Let T be a finite set of tradeable goods, and $\Re^T$ the corresponding Euclidean space of net trade vectors with typical member denoted by x or t. In addition, suppose that allocations of goods which individuals cannot freely exchange are described by the points z of an abstract locally compact and separable metric space Z. The relevant commodity or goods space is therefore the locally compact and separable product metric space $G = \Re^T \times Z$, of which the typical member is (x, z) where $x \in \Re^T$ and $z \in Z$.

The set of agents is assumed to be the measure space $(I, \mathcal{B}, \mu)$. Here I is the interval $[0, 1]$ of the real line, whose members $i \in I$ are individual *identifiers*. Then $\mathcal{B}$ is the real Borel σ-field, and μ is Lebesgue measure.

Let Θ be the domain of possible individual *characteristics*. Each typical member $\theta \in \Theta$ determines a non-empty closed *feasible set* $F(\theta) \subset G$ of net trade vectors, together with a (reflexive, complete and transitive binary weak) preference ordering $\succsim_\theta$ on $F(\theta)$ which is *continuous* in the sense of having a closed graph in $G \times G = G^2$ given by

$$\mathrm{Graph}(\succsim_\theta) := \{ \big((x, z), (x', z')\big) \in F(\theta) \times F(\theta) \mid (x, z) \succsim_\theta (x', z') \}$$

Let $\succ_\theta$ be the strict preference relation corresponding to $\succsim_\theta$. Note that, because $\succsim_\theta$ is reflexive, once the graph of $\succsim_\theta$ is known, so is the feasible set, which must be

$$F(\theta) = \{ (x, z) \in G \mid \big((x, z), (x, z)\big) \in \text{Graph}(\succsim_\theta) \}$$

Following Hildenbrand [1974], who credits Mertens [1970] with the idea, the space Θ of (non-empty) closed subsets of G^2 representing possible consumer characteristics will be given the *closed convergence* topology. This corresponds to the metric ρ on the space Θ defined by $\rho(E, F) = d_H(E \cup \{\infty\}, F \cup \{\infty\})$ for all non-empty closed subsets $E, F \subset G^2$, where d_H denotes the Hausdorff metric, and ∞ denotes an added point at infinity which makes G^2 compact.footnoteMore formally, $G^2 \cup \{\infty\}$ is the Alexandroff one-point compactification of G^2. For interesting criticism of the closed convergence topology, see Anderson [1994, p. 448] and Manelli's work cited there. Because the space G is locally compact and separable, it follows from Hildenbrand [1974, Theorem 2, p. 19] that Θ is compact. Also, give the product space $I \times \Theta$ any reasonable metric such as $d\big((h, \theta), (i, \eta)\big) := |h - i| + d_H(\theta, \eta)$. Because the spaces I and G are respectively compact and locally compact, $I \times \Theta$ is a compact and so separable metric space whose Borel σ-field is equal to the product of the Borel σ-fields for the two component metric spaces.

Each individual $i \in I$ is assumed to have a range $\Theta_i \subset \Theta$ of possible characteristics. Furthermore, the correspondence $i \mapsto\!\!\!\rightarrow \Theta_i$ is assumed to have a graph

$$A := \{ (i, \theta) \in I \times \Theta \mid \theta \in \Theta_i \}$$

which is Borel measurable in the product space $I \times \Theta$. Let $\mathcal{A}$ denote the restriction to the measurable set A of the Borel σ-field on $I \times \Theta$. Let $\mathcal{M}(A)$ denote the set of probability measures on the measurable space $(A, \mathcal{A})$. Give $\mathcal{M}(A)$ the topology of weak convergence.

The above formulation has been chosen because it allows a number of interesting different cases of private information, such as when agent i's feasible set F_i is known but i's preference ordering $\succsim_i$ is not. It even allows for the complete absence of any relevant private information when both F_i and $\succsim_i$ are completely known and so Θ_i is a singleton.

Following Aumann [1964] or more especially Hildenbrand [1974], it has become standard practice to define a continuum economy as a measurable mapping $\mathcal{E} : I \to \Theta$ or $i \mapsto \theta_i$ from agents' identifiers to their characteristics. The measurability property appears to be unnecessary, however, as well as rather artificial. Worse, Lusin's theorem states that, except on a set of arbitrarily small measure, any measurable function on a compact domain is equal to some continuous function — see, for instance, Halmos [1950, pp. 242–3] or Royden [1968, p. 72, Problem 31]. More precisely, if $i \mapsto \theta_i$ is measurable, then for every $\epsilon > 0$ there exists a measurable set $I_\epsilon \subset I$ with $\mu(I_\epsilon) < \epsilon$ and a continuous function $i \mapsto \theta_i^\epsilon$ such that $\theta_i^\epsilon = \theta_i$ for all $i \in I \setminus I_\epsilon$. Thus, any measurable $i \mapsto \theta_i$ can be approximated rather well by a sequence of continuous mappings. Moreover, if $i \mapsto \theta_i$ is continuous, then any particular individual h's characteristic θ_h can be inferred from knowledge of θ_i for all $i \in N_h \setminus \{h\}$, where N_h is any neighbourhood of h. As Guesnerie [1995, Section 1.6] points out, this implies that some otherwise unintuitive or rather unappealing first-best mechanisms may be incentive compatible. For example, suppose that the announcements of θ_i for $i \in N_h \setminus \{h\}$ are such that $\theta_h^* := \lim_{h \to i} \theta_i$ exists. Then the mechanism might ignore entirely whatever θ_h is announced by h and simply generate whatever allocation would be appropriate if h's characteristic really were θ_h^*. After all, if the mapping $i \mapsto \theta_i$ really is continuous at $i = h$, and if all individuals $i \in N_h \setminus \{h\}$ are announcing their true characteristics, then θ_h^* must be h's true characteristic. Also, if necessary a severe punishment can be used to deter individuals from announcing a characteristic $\theta_h \neq \theta_h^*$. For such a mechanism, then, truthful revelation by almost all agents is at least a Nash equilibrium whenever $i \mapsto \theta_i$ is continuous. To avoid such awkward possibilities, I shall instead consider random economies in which all individuals' characteristics are i.i.d. random variables. Then, of course, knowing θ_i for all $i \in N_h \setminus \{h\}$ gives no information about θ_h.

Accordingly, a *random continuum economy* E is defined as a probability measure $\alpha \in \mathcal{M}(A)$ on the measurable set $A \subset I \times \Theta$ whose marginal distribution $\text{marg}_I\, \alpha$ on $I = [0, 1]$ is assumed to be uniform — i.e., equal to the Lebesgue measure μ. Let $\mathcal{E}$ be the domain of all such random continuum economies. The intended interpretation is that any $\alpha \in \mathcal{E}$ is the limit as $n \to \infty$ of a finite sample of n individuals whose identifiers and characteristics are drawn independently from the set A according to the joint probability measure α. For $n = 1, 2, \ldots$, let α_n denote the random empirical distribution from a sample of n agents. Because $I \times \Theta$ is a separable metric space, one can apply the Glivenko–Cantelli theorem in the form due to Parthasarathy [1967, Theorem 7.1, p. 53], as cited also by Hildenbrand [1974, pp. 52–3]. This result is a suitable law of large numbers ensuring that, almost surely as $n \to \infty$, the sequence of measures α_n converges weakly to α.

Note that this formulation of a random economy differs from that due to Hildenbrand [1971] and to Bhattacharya and Majumdar [1973] because they assume a fixed set of agents, independent of the state of the world. Here, by contrast, in any approximation to the continuum economy, the set of agent identifiers appearing in the finite random sample is itself random.

As discussed by Aumann [1964b], Feldman and Gilles [1985], Judd [1985] and Al-Najjar [1995], a continuum of independent random variables θ_i $(i \in I)$ almost never produces a measurable mapping from I to Θ. As an illustration, suppose that each $i \in I$ consists of a (nine-digit) U.S. Social Security number multiplied by 10^{-9}, and that $\theta_i \in \Re_+$ indicates the height of the person with number i. Then the mapping $i \mapsto \theta_i$ surely has a very messy graph, thus suggesting that it would not be measurable in the limit as $n \to \infty$.

For this reason, any realization of a random continuum economy almost surely does not meet the measurability requirements imposed by Aumann or Hildenbrand. This creates no difficulties, however, because there will never be a need to integrate any mapping whose domain is I instead of A; for the latter domain, all the functions that need to be considered will be measurable. In particular, for the example considered above, the mean height of people with U.S. Social Security numbers can be calculated by integrating w.r.t. the joint distribution α the (measurable) projection mapping from $A \subset I \times \Theta$ onto Θ. Equivalently, one could integrate w.r.t. the marginal measure $\text{marg}_\Theta\, \alpha$ on Θ. Of course, this marginal measure describes the distribution of heights precisely.

3. Multilaterally Strategy-Proof Allocation Mechanisms

An *allocation mechanism* $(\mathbf{x}, \mathbf{z})$ is a pair of mappings $\mathbf{x} : \mathcal{E} \times A \to \Re^T$ and $\mathbf{z} : \mathcal{E} \times A \to Z$ with values denoted by $x_i(\alpha, \theta)$ and $z_i(\alpha, \theta)$ respectively which, for all fixed $\alpha \in \mathcal{E}$, are measurable w.r.t. (i, θ). Note that the distribution α is treated as a variable of the allocation mechanism. This reflects the obvious requirement that an economic system should function and produce a feasible allocation even when α can only be discovered by inducing individuals to reveal their privately known true characteristics either directly or indirectly.

Overall feasibility typically requires that the mechanism should satisfy an additional *resource balance* constraint. In the case when z represents a vector of private goods that individuals cannot exchange on the side, Z should be a normed linear space. In this case, the mean net trade vector per head is given by $y(\alpha) := \int_A (x_i(\alpha, \theta), z_i(\alpha, \theta)) d\alpha$. Then the resource balance constraint would be $y(\alpha) = 0$, or alternatively $y(\alpha) \leqq 0$ if free disposal of every good were possible. On the other hand, if each point z represents a collection of public goods, then overall feasibility would require that $z_i(\alpha, \theta) = z(\alpha)$ for α-almost all $(i, \theta) \in A$, and that $\int_A x_i(\alpha, \theta)\, d\alpha = x(\alpha)$ where $(x(\alpha), z(\alpha)) \in Y$ for some known production possibility set $Y \subset G$. In this case, the space Z does not need a linear structure. No such resource balance constraint plays any role in the subsequent analysis of this paper, however, because only individual feasibility and incentive constraints will be considered. For this reason, say that the mechanism $(\mathbf{x}, \mathbf{z})$ is *feasible* provided that, for all $\alpha \in \mathcal{E}$, one has $(x_i(\alpha, \theta), z_i(\alpha, \theta)) \in F(\theta)$ for α-almost all $(i, \theta) \in A$.

Individual incentive constraints arise because individuals are privately informed about their characteristics. The constraints require that no individual i with true characteristic θ acting alone can benefit from manipulating the mechanism by simulating how i would be expected to behave with characteristic η instead of θ. Hence, for each distribution $\alpha \in \mathcal{E}$ and combination (i, θ, η) with $i \in I$ and $\theta, \eta \in \Theta_i$, the *individual incentive constraint*

$$(x_i(\alpha, \theta), z_i(\alpha, \theta)) \succsim_\theta (x_i(\alpha, \eta), z_i(\alpha, \eta)) \tag{3.1}$$

must be satisfied. Note how (3.1) reflects the fact that one individual acting alone in a continuum economy is powerless to change the apparent distribution α. In fact, these incentive constraints are natural adaptations of those that apply to the anonymous or symmetric mechanisms considered by Hammond [1979, 1987], Gale [1980, 1982] and Guesnerie [1981, 1995].

For manipulation to be feasible, it should be true that $(x_i(\alpha, \eta), z_i(\alpha, \eta)) \in F(\theta)$. Otherwise individual i would be unable to carry out the expected net trade, in which case the presumption is that i will be caught in time and required to behave in some other way that does allow the mechanism to allocate a feasible net trade vector. If i does not get caught in time, however, there may still be scope for manipulation unless additional incentive constraints are satisfied, such as those discussed in Hammond [1992]. Of course, the difficulties surrounding this extra restriction disappear if $F(\theta) = F_i$ for all $\theta \in \Theta_i$ because i's feasible set of net trades is known to be F_i.

The ensuing analysis will make use of the notation

$$A(\theta) := \{\, (i, \eta) \in A \mid (x_i(\alpha, \eta), z_i(\alpha, \eta)) \in F(\theta) \,\}$$

for the set of pairs (i, η) which any agent i with true characteristic θ can credibly claim without violating individual feasibility.

After these preliminaries, the allocation mechanism $(\mathbf{x}, \mathbf{z})$ is said to be *almost strategy-proof* provided that, for all fixed $\alpha \in \mathcal{E}$, for α-almost all pairs $(i, \theta) \in A$, and for α-almost all $(i, \eta) \in A(\theta)$, the incentive constraint (3.1) is satisfied. Hence, there must exist a *full* measurable set $A^1_\alpha \subset A$ with $\alpha(A^1_\alpha) = 1$ such that (3.1) is true for all (i, θ, η) satisfying $(i, \theta) \in A^1_\alpha$ and $(i, \eta) \in A(\theta)$.

The above definition is really too weak, however, because a non-null set of potential agents $(i, \theta) \in A^1_\alpha$ might still be able to manipulate by behaving as though they have a characteristic η satisfying $(i, \eta) \in A \setminus A^1_\alpha$. For this reason, say that $(\mathbf{x}, \mathbf{z})$ is *strategy-proof* provided that it is almost strategy-proof, but in addition (3.1) is true for *all* $(i, \eta) \in A(\theta)$ without exception, not just for α-almost all such (i, η). More precisely, (3.1) must be true for all $(\alpha, i, \theta, \eta) \in \mathcal{E} \times I \times \Theta \times \Theta$ which, for some full set $A^1_\alpha \subset A$, satisfy $(i, \theta) \in A^1_\alpha$ and $(i, \eta) \in A(\theta)$. Note that a mechanism is almost strategy-proof if it is equal to a strategy-proof mechanism for α-almost all $(i, \theta) \in A$.

Also, for the game of direct revelation corresponding to the mechanism in which each individual $i \in I$ has a strategy set Θ_i, strategy-proofness implies that, for all fixed $\alpha \in \mathcal{E}$, almost every individual has truthfulness as at least a weakly dominant strategy; almost strategy-proofness only implies that almost every individual has truthfulness as an "almost" weakly dominant strategy, in some obvious sense.

Next, a *potential finite coalition* is defined as a finite set C of pairs $(i, \theta) \in A$ with the property that each individual $i \in I$ features at most once in C. Thus, a potential finite coalition C is equivalent to a particular finite subset $I_C \subset I$ of individuals i whose respective characteristics $\theta_i \in \Theta_i$ are such that $C = \{\, (i, \theta_i) \mid i \in I_C \,\}$. The idea is that the potential finite coalition could actually form whenever the individuals $i \in I_C$ all happen to have exactly the right characteristics.

Given the joint distribution $\alpha \in \mathcal{E}$, say that the potential finite coalition $C \subset A$ can *manipulate* the allocation mechanism $(\mathbf{x}, \mathbf{z})$ provided that, when all individuals $i \in I_C$ have respective characteristics $\theta_i \in \Theta_i$ satisfying $(i, \theta_i) \in C$, they can jointly find "manipulative" characteristics $\eta_i \in \Theta_i$ and net trade vectors $t_i \in \Re^T$ of tradeable goods satisfying $\sum_{i \in I_C} t_i = 0$ with the property that both $(x_i(\alpha, \eta_i) + t_i, z_i(\alpha, \eta_i)) \in F(\theta_i)$ and

$$x_i(\alpha, \eta_i) + t_i, z_i(\alpha, \eta_i)) \succ_{\theta_i} (x_i(\alpha, \theta_i), z_i(\alpha, \theta_i)) \tag{3.2}$$

Note that in the special case when $\#C = 1$, or when there are no tradeable goods ($T = \emptyset$) — or even only one ($\#T = 1$) provided that individuals prefer more of it to less — then (3.2) holds iff the individual incentive constraint (3.1) is violated. As Guesnerie (1981, 1995) and I have both shown, outside these special cases, multilateral strategy-proofness is considerably more demanding than ordinary individual strategy-proofness. Gale [1980, 1982] also has similar but less general results.

Of course one is only interested in manipulation by potential finite coalitions who have a positive probability of being able to form. In particular, potential manipulation by finite coalitions does not matter if one could eliminate any possibility of it by removing a null set of agents. Accordingly, the allocation mechanism $(\mathbf{x}, \mathbf{z})$ is said to be *multilaterally strategy-proof* provided that, for any fixed $\alpha \in \mathcal{E}$, there exists a full measurable set $A^1_\alpha \subset A$ with $\alpha(A^1_\alpha) = 1$ such that no potential finite coalition $C \subset A^1_\alpha$ can manipulate the mechanism. Hence, if a mechanism is not multilaterally strategy-proof, it is because any full set $A^1_\alpha \subset A$ contains at least one potential finite coalition with the power to manipulate; excluding any null set from A cannot remove all such manipulative potential coalitions. Evidently, a mechanism is multilaterally strategy-proof only if it is (individually) strategy-proof.

If $\Theta_i = \{\theta_i\}$ for all $i \in I$ and so the economy has no private information, then the above definition reduces to a condition called "f-Pareto efficiency" in Hammond

[1995a]. When in addition $G = \Re^T$, in effect, because there are only traded goods, the definition in Hammond, Kaneko and Wooders [1989] and in Hammond [1995a] of an f-core allocation $\mathbf{x}$ for the special case of an economy without widespread externalities can be obtained by replacing (3.2) with $t_i \succ_{\theta_i} x_i(\alpha, \theta_i)$ for all $i \in C$.

A rather weaker notion of multilateral strategy-proofness arises if one insists that the set of agents who belong to manipulative potential coalitions is more evidently of positive measure. Specifically, say that a finite family A_k $(k \in K)$ of measurable non-null subsets of A, not necessarily pairwise disjoint, can *discernibly manipulate* the allocation mechanism $(\mathbf{x}, \mathbf{z})$ provided that there exist corresponding natural numbers n_k $(k \in K)$ such that manipulation is possible by every potential finite coalition C which, for each $k \in K$, contains exactly n_k different individuals $i \in I$ with appropriate characteristics θ_i such that $(i, \theta_i) \in A_k$. And say that the allocation mechanism $(\mathbf{x}, \mathbf{z})$ is *weakly multilaterally strategy-proof* if there is no such finite family that can discernibly manipulate.

4. Budget Decentralizations with Linear Prices for Tradeable Goods

A *budget correspondence* $B : I \times \mathcal{E} \twoheadrightarrow G$ specifies the budget set $B_i(\alpha)$ of each agent $i \in I$ as a function of the distribution α. But $B_i(\alpha)$ must be independent of agent i's own characteristic $\theta_i \in \Theta_i$. Such a correspondence *almost decentralizes* the allocation mechanism $(\mathbf{x}, \mathbf{z})$ provided that for all $\alpha \in \mathcal{E}$ and α-almost all $(i, \theta) \in A$, one has $(x_i(\alpha, \theta), z_i(\alpha, \theta)) \in B_i(\alpha)$ and

$$(x, z) \in B_i(\alpha) \cap F(\theta) \implies (x_i(\alpha, \theta), z_i(\alpha, \theta)) \succsim_\theta (x, z) \tag{4.1}$$

In other words, for all (i, θ) in some full set A^1_α with $\alpha(A^1_\alpha) = 1$, the pair of vectors $(x_i(\alpha, \theta), z_i(\alpha, \theta))$ must together maximize the preference ordering $\succsim_\theta$ w.r.t. (x, z) subject to the budget constraint $(x, z) \in B_i(\alpha)$ combined with the individual feasibility constraint $(x, z) \in F(\theta)$.

Second, the budget correspondence B *decentralizes* the mechanism $(\mathbf{x}, \mathbf{z})$ provided that it almost decentralizes $(\mathbf{x}, \mathbf{z})$, but in addition $(x_i(\alpha, \theta), z_i(\alpha, \theta)) \in B_i(\alpha)$ for all $\alpha \in \mathcal{E}$ and for *all* $(i, \theta) \in A$ without exception; it is still possible to have a null subset of pairs $(i, \theta) \in A$ for which $(x_i(\alpha, \theta), z_i(\alpha, \theta))$ is not the best point of $B_i(\alpha) \cap F(\theta)$.

Following Hammond [1979], it is obvious that if a mechanism is almost decentralizable then it must be almost strategy-proof, and if it is decentralizable then it must be strategy-proof. Conversely, by choosing the decentralization

$$B_i(\alpha) := \{ (x, z) \in G \mid \exists \theta \in \Theta : (x, z) = (x_i(\alpha, \theta), z_i(\alpha, \theta)) \}$$

it is easy to see that a mechanism is almost decentralizable if it is almost strategy-proof, and decentralizable if it is strategy-proof.

Next, the budget correspondence $B : I \times \mathcal{E} \twoheadrightarrow G$ is said to have *linear prices for tradeable goods* provided that there exists a price mapping $p : \mathcal{E} \to \Re^T \setminus \{0\}$ and a (generally non-linear) *allowable expenditure function* $m : I \times \mathcal{E} \times Z \to \Re$ such that

$$B_i(\alpha) = \{ (x, z) \in G \mid p(\alpha)\, x \leq m_i(\alpha, z) \} \tag{4.2}$$

Notice that $m_i(\alpha, z)$ specifies how much agent i is allowed to spend on tradeable goods as a function of the distribution α and the non-tradeable good vector z chosen by the agent. Thus, $m_i(\alpha, z)$ reflects any non-linear pricing schedule that applies to collections of non-tradeable goods in the space Z.

If the allocation mechanism $(\mathbf{x}, \mathbf{z})$ is decentralizable by a budget correspondence $B_i(\alpha)$ satisfying (4.2) everywhere, then for almost all $(i, \theta) \in A$ and for all $(x, z) \in F(\theta)$, one has

$$(x, z) \succ_\theta (x_i(\alpha, \theta), z_i(\alpha, \theta)) \implies p(\alpha)\, x > m_i(\alpha, z) \tag{4.3}$$

On the other hand, if the allocation mechanism $(\mathbf{x}, \mathbf{z})$ is decentralizable or almost decentralizable by such a budget correspondence, then (4.3) is true for all (i, θ) belonging to some set A^1_α of full measure.

For the second lemma below, say that the budget set $B_i(\alpha)$ given by (4.2) is *almost a compensated linear decentralization* provided that for all $\alpha \in \mathcal{E}$ and α-almost all $(i, \theta) \in A$, one has $(x_i(\alpha, \theta), z_i(\alpha, \theta)) \in B_i(\alpha)$ and

$$(x, z) \succsim_\theta (x_i(\alpha, \theta), z_i(\alpha, \theta)) \implies p(\alpha)\, x \geq m_i(\alpha, z)$$

whenever $(x, z) \in F(\theta)$. Also, say that $B_i(\alpha)$ is a *compensated linear decentralization* provided that it is almost a compensated linear decentralization with $(x_i(\alpha, \theta), z_i(\alpha, \theta)) \in B_i(\alpha)$ for all $\alpha \in \mathcal{E}$ and all $(i, \theta) \in A$.

Theorem 4.1. First Decentralization Theorem. *An allocation mechanism $(\mathbf{x}, \mathbf{z})$ that is decentralizable by a linear budget constraint $p(\alpha)\, x \leq m_i(\alpha, z)$ is multilaterally strategy-proof.*

Proof. Fix any $\alpha \in \mathcal{E}$. Suppose there is such a decentralization, with (4.1) holding for all $(i, \theta) \in A^1_\alpha$ where $\alpha(A^1_\alpha) = 1$. Suppose also that (3.2) is true for some pair $(i, \theta_i) \in A^1_\alpha$ and characteristic $\eta_i \in \Theta_i$. Then, faced with the budget set $B_i(\alpha)$ given by (4.2), individual i with characteristic η_i must be able to afford $(x_i(\alpha, \eta_i), z_i(\alpha, \eta_i))$, whereas i with characteristic θ_i must be unable to afford $(x_i(\alpha, \eta_i) + t_i, z_i(\alpha, \eta_i))$. Therefore

$$p(\alpha)\, [x_i(\alpha, \eta_i) + t_i] > m_i(\alpha, z_i(\alpha, \eta_i)) \geq p(\alpha)\, x_i(\alpha, \eta_i)$$

implying that $p(\alpha)\, t_i > 0$. Hence, if there exists a potential finite coalition C of pairs $(i, \theta_i) \in A^1_\alpha$ for individuals $i \in I_C$, together with characteristics $\eta_i \in \Theta_i$ and net trade vectors $t_i \in \Re^T$ satisfying (3.2) for each $i \in I_C$, then $\sum_{i \in I_C} p(\alpha)\, t_i > 0$. This excludes the possibility that $\sum_{i \in I_C} t_i = 0$, so no finite coalition $C \subset A^1_\alpha$ can manipulate. Hence, $(\mathbf{x}, \mathbf{z})$ must be multilaterally strategy-proof. □

This first result is similar to Arrow's [1951] first efficiency theorem of welfare economics, saying that any Walrasian equilibrium allocation is (weakly) Pareto efficient. The theorem even has a similar proof. Its converse would be a second decentralization theorem, saying that any (weakly) multilaterally strategy-proof mechanism is decentralizable by a linear budget constraint. Similarly, Arrow's [1951] second efficiency theorem of welfare economics says that any (weakly) Pareto efficient allocation can be decentralized as a Walrasian equilibrium with suitable lump-sum redistribution. In fact this result is not true without several extra assumptions. Moreover, at least for an economy with a continuum of agents, the extra assumptions are somewhat similar, as is the proof.

Let $Q \subset \Re$ denote the set of rational numbers. Let Q^T denote the subset of vectors in $\Re^T$ whose coordinates are all rational.

Assumption 1. For each $\theta \in \Theta$, the feasible set $F(\theta)$ and strict preference relation $\succ_\theta$ satisfy *rational local non-satiation in tradeable goods* in the following sense: given any $(x, t, z) \in \Re^T \times \Re^T \times Z$ with $(x+t, z) \in F(\theta)$ and any neighbourhood V of t in $\Re^T$, there exists $t' \in V \cap Q^T$ such that $(x+t', z) \in F(\theta)$ with $(x+t', z) \succ_\theta (x+t, z)$.

Lemma 4.2. Budget Exhaustion. *Suppose that Assumption 1 is satisfied, and that $B_i(\alpha)$ given by (4.2) is almost a compensated linear decentralization. Then $p(\alpha)\, x_i(\alpha,\theta) = m_i(\alpha, z_i(\alpha,\theta))$ for all $\alpha \in \mathcal{E}$ and α-almost all $(i,\theta) \in A$.*

Proof. Suppose that $\alpha \in \mathcal{E}$ and $p(\alpha)\, x_i(\alpha,\theta) < m_i(\alpha, z_i(\alpha,\theta))$ for some $(i,\theta) \in A$. By Assumption 1, there must then exist $t \in \Re^T$ such that $(x_i(\alpha,\theta) + t, z_i(\alpha,\theta)) \in F(\theta)$ with $(x_i(\alpha,\theta) + t, z_i(\alpha,\theta)) \succ_\theta (x_i(\alpha,\theta), z_i(\alpha,\theta))$ and $p(\alpha)\,[x_i(\alpha,\theta) + t] < m_i(\alpha, z_i(\alpha,\theta))$. But because $B_i(\alpha)$ is almost a compensated linear decentralization, this can be true for at most an α-null set of potential agents $(i,\theta) \in A$. □

Consider a given mechanism $(\mathbf{x}, \mathbf{z})$ and any fixed $\alpha \in \mathcal{E}$. Then, given any $a = (i,\theta) \in A$, define the two sets

$$T_a := \{\, t \in \Re^T \mid \exists \eta \in \Theta_i : (x_i(\alpha,\eta) + t, z_i(\alpha,\eta)) \succ_\theta (x_i(\alpha,\theta), z_i(\alpha,\theta)) \,\} \tag{4.4}$$

$$V_a := \{\, t \in \Re^T \mid \exists \eta \in \Theta_i : (x_i(\alpha,\eta) + t, z_i(\alpha,\eta)) \succsim_\theta (x_i(\alpha,\theta), z_i(\alpha,\theta)) \,\} \tag{4.5}$$

of trade vectors t which can be used by i with characteristic θ to manipulate (or manipulate weakly) the mechanism by behaving like an agent with some suitable characteristic $\eta \in \Theta_i$.

Lemma 4.3. Compensated Linear Decentralization. *Under the hypothesis that preferences are rationally local non-satiated in tradeable goods, a feasible allocation mechanism $(\mathbf{x}, \mathbf{z})$ is weakly multilaterally strategy-proof only if, for each $\alpha \in \mathcal{E}$, there exist $p(\alpha) \in \Re^T \setminus \{0\}$ and a mapping $m_i(\alpha, z)$ such that the budget constraint $p(\alpha)\, x \leq m_i(\alpha, z)$ is almost a compensated linear decentralization.*

Proof. Because $(\mathbf{x}, \mathbf{z})$ is feasible, it loses no generality to replace A with the measurable subset $\{\, (i,\theta) \in A \mid (x_i(\alpha,\theta), z_i(\alpha,\theta)) \in F(\theta) \,\}$ whose measure is 1. Let $\alpha \in \mathcal{E}$ be any fixed continuum economy. Following the beginning of Aumann's [1964a, p. 45] and Hildenbrand's [1982, pp. 843–4] proof of core equivalence, define for every $t \in Q^T$ the set

$$A(t) := \{\, a \in A \mid t \in T_a \,\} \tag{4.6}$$

Claim. The set $A(t)$ is measurable, for all $t \in Q^T$.

Proof of claim. First, because preferences are continuous and the space Θ has been given the topology of closed convergence, the two sets

$$\begin{aligned} S_1 &:= \{\, (\theta, x, z, x', z') \in \Theta \times G \times G \mid (x,z), (x',z') \in F(\theta) \,\} \\ S_2 &:= \{\, (\theta, x, z, x', z') \in \Theta \times G \times G \mid (x,z) \succsim_\theta (x',z') \,\} \end{aligned}$$

are both closed in the relevant product topology. So the set

$$S_3 := S_1 \setminus S_2 = \{\, (\theta, x, z, x', z') \in \Theta \times G \times G \mid (x',z') \succ_\theta (x,z) \,\}$$

is measurable in the corresponding Borel σ-field. Next, define the two sets

$$\begin{aligned} H_1(t) &:= \{\, (h, \eta^1, x^1, z^1) \in A \times G \mid x^1 = x_h(\alpha, \eta^1) + t,\ z^1 = z_h(\alpha, \eta^1) \,\} \\ H_2 &:= \{\, (i, \eta^2, x^2, z^2) \in A \times G \mid x^2 = x_i(\alpha, \eta^2),\ z^2 = z_i(\alpha, \eta^2) \,\} \end{aligned}$$

Both are clearly measurable in the relevant product σ-field because the allocation mechanism $(\mathbf{x}, \mathbf{z})$ involves measurable mappings $(i,\theta) \mapsto x_i(\alpha,\theta)$ and $(i,\theta) \mapsto$

$z_i(\alpha,\theta)$ whose graphs must therefore be measurable. On the other hand, the set

(4.7)
$$E := \{ (\theta, x, z, x', z', h, \eta^1, x^1, z^1, i, \eta^2, x^2, z^2) \in \Theta \times G \times G \times A \times G \times A \times G \mid \theta = \eta^2,\ x' = x^1,\ z' = z^1,\ x = x^2,\ z = z^2,\ h = i \}$$

is evidently closed. Now let $S(t) := S_3 \times H_1(t) \times H_2$, which is also measurable in the relevant product σ-field, as is $S(t) \cap E$. But then

(4.8)
$$A(t) = \{ (i,\theta) \in A \mid \exists (x, z, x', z', h, \eta^1, x^1, z^1, \eta^2, x^2, z^2) \in G \times G \times A \times G \times \Theta \times G : (\theta, x, z, x', z', h, \eta^1, x^1, z^1, i, \eta^2, x^2, z^2) \in S(t) \cap E \}$$

which is the projection of $S(t) \cap E$ onto A. Therefore, $A(t)$ must be measurable. □

Having proved the claim, we now prove the lemma. To do so, first define

$$A^* := A \setminus \left[\bigcup \{ A(t) \mid t \in Q^T,\ \alpha(A(t)) = 0 \} \right] \tag{4.9}$$

Because Q^T is countable and each set $A(t)$ is measurable, the set A^* is also measurable and satisfies $\alpha(A^*) = \alpha(A) = 1$. Finally, define the convex hull

$$K := \text{co} \left[\bigcup \{ T_a \cap Q^T \mid a \in A^* \} \right] \tag{4.10}$$

Suppose that $0 \in K$. Then there exists a finite collection of m individual–characteristic pairs $a_k = (i_k, \theta_{i_k}) \in A^*$, together with associated rational net trade vectors $\tilde{t}_k \in T_{a_k} \cap Q^T$ $(k = 1, \ldots, m)$ such that, for some positive convex weights $r_k \in \Re_+$, one has

$$\sum_{k=1}^{m} r_k = 1 \quad \text{and} \quad \sum_{k=1}^{m} r_k \, \tilde{t}_k = 0 \tag{4.11}$$

Equations (4.11) can be combined into the single matrix equation $\mathbf{T}\,\mathbf{r} = \mathbf{e}_1$, where $\mathbf{T}$ is a $(\#T + 1) \times m$ matrix, $\mathbf{r}$ is an m-vector, and $\mathbf{e}_1$ is the first unit vector in $\Re \times \Re^T$. This equation has a solution $\mathbf{r} \gg 0$. Because each $\tilde{t}_k \in Q^T$ has rational co-ordinates, the matrix $\mathbf{T}$ has rational elements. Now, if a positive solution exists, it can be found by Gaussian elimination or some other sequence of elementary row operations involving addition, subtraction, multiplication and division. Hence, there is at least one strictly positive solution $\mathbf{r}$ with only rational co-ordinates, which therefore involves *rational* positive convex weights $r_k \in Q_+$ $(k = 1, \ldots, m)$. After multiplying by a large enough common multiple of the denominators in the rational numbers r_k, it follows that there exist natural numbers n_k $(k = 1, \ldots, m)$ for which $\sum_{k=1}^{m} n_k \, \tilde{t}_k = 0$.

Because $a_k \in A^*$, there is no $t \in Q^T$ such that $a_k \in A(t)$ and $\alpha(A(t)) = 0$. Now, $\tilde{t}_k \in T_{a_k}$ and so $a_k \in A(\tilde{t}_k)$ $(k = 1, \ldots, m)$. But $\tilde{t}_k \in Q^T$, from which it follows that $\alpha(A(\tilde{t}_k)) > 0$ $(k = 1, \ldots, m)$. In particular, each set $A(\tilde{t}_k)$ has infinitely many members.

Now let $C \subset \cup_{k=1}^{m} A(\tilde{t}_k)$ be any potential finite coalition consisting of $\sum_{k=1}^{m} n_k$ different individuals $i \in I_C \subset I$ with characteristics $\theta_i \in \Theta_i$ such that, for each $k = 1, \ldots, m$, the set $A(\tilde{t}_k) \cap C$ has exactly n_k members. Also, for all $i \in I_C$, define $t_i := \tilde{t}_k$ for the unique k such that $(i, \theta_i) \in A(\tilde{t}_k) \cap C$. Then $\tilde{t}_k \in T_a$ whenever $a \in A(\tilde{t}_k) \cap C$. Also, $\sum_{i \in I_C} t_i = \sum_{k=1}^{m} n_k \, \tilde{t}_k = 0$. It follows that any such coalition can manipulate the mechanism $(\mathbf{x}, \mathbf{z})$. This shows that the finite family $A(\tilde{t}_k)$ ($k = 1$ to m) can discernibly manipulate the mechanism $(\mathbf{x}, \mathbf{z})$. So $0 \in K$ implies that the mechanism $(\mathbf{x}, \mathbf{z})$ cannot be weakly multilaterally strategy-proof.

On the other hand, if $(\mathbf{x}, \mathbf{z})$ is weakly multilaterally strategy-proof, then $0 \notin K$. By construction, K is convex, so the point 0 can be separated from K by a hyperplane. Moreover, this hyperplane can be chosen so it passes through 0. Hence, there must exist $p(\alpha) \in \Re^T \setminus \{0\}$ for which $p(\alpha)\, t \geq 0$ whenever $t \in K$.

Now, given any $a \in A^*$ and any $t \in U_a$, rational local non-satiation in tradeable goods and transitivity of $\succsim_\theta$ together imply that t is the limit of a sequence t^q ($q = 1, 2, \ldots$) of net trade vectors in $T_a \cap Q^T$. Then $t^q \in K$ and so $p(\alpha)\, t^q \geq 0$ ($q = 1, 2, \ldots$), implying that $p(\alpha)\, t \geq 0$ in the limit as $q \to \infty$.

To complete the proof, first introduce the additional notation

$$\Theta_i^* := \{\, \theta \in \Theta_i \mid (i, \theta) \in A^* \,\}; \quad \begin{aligned} Z_i^* &:= \{\, z \in Z \mid \exists \theta \in \Theta_i^* : z = z_i(\alpha, \theta) \,\}; \\ Z_i &:= \{\, z \in Z \mid \exists \theta \in \Theta_i : z = z_i(\alpha, \theta) \,\}. \end{aligned}$$

Suppose now that the combination of $z \in Z_i^*$ with $a = (i, \theta) \in A^*$ and $\eta \in \Theta_i$ together satisfy $z_i(\alpha, \theta) = z_i(\alpha, \eta) = z$. Then (4.5) implies that $x_i(\alpha, \theta) - x_i(\alpha, \eta) \in U_a$ and so $p(\alpha)\, x_i(\alpha, \theta) \geq p(\alpha)\, x_i(\alpha, \eta)$. If $\eta \in \Theta_i^*$ also, then reversing θ and η in the above argument shows that $p(\alpha)\, x_i(\alpha, \theta) \leq p(\alpha)\, x_i(\alpha, \eta)$, and so $p(\alpha)\, x_i(\alpha, \theta) = p(\alpha)\, x_i(\alpha, \eta)$. Hence, one can define $m_i(\alpha, z)$ for all $z \in Z_i^*$ in order to satisfy $m_i(\alpha, z) = p(\alpha)\, x_i(\alpha, \theta)$ for all $\theta \in \Theta_i^*$ such that $z = z_i(\alpha, \theta)$. This already implies that $p(\alpha)\, x_i(\alpha, \theta) \leq m_i(\alpha, z_i(\alpha, \theta))$ for all $(i, \theta) \in A^*$.

To complete the definition of the function $m_i(\alpha, \cdot)$, for $z \in Z_i \setminus Z_i^*$ it is enough to define

$$m_i(\alpha, z) := \inf_{x, \theta} \{\, p(\alpha)\, x \mid z = z_i(\alpha, \theta);\ (x, z) \in F(\theta);\ (x, z) \succsim_\theta (x_i(\alpha, \theta), z) \,\}$$

For $z \in Z \setminus Z_i$, one can define

$$m_i(\alpha, z) := \inf_x \left\{ p(\alpha)\, x \mid (x, z) \in \bigcup_{\theta \in \Theta_i} F(\theta) \right\} - 1$$

Suppose that $(i, \theta) \in A^*$ and $(x, z) \in F(\theta)$ satisfy $(x, z) \succsim_\theta (x_i(\alpha, \theta), z_i(\alpha, \theta))$. Clearly, in the case when $z \in Z \setminus Z_i$, it must be true that $p(\alpha)\, x \geq m_i(\alpha, z)$ by construction; the same is also true when $z \in Z_i \setminus Z_i^*$ because $z = z_i(\alpha, \eta)$ for some $\eta \in \Theta_i \setminus \Theta_i^*$. The other case is when $z \in Z_i^*$ because $z = z_i(\alpha, \eta)$ for some $\eta \in \Theta_i^*$. But then $p(\alpha)\, x_i(\alpha, \eta) = m_i(\alpha, z)$. Also, (4.5) implies that $x - x_i(\alpha, \eta) \in U_a$, so $p(\alpha)\, x \geq p(\alpha)\, x_i(\alpha, \eta) = m_i(\alpha, z)$ in this case as well. This completes the proof that the budget constraint $p(\alpha)\, x \leq m_i(\alpha, z)$ is almost a compensated linear decentralization. □

5. Second Decentralization Theorem

To go from a compensated linear decentralization to an ordinary linear decentralization requires at least two additional assumptions. The two presented below are alternatives to the three that were used in Hammond [1987]. The extra conditions will be stated for each separate economy $\alpha \in \mathcal{E}$. Before doing so, first define, for each $\theta \in \Theta$ and $z \in Z$, the *conditional* feasible set given z as

$$X(z; \theta) := \{\, x \in \Re^T \mid (x, z) \in F(\theta) \,\}$$

Assumption 2. For all $\theta \in \Theta$ and $z \in Z$, the set $X(z;\theta)$ is equal to the union $\cup_{k \in K(z;\theta)} X^k(z;\theta)$ of a finite or countably infinite family of closed convex subsets of $\Re^T$.

Note that the different sets $X^k(z;\theta)$ need not be disjoint. Assumption 2 is trivially satisfied when each set $X(z;\theta)$ is convex. It certainly excludes some forms of non-convexity in $X(z;\theta)$, but it does allow indivisible goods or other particular kinds of non-convexity. Assumption 2 leads to the following generalization of a well known result:

Lemma 5.1. Cheaper Point. *Suppose $(i,\theta) \in A$ is such that $p(\alpha)\, x \geq m_i(\alpha, z)$ whenever $(x,z) \in F(\theta)$ satisfies $(x,z) \succsim_\theta (x_i(\alpha,\theta), z_i(\alpha,\theta))$. Then for any $z \in Z$ and $k \in K(z;\theta)$ such that there exists some "cheaper point" $\underline{x} \in X^k(z;\theta)$ with $p(\alpha)\, \underline{x} < m_i(\alpha, z)$, one has $p(\alpha)\, x > m_i(\alpha, z)$ whenever $x \in X^k(z;\theta)$ satisfies $(x,z) \succ_\theta (x_i(\alpha,\theta), z_i(\alpha,\theta))$.*

Proof. Fix any $(i,\theta) \in A$, $z \in Z$ and $k \in K(z;\theta)$. Suppose that $x \in X^k(z;\theta)$ satisfies $(x,z) \succ_\theta (x_i(\alpha,\theta), z_i(\alpha,\theta))$, and that there is a cheaper point $\underline{x} \in X^k(z;\theta)$. By continuity of preferences and Assumption 2, there must exist a scalar $\lambda \in (0,1)$ and a convex combination $\lambda\, x + (1-\lambda)\, \underline{x} \in X^k(z;\theta)$ such that $\lambda\, x + (1-\lambda)\, \underline{x} \succsim_\theta (x_i(\alpha,\theta), z_i(\alpha,\theta))$. By the hypotheses of the lemma, it follows that $p(\alpha)\,[\lambda\, x + (1-\lambda)\, \underline{x}] \geq m_i(\alpha, z)$. Therefore $\lambda\,[p(\alpha)\, x - m_i(\alpha, z)] \geq (1-\lambda)\,[m_i(\alpha, z) - p(\alpha)\, \underline{x}]$. Because $p(\alpha)\, \underline{x} < m_i(\alpha, z)$ and $0 < \lambda < 1$, it follows that $p(\alpha)\, x > m_i(\alpha, z)$. □

For each $\alpha \in \mathcal{E}$, $(i,\theta) \in A$, $z \in Z$ and $k \in K(z;\theta)$, define the minimum wealth level

$$\underline{m}_i^k(\alpha, z;\theta) := \min_x \{\, p(\alpha)\, x \mid x \in X^k(z;\theta) \,\}$$

Also, for each $\alpha \in \mathcal{E}$ and $z \in Z$, define

$$A_\alpha(z) := \{\, (i,\theta) \in A \mid \exists k \in K(z;\theta) : m_i(\alpha, z) = \underline{m}_i^k(\alpha, z;\theta) \,\}$$

Assumption. 3* The distribution $\alpha \in \mathcal{E}$ satisfies $\alpha\,(\cup_{z \in Z} A_\alpha(z)) = 0$.

At least when Z is a finite or countably infinite set, assumption 3* seems a fairly obvious generalization of the "dispersion" condition due to Yamazaki [1978, 1981]. When Z is a continuum, it is not nearly so weak. Nevertheless, it is sufficient to use the following obviously weaker version of assumption 3*:

Assumption 3. There is at most a null set A^0 of potential agents $(i,\theta) \in A$ having points $z \in Z_i$ with the property that, for some $k \in K(z;\theta)$, the set $X^k(z;\theta)$ has a cheapest net trade vector $\underline{x}$ satisfying $(\underline{x}, z) \succ_\theta (x_i(\alpha,\theta), z_i(\alpha,\theta))$, as well as $p(\alpha)\, x \geq p(\alpha)\, \underline{x} = m_i(\alpha, z)$ for all $x \in X^k(z;\theta)$.

Theorem 5.2. Second Decentralization Theorem. *In any economy $\alpha \in \mathcal{E}$ for which Assumptions 1–3 all hold, a feasible allocation mechanism $(\mathbf{x}, \mathbf{z})$ is weakly multilaterally strategy-proof only if it is almost decentralizable by the linear budget constraint $p(\alpha)\, x \leq m_i(\alpha, z)$ that was constructed in proving the Compensated Decentralization Lemma.*

Proof. By the Compensated Linear Decentralization Lemma 4.3, there must exist $p(\alpha) \in \Re^T \setminus \{0\}$ and a mapping $m_i(\alpha, z)$ such that, for all $(i,\theta) \in A^*$, whenever $(x,z) \in F(\theta)$ satisfies $(x,z) \succsim_\theta (x_i(\alpha,\theta), z_i(\alpha,\theta))$, then $p(\alpha)\, x \geq m_i(\alpha, z)$.

Suppose that $(i, \theta) \in A^*$ and $(\underline{x}, z) \in F(\theta)$ satisfy $(\underline{x}, z) \succ_\theta (x_i(\alpha, \theta), z_i(\alpha, \theta))$ with $p(\alpha)\, \underline{x} = m_i(\alpha, z)$. By construction of the function $m_i(\alpha, z)$, it must be true that $z \in Z_i$. Also, by the Cheaper Point Lemma 5.1, there must exist $k \in K(z; \theta)$ such that $\underline{x}$ is the cheapest net trade vector of $X^k(z; \theta)$ at prices $p(\alpha)$. So $(i, \theta) \in A^0$. By Assumption 3, it follows that $(\mathbf{x}, \mathbf{z})$ is almost decentralizable, as claimed. □

References

[1] Aghion, P. and P. Howitt [1995] *Technical Progress in the Theory of Economic Growth*, in *Economics in a Changing World, Vol. V: Economic Growth and Capital and Labour Markets* (J.-P. Fitoussi, ed.), Macmillan, London, 101–122.

[2] Al-Najjar, N.I. [1995] *Decomposition and Characterization of Risk with a Continuum of Random Variables*, Econometrica, **63**, 1195–1224.

[3] Anderson, R.M. [1991] *Non-Standard Analysis with Applications to Economics*, in *Handbook of Mathematical Economics, Vol. IV* (W. Hildenbrand and H. Sonnenschein, eds.), North-Holland, Amsterdam, 2145–2208.

[4] Anderson, R.M. [1994] *The Core in Perfectly Competitive Economies*, in *Handbook of Game Theory with Economic Applications, Vol. I* (R.J. Aumann and S. Hart, eds.), North-Holland, Amsterdam, 413–457.

[5] Arrow, K.J. [1951] *An Extension of the Basic Theorems of Classical Welfare Economics*, in *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, (J. Neyman, ed.), University of California Press, Berkeley, 507–532.

[6] Artstein, Z. [1980] *Discrete and Continuous Bang-Bang and Facial Spaces, or: Look for the Extreme Points*, SIAM Review, **22**, 172–185.

[7] Aumann, R.J. [1964a] *Markets with a Continuum of Traders*, Econometrica, **32**, 39–50.

[8] Aumann, R.J. [1964b] *Mixed vs. Behavior Strategies in Infinite Extensive Games*, Annals of Mathematics Studies, **52**, 627–630.

[9] Aumann, R.J. [1965] *Integrals of Set-Valued Functions*, Journal of Mathematical Analysis and Applications, **12**, 1–12.

[10] Aumann, R.J. [1966] *Existence of Competitive Equilibria in Markets with a Continuum of Traders*, Econometrica, **34**, 1–17.

[11] Aumann, R.J. and L.S. Shapley [1974] *Values of Non-Atomic Games*, Princeton University Press, Princeton.

[12] Barro, R.J. and X. Sala-i-Martin [1995] *Economic Growth*, McGraw Hill, New York.

[13] Bhattacharya, R.N. and M. Majumdar [1973] *Random Exchange Economies*, Journal of Economic Theory, **6**, 37–67.

[14] Blackorby, C. and D. Donaldson [1988] *Cash versus Kind, Self-Selection, and Efficient Transfers*, American Economic Review, **78**, 691–700.

[15] Brown, D.J. and A. Robinson [1975a] *The Cores of Large Standard Exchange Economies*, Journal of Economic Theory, **9**, 245–254.

[16] Brown, D.J. and A. Robinson [1975b] *Nonstandard Exchange Economies*, Econometrica, **43**, 41–55.

[17] Clarke, F.H. [1983] *Optimization and Nonsmooth Analysis*, John Wiley, New York.

[18] Feldman, M. and C. Gilles [1985] *An Expository Note on Individual Risk without Aggregate Uncertainty*, Journal of Economic Theory, **35**, 26–32.

[19] Gale, D.M [1980] *Money, Information and Equilibrium in Large Economies*, Journal of Economic Theory, **23**, 28–65.

[20] Gale, D.M. [1982] *Money: In Equilibrium*, James Nisbet, Welwyn.

[21] Grodal, B. [1972] *A Second Remark on the Core of an Atomless Economy*, Econometrica, **40**, 581–583.

[22] Guesnerie, R. [1981] *On Taxation and Incentives: Further Reflections on the Limits of Redistribution*, Discussion Paper No. 89, Sonderforschungsbereich 21, University of Bonn; revised as ch. 1 of Guesnerie [1995].

[23] Guesnerie, R. [1995] *A Contribution to the Theory of Taxation*, Cambridge University Press, Cambridge.

[24] Hahn, F.H. [1995] *On Growth Theory*, in *Economics in a Changing World, Vol. V: Economic Growth and Capital and Labour Markets* (J.-P. Fitoussi, ed.), Macmillan, London, 85–100.

[25] Halmos, P.R. [1950] *Measure Theory*, Van Nostrand Reinhold, New York.

[26] Hammond, P.J. [1979] *Straightforward Individual Incentive Compatibility in Large Economies*, Review of Economic Studies, **46**, 263–282.

[27] Hammond, P.J. [1987] *Markets as Constraints: Multilateral Incentive Compatibility in a Continuum Economy*, Review of Economic Studies, **54**, 399–412.

[28] Hammond, P.J. [1989] *The Power of Small Coalitions in Large Economies*, Stanford University, Institute of Mathematical Studies in the Social Sciences, Economics Technical Report No. 501; to appear in M.O.L. Bacharach, M.A.H. Dempster and J.L. Enos (eds.) *Mathematical Models in Economics: Oxford University Mathematical Economic Seminar 25th Anniversary Volume*, Oxford University Press, Oxford:.

[29] Hammond, P.J. [1992] *On the Impossibility of Perfect Capital Markets*, in *Economic Analysis of Markets and Games: Essays in Honor of Frank Hahn* (P. Dasgupta, D. Gale, O. Hart, and E. Maskin, eds.), M.I.T. Press, Cambridge, Mass., 527–560.

[30] Hammond, P.J. [1993a] *Irreducibility, Resource Relatedness, and Survival in Equilibrium with Individual Non-Convexities*, in *General Equilibrium, Growth, and Trade II: The Legacy of Lionel W. McKenzie* (R. Becker, M. Boldrin, R. Jones, and W. Thomson, eds.), Academic Press, San Diego, 73–115.

[31] Hammond, P.J. [1993b] *History as a Widespread Externality in Some Arrow–Debreu Market Games*, University of Munich, Center for Economic Studies Working Paper No. 47; for inclusion in *The Formulation of Economic Theory and Other Essays in Honor of Kenneth Arrow*, (G. Chichilnisky ed.), Cambridge University Press, Cambridge (in preparation).

[32] Hammond, P.J. [1995] *Four Characterizations of Constrained Pareto Efficiency in Continuum Economies with Widespread Externalities*, Japanese Economic Review, **46**, 103–124.

[33] Hammond, P.J. [1998] *On f-Core Equivalence with General Widespread Externalities*, Working Paper 95-004, Department of Economics, Stanford University; to appear in the Journal of Mathematical Economics.

[34] Hammond, P.J., Kaneko, M. and M.H. Wooders [1989] *Continuum Economies with Finite Coalitions: Core, Equilibrium and Widespread Externalities*, Journal of Economic Theory, **49**, 113–134.

[35] Hammond, P.J. and A. Rodríguez-Clare [1993] *On Endogenizing Long-Run Growth*, Scandinavian Journal of Economics, **95**, 391–425; also in *Endogenous Growth*, (T.M. Andersen and K.O. Moene, eds.), Blackwell, Oxford, 1–35.

[36] Haubrich, J.G. [1988] *Optimal Financial Structure in Exchange Economies*, International Economic Review, **29**, 217–235.

[37] Haubrich, J.G. and R.G. King [1990] *Banking and Insurance*, Journal of Monetary Economics, **26**, 361–386.

[38] Helpman, E. [1992] *Endogenous Macroeconomic Growth Theory*, European Economic Review, **36**, 237–267.

[39] Hildenbrand, W. [1971] *Random Preferences and General Economic Equilibrium*, Journal of Economic Theory, **3**, 414–429.

[40] Hildenbrand, W. [1974] *Core and Equilibria of a Large Economy*, Princeton University Press, Princeton.

[41] Hildenbrand, W. [1982] *Core of an Economy*, in *Handbook of Mathematical Economics, Vol. II*, (K.J. Arrow and M.D. Intriligator, eds.), North-Holland, Amsterdam, 831–877.

[42] Jacklin, C.J. [1987] *Demand Deposits, Trading Restrictions, and Risk Sharing*, in *Contractual Arrangements for Intertemporal Trade* (E.C. Prescott and N. Wallace, eds.), University of Minnesota Press, Minneapolis, 26–47.

[43] Judd, K. [1985] *The Law of Large Numbers with a Continuum of Random Variables*, Journal of Economic Theory, **35**, 19–25.

[44] Kaneko, M. and M.H. Wooders [1986] *The Core of a Game with a Continuum of Players and Finite Coalitions: The Model and Some Results*, Mathematical Social Sciences, **12**, 105–137.

[45] Kaneko, M. and M.H. Wooders [1989] *The Core of a Continuum Economy with Widespread Externalities and Finite Coalitions: From Finite to Continuum Economies*, Journal of Economic Theory, **49**, 135–168.

[46] Kaneko, M. and M.H. Wooders [1994] *Widespread Externalities and Perfectly Competitive Markets: Examples*, in *Imperfections and Behavior in Economic Organizations*, (R.P. Gilles and P.H.M. Ruys, eds.), Kluwer Academic Publishers, Boston, 71–87.

[47] Lucas, R.E. [1993] *Making a Miracle*, Econometrica, **61**, 251–272.

[48] Lyapunov, A.A. [1940] *On Completely Additive Vector-Functions*, Proceedings of the Academy of Sciences of the USSR, **4**, 465–478.
[49] Mas-Colell, A. [1989] *An Equivalence Theorem for a Bargaining Set*, Journal of Mathematical Economics, **18**, 129–139.
[50] Mertens, J.-F. [1970] *Upper Semi-Continuity of the Walras–Correspondence: A Probabilistic Proof*, CORE Discussion Paper No. 7003.
[51] Mirrlees, J.A. [1971] *An Exploration in the Theory of Optimum Income Taxation*, Review of Economic Studies, **38**, 175–208.
[52] Parthasarathy, K.R. [1967] *Probability Measures on Metric Spaces*, Academic Press, New York.
[53] Richter, H. [1963] *Verallgemeinerung eines in der Statistik benötigten Satzes der Masstheorie*, Mathematische Annalen, **150**, 85–90; correction 440–441.
[54] Romer, P.M. [1991] *Increasing returns and new developments in the theory of growth*, in *Equilibrium Theory and Applications: Proceedings of the 6th International Symposium in Economic Theory and Econometrics*, (W. Barnett, ed.), Cambridge University Press, Cambridge, 83–110.
[55] Royden, H.L. [1968] *Real Analysis*, 2nd ed., Macmillan, New York.
[56] Schmeidler, D. [1972] *A Remark on the Core of an Atomless Economy*, Econometrica, **40**, 579–580.
[57] Stern, N. [1991] *The Determinants of Growth*, Economic Journal, **101**, 122–133.
[58] Vickrey, W.S. [1945] *Measuring Marginal Utility by the Reactions to Risk*, Econometrica, **13**, 319–333.
[59] Vind, K. [1972] *A Third Remark on the Core of an Atomless Economy*, Econometrica, **40**, 585–586.
[60] Weiss, E.-A. [1981] *Finitely Additive Exchange Economies*, Journal of Mathematical Economics, **8**, 221–240.
[61] Yamazaki, A. [1978] *An Equilibrium Existence Theorem without Convexity Assumptions*, Econometrica, **46**, 541–555.
[62] Yamazaki, A. [1981] *Diversified Consumption Characteristics and Conditionally Dispersed Endowment Distribution: Regularizing Effect and Existence of Equilibria*, Econometrica, **49**, 639–654.

Department of Economics, Stanford University, CA 94305-6072, U.S.A.
E-mail address: `hammond@leland.stanford.edu`