PERFECTED OPTION MARKETS IN ECONOMIES WITH ADVERSE SELECTION

Peter J. Hammond

Department of Economics, European University Institute,

Badia Fiesolana, 50016 S. Domenico (FI), Italy;

and Stanford University, CA 94305–6072, U.S.A.

[December 1989 version for the European University Institute working paper series; based on the presentation to the Econometric Society European Meeting, Munich.]

ABSTRACT

In economies with adverse selection, Arrow-Debreu contingent commodity contracts must satisfy incentive constraints. Following Prescott and Townsend (in *Econometrica*, 1984), an Arrow-Debreu economy is considered with a continuum of agents whose feasible sets are artificially restricted by imposing these incentive constraints. Equilibria in such an economy must be incentive-constrained Pareto efficient. It is shown that deterministic equilibria of this kind are achievable through "perfected" option markets with non-linear pricing in a way which makes the incentive constraints self-enforcing. Rothschild, Stiglitz and others have shown, however, that these equilibria must be vulnerable to free entry by profit seeking firms.

PERFECTED MARKETS

1. Introduction.

It is becoming generally well understood how adverse selection and moral hazard create incentive constraints which restrict what is truly feasible in economies with private information. An adequate revised general equilibrium theory is still lacking, however. So far, the most comprehensive attempt in this direction is that of Prescott and Townsend (1984a, b). They undertook a systematic exploration of what allocation mechanisms in a general continuum economy are *ex ante* Pareto efficient among the class of those which can be implemented in dominant strategies, and sought a price system by means of which such allocation mechanisms can be decentralized. When they tried to deal with adverse selection problems in which agents could trade after acquiring private information, however, they were unable to construct a satisfactory price system for decentralizing constrained efficient allocations of resources. In unpublished preliminary versions of their paper, they did consider various new expansions of the commodity space and were able to prove some decentralization results. But as Prescott and Townsend (1984a, p. 44) frankly conclude, "Thus equilibria of this kind fail to provide much predictive content and have undesirable normative properties as well." Nor is this too surprising, given the work of Rothschild and Stiglitz (1976) and many successors on the necessary inefficiencies which competition can create when there is adverse selection.

Here I will present a form of decentralization which differs from those considered by Prescott and Townsend — even, it seems, from those considered in the earlier unpublished versions of Prescott and Townsend (1984a). The decentralization relies on a very special form of option market with non-linear pricing. No predictive content is claimed for this kind of "perfected" market. But the normative properties will be everything that could be desired. Both the first and the second fundamental efficiency theorems of welfare economics will be true, subject to the usual kind of qualifications regarding the second theorem, and allowing for some difficulties which arise because incentive constraints can cause local satiation in the relevant feasible sets. Moreoever, perfected option market equilibria (POME's) will lie in a suitably defined version of the core, and that core will be equivalent to the set of POME allocations for the continuum economy which I shall consider throughout.

Section 2 below presents the basic model of an adverse selection economy with a continuum of agents. In fact the model is derived by introducing adverse selection into a version with a continuum of agents of Malinvaud's (1972, 1973) description of a large economy with individual uncertainty. It reduces to the Rothschild and Stiglitz (1976) model of insurance markets as a special case when there is a single physical commodity and two different types of agent with differing probabilities of experiencing each of two individual states.

Thereafter Section 3 explains why the incentive constraints caused by adverse selection create problematic externalities between different types of the same agent. Such externalities, moroever, generate fundamental non-convexities of a kind which typically prevent decentralization of efficient allocations through any linear price system. Individuals may also have preferences which are locally satiated at some points. Section 4, however, shows that a certain form of "coalitional local non-satiation" is satisfied.

Section 5 then takes us part way toward resolving the problems posed by these externalities. It describes both uncompensated and compensated Walrasian equilibria in which agents are artificially constrained to internalize the externalities which they would otherwise create. As shown in Sections 6 and 7, these artificially constrained equilibria have most of the usual efficiency and even core properties of Walrasian allocations in continuum economies — the only difference comes in the first efficiency theorem of welfare economics, which holds only in a somewhat weakened form because individuals' preferences may be locally satiated in the space of incentive compatible allocations. Under standard assumptions, Section 8 demonstrates the existence of a weakened form of Walrasian equilibrium — indeed, it shows how there are likely to be a vast profusion of such equilibria, in the presence of adverse selection. Again, a weakening of the standard definition is necessary because individual agents' preferences may be locally satiated.

The heart of the paper comes in Section 9. This demonstrates how agents can be induced to internalize voluntarily the externalities due to incentive constraints. This is done by means of suitably designed markets for "insurance securities" which pay a specified number of units of the *numéraire* commodity contingent upon each observed individual state of the world. These are "perfected" option markets.

Section 10 concludes by discussing possible extensions and limitations to the scope of perfected markets.

2. A Continuum Economy with Adverse Selection

Consider a continuum economy with an atomless measure space of agents $(A, \mathcal{A}, \alpha)$, as in Aumann (1964, 1966) and Hildenbrand (1974).

Next, as in the individual uncertainty model of Malinvaud (1972, 1973) and the insurance model of Rothschild and Stiglitz (1976), suppose that there is a finite set S of possible individual states s, each of which is observable — for example, whether the individual suffers an accident or has some other reason to make a legitimate insurance claim. Indeed, where different possible kinds of accident are possible, s should determine what kind occurs, and so what an insurance company's liability must be. It will be assumed that these are *ex post* states of the world, determined after all market transactions have taken place.

Suppose that there is also a finite set T of possible individual types or characteristics t which are private information, so not observable by any other agent such as an insurance company. For example, t could determine how likely the individual is to experience an accident of each possible degree of severity, depending on the value of s. For simplicity it is assumed that each agent has the same type space T. These are like *ex ante* states of the world, in that they are supposed to be determined before any market transactions have taken place.

Combining the elements of the last three paragraphs gives us three different sets of "contingent" economic agents. First there is A, which can be regarded as the set of possible agent labels. Second there is $A \times T$, which is the set of possible type-contingent agents, indexed by both a personal label a and an unobservable type t. Third there is the set $\Theta := A \times T \times S$, which is the set of possible state- and type-contingent agents, indexed by a personal label a, by an unobservable *ex ante* type t, and also by an observable *ex post* individual state s.

Actually, in order to resolve the continuum of independent random variables problem noticed by Gale (1979), Feldman and Gilles (1985) and Judd (1985), the set of agents is allowed to be random. Specifically, suppose that there is a joint probability measure $\nu \in \mathcal{M}(\Theta)$ over the set of all possible state- and type-contingent agents, and that (with probability one) ν is equal to the product measure $\alpha \times \mu$ for some well-defined distribution $\mu \in \Delta(T \times S)$ on the finite set $T \times S$ of possible pairs (t, s) consisting of agents' types and states. Moreover, suppose that there is a well-defined distribution $\lambda \in \Delta(T)$ on the finite set T of possible agents' types t, and that each $t \in T$ determines conditional probabilities $\pi(s|t)$ over the finite set S of possible agents' states s. Thus it is being assumed that $\mu(t,s) = \pi(s|t)\,\lambda(t)$ for all pairs (t,s) in the finite set $T \times S$.

It will now simply be assumed that the law of large numbers holds and induces perfect risk-pooling. So there can be no aggregate uncertainty about the distribution of types and states for the agents in any non-null measurable set $K \in \mathcal{A}$. In fact, with probability one, a fraction $\alpha(K)$ of all the agents in the economy have labels in K. Of these, a fraction $\lambda(t)$ are of unobservable type t, and a fraction $\mu(t,s)$ have the type-state pair (t,s). This, it should be noted, is a different formulation of a random economy from that due to Hildenbrand (1971) and to Bhattacharya and Majumdar (1973) because they had a fixed set of agents, independent of the state of the world. Here, by contrast, in any finite approximation to the continuum economy, the set of agents itself is random.

Now let there be a finite set G of physical commodities. Then each agent a of type t in (individual) state s has three different possible contingent net trade vectors to be considered. The first and most limited is the actual *ex post* (t,s)-contingent net trade vector $x_{ats} \in \Re^G$, which is simply a single vector of physical commodities. The second is the *ex ante* state-contingent net trade vector $x_{at} \in \Re^{GS}$, which applies when agent a is of type t. The third and most extensive is the complete contingent net trade vector $x_a \in \Re^{GTS}$, which is the type- and state-contingent net trade vector for agent a from the point of view of any outside observer, or other agent in the economy, who does not know agent a's true type.

It is assumed that each agent $a \in A$ has a type-independent set $X_a \subset \Re^{GS}$ of physically feasible state-contingent net trade vectors, whose typical member is x_{at}, for any $t \in T$. Moreover, make the standard assumption that the graph $\{\,(a,x) \in A \times \Re^{GS} \mid x \in X_a\,\}$ of the feasible set correspondence $X : A \twoheadrightarrow \Re^{GS}$ is measurable. Then there is also a Cartesian product set $X_a^T := \prod_{t \in T} X_{at} \subset \Re^{GTS}$ of physically feasible type- and state-contingent net trade vectors, where each X_{at} is a copy of X_a. A typical member of X_a^T is x_a. Note that endowments are not considered explicitly. Also, the assumption that the feasible net trade set X_a is independent of t avoids the complications which arise when feasible sets or endowments are unknown — see, for instance, Hammond (1989).

Suppose too that each agent $a \in A$ has, for each type $t \in T$, a continuous type-contingent utility function $U_{at}(x_{at})$, defined for all state-contingent net trade vectors x_{at} in the feasible net trade set X_a, and taking the expected utility form $U_{at}(x_{at}) \equiv \sum_{s \in S} \pi(s|t)\, u_{ats}(x_{ats})$ for some state- and type-dependent von Neumann-Morgenstern utility function $u_{ats}(\cdot)$. Suppose also that, for each fixed type $t \in T$ and each fixed state-contingent commodity vector $x^S \in \Re^{GS}$, $U_{at}(x^S)$ is measurable as a function of a wherever it is defined because $x^S \in X_a$.

To complete the description of the economy, it suffices to specify the resource balance constraint. In fact, after allowing aggregate free disposal, a physically feasible allocation in the economy is given by an $\mathcal{A}$-measurable and so α-integrable function $\mathbf{x} : A \mapsto \Re^{GTS}$ satisfying the weak vector inequality

$$\int_A \sum_{t \in T} \sum_{s \in S} \mu(t,s)\, x_{ats}\, \alpha(da) \leqq 0 \in \Re^G,$$

where $\alpha(da)$ indicates integration with respect to a, using the measure α on A. This expresses the restriction that the mean over all agents of each agent's expected net trade vector cannot have any positive component. Note that the law of large numbers ensures that, with probability one, any allocation satisfying this inequality is feasible *ex post*, after all the agents' states have become known, and no matter what their true types happen to be.

In order to permit inequalities such as the last to be written more succinctly, it is useful to introduce the notation $\mu \bullet z$ to indicate the double sum $\sum_{t \in T} \sum_{s \in S} \mu(t,s)\, z_{ts}$ whenever $z \in \Re^{GTS}$. If z is regarded as a function $z : T \times S \mapsto \Re^G$ and so a random variable with distribution corresponding to the probabilities $\mu(t,s)$ on $T \times S$, then $\mu \bullet z$ is just the expected value of z. With this notation, the above resource balance constraint reduces to

$$\int_A (\mu \bullet x_a)\, \alpha(da) \leqq 0.$$

The *R-S model* of Rothschild and Stiglitz (1976) is an example of such a continuum economy with adverse selection. Indeed, suppose that there is a single physical commodity. Suppose that $t \in T$ denotes the probability of having an accident, where T is some finite subset of the line interval $[0,1]$. Let S consist of the two states $s = 0$ and $s = 1$, where $s = 1$ signifies that an accident has occurred, and $s = 0$ signifies that it has not. Then let c_0 denote consumption if there is no accident, and let c_1 denote consumption if there is an accident. Suppose also that all individuals of all types have the same endowments e_0, e_1 of the only consumption good, depending on whether or not they suffer an accident. Then, provided that agent a's type is t, a's net trade vector is the pair $(x_{at0}, x_{at1}) := (c_{at0} - e_0, c_{at1} - e_1)$. Because the agent's type is private information, however, it is appropriate to consider the entire state contingent net trade vector $x_a = \langle\, (x_{at0}, x_{at1})\, \rangle_{t \in T}$. This is a point in the space $\Re^{2T}$ of complete insurance contracts contingent on not only whether there is an accident, but on now susceptible agent a is to having an accident. All agents are assumed to have the same type-contingent expected utility function

$$U_t(x_0, x_1) \equiv (1 - t)\, v_0(x_0 + e_0) + t\, v_1(x_1 + e_1).$$

This function is defined for all (x_0, x_1) satisfying the two constraints $x_0 \geq -e_0$ and $x_1 \geq -e_1$ which together ensure that consumption in each possible individual state is non-negative. Here $v_0(\cdot)$ and $v_1(\cdot)$ are the two state-contingent utility functions. Finally, the allocation $\mathbf{x} : A \mapsto \Re^{2T}$ satisfies the aggregate resource balance constraint in the economy if and only if

$$\int_A \sum\nolimits_{t \in T} \lambda(t)[(1 - t)\, x_{at0} + t\, x_{at1}]\, \alpha(da) \leq 0,$$

where $\lambda(t)$ denotes the proportion of agents in the economy whose probability of an accident is t.

3. Incentive Constraints and Externalities

Because each agent's type is private information, not every physically feasible allocation is truly feasible. True feasibility requires in addition that the physically feasible allocation $\mathbf{x}$ in the adverse selection economy should also satisfy the incentive constraints $U_{at}(x_{at}) \geq U_{at}(x_{at'})$ for all pairs $t, t' \in T$ and for almost all $a \in A$. These constraints express the requirement that, if agent a is really of type t, then agent a cannot gain by acting deceptively in the economic system in a way which obtains the net trade vector of an agent whose true type is t'. In the special case of the R-S model described above, the incentive constraints take the form

$$(1 - t)\ v_0(x_{at0} + e_0) + t\ v_1(x_{at1} + e_1) \geq (1 - t)\ v_0(x_{at'0} + e_0) + t\ v_1(x_{at'1} + e_1)$$

for all pairs t, t' of types in T.

In future, F_a will be used to denote the set of type- and state-contingent net trade vectors for agent a which are physically feasible and also satisfy the incentive constraints above — i.e.,

$$F_a := \{\, x_a \in X_a^T \mid \forall t, t' \in T : U_{at}(x_{at}) \geq U_{at}(x_{at'}) \,\}.$$

In reality, then, F_a denotes the set of feasible net trades for agent a, and so for the whole set $\{a\} \times T$ of different type-contingent versions of agent a. Of course, only one agent in the set $\{a\} \times T$ actually participates in the economy. Nevertheless, other agents, insurance companies, etc., are unable to distinguish between the different type-contingent versions of agent a, and so all must be included as potential participants in the economy.

In addition, all type-contingent agents in the set $\{a\} \times T$ must effectively be consuming the same type- and state-contingent net trade vector $x_a \in F_a$, since that is all that other agents, insurance companies, etc., can observe. In this respect there is a form of externality because the incentive constraints can only be satisfied if the net trade vector x_{at} of any one type-contingent agent (a, t) $(t \in T)$ puts an upper bound on the net trade vectors of all the other type-contingent agents (a, t') $(t' \in T \setminus \{t\})$. So the economy is very like one with local public goods being provided at a continuum of localities indexed by $a \in A$, with a set $\{a\} \times T$ of individuals living in each locality a. Here, however, individuals have no opportunity of changing their label $a \in A$, whereas in most models of local public goods since the work of Tiebout (1956) — see also Bewley (1981) and the literature cited there — individuals do have the option of changing their locality.

The extra incentive constraints typically create nonconvexities in the feasible sets F_a, as Prescott and Townsend (1984a), amongst others, have pointed out. Indeed, such nonconvexities are illustrated in Figure 1, for the case of one physical commodity, two observable individual states of the world, and two unobservable types t_1 and t_2. The two axes indicate net trade quantities in each of the two different observable states s_1 and s_2. The point X_1 represents the state contingent net trade vector of a typical agent a when of type t_1; the two points X_2 and X_2' represent two different state contingent net trade vectors of a type t_2 agent. The indifference curves I_1 and I_2 of these two types of agent through the respective net trade allocations X_1 and X_2 are drawn in. The two type- and state-contingent allocations $x_a = (X_1, X_2)$ and $x_a' = (X_1, X_2')$ in the space $\Re^4$ of such allocations are both clearly incentive compatible when there are only these two possible types. Yet the convex combination $(X_1, \frac{1}{2}X_2 + \frac{1}{2}X_2')$ is incentive incompatible, because an agent of type t_1 could then gain by claiming to be a type t_2 agent. Thus neither the set F_a of incentive compatible

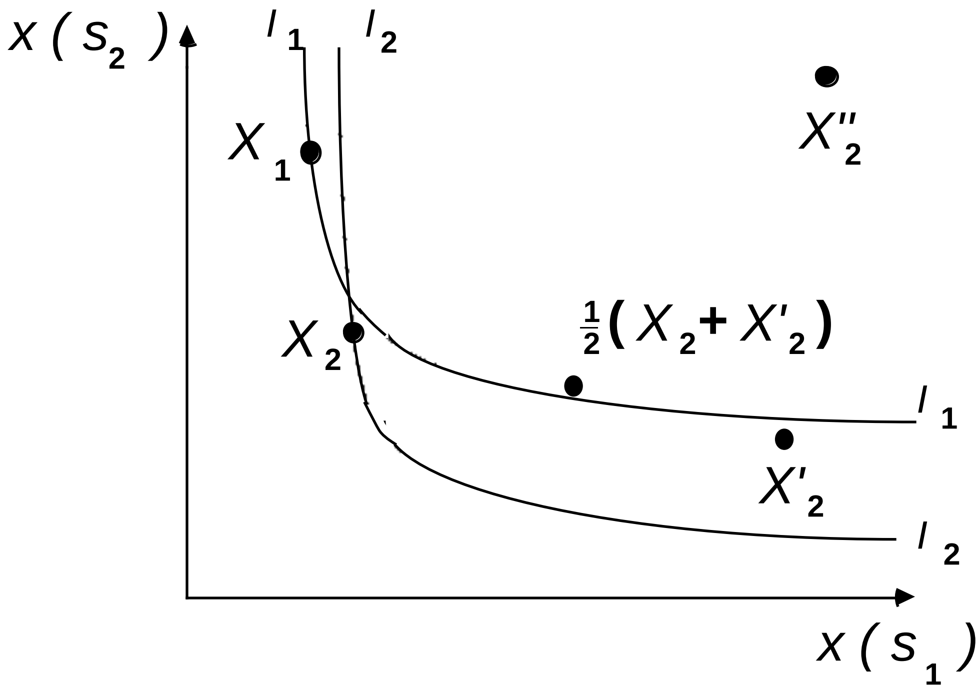

FIGURE 1

allocations, nor the subset $F_a(x_a)$ of such allocations which make neither type worse off than at x_a, is convex. The same diagram also shows how neither set may satisfy the usual free disposal assumption of general equilibrium theory; the type- and state-contingent allocation $x''_a = (X_1, X''_2) \in \Re^4$ has $X''_2 >> X_2$ and yet is not incentive compatible. Indeed, there obviously exists an $X'_1 >> X_1$ near enough to X_1 so that the type- and state-contingent allocation (X'_1, X''_2) has $(X'_1, X''_2) >> (X_1, X_2)$ in $\Re^4$ and yet (X'_1, X''_2) is not incentive compatible.

This failure of the free disposal assumption in the set F_a of incentive compatible contracts means that we should carefully reconsider whether agents' preferences satisfy local non-satiation. In fact, they need not. To see why, consider a specific example in which the economy has two physical commodities, one individual state (i.e., no individual uncertainty at all) and two types, $T = \{1, 2\}$. Suppose also that the two possible types of an individual have piecewise linear, concave, and strictly increasing utility functions on $\Re^2$ which are given by

$$U_1(x) \equiv \min \{\, p\,(x - a),\; 2q\,(x - b)\,\}$$

$$U_2(x) \equiv \min \{\, 2p\,(x - a),\; q\,(x - b)\,\}$$

where $a = (1, -1)$, $b = (-1, 1)$, $p = (1, 2)$, and $q = (2, 1)$. Then I claim that the allocation $\hat{x} \in \Re^4$ with $\hat{x}_1 = a$ and $\hat{x}_2 = b$ is a point of local satiation in the relevant incentive-constrained feasible set. For $p\,(b - a) = q\,(a - b) = 2 > 0$ and so, for all (x_1, x_2) in the neighbourhood of $\hat{x}$, the utility functions take the approximate linear forms

$$U_1(x_1) \equiv p\,(x_1 - a); \quad U_1(x_2) \equiv 2q\,(x_2 - b);$$

$$U_2(x_1) \equiv 2p\,(x_1 - a); \quad U_2(x_2) \equiv q\,(x_2 - b).$$

Therefore the two incentive constraints $U_1(x_1) \geq U_1(x_2)$ and $U_2(x_2) \geq U_2(x_1)$ become

$$p\,(x_1 - a) \geq 2q\,(x_2 - b); \quad q\,(x_2 - b) \geq 2p\,(x_1 - a)$$

in such a neighbourhood. These inequalities clearly imply that

$$p\,(x_1 - a) \geq 2q\,(x_2 - b) \geq 4p\,(x_1 - a) \geq 8q\,(x_2 - b)$$

which can only be satisfied, of course, if both $p\,(x_1 - a) \leq 0$ and $q\,(x_2 - b) \leq 0$. Therefore the incentive constraints imply that $U_1(x_1) \leq U_1(a)$ and $U_2(x_2) \leq U_2(b)$ for all allocations (x_1, x_2) in the neighbourhood of $\hat{x} = (\hat{x}_1, \hat{x}_2) = (a, b)$ — i.e., they imply local satiation.

Globally, however, any type-contingent net trade vector $\tilde{x} \in \Re^4$ with $\tilde{x}_1 = \tilde{x}_2 \geq (1, 1)$ is both Pareto superior and incentive compatible. So the satiation is local and not global. And in a continuum economy, any coalition with a few extra resources can use them to move a small proportion of its members up to $\tilde{x}$, while leaving all the others at $\hat{x}$. Accordingly, even though individuals may be locally satiated, coalitions are usually not, as will be shown in the next section.

Part of Prescott and Townsend's (1984a, 1984b) distinctive contribution was to show how lotteries are useful in overcoming the individual non-convexities which incentive constraints create. Of course lotteries are indeed likely to be part of any ex-ante Pareto efficient allocation. To avoid working with infinite dimensional spaces of measures over the commodity space $\Re^{GTS}$, however, I shall consider here only "pure" rather than "mixed" or randomized allocations. Apart from the need to allow for such individual non-convexities, this section has shown how incentive constraints typically mean that, for individuals, we cannot just assume free disposal as usual, or even the much weaker standard condition of local non-satiation.

4. Coalitionally Monotone Preferences

Nevertheless, in this continuum economy, make the standard assumption that the preferences of each type-contingent agent in $A \times T$ are weakly monotone in the space $\Re^{GS}$ of state-contingent net trade vectors — i.e., that if $x_{at} \in X_a$ and if z is any strictly positive vector of $\Re^{GS}_{++}$, then $x_{at} + z \in X_a$ and also $U_{at}(x_{at} + z) > U_{at}(x_{at})$. Then it can be shown that, for any coalition $K \subset A$ of agents with $\alpha(K) > 0$, their aggregate preferences in the space $\Re^{GTS}$ are weakly monotone in the following sense. Given any measurable function $\mathbf{x} : K \mapsto \Re^{GTS}$ satisfying $x_a \in F_a$ (a.e. in K) which defines an incentive compatible type- and state-contingent allocation to the members of K, and given any strictly positive vector $y \in \Re^{G}_{++}$, no matter how small, there exists an alternative Pareto superior allocation $\mathbf{x}' : K \mapsto \Re^{GTS}$ to the members of K, with: (i) $x'_a \in F_a$; (ii) $U_{at}(x'_{at}) \geq U_{at}(x_{at})$ (for all $t \in T$, for a.e. $a \in K$); (iii) for some subset $K' \subset K$ of positive measure, $U_{at}(x'_{at}) > U_{at}(x_{at})$ (for a.e. $a \in K'$ and all $t \in T$); and (iv) $\int_K [\mu \bullet (x'_{ats} - x_{ats})] \, \alpha(da) \leqq y$. Thus, if any coalition K is allowed to have a little bit more of every physical commodity on aggregate, then it can generate a strict Pareto improvement for almost all its own members.

To show how this follows from weak monotonicity in the space $\Re^{GS}$, consider any combination of: (i) a coalition $K \subset A$ of agents with $\alpha(K) > 0$; (ii) any measurable function $\mathbf{x} : K \mapsto \Re^{GTS}$ satisfying $x_a \in F_a$ (a.e. $a \in K$); (iii) any strictly positive vector $y \in \Re^{G}_{++}$. Then take two small positive numbers ϵ and δ, and construct both a measurable subset $K_\epsilon \subset K$ for which $\alpha(K_\epsilon) < \epsilon \, \alpha(K)$, as well as a new allocation $\hat{\mathbf{x}} : K \mapsto \Re^{GTS}$, by taking

$$\hat{x}_{at} := \begin{cases} \arg\max_{t' \in T} \, \{ \, U_{at}(x_{at'} + \delta y) \, \} & \text{if } a \in K_\epsilon; \\ x_{at} & \text{otherwise.} \end{cases}$$

This construction obviously gives rise to a new allocation satisfying the incentive constraints which are embodied in the definition of each set F_a. Moreover, the new allocation to the members of K is feasible, and makes all the members of the coalition K_ϵ better off without changing the net trade vector of any member of $K \setminus K_\epsilon$. Finally, because both S and T are finite sets, the new allocation also satisfies the feasibility constraint

$$\int_K [\mu \bullet (\hat{x}_{ats} - x_{ats})]\, \alpha(da) \leqq y$$

provided that both ϵ and δ are small enough positive numbers. This confirms coalitional local monotonicity.

5. Equilibrium with Internalized Externalities

The usual way of achieving Pareto efficient equilibria with externalities and public goods is to price those externalities (with "Pigovian taxes") and those public goods (with Lindahl prices), as well as the usual physical commodities. In our economy with adverse selection, this approach would require each type-contingent agent $(a, t) \in A \times T$ to pay for the net trade vectors of all the type-contingent agents in the set $\{a\} \times T$. Usually this will not work, however, because of the non-convexities which arise in the incentive-constrained feasible sets F_a for each agent $a \in A$. Accordingly, a more restrictive approach must be adopted, resembling Foley's (1967) "political-economic equilibrium" for an economy with public goods. Instead of having Lindahl prices to clear markets for public goods, Foley's definition restricts changes in the vector of public goods and in the lump-sum taxes used to finance their production to those which are agreed unanimously because they appear to generate Pareto improvements when each individual regards private goods prices as fixed. In similar fashion, given any specific contingent reference allocation $\hat{x}_a \in \Re^{GTS}$ to the different possible types of agent a, it will be assumed that agent a is then restricted to the set

$$F_a(\hat{x}_a) := \{\, x_a \in F_a \mid \forall t \in T : U_{at}(x_{at}) \geq U_{at}(\hat{x}_{at}) \,\}$$

of incentive compatible contingent allocations $x_a \in \Re^{GTS}$ which are Pareto non-inferior for all the type-contingent agents $(a, t) \in A \times T$. Thus the earlier incentive constraints on type-contingent allocations have now been supplemented by what amount to "efficiency constraints".

Now all the ingredients are available for the modified Arrow-Debreu economy which I shall construct. The commodity space is $\Re^{GTS}$, the set of all type- and individual state-contingent net trade vectors in the set G of physical commodities. There is a continuum of (random) type-contingent agents in the set $A \times T$. A given incentive-compatible and physically feasible reference allocation $\hat{\mathbf{x}} : A \mapsto \Re^{GTS}$ will always be postulated — it is assumed that all the different possible types of agent a share the same reference allocation $\hat{x}_a \in F_a$. Then the different type-contingent agents in $\{a\} \times T$ all have the same set $F_a(\hat{x}_a)$ of feasible net trades in $X_a \subset \Re^{GS}$, but different preferences represented by a continuous and weakly increasing utility function $U_{at} : X_a \mapsto \Re$.

In general, price vectors would be arbitrary non-negative vectors in $\Re^{GTS}_+$, although possibly normalized because only relative prices matter. In the particular insurance economy being considered here, however, the continuum of agents ensures perfect risk pooling, so it suffices to have a single price vector $p \in \Re^G_+$, with type- and state-contingent prices being derived by multiplying this price vector by the appropriate probabilities $\mu(t, s)$. Thus the value of an agent a's typical type-contingent net trade vector $x_a \in \Re^{GTS}$ is given by $p\,(\mu \bullet x_a)$, which is just the expected value at prices p of agent a's physical net trade vector.

Various concepts of Walrasian equilibrium can now be defined. The definitions are slightly different from the standard ones. This is because of the incentive-constraints, because of the externalities which those constraints cause, and also because of the efficiency constraints which are being used to internalize these externalities. In addition, the following definitions will allow for lump-sum redistribution of income, as required in the usual second efficiency theorem of welfare economics. The usual definition without lump-sum redistribution reduces to a special case.

A *compensated (Walrasian) equilibrium (with transfers)* is an allocation $\hat{\mathbf{x}} : A \mapsto \Re^{GST}$, together with a price vector $p \in \Re^G_+$, such that:

(i) for a.e. $a \in A$, both $\hat{x}_a \in F_a$ and also $x_a \in F_a(\hat{x}_a) \Longrightarrow p\,(\mu \bullet x_a) \geq p\,(\mu \bullet \hat{x}_a)$;

(ii) $\int_A (\mu \bullet \hat{x}_a)\, \alpha(da) \leqq 0,\ p \geqq 0$ (comp).

Part (i) of the above definition has (almost) every agent a minimize expenditure at the appropriate price system, subject to the incentive constraints and the constraint that no type of agent a can be made worse off than with the allocation $\hat{\mathbf{x}}$; part (ii) is the usual market clearing condition, including the rule of free goods — i.e., if any good is in excess supply, its price must be zero. The above definition also presumes that lump-sum redistribution is allowed; if it is not, then the additional requirement that $p\,(\mu \bullet \hat{x}_a) = 0$ (a.e. $a \in A$) would have to be imposed.

Compensated equilibria are useful because it is easier to prove results such as price characterizations of efficient or core allocations using such equilibria instead of the more customary uncompensated equilibria which will be defined next. Indeed, an *uncompensated (Walrasian) equilibrium (with transfers)* is an allocation $\hat{\mathbf{x}} : A \mapsto \Re^{GST}$, together with a price vector $p \in \Re^G_+$, such that the market clearing and rule of free goods condition (ii) above is satisfied, as well as the following strengthened version of (i):

(i*) for almost every $a \in A$, both $\hat{x}_a \in F_a$ and also, for any other $x_a \in F_a(\hat{x}_a)$, if there exists some $t^* \in T$ for which $U_{at^*}(x_{at^*}) > U_{at^*}(\hat{x}_{at^*})$, then it must be true that $p\,(\mu \bullet x_a) > p\,(\mu \bullet \hat{x}_a)$.

This is now a form of preference maximization, using the incomplete Pareto preference relation for the entire set of type-contingent agents $\{a\} \times T$. It requires that any Pareto improvement to the allocation $\hat{\mathbf{x}}$ for this set of agents must be too expensive, given the appropriate price system. If there is no lump-sum redistribution, then the additional restriction that $p\,(\mu \bullet \hat{x}_a) = 0$ (a.e. $a \in A$) must be met, just as for compensated equilibrium.

In standard general equilibrium theory, any uncompensated equilibrium will also be a compensated equilibrium because consumers have locally non-satiated preferences everywhere in their feasible sets. Unfortunately, incentive constraints make this standard implication false by allowing points of local satiation in agents' feasible sets. This will complicate the statement of the first efficiency theorem in the next section. It also means that not even compensated equilibrium can be shown to exist, in general. Instead, the following concept is used in the existence proof presented in Section 8 below.

A *weak (Walrasian) equilibrium (with transfers)* is an allocation $\hat{\mathbf{x}} : A \mapsto \Re^{GST}$, together with a price vector $p \in \Re^G_+$, such that the market clearing and rule of free goods condition (ii) above is satisfied, as well as the following weakened version of (i):

(i$'$) for almost every $a \in A$, both $\hat{x}_a \in F_a$ and also, for any other $x_a \in F_a(\hat{x}_a)$, if there exists some $t^* \in T$ for which $U_{at^*}(x_{at^*}) > U_{at^*}(\hat{x}_{at^*})$, then it must be true that $p\,(\mu \bullet x_a) \geq p\,(\mu \bullet \hat{x}_a)$.

This is a weakening of the corresponding (i) and (i*) in the definitions of both compensated and uncompensated equilibrium. The hypothesis of (i$'$) is the same as that of (i*), but stronger than that of (i), because at least one weak inequality has become strict. The conclusion of (i′) is the same as that of (i), but weaker than that of (i*), because the inequality has become weak instead of strict. If preferences for any agent happen to be locally non-satiated everywhere in the budget set

$$\{ x_a \in F_a \mid p\,(\mu \bullet x_a) \leq p\,(\mu \bullet x_a) \},$$

then a weak equilibrium must be a compensated equilibrium for that agent, but need not be an uncompensated equilibrium. If there is no lump-sum redistribution, then the additional restriction that $p\,(\mu \bullet \hat{x}_a) = 0$ (a.e. $a \in A$) must be met, just as it must be for compensated and for uncompensated equilibrium without transfers.

6. Efficiency Theorems

An allocation $\hat{\mathbf{x}} : A \mapsto \Re^{GTS}$ is said to be *Pareto efficient* if there is no other physically feasible allocation $\mathbf{x} : A \mapsto \Re^{GTS}$ satisfying both the incentive and efficiency constraints which makes a non-null set of type-contingent agents $(a, t) \in A \times T$ strictly better off. In other words, there must be no alternative allocation $\mathbf{x} : A \mapsto \Re^{GTS}$ such that:

(i) $x_a \in F_a(\hat{x}_a)$ (a.e. $a \in A$);

(ii) $\int_A (\mu \bullet x_a)\, \alpha(da) \leqq 0$;

(iii) there is a measurable subset $K \subset A \times T$, whose measure $[\alpha \times \lambda](K)$ is positive, such that $U_{at}(x_{at}) > U_{at}(\hat{x}_{at})$ for all $(a, t) \in K$.

Any allocation $\mathbf{x}$ satisfying these three conditions is said to be *Pareto superior* or a *Pareto improvement* to $\hat{\mathbf{x}}$.

Note that these are the suitable definitions of Pareto efficiency and of Pareto improvements, bearing in mind the incentive constraints which arise because of private information. Also, all potential type-consistent agents should be considered separately, in an economy where each agent already knows his type when transactions are being arranged. An alternative concept of *ex ante* Pareto efficiency, taking into account uncertainty about each agent's type, would be appropriate if agents were still uncertain about their own types.

Because individual type-contingent agents may have locally satiated preferences, even this concept of Pareto efficiency is rather too strong to be satisfied by all (uncompensated)

Walrasian equilibria, with or without lump-sum redistribution. Instead, given any measurable set of agents $K \subset A$, the allocation $\hat{\mathbf{x}} : A \mapsto \Re^{GTS}$ is said to be *weakly K-Pareto efficient* if there is no Pareto superior allocation $\mathbf{x} : A \mapsto \Re^{GTS}$ such that, a.e. $a \in K$, there exists some type $t \in T$ for which $U_{at}(x_{at}) > U_{at}(\hat{x}_{at})$. In other words, no Pareto improvement can possibly benefit (almost) every agent of the set K. When K is empty, this is the usual definition of Pareto efficiency. When $K = A$, this is a standard definition of weak Pareto efficiency, requiring the absence of any alternative feasible allocation which makes (almost) every agent better off simultaneously.

First Efficiency Theorem. *Suppose that the allocation $\hat{\mathbf{x}} : A \mapsto \Re^{GST}$ and the price vector $p \in \Re^G_+$ together constitute an uncompensated equilibrium. Then the allocation $\hat{\mathbf{x}}$ must be weakly K-Pareto efficient, where K denotes the set of agents who are not in compensated equilibrium (even though almost all of them are in uncompensated equilibrium) — i.e.,*

$$K := \{ a \in A \mid \exists x_a \in F_a(\hat{x}_a) : p\,(\mu \bullet x_a) < p\,(\mu \bullet \hat{x}_a) \}.$$

Proof: Suppose that the allocation $\hat{\mathbf{x}} : A \mapsto \Re^{GST}$ is not weakly K-Pareto efficient. Then there must be an alternative allocation $\mathbf{x} : A \mapsto \Re^{GTS}$, and also some measurable set K' for which both $\alpha(K') > 0$ and $K \subset K' \subset A$, such that $x_a \in F_a(\hat{x}_a)$ (a.e. $a \in A$) and also, a.e. $a \in K'$, there exists some type $t \in T$ for which $U_{at}(x_{at}) > U_{at}(\hat{x}_{at})$. By definition of uncompensated equilibrium, it follows that $p\,(\mu \bullet x_a) > p\,(\mu \bullet \hat{x}_a)$ (a.e. $a \in K'$). But by definition of the set K, almost all agents a outside it, and so also outside K', are in compensated equilibrium at $\hat{x}_a$. Therefore $p\,(\mu \bullet x_a) \geq p\,(\mu \bullet \hat{x}_a)$ (a.e. $a \in A \setminus K'$). Then it follows that

$$\begin{aligned} 0 &< \int_{K'} p\,[(\mu \bullet x_a) - (\mu \bullet \hat{x}_a)]\,\alpha(da) + \int_{A\setminus K'} p\,[(\mu \bullet x_a) - (\mu \bullet \hat{x}_a)]\,\alpha(da) \\ &= \int_A p\,[(\mu \bullet x_a) - (\mu \bullet \hat{x}_a)]\,\alpha(da). \end{aligned}$$

Yet $\int_A p\,(\mu \bullet \hat{x}_a)\,\alpha(da) = 0$, as an implication of the rule of free goods in the definition of uncompensated equilibrium, and so $\int_A p\,(\mu \bullet x_a)\,\alpha(da) > 0$. This, however, contradicts the pair of vector inequalities $\int_A (\mu \bullet x_a)\,\alpha(da) \leqq 0$ and $p \geqq 0$. So the allocation $\hat{\mathbf{x}}$ must be weakly K-Pareto efficient after all. ▮

Second Efficiency Theorem. *If the condition of coalitionally monotone preferences is satisfied, then any Pareto efficient allocation is a compensated equilibrium at some suitable price vector.*

PROOF: Let $\hat{\mathbf{x}} : A \mapsto \Re^{GTS}$ be any Pareto efficient allocation. For each $a \in A$, define the set

$$\phi(a) := \{ \bar{x}_a \in \Re^G \mid \exists x_a \in F_a(\hat{x}_a) : \bar{x}_a = \mu \bullet x_a \}$$

of net demand vectors which are equal to the expected value of some type- and state-contingent allocation to agent a that is both incentive compatible and Pareto non-inferior to $\hat{x}_a$. Because of our earlier assumptions, $\phi : A \mapsto\!\!\!\!\rightarrow \Re^G$ is a measurable correspondence whose values are non-empty. Because A is a continuum, it follows from Hildenbrand (1974, p. 62, Theorem 1) that $\int_A \phi(a)\alpha(da)$ is a non-empty convex set. This set cannot intersect $\Re^G_{--}$, otherwise there would be an allocation $\mathbf{x} : A \mapsto \Re^{GTS}$ for which $x_a \in \phi(a)$ (a.e. $a \in A$) and also $\int_A (\mu \bullet x_a)\, \alpha(da) << 0$. Then, because individual preferences are transitive and those of the grand coalition A are coalitionally monotone, $\hat{\mathbf{x}}$ could not be Pareto efficient. This shows that $\int_A \phi(a)\, \alpha(da)$ and $\Re^G_{--}$ are non-empty disjoint convex subsets of $\Re^G$, and so they can be separated by a hyperplane $p\, x = 0$ through the origin of $\Re^G$. So there exists a non-zero price vector $p \in \Re^G$ for which: (i) $p\, x \geq 0$ whenever $x \in \int_A \phi(a)\, \alpha(da)$; and (ii) $p\, x \leq 0$ whenever $x << 0$. From (ii) it follows that p must be semi-positive.

Now let $\hat{x}$ denote $\int_A (\mu \bullet \hat{x}_a)\, \alpha(da)$. Then $\hat{x} \leqq 0$ because of feasibility, and also $p \geqq 0$, so $p\, \hat{x} \leq 0$. Yet also $p\, \hat{x} \geq 0$ because $\hat{x}_a \in F_a(\hat{x}_a)$ and so $\mu \bullet \hat{x}_a \in \phi(a)$ (a.e. $a \in A$). Therefore $p\, \hat{x} = 0$. This confirms the rule of free goods. Finally, again because $\hat{x}_a \in F_a(\hat{x}_a)$ (a.e. $a \in A$), it also follows that, for every measurable subset $K \subset A$, whenever $x_a \in F_a(\hat{x}_a)$ (a.e. $a \in K$), then $\mu \bullet x_a \in \phi(a)$ (a.e. $a \in K$). Therefore

$$\int_K p\, (\mu \bullet x_a)\, \alpha(da) + \int_{A \setminus K} p\, (\mu \bullet \hat{x}_a)\, \alpha(da) \geq 0 = p\, \hat{x} = \int_A p\, (\mu \bullet \hat{x}_a)\, \alpha(da)$$

from which it follows that

$$\int_K p\, [\mu \bullet (x_a - \hat{x}_a)]\, \alpha(da) \geq 0.$$

Since this is true for every measurable subset $K \subset A$, it follows that

$$x_a \in F_a(\hat{x}_a) \Longrightarrow p\, [\mu \bullet (x_a - \hat{x}_a)] \geq 0$$

for almost all $a \in A$, as required for compensated equilibrium (with transfers). ∎

7. A Core Equivalence Theorem

A *coalition* is a measurable set $K \in \mathcal{A}$ whose measure $\alpha(K)$ is positive. The allocation $\hat{\mathbf{x}} : A \mapsto \Re^{GTS}$ is said to be *blocked* by such a coalition K if there exists an alternative allocation $\mathbf{x} : K \mapsto \Re^{GTS}$ to its members that satisfies: (i) $x_a \in F_a(\hat{x}_a)$ (a.e. $a \in K$); (ii) $U_{at}(x_{at}) > U_{at}(\hat{x}_{at})$ for all (a, t) in a subset of $K \times T$ which has positive measure; (iii) $\int_K (\mu \bullet x_a)\, \alpha(da) \leqq 0$. Thus an allocation is blocked by a coalition K provided that K can generate a Pareto improvement for all its own type-contingent members in the set $K \times T$ by using its own resources, while also satisfying the incentive constraints. An allocation $\hat{\mathbf{x}} : A \mapsto \Re^{GTS}$ is in the *core* if and only if it is not blocked by any coalition.

Core Equivalence Theorem. *If the condition of coalitionally monotone preferences is satisfied, then any allocation in the core is a compensated equilibrium without transfers at some suitable price vector.*

Proof: Let $\hat{\mathbf{x}} : A \mapsto \Re^{GTS}$ be any allocation in the core. For each $a \in A$, define the set $\phi(a)$ as in the proof of the second efficiency theorem above, and then let $\psi(a) := \phi(a) \cup \{0\}$. Once again, like ϕ, the correspondence $\psi : A \twoheadrightarrow \Re^G$ is measurable and has values which are non-empty sets, and so $\int_A \psi(a)\, \alpha(da)$ is a non-empty convex set. This set cannot intersect $\Re^G_{--}$, otherwise there would be a measurable set $K \subset A$ and an allocation $\mathbf{x} : K \mapsto \Re^{GTS}$ to the members of K for which $x_a \in \phi(a)$ (a.e. $a \in K$) and also $\int_A (\mu \bullet x_a)\, \alpha(da) << 0$. Then, because individual preferences are transitive and those of K are coalitionally monotone, $\hat{\mathbf{x}}$ could not be in the core because K could block it.

Therefore, as in the proof of the second efficiency theorem, $\int_A \psi(a)\, \alpha(da)$ and $\Re^G_{--}$ are non-empty disjoint convex subsets of $\Re^G$, so there exists a semi-positive price vector $p \in \Re^G$ for which $p\, x \geq 0$ whenever $x \in \int_A \psi(a)\, \alpha(da)$. Moreover it must be true, as before, that the rule of free goods $\int_A p\, (\mu \bullet \hat{x}_a)\, \alpha(da) = 0$ is satisfied.

As before, because $\hat{x}_a \in F_a(\hat{x}_a)$ (a.e. $a \in A$), it follows that, for every measurable subset $K \subset A$, whenever $x_a \in F_a(\hat{x}_a) \cup \{0\}$ (a.e. $a \in K$), then $\mu \bullet x_a \in \psi(a)$ (a.e. $a \in K$), and so

$$\int_K p\, (\mu \bullet x_a)\, \alpha(da) + \int_{A \setminus K} p\, (\mu \bullet \hat{x}_a)\, \alpha(da) \geq 0 = \int_A p\, (\mu \bullet \hat{x}_a)\, \alpha(da).$$

In particular, for every measurable subset $K \subset A$, it must then be true that

$$\int_K p\, [\mu \bullet (x_a - \hat{x}_a)]\, \alpha(da) \geq 0.$$

From this it follows that

$$x_a \in F_a(\hat{x}_a) \Longrightarrow p\,[\mu \bullet (x_a - \hat{x}_a)] \geq 0$$

for almost all $a \in A$, as required for compensated equilibrium (with transfers). But now, in addition, taking $x_a = 0$ (a.e. $a \in K$) implies that

$$\int_{A \setminus K} p\,(\mu \bullet \hat{x}_a)\,\alpha(da) \geq 0 = \int_A p\,(\mu \bullet \hat{x}_a)\,\alpha(da)$$

or that $\int_K p\,(\mu \bullet \hat{x}_a) \leq 0$ for every measurable subset $K \subset A$. Therefore $p\,(\mu \bullet \hat{x}_a) \leq 0$ for almost all $a \in A$. But then the rule of free goods $\int_A p\,(\mu \bullet \hat{x}_a)\,\alpha(da) = 0$ implies that $p\,(\mu \bullet \hat{x}_a) = 0$ for almost all $a \in A$, thus showing that the core allocation $\hat{\mathbf{x}}$ is actually a compensated equilibrium without transfers. ▮

8. Existence and Multiplicity of Weak Walrasian Equilibria

There is one special case in which both the existence of Walrasian equilibrium is easy to explain, and the set of all such equilibria is easy to describe. This occurs when there is only one physical commodity, in which case the only price vector to consider is just the single real number 1. In this special case it is easy to see that the set of Walrasian equilibria consists precisely of those incentive compatible allocations $\hat{\mathbf{x}} : A \mapsto \Re^{TS}$ with the property that almost every agent $a \in A$ has a type- and state-contingent net trade vector $\hat{x}_a$ which is Pareto efficient among the set $\{\, x_a \in F_a \mid \mu \bullet x_a = 0 \,\}$ of incentive compatible and actuarially fair feasible allocations to all type-contingent agents in the set $\{a\} \times T$.

In order to prove existence of weak Walrasian equilibrium in general, it is necessary to make one further assumption, to be used in this section only. It is that almost every agent $a \in A$ has the physically feasible net trade set X_a bounded below by a vector $\underline{x}_a \in \Re^{GTS}$ — i.e., it must be true that $x_a \in X_a$ implies $x_a \geqq \underline{x}_a$. Moreover, it must also be true that the mean lower bound $\int_A \underline{x}_a \alpha(da)$ exists and is finite.

With this extra assumption, it is then not too difficult to prove that weak equilibrium exists, and that there is actually a great multiplicity of such equilibria. Indeed, consider any measurable function $\beta : A \mapsto$ interior Δ^T, determining positive welfare weights β_{at} for each type-contingent agent $(a,t) \in A \times T$, which sum to one for each agent $a \in A$. Then it will actually be shown that, in the continuum economy, there exists a weak equilibrium consisting of a physically feasible and incentive compatible allocation $\hat{\mathbf{x}} : A \mapsto \Re^{GTS}$ together with a price vector $p \in \Re^G_+$, having the following special property: for almost every agent $a \in A$, any alternative type- and state-contingent net trade vector $x_a \in F_a(\hat{x}_a)$ which satisfies $\sum_{t \in T} \beta_{at} \left[U_{at}(x_{at}) - U_{at}(\hat{x}_{at}) \right] > 0$ must also satisfy $p\,(\mu \bullet x_a) \geq p\,(\mu \bullet \hat{x}_a) = 0$. Thus, if the first inequality in the hypothesis were weak, there would be a compensated equilibrium in the economy where the feasible set of net trades for each agent $a \in A$ is F_a, and where each agent's preferences are represented by the welfare weighted utility function $W_{\beta_a}(x_a) \equiv \sum_{t \in T} \beta_{at}\, U_{at}(x_{at})$. And if the second inequality in the implication were strong, there would be an uncompensated equilibrium in the same economy. Because of possible local non-satiation, however, the inequality in the hypothesis has to be strict, thus giving a weak equilibrium in this economy. This is then also a weak equilibrium in the sense of Section 5, since for almost every agent $a \in A$ it must be true that whenever $x_a \in F_a(\hat{x}_a)$ with $U_{at}(x_{at}) > U_{at}(\hat{x}_{at})$ for at least one $t \in T$, then $W_\beta(x_a) > W_\beta(\hat{x}_a)$, and so $p\,(\mu \bullet x_a) \geq p\,(\mu \bullet \hat{x}_a)$. In general, of course, provided that at least one such weak equilibrium exists for each different welfare weight function β_{at}, varying this function will give rise to different weak equilibrium allocations, so there is indeed a great multiplicity of such equilibria.

In fact the existence of such weak equilibria is easy to show. One considers the "weak compensated demand correspondence" defined by

$$\xi^C_a(p; \beta_a) := \{\, \hat{x}_a \in F_a \;\mid p\,(\mu \bullet \hat{x}_a) = 0 \\ \& \; W_{\beta_a}(x_a) > W_{\beta_a}(\hat{x}_a) \Longrightarrow p\,(\mu \bullet x_a) \geq p\,(\mu \bullet \hat{x}_a) \,\}.$$

It is then straightforward to adapt the arguments which Khan and Yamazaki (1981) used to prove their existence result (Proposition 2, part (a)). Their arguments show that, for each fixed set of welfare weights $\beta_a \in \Delta^T$, the correspondence ξ^C_a is upper hemi-continuous as prices vary, and then establish that there is a convergent subsequence of fixed points whose limit is a price vector giving rise to a weak equilibrium. Their arguments remain valid even when there may be local satiation.

9. Perfected Option Markets

It has been seen how adverse selection, and the incentive constraints to which it gives rise, create a form of externality. For general continuum economies, Prescott and Townsend (1984a, b) sought to internalize such externalities by restricting agents to contingent commodity allocations which satisfy the incentive constraints. When agents can commit themselves to contingent contracts before knowing their own true type, or when their type is something which they themselves choose (as in a moral hazard problem), this idea works admirably. If agents trade when they are still symmetrically informed because none of them yet knows their own type, then the incentive constraints really are self-enforcing to the extent that agents see the need to have incentive compatible contracts. Similarly if there is moral hazard.

When agents already know their own type before they begin to trade, however, there is no reason why they should choose to satisfy the incentive constraints. Nor does the Prescott and Townsend approach succeed in this case. Unfortunately this seems to be the usual adverse selection problem in practice. Yet there is an alternative form of market decentralization, based on option contracts rather similar to those which Prescott and Townsend considered for the cases where they did succeed in making the incentive constraints self-enforcing.

To introduce this new approach, notice that so far we have been considering a fictitious economy in which there are complete markets for Arrow-Debreu contingent commodity contracts, and in which, for some reason, individuals' own exogenous feasibility constraints force them to satisfy the incentive constraints. On moving from the above fictitious economy to the original economy in which the incentive constraints cannot simply be imposed exogenously, special option markets will be devised which produce equivalent outcomes. The idea here is that an incentive compatible type-contingent contract $x_a = \langle x_{at} \rangle_{t \in T} \in F_a$ is itself really an option contract. The agent of type t who buys x_a will exercise whichever option in the set $\{ x_{at} \mid t \in T \}$ is preferred by such an agent. And, if x_a is indeed incentive compatible, this can be the option x_{at} if the individual is indeed of type t. Thus the option contract which is represented by the set $\{ x_{at} \mid t \in T \}$ is effectively the same as the self-enforcing type-contingent contract $x_a = \langle x_{at} \rangle_{t \in T} \in F_a$.

Accordingly it is natural to think of an option contract for agent a as some (non-empty and finite) subset C_a of the space $\Re^{GS}$ of individual state-contingent net trade vectors. Given any such contract C_a, when agent a has type t he will choose an option $\xi_{at}(C_a)$ which maximizes his actual expected utility $U_{at}(x)$ with respect to x subject to $x \in C_a$. So the option contract C_a can be regarded as equivalent to the state contingent contract $\langle\, \xi_{at}(C_a)\,\rangle_{t \in T}$. Indeed, provided that the option rule $\xi_{at}(\cdot)$ is allowed to vary in an appropriate manner for each type-contingent agent (a,t) and for each contract C_a, there is an obvious one-to-one correspondence between the set of all allocations which can result from such option contracts and the set of all incentive compatible contingent commodity contracts.

Formally, then, an *option contract* for agent a is a non-empty finite set $C_a \subset \Re^{GS}$. An *option rule* for the type-contingent agent (a,t) is a mapping $\xi_a \equiv \langle \xi_{at}(\cdot) \rangle_{t \in T}$ defined for all non-empty finite subsets of $\Re^{GS}$ and satisfying the requirement that, for all $t \in T$,

$$\xi_{at}(C_a) \in \arg\max_x \{\, U_{at}(x) \mid x \in C_a \,\}$$

everywhere in its domain. As remarked above the following result is then obvious.

Decentralization Lemma. *$x_a \in F_a$ if and only if there exists a contract C_a and an option rule ξ_a for which $x_{at} = \xi_{at}(C_a)$ (all $t \in T$).*

Proof: See Hammond (1979, Theorem 2, p. 268), for instance. ∎

We shall now see how the compensated, uncompensated, and weak equilibria which were defined in Section 5 above can be implemented through "perfected" markets for such option contracts. To this end, consider any incentive-compatible allocation $\hat{x}_a \in F_a$ to agent a. The corresponding option contract is then the set $\hat{C}_a := \{\, \hat{x}_{at} \mid t \in T \,\} \subset \Re^{GS}$. And the corresponding option rule can be any rule $\hat{\xi}_a$ satisfying $\hat{\xi}_{at}(\hat{C}_a) = \hat{x}_{at}$ for all $t \in T$.

Option contracts of this kind can obviously be used to deal with the incentive constraints. The efficiency constraints introduced in Section 5 require extra restrictions, however. In fact, it is enough to restrict each agent a to choose option contracts C_a satisfying $C_a \supset \hat{C}_a$. Then none of agent a's possible types can possibly be made worse off, because the set of options available to each never shrinks. In fact, the following is obvious.

Lemma. *If both* $x_{at} = \hat{\xi}_{at}(C_a)$ *and* $\hat{x}_{at} = \hat{\xi}_{at}(\hat{C}_a)$ *for all* $t \in T$*, where* $C_a \supset \hat{C}_a$*, then* $x_a \in F_a(\hat{x}_a)$.

Proof: The above decentralization lemma shows that $x_a \in F_a$. But then, by definition of an option rule,

$$U_{at}(x_{at}) = \max_x \{ U_{at}(x) \mid x \in C_a \} \geq \max_x \{ U_{at}(x) \mid x \in \hat{C}_a \} = U_{at}(\hat{x}_{at})$$

and so $x_a \in F_a(\hat{x}_a)$, as required. ∎

Thus agents in our perfected option market economy will be restricted to choosing option contracts C_a satisfying $C_a \supset \hat{C}_a$. Their derived utilities for such contracts are given by

$$V_{at}(C_a) \equiv U_{at}(\xi_{at}(C_a)) = \max_x \{ U_{at}(x) \mid x \in C_a \}$$

which is actually independent of the option rule ξ_a, not surprisingly. Given any physical commodity price vector $p \in \Re^G_+$, the corresponding price of a contract C_a will depend upon the precise option rule ξ_a, and be given by the function

$$v_a(p, C_a; \xi_a) := p\,[\mu \bullet \xi_a(C_a)],$$

where $\xi_a(C_a) \in \Re^{GTS}$ denotes the type-contingent net trade vector $\langle\, \xi_{at}(C_a)\, \rangle_{t \in T}$. Thus $v_a(p, C_a; \xi_a)$ is precisely the expected value at prices p of $\xi_a(C_a)$. The corresponding budget constraint, when there is lump-sum redistribution, is given by

$$v_a(p, C_a; \xi_a) \leq v_a(p, \hat{C}_a; \xi_a) = p\,(\mu \bullet \hat{x}_a).$$

And when there is no lump-sum redistribution, it is given by

$$v_a(p, C_a; \xi_a) \leq 0 = v_a(p, \hat{C}_a; \xi_a) = p\,(\mu \bullet \hat{x}_a).$$

The valuation function $v_a(p, C_a; \xi_a)$ is generally not in any sense a linear function of the set C_a. Indeed, unless preferences happen to be homothetic it will not even be true in general that $v_a(p, \lambda C_a; \xi_a) = \lambda\, v_a(p, C_a; \xi_a)$ for all $\lambda > 0$. So the budget constraint is nearly always non-linear. Of course, in economies with more than one physical commodity, such non-linear pricing will be vulnerable to manipulation by small coalitions, as in Hammond (1987). Or it will be vulnerable to the same agent entering the market several times, as in

Mas-Colell (1987). Then linear pricing may have to be imposed as an additional constraint. But there can still be non-linear pricing of many insurance contracts. After all, except in the special case of life insurance, the insurance industry naturally makes it hard for a claimant to collect full compensation simultaneously on more than one insurance policy covering the same risk.

The perfected option markets will clear when the contracts C_a chosen by the different agents $a \in A$ satisfy

$$\int_A [\mu \bullet \hat{\xi}_a(C_a)] \alpha(da) \leqq 0; \; p \geqq 0 \quad \text{(comp)}.$$

Note how there is a one-to-one correspondence between:

(i) the set of all allocations $x_a \in \Re^{GST}$ which, for some option rule ξ_a, result from an option contract C_a satisfying both the budget constraint $v_a(p, C_a; \xi_a) \leq v_a(p, \hat{C}_a; \xi_a)$ and the restriction that $C_a \supset \hat{C}_a$;

(ii) the set of all incentive compatible and efficiency constrained type-contingent net trade vectors in $F_a(\hat{x}_a)$ satisfying the usual Arrow-Debreu linear budget constraint $p\,(\mu \bullet x_a) \leq p\,(\mu \bullet \hat{x}_a)$.

All this motivates the following three definitions. First, an *uncompensated perfected option market equilibrium* is a type-independent allocation of option contracts $\hat{C}_a$ to each agent, together with an *option rule* ξ_a, and also a physical commodity price vector $p \in \Re^G_+$, such that:

(i) for almost every agent $a \in A$, and for each finite set $C_a \subset \Re^G$ and type $t \in T$, the option rule ξ_a satisfies $\xi_{at}(C_a) \in \arg\max_x \{ U_{at}(x) \mid x \in C_a \}$;

(ii) for almost every agent $a \in A$, and for each type $t \in T$, the same option contract $\hat{C}_a$ maximizes each type's derived utility function $V_{at}(C_a)$ with respect to C_a, subject to the budget constraint $v_a(p, C_a; \xi_a) \leq v_a(p, \hat{C}_a; \xi_a)$ and the restriction that $C_a \supset \hat{C}_a$;

(iii) the resulting allocation $\xi_{at}(C_a)$ satisfies the market clearing condition

$$\int_A [\mu \bullet \xi_a(C_a)]\, \alpha(da) \leqq 0; \, p \geqq 0 \text{ (comp)}.$$

Second, a *compensated perfected option market equilibrium* is defined similarly as a collection $\hat{C}_a$, ξ_a (all $a \in A$), and $p \in \Re^G_+$ which together satisfy (i) and (iii) above, as well as the following modified form of condition (ii):

(ii′) for almost every agent $a \in A$, and for each type $t \in T$, the same option contract $\hat{C}_a$ minimizes each type's net expenditure $v_a(p, C_a; \xi_a)$ with respect to C_a, subject to the constraint that the derived utility $V_{at}(C_a)$ is no less than $V_{at}(\hat{C}_a)$, as well as the restriction that $C_a \supset \hat{C}_a$.

Third, a *weak perfected option market equilibrium* is defined similarly as a collection $\hat{C}_a$, ξ_a (all $a \in A$), and $p \in \Re^G_+$ which together satisfy (i) and (iii) above, as well as the following modified form of condition (ii):

(ii″) for almost every agent $a \in A$, and for each type $t \in T$, the same option contract $\hat{C}_a$ has the property that, whenever C_a satisfies both $C_a \supset \hat{C}_a$ and $V_{at}(C_a) > V_{at}(\hat{C}_a)$, then $v_a(p, C_a; \xi_a) \geq v_a(p, \hat{C}_a; \xi_a)$.

Note that the allocation which corresponds to a perfected option market equilibrium — uncompensated, compensated, or weak — is given by $\hat{x}_{at} := \xi_{at}(\hat{C}_a)$ for all $t \in T$. This allocation is certainly physically feasible, because of (iii) above. It is also incentive compatible because of the decentralization lemma above, and in fact it must also be true that $\hat{x}_a \in F_a(\hat{x}_a)$.

It has already been noted that there is a one-to-one correspondence between, on the one hand, the set of all type-contingent net trade vectors in $F_a(\hat{x}_a)$ which satisfy the usual Arrow-Debreu linear budget constraint, and on the other, the set of all those allocations which, for some suitable option rule ξ_a, are generated by option contracts C_a which satisfy the non-linear budget constraint $v_a(p, C_a; \xi_a) \leq v_a(p, \hat{C}_a; \xi_a)$ as well as the restriction that $C_a \supset \hat{C}_a$. So the earlier efficiency, core equivalence, and existence theorems, when applied to the fictitious Arrow-Debreu economy with exogenous incentive and efficiency constraints, imply similar results for perfected option market equilibria. In particular, as already mentioned above, apart from the modifications which are needed because of possible local satiation, such equilibria usually exist and are incentive constrained Pareto efficient, while any incentive constrained Pareto efficient allocation can be achieved through perfected option markets in equilibrium, and the usual core equivalence theorem is true as well.

10. Comparison with Other Work

At first sight, these results may seem clearly to contradict Rothschild and Stiglitz (1976) and the extensive work which has grown out of that article on non-existence and inefficiency of equilibrium in insurance markets subject to adverse selection. It is important, in fact, to see why there is no such contradiction. The reason is that their definition of competitive equilibrium is different. Rothschild and Stiglitz look for allocations in which no firm could enter and make a profit by offering new insurance contracts. Their work shows that the efficient self-enforcing Arrow-Debreu option equilibria just discussed are often vulnerable to entry by firms seeking profit opportunities. If entry into the insurance industry cannot in fact be restricted, that imposes an additional constraint which should really be reflected in the notion of constrained Pareto efficiency that we use.

In standard general equilibrium theory, a counterpart to the Rothschild-Stiglitz notion of equilibrium with free entry is the core. Typically, in an economy whose consumers have continuous locally non-satiated preferences and in which free disposal is possible, an allocation can be blocked or improved by a coalition if and only if a firm could enter and earn a profit from arranging net trades close to, but slightly less than, those which the blocking coalition uses. So any Walrasian equilibrium is an equilibrium with free entry, and in an economy with a continuum of agents, the usual core equivalence theorem of Aumann (1964) and Hildenbrand (1974) shows that any equilibrium with free entry is a Walrasian equilibrium. Because there is usually a Walrasian equilibrium, the core is non-empty, so there is an equilibrium with free entry. Because all core allocations are Pareto efficient, so are all equilibria with free entry. These results even remain true for an economy with adverse selection, provided that the core is defined in a way which takes account of the need for each blocking coalition to respect the incentive constraints due to adverse selection, and provided that the distinction between compensated and uncompensated equilibria remains unimportant.

With adverse selection, however, there is a crucial difference between the core and the Rothschild-Stiglitz concept of equilibrium with free entry. The latter allows coalitions to form whose membership is not a fixed set of individuals, but rather a group who select themselves. An entering firm offers new option contracts, in effect, and can do so in a way which implies that only good risks become the new firm's customers. This makes it easier to earn a profit from a new contract than it would be if every entering insurance firm were forced to offer new contracts which all members of a fixed set of individuals wanted to accept, regardless of their true type. Yet the latter is what is involved in finding a blocking coalition in the usual sense. The easier notion of blocking used by Rothschild and Stiglitz explains why they find that there may be no equilibrium with free entry.

Finally, these results should be compared with those of Greenwald and Stiglitz (1986, 1988), who demonstrate the generic Pareto inefficiency of competitive equilibria with adverse selection and moral hazard. They do so, however, after assuming linear commodity pricing, whereas the self-enforcing Arrow-Debreu option equilibria considered here involve non-linear pricing. If linear pricing is necessary to ensure multilateral incentive compatibility when trading on the side is possible, as in Gale (1980, 1982), Hammond (1987), then our notion of incentive-constrained efficiency should change accordingly.

Perfect option market equilibria, however, clearly remain far from reality. One can immediately think of many reasons for this, but some of the most interesting are perhaps additional incentive constraints. These may be due to difficulties in enforcing non-linear pricing and credit rationing schemes, as mentioned in the previous paragraph, or to difficulties in regulating entry by competing firms, in arranging effective monitoring without wasteful duplication, as well as in regulating the regulators, etc. These all seem important issues. They deserve attention in future work which goes beyond the scope of this paper.

ACKNOWLEDGEMENTS

Earlier versions of this paper formed the basis of presentations to the Institute of Mathematical Studies in the Social Sciences summer workshop in Mathematical Economics at Stanford in August 1987, as well as to conferences on "Normative Economics" at the Nara Hotel, Japan in October 1987, and on "Recent Advances in Macroeconomics" at the Universidad Internacional Menéndez Pelayo, Santander, in June 1989. I am grateful to Robert Townsend and especially the discussants Miquel Faig, Joan Ketterer, and Dilip Mookherjee for their comments, and also to Claude d'Aspremont, Jacques Drèze, Bruce Greenwald, Frank Hahn, Ramon Marimon, David Starrett, and Joseph Stiglitz, whose justified scepticism about my earlier inadequate attempts to "perfect" markets through suitable option contracts has forced me to make big improvements. The present revised version has benefited from comments by participants in the European Meeting of the Econometric Society in Munich, as well as in seminars at G.R.E.Q.E., Marseille and at the University of Bologna.

REFERENCES

R.J. Aumann (1964), "Markets with a Continuum of Traders," *Econometrica*, **32**: 39–50.

R.J. Aumann (1966), "Existence of Competitive Equilibria in Markets with a Continuum of Traders," *Econometrica*, **34**: 1–17.

T.F. Bewley (1981), "A Critique of Tiebout's Theory of Local Public Expenditures," *Econometrica*, **49**: 713–740.

R.N. Bhattacharya and M. Majumdar (1973), "Random Exchange Economies," *Journal of Economic Theory*, **6**: 37–67.

M. Feldman and C. Gilles (1985), "An Expository Note on Individual Risk without Aggregate Uncertainty," *Journal of Economic Theory*, **35**: 26–32.

D. Foley (1967), "Resource Allocation and the Public Sector," *Yale Economic Essays*, **7**: 45–98.

D.M. Gale (1979), "Large Economies with Trading Uncertainty," *Review of Economic Studies*, **46**: 319–338.

D.M. Gale (1980), "Money, Information and Equilibrium in Large Economies," *Journal of Economic Theory*, **23**: 28–65.

D.M. Gale (1982), *Money: In Equilibrium*. Welwyn: James Nisbet.

B. Greenwald and J.E. Stiglitz (1986), "Externalities in Economies with Imperfect Information and Incomplete Markets," *Quarterly Journal of Economics*, **100**: 229–264.

B. Greenwald and J.E. Stiglitz (1988), "Pareto Inefficiency of Market Economies: Search and Efficiency Wage Models," National Bureau of Economic Research, Working Paper No. 2651, Cambridge, MA.

P.J. Hammond (1979), "Straightforward Individual Incentive Compatibility in a Large Economies," *Review of Economic Studies*, **46**: 263–282.

P.J. Hammond (1987), "Markets as Constraints: Multilateral Incentive Compatibility in a Continuum Economy," *Review of Economic Studies*, **54**: 399–412.

P.J. Hammond (1989), "On the Impossibility of Perfect Capital Markets," European University Institute, working paper (to appear).

W. Hildenbrand (1971), "Random Preferences and General Economic Equilibrium," *Journal of Economic Theory*, **3**: 414–429.

W. Hildenbrand (1974), *Core and Equilibria of a Large Economy*. Princeton: Princeton University Press.

K. Judd (1985), "The Law of Large Numbers with a Continuum of Random Variables," *Journal of Economic Theory*, **35**: 19–25.

M.A. Khan and A. Yamazaki (1981), "On the Cores of Economies with Indivisible Commodities and a Continuum of Traders," *Journal of Economic Theory*, **24**: 218–225.

E. Malinvaud (1972), "The Allocation of Individual Risks in Large Markets," *Journal of Economic Theory*, **4**: 312–328.

E. Malinvaud (1973), "Markets for an Exchange Economy with Individual Risks," *Econometrica*, **41**: 383–410.

A. Mas-Colell (1987), "On the Second Welfare Theorem for Anonymous Net Trades in Exchange Economies with Many Agents," in *Information, Incentives, and Economic Mechanisms: Essays in Honor of Leonid Hurwicz* edited by T. Groves, R. Radner and S. Reiter (Minneapolis: University of Minnesota Press), ch. 9, pp. 267–292.

E.C. Prescott and R.M. Townsend (1984a), "Pareto Optima and Competitive Equilibria with Adverse Selection and Moral Hazard," *Econometrica*, **52**: 21–45.

E.C. Prescott and R.M. Townsend (1984b), "General Competitive Analysis in an Economy with Private Information," *International Economic Review*, **25**: 1–20.

M. Rothschild and J.E. Stiglitz (1976), "Equilibrium in Competitive Insurance Markets: An Essay in the Economics of Imperfect Information," *Quarterly Journal of Economics*, **90**: 629–649.

C. Tiebout (1956), "A Pure Theory of Local Expenditures," *Journal of Political Economy*, **64**: 416–424.