Variance Based Tests of Market Demand

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April 21, 2021

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Abstract

In this paper, we generalize the Sonnenchein-Mantel-Debreu (SMD) framework to one where the analyst can observe both aggregate demand and the variance of demand. We show that the SMD framework's impossibility results do not survive this strengthening of the premise. We prove observable restrictions on data that individual utility maximization places. Then we find that measuring the variance of consumption as a function of price allows the analyst to improve estimates of welfare changes due to price changes. Lastly, we consider setups in which only cross-sectional data can be observed and show that many of our results survive in even this situation.

Key words: aggregation, market demand, observable restrictions, identification.

JEL classification: D11; D50; C43

1 Introduction

Does individual utility maximization generate observable restrictions on aggregate variables?

This one question has plagued economists for a very long time and particularly since the 1970s. A series of results early in the decade [1] made economists believe that as long a population of individuals is as Arrow (1990) put it," *different in unspecifiable ways*", any function can arise as an average of individually rational demand functions. This belief was profoundly negative and discouraging. However, we claim that this problem of heterogeneity is not insurmountable.

The above result does hold if the analyst observes only average demand, but the average is not the only "aggregate" information to which one may have access. Suppose the analyst observes the average demand, "the first moment", and the variance of demand, the "second moment". We show that individual utility maximization does place sharp restrictions on *possible* observable data. Not only that, but this lets the analyst conduct several counterfactual welfare exercises. By presenting findings in sharp contrast to the negative results discussed above, we hope to highlight their *fragility*.

Before our formal results, we briefly summarize the history of the problem and highlight the contributions made by different people. Beginning with (Sonnenschein, 1973a), Hugo Sonnenschien showed that with 2 goods, Walras law and homogeneity of degree zero characterize excess demand. Debreu (1974) and Mantel (1974),Mantel (1976) then extended the argument to general economies. These results are collectively known as the Sonnenschein-Mantel-Debreu (SMD) theorem.

This result had a significant and resounding impact on economic theory. To quote Kenneth Arrow *"in the aggregate, the hypothesis of rational behavior has in general no implications"* he then went on to say that *"if agents are different in unspecifiable ways, then [...] very few, if any, inferences can be made"* Arrow (1990). To further emphasize the point James Tobin, who firmly held that economics can and should alleviate need and improve general welfare, considered the Sonnenschein-Mantel-Debreu theorem as a result that should not have been proved.

However, recent work has shown that this view is, as Chiappori and Ekeland put in their 2011 piece, *"overly pessimistic"*. Starting from Brown and Matzkin (1996), there was a body of work such as Kubler (2003), Chiappori and Ekeland (2002, 2004); Chiappori, Ekeland, Kubler, and Polemarchakis (2004). This body generates a coherent response to the criticism made by

[1]For a review, see Rizvi (2006)

Arrow. It shows that while General Equilibrium theory does generate robust predictions, it does so only with access to individual-level data.

We take one step further, showing that even access to individual data is unnecessary. We claim that SMD theory's negative results only hold when there is *no* aggregate data about the distribution of consumption. If *some* data about the aggregate distribution of consumption (its variance in our setup) is observable, there still are sharp predictions of utility maximization, contrary to the views held by Arrow and Tobin.
SMD type results are studied in 2 similar settings.

1. Excess demand:- in this setup, *endowments* are fixed, and we observe the sum of individual demands as prices vary, and all individuals face identical prices

2. Market Demand:- in this setup, *nominal incomes* are fixed, we observe the sum of individual demands as prices vary, and all individuals face identical prices.

In this article, we only consider the specific case of market demand. In the tradition of Sonnenschein, Chiappori and Ekeland (1999a) show that any analytic "market demand" function which satisfies homogeneity and Walras' law, can locally be decomposed into finitely many rational demand functions. One can think of market demand as the "average" or the "first moment" demand of a population of individuals at some given price. Once we think of market demand as the first moment of the population demand function, The following might be a natural extension.

Suppose the analyst had access to both the first and second moment of market demand as a function of price, are there local restrictions of these moments?

We go on to answer this question in the affirmative emphatically. Finding that, counter-intuitively, not only are there sharp(local) restrictions of these moments, but there are welfare-relevant quantities which they can be used to identify. To do this, we first compute a novel decomposition of society's (average) substitution(matrix), decomposing it into a "naive" substitution term which essentially treats society as a single agent and a "variance correction" (VC henceforth) which depends **only** on the second moment of consumption.

Using this decomposition, we can directly test a mean-variance pair for "rationality". These restrictions are *some* of the standard ones which arise from a neo-classical consumer i.e. negative semi-definiteness of the substitution matrix, it lying in the kernel of price, and its negative definiteness in the subspace orthogonal to price.

We then assume that the mean demand itself satisfies the Slutsky equation. This condition is sufficient for the mean demand function to behave like a single individual. However, society's representative agent may not be welfare-relevant; he may even be Pareto inconsistent, i.e., preferring one situation to another even though all agents in society prefer the reverse. cf(Dow and da Costa Werlang, 1988; Jerison, 1984). We construct a test of the mean demand for the representative agent to be truly "representative".

After this, we show that our result can be used to construct estimates of "exact consumer surplus", allowing us to construct its second order approximations. These estimates constitute an improvement to the approach taken in Schlee (2007),Blundell, Browning, and Crawford (2003), and Foster and Hahn (2000), which treat unobserved heterogeneity as a regression error term. These papers construct first-degree approximations of welfare using mean demand, which are bettered by the second-degree approximations we construct. On a prescriptive note, we recommend explicitly estimating heteroskedasticity in demand estimation, which can significantly improve the accuracy of estimated changes in welfare.

In the literature concerned with "demand analysis", another paper close to ours is Lewbel (2001). This article characterizes the nature of the "error term" for aggregate demand to be integrable. However, in that paper, the author assumes that the co-variance between income effects and consumption is observable, then there are restrictions on this co-variance. We *only* observe aggregate data about consumption, which leads to our weaker but more robust restrictions. Also, the part of our paper which deals with identification has no analogue in the above work.

The observation that restrictions on the variance of population demand lead to restrictions on market demand behaviour has been made in various contexts. A series of papers (Hildenbrand, 1983, 1994; Härdle, Hildenbrand, and Jerison, 1991) showed that if the average variance of demand increases with income, market demand obeys the "law of demand". In a similar setting, Grandmont (1987b, 1992); Quah (1997) also showed that restrictions of this form lead to the "law of market demand" and thus local stability of Walrasian equilibrium.

In a sense, we tackle the inverse problem; rather than putting restrictions on the fundamentals to discipline market demand, we observe the variance directly and see if we can test market demand directly for utility maximization.

We differ from the above literature in two ways. Firstly, the restrictions we can impose are much stronger than those implied above and hold even when the above literature cannot impose any restrictions at all. Secondly, we show that beyond imposing testable restrictions observing the variance allows the analyst to *identify* parameters of interest (such as the average substitution effect), which help conduct counterfactual welfare analysis.

Our paper is also related to the stochastic revealed preference literature; for a review see McFadden (2005). This literature branch tries to impose restrictions on the whole distribution of demand as a function of price. Similarly, a series of papers, such as Hoderlein and Stoye (2014, 2015); Dette, Hoderlein, and Neumeyer (2016) even find restrictions on distributions of demand which arise from utility maximization when only the marginal quantile distribution of consumption is observable.

However, the data the analyst needs to observe in the above strand of literature is far richer than what we require. Also, in the case of additively separable measurement error, the exact quantile distribution may be affected, but the parameters of interest to our analysis remain unaltered. One can also think of our paper as establishing necessary conditions for stochastic rationalizability[2] for the quantile distribution of demand, as prices vary locally.

Lastly, we explore situations where the analyst is only able to observe cross-sectional variances. We show that we can salvage many of the above results even in this very specialized setup. Observable restrictions correspond to the variance of every good we do observe. Approximations for welfare changes can also be computed up to the second degree as long as prices change only for one good.[3]. Further, it is also possible to put bounds on all the entries of the variance matrix even if we observe only a few.[4]

This paper is laid out as follows. In the introduction, we briefly survey the results in this paper and compare them to the existing literature. We go on to describe our basic setup in section 2. We then briefly summarize the "negative" results the literature proves in Section 3. Section 4 lays out the methods used to overcome the negative re- results, proves our first original results and how they lead to testable restrictions of aggregate data. Sections 5 and 6 discuss our results' natural applications, tests for the representative consumer assumption, and counterfactual welfare analysis for small price changes. Section 7 talks about extending our results when the analyst has access to only a limited data-set. Finally, Section 8 discusses possible directions of future research and then concludes.

[2]See McFadden and Richter (1991); McFadden (2005)

[3]This was an observation first made by Hausman (1981)

[4]This is similar in spirit but not in method to Hausman and Newey (2016)

2 Setup

We consider a simple market demand setup, where there is a finite number of individuals, each of whom has an identical income I. They all maximize utility subject to facing common prices. We go on to lay out the basic setup.

1. The commodity space is $\mathcal{X} := \mathbf{R}^l_+$, the analyst observes an open set of prices $\mathcal{P} \subset \mathbf{R}^l_+$

2. There are n individuals, each of whom has a (smooth) demand function $x_i : \mathcal{P} \times \mathbf{R}_+ \rightarrow \mathcal{X}$, such that

$$x_i(\alpha p, \alpha I) = x_i(p, I) \quad \text{(homogeneity of Degree Zero)}$$

$$p \cdot x_i(p, I) = I \quad \text{(Walras Law)}$$

   Remark. We refer to the n individuals collectively as a *population*.

3. We say that these demand functions are *rational*; if there are smooth, strongly quasi-concave utilities, $U_i : \mathcal{X} \rightarrow \mathbf{R}$ such that:-

$$x_i(p, I) = \arg\max_{p.Y \leq I} U_i(Y)$$

4. For every rational demand function $x_i(p, I)$ there exists compensated or hicksian demand function $h_i(p, u)$ defined as

$$h_i(p, u) = \arg\min_{h \in \mathbf{R}^l} \{p \cdot h | U_i(h) \geq u\}$$

   Remark. Throughout the paper, we assume that expenditure or income varies independently of price; this is in keeping with the extensive theoretical literature on market demand analysis starting from Hildenbrand (1983, 1994)[5], the literature on demand estimation, and non-parametric welfare analysis.[6]

   Once we make this assumption, we assume that each individual has an identical income for simplicity.

5. The analyst observes mean demand.

$$\mu(p, I) = \frac{1}{n} \sum_{i=1}^{n} x_i(p, I)$$

[5]For other papers see Grodal and Hildenbrand (1992); Zambrano and Vogelsang (2000); Grandmont (1987a); Marhuenda (1995)

[6]See Schlee (2007); Blundell, Browning, and Crawford (2003); Lewbel (2001); Hausman and Newey (2016, 2017)

6. We deviate from the existing literature, assuming that the analyst also observes the population(or sample) second moment.

$$\sigma^2(p, I) = \frac{1}{n}\left[\sum_{i=1}^{n} x_i(p, I)x_i^\intercal(p, I)\right] - [\mu(p, I)][\mu(p, I)]^\intercal$$

We say that mean demand is rationalizable if there exist n rational demand functions such that

$$\mu(p, I) = \frac{1}{n}\sum x_i(p, I)$$

Definition 1. We say that a demand variance pair $(\mu(p, I), \sigma^2(p, I))$ is *jointly rationalizable* if there are n rational demand functions such that

$$\mu(p, I) = \frac{1}{n}\sum x_i(p, I)$$

and

$$\sigma^2(p, I) = \frac{1}{n}\sum x_i(p, I)x_i^\intercal(p, I) - [\mu(p, I)][\mu(p, I)]^\intercal$$

Remark. Our setup is similar to those in Schlee (2007) and Lewbel (2001), if we assume that the population's underlying distribution has finite support. However, this difference is mainly stylistic, and all results go through if we smooth distributions of unobserved individual heterogeneity we the above papers. In the same vein, our approach can also be justified by assuming *preferences are independent of income levels* in the population.

Remark. It is clear from a preliminary inspection that mean demand must satisfy the following 2 conditions.

$$\mu(\alpha p, \alpha I) = \mu(p, I) \qquad \text{(homogeneity of Degree Zero)}$$

$$p \cdot \mu(p, I) = I \qquad \text{(Walras' Law)}$$

The question which Sonnenschein (1972, 1973a) posed was, that abstracting from non-negativity considerations, are these the only restrictions that can be placed on mean demand.[7]

3 Mean Demand and the "Income Effect"

We first give a brief account of the disappearance of restrictions that apply to individual demand on aggregation. It is well known from Slutsky (1915) that

[7] Chiappori and Ekeland (1999b) showed that this is indeed the case.

a demand function arises from utility maximization if and only if it satisfies negative semi-definiteness and symmetry of the Substitution matrix.

However, these restrictions do not survive when there is a population of individuals. To demonstrate this, we describe a theorem from Geanakoplos and Polemarchakis (1980), about testable restrictions "at a point".

Theorem 3.1. *Let the analyst observe an individual demand x(p,I) and the Jacobian of x, call it J(p,I) at one price income pair (p,I), this Jacobian and demand is rationalizable if and only if there exists a vector $v \in \mathbb{R}^l$ and a matrix $\mathcal{K} \in \mathbb{R}^{n^2}$ such that.*

$$J = \mathcal{K} - vx^{\intercal}$$

and

1. *$\mathcal{K}(p, I)$ is symmetric and negative semidefinite.*
2. *$\mathcal{K}(p, I)$ has rank (l-1) and $p\mathcal{K} = \mathcal{K}p = 0$*

Refer to the subspace of vectors orthogonal to $x(p, I)$ as $[x(p, I)]^{\perp}$, And the Jacobian of x as $\mathbf{D}_p x(p, I)$

The above theorem can be restated by as can be restated as follows

$$\forall V \in [x(p, I)]^{\perp}, \quad V^{\intercal}\mathbf{D}_p x(p, I)V \leq 0$$

and

$$\forall V \in [x(p, I), p]^{\perp}, \quad V^{\intercal}\mathbf{D}_p x(p, I)V < 0$$

The problem is that when we sum the demands of individuals, the individual demands need not be co-linear. Thus, the subspaces where the Jacobian is negative semi-definite are different. Given enough individuals, the intersection of these subspaces may be empty. Formally,

$$\bigcap_{i=n}^{n} [x_i(p, I)]^{\perp} = \phi$$

This problem causes the structure rationality places on individual demand to break down when aggregated.

In this vein, the following result was first demonstrated by Sonnenschein (1973b), and then generalized by Diewert (1977) and Mantel (1975).

Theorem 3.2. *Let $\hat{X}(p, I)$ be any function that satisfies Walras' law and homogeneity. At a point $(\overline{p}, \overline{I})$ it "looks like" a mean demand function, meaning, there exist l individual, rationalizable demand functions $(x_1(p, I)...x_l(p, I))$, such that:-*

1. $\frac{1}{n}\sum x(\overline{p},\overline{I}) = \hat{X}(\overline{p},\overline{I})$

2. $\frac{1}{n}\sum \mathbf{D}_p x_i(p) = \mathbf{D}_p\hat{X}(\overline{p},\overline{I})$

This theorem encapsulates the spirit of what we referred to in our intro as SMD theory. It shows that an analyst observing market demand "at a point", can never falsify the hypothesis of utility maximization. This problem occurs mainly because of the misbehavior of "income effects," as all observable restrictions are placed on the substitution matrix. Andreu (1983) then showed that abstracting from non-negativity considerations allows one to extend the same result to finite price demand data.

We state one final result from Chiappori and Ekeland (1999a) which significantly generalizes this theorem to a small open set around a point, however, this requires the observed mean demand to be analytic.

Theorem 3.3. *Consider some open set* $\mathcal{U} \in \mathbf{R}^l_+ \times \mathbf{R}$ *and an analytic mapping* $\hat{X} : \mathcal{U} \to \mathbf{R}^l_+$ *which satisfies Walras' law and homogeneity. For all* $(\overline{p},\overline{I}) in \mathcal{U}$ *there exist* n *rationalizable individual demand function* $(x_1, ...x_n)$ *such that:-*

$$\sum x_i(\overline{p},\overline{I}) = \hat{X}(\overline{p},\overline{I})$$

for all p *In some convex neighbourhood* $\mathcal{V}$ *of* $\overline{p}$

This last theorem ends our short review of the main negative results in the specific market demand case we consider. We now state our main results and show how the (sample) variance rids us of the income effect problem.

4 Decomposition of Pointwise Substitution

Now we go on to describe our main insight, Let $x(p, I)$ represent a demand function, the Slutsky equation is as follows:-

$$\mathbf{D}_p x(p, I) = \mathbf{D}_p h(p, u) - \left[\frac{\partial}{\partial I} x(p, I)\right] x(p, I)^\intercal \tag{1}$$

Notice that the second term on the right-hand side, the "income effect," resembles a square's derivative. The second moment of demand's derivative with respect to income is the "symmetrized" income effect.

$$\frac{\partial}{\partial I}\left[x(p, I)x^\intercal(p, I)\right] = \left[\frac{\partial}{\partial I} x(p, I)\right] x(p, I)^\intercal + x(p, I)\left[\frac{\partial}{\partial I} x(p, I)\right]^\intercal$$

The problem is that the matrix $\left(xx^\intercal\right)$ is symmetric by construction, and so must be its derivative. The income effect, on the other hand, need not be

symmetric.
In an attempt to solve the problem, we consider the "symmetrized" Slutsky equation. First, we transpose equation 1, fetching:-

$$[\mathbf{D}_p x(p,I)]^\intercal = [\mathbf{D}_p h(p,u)]^\intercal - \left[\left[\frac{\partial}{\partial I} x(p,I)\right] x(p,I)^\intercal\right]^\intercal$$

Now, we already know that utility maximization implies Slutsky symmetry, which means

$$\mathbf{D}_p h(p,u) = [\mathbf{D}_p h(p,u)]^\intercal$$

plugging this back into 1 gives us.

$$\implies [\mathbf{D}_p x(p,I)]^\intercal = \mathbf{D}_p h(p,u) + x(p,I)\left[\frac{\partial}{\partial I} x(p,I)\right]^\intercal \tag{2}$$

Now, we can add equations 1 and 2 to construct the symmetric matrix we want; this gives us:-

$$[\mathbf{D}_p x(p,I)]^\intercal + \mathbf{D}_p x(p,I) = 2\mathbf{D}_p h(p,u) - \left[x(p,I)\left[\frac{\partial}{\partial I} x(p,I)\right]^\intercal + \left[\frac{\partial}{\partial I} x(p,I)\right] x(p,I)^\intercal\right]$$

$$\implies [\mathbf{D}_p x(p,I)]^\intercal + \mathbf{D}_p x(p,I) = 2\mathbf{D}_p h(p,u) - \frac{\partial}{\partial I}\left[x(p,I)x^\intercal(p,I)\right]$$

Which brings us to our first main equation.

$$\mathbf{D}_p h(p,u) = \frac{1}{2}\left[[\mathbf{D}_p x(p,I)]^\intercal + \mathbf{D}_p x(p,I) + \frac{\partial}{\partial I}\left[x(p,I)x^\intercal(p,I)\right]\right] \tag{3}$$

This trick gets rid of the dreaded income effects!! It also gives us a new way to decompose the derivative of demand into substitution and a term which "resembles" the second moment.

Now for a single individual, this seems to be uninformative. However, this lends itself very well to analyzing the aggregation problem. To show this, we first give one definition.

Definition 2 (Naive Substitution)**.** Let $\mu(p,I)$ be a mean demand function, define the *Naive Substitution Matrix* $\hat{S}$ as

$$\hat{S}(p,I) = \mathbf{D}_p \mu(p,I) + \frac{\partial \mu}{\partial I}\mu^\intercal$$

We call this term naive because it constructs the substitution matrix "as if" a single consumer generates the mean demand. Observe that, in general, $\hat{S}$ is **not** symmetric and negative semi-definite.

Now we' define the "true" average substitution as the average of the substitution matrix of society.

Definition 3 ((True)Average Substitution)**.** Let a mean demand $\mu(p, I)$ be generated by a population $\{1, ...n\}$ of rational utility maximizers. Define the average substitution S as.

$$S = \sum_{i=1}^{n} \mathbf{D}_p h_i(p, u) = \sum_{i=1}^{n} \left[\mathbf{D}_p x_i(p, I) + \frac{\partial x_i(p, I)}{\partial I} x_i(p, I)^\intercal \right]$$

Decomposition

We now use Equation 3 to analyse the relationship between the 'true' average substitution and the naive average substitution. We can decompose the true substitution into the naive term and the "variance correction".

The remarkable thing we show is that the correction *only* depends on the variance, rather *its rate of increase*, at a given point. In this endeavor, we prove the following theorem.

Theorem 4.1. *Let $(\mu(p, I), \sigma^2(p, I))$ be a mean-variance pair, and let it be generated by a population $\{1, ...n\}$ of rational utility maximizers. At every point, the "true" average substitution is a sum of the "symmetrized" naive substitution and a correction term which depends only on the variance, Formally:-*

$$\underbrace{S(p, I)}_{\text{“True” Substitution}} = \overbrace{\frac{1}{2} \left\{ \hat{S}(P, I) + \hat{S}(P, I)^\intercal \right\}}^{\text{“Symmetrized” Substitution}} + \underbrace{\frac{1}{2} \left\{ \frac{\partial}{\partial I} \sigma^2(p, I) \right\}}_{\text{Variance Correction}} \tag{4}$$

Proof. We first rewrite equation 3, adding subscripts for each individual in the population.

$$\mathbf{D}_p h_i(p, u) = \frac{1}{2} \left[[\mathbf{D}_p x_i(p, I)]^\intercal + \mathbf{D}_p x_i(p, I) + \frac{\partial}{\partial I} \left[x_i(p, I) x_i^\intercal(p, I) \right] \right]$$

Now we aggregate this by taking the average across all individuals, which gives us,

$$\frac{1}{n} \sum_{i=1}^{n} \mathbf{D}_p h_i(p, u) = \frac{1}{2n} \sum_{i=1}^{n} \left[[\mathbf{D}_p x_i(p, I)]^\intercal + \mathbf{D}_p x_i(p, I) + \frac{\partial}{\partial I} \left[x_i(p, I) x_i^\intercal(p, I) \right] \right]$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left[\frac{[\mathbf{D}_p x_i(p, I)]^\intercal + \mathbf{D}_p x_i(p, I)}{2} \right] + \frac{1}{n} \sum_{i=1}^{n} \left[\frac{1}{2} \frac{\partial}{\partial I} \left[x_i(p, I) x_i^\intercal(p, I) \right] \right]$$

Where the left-hand side represents the average substitution effect.

Now, because differentiation is a linear operation, we can take the sum inside the derivative operator.

$$= \frac{1}{2}\left\{\mathbf{D}_p\left[\frac{1}{n}\sum_{i=1}^{n} x_i(p,I)\right] + \mathbf{D}_p\left[\frac{1}{n}\sum_{i=1}^{n} x_i(p,I)\right]^{\intercal}\right\} + \frac{1}{2}\left\{\frac{\partial}{\partial I}\left[\frac{1}{n}\sum_{i=1}^{n} x_i(p,I)x_i^{\intercal}(p,I)\right]\right\}$$

On an aside we can now implement a simple substitution, we know that

$$\sigma^2(p,I) = \frac{1}{n}\sum_{i=1}^{n} x(p,I)x(p,I)^{\intercal} - \mu(p,I)\mu(p,I)^{\intercal}$$
$$\implies \frac{1}{n}\sum_{i=1}^{n} x(p,I)x(p,I)^{\intercal} = \sigma^2(p,I) + \mu(p,I)\mu(p,I)^{\intercal}$$

Substituting this into the second term of the above equation gives us.

$$= \frac{1}{2}\left\{\mathbf{D}_p\left[\mu(p,I)\right] + \mathbf{D}_p\left[\mu(p,I)\right]^{\intercal}\right\} + \frac{1}{2}\left\{\frac{\partial}{\partial I}\left[\sigma^2(p,I) + [\mu(p,I)][\mu(p,I)]^{\intercal}\right]\right\}$$

This can be rearranged as

$$\frac{1}{2}\left\{\left[\mathbf{D}_p\mu(p,I) + \mu(p,I)\frac{\partial\mu(p,I)}{\partial I}\right] + \left[\mathbf{D}_p\mu(p,I) + \mu(p,I)\frac{\partial\mu(p,I)}{\partial I}\right]^{\intercal}\right\} + \frac{1}{2}\left\{\frac{\partial\sigma^2(p,I)}{\partial I}\right\}$$

Notice that from 3 this can be substituted to get

$$S = \frac{1}{2}\left\{\hat{S} + \hat{S}^{\intercal}\right\} + \frac{1}{2}\left\{\frac{\partial}{\partial I}\left[\sigma^2(p,I)\right]\right\} \tag{5}$$

Which is what we set out to show. □

The above theorem has several consequences for analyzing mean-variance pairs. Assuming the mean-variance pair does indeed arise from individual utility maximization, the above theorem gives us a remarkable decomposition of society's average substitution matrix.

The first term, the "symmetrized substitution", represents what the substitution matrix of society would look like if the mean demand of the population represented the population's preferences accurately.

The second term represents the unit increase in the variance of consumption for a unit increase in income for everyone in the population. It measures how far the mean demand departs from representing the population.

Remark (**Magnitude Independence**). Notice that the decomposition does not depend on the exact magnitude of the variance but only its rate of change. This independence leads to 2 fundamental properties.

1. If there is any additively separable error in observations of demand data which we use to compute the variance, there is no effect on the restrictions. This observation provides a potential reason for using these restrictions even if micro-level data is available.

2. None of the results in this paper depend on "positivity" or the constraints that arise from the variance being close to zero and Chebyshev type tail inequalities.

Remark. One would expect that the divergence of average demand from representing society would depend on the income effects' variance. It turns out that we can reverse the order of differentiation, and what matters is the income effect of the variance, which changes an individual-level variable to a population level aggregate variable.

Another critical application of 4 is testable implications for mean variance data. In order to find these testable restrictions, we first prove a simple lemma.

Lemma 4.1. *In any population, mean demand $\mu(p, I)$ and variance of demand $\sigma^2(p, I)$ identify the average substitution effect.*

Proof. Equation 4 gives us

$$S = \frac{1}{2}\left\{\hat{S} + \hat{S}^{\intercal}\right\} + \frac{1}{2}\left\{\frac{\partial}{\partial I}\left[\sigma^2(p, I)\right]\right\}$$

Observe that the analyst observes each term of the RHS and can use that information to compute the S matrix. □

We will first use the above results to generate restrictions on mean-variance pairs arising from demand.

Tests of Demand

Theorem 4.2. *If let $(\mu(p, I), \sigma^2(p, I))$ be a mean-variance pair generated by a population of individuals, who are all maximising utility. Further, define*

$$S = \frac{1}{2}\left\{\hat{S} + \hat{S}^{\intercal}\right\} + \frac{1}{2}\left\{\frac{\partial}{\partial I}\left[\sigma^2(p, I)\right]\right\}$$

Like in eqn 4 The matrix S must be negative semi-definite, and further at each price S must lie and the kernel of price. Formally

$$\forall (p, I) \in (R^n, R)$$

1. $\forall V \in R^n \quad V^\intercal S(p, I)V \leq 0$
2. $p \cdot S(p, I) = S(p, I) \cdot p = \mathbf{0}$
3. $S(p, I)$ *has rank (n-1)*

Proof. By Lemma 4.1, if the mean variance pair is indeed generated by a population of n rational individuals, it must be that $S = \frac{1}{n} \sum_{i=1}^{n} \mathbf{D}_p h(p, u)$.

We know from Slutsky (1915) that:-

$$\forall V \in R^n \quad V^\intercal \mathbf{D}_p h(p, u)V \leq 0$$

$$\implies \sum V^\intercal \mathbf{D}_p h(p, u)V \leq 0$$

$$\implies V^\intercal \sum \mathbf{D}_p h(p, u)V \leq 0$$

$$\implies V^\intercal SV \leq 0$$

Similarly, for (2) $\mathbf{D}_p h(p, u)p = 0$

$$\implies \sum \mathbf{D}_p h(p, u)p = 0 \implies S(p, I) \cdot p = 0$$

When it comes to (3), notice that in the subspace $\mathbf{P} = [p]^\perp$, each individual substitution matrix must be **negative definite**, meaning

$$\forall V \in \mathbf{P} \quad V^\intercal \mathbf{D}_p h(p, I)V \lneq 0$$

which means that

$$\left[\sum_{i=1}^{n} V^\intercal \mathbf{D}_p h(p, I)V\right] = V^\intercal \left[\sum_{i=1}^{n} \mathbf{D}_p h(p, I)\right] V$$

$$= V^\intercal nS(p, I)V \lneq 0 \implies V^\intercal S(p, I)V \lneq 0$$

The above line implies that in the subspace **P**, the matrix S has full rank; as the dimension of **P** is (l-1), so must be the rank of S. However, from(2) we know that the rank of S cannot be more than (l-1) because at least one non zero vector lies in the kernel of S. Combining these 2 results, we can conclude that the rank of S must be exactly (l-1).

□

Remark. Out of the 3 restrictions we have in Theorem 4.2, the second translates to a restriction purely on the second term, i.e., the income derivative of variance, giving us the following lemma.

Lemma 4.2 (Variance Restrictions)**.** *Let $\sigma^2(p, I)$ be the variance of demand for a population of rational utility maximisers, then, following conditions must hold.*

$$\left[\frac{\partial}{\partial I}\sigma^2(p, I)\right] \cdot p = p \cdot \left[\frac{\partial}{\partial I}\sigma^2(p, I)\right] = 0$$

Remark. Even though there are "Slutsky type" restrictions on the data, we lose some power in terms of the symmetry conditions. Notice that the very construction we highlight above *necessitates* the symmetry of S because it is the sum of the "symmetrized" price derivative and a symmetric matrix. So, the only restrictions we have are those in Theorem 4.2.

Remark (Nature of Restrictions). Notice that restrictions (1) and (3) provided by the above tests are **open** in nature. Which is to say the following:-

Lemma 4.3. *Suppose a mean-variance pair (μ, σ^2) is jointly rationalizable. Further, suppose there is another mean variance pair $(\overline{\mu}, \overline{\sigma}^2)$ such that*

1. *$\overline{\mu}$ satisfies Walras law and homogeneity.*
2. $p \cdot \frac{\partial \overline{\sigma}^2}{\partial I} = \frac{\partial \overline{\sigma}^2}{\partial I} \cdot p = 0$

There exists an $\epsilon > 0$ such that for all $\epsilon' \leq \epsilon$, $(\mu + \epsilon'\overline{\mu}, \sigma^2 + \epsilon'\overline{\sigma}^2)$ must also satisfy the conditions in Theorem 4.2

Proof. The above statement holds because NSD and rank are open conditions on matrices. □

These restrictions are not as powerful as the ones on individual demand, where the Slutsky matrix must be *symmetric*, which is not an open condition. However, the very existence of restrictions is rather counter-intuitive because Chiappori and Ekeland (1999a)demonstrate the absence of any local restrictions to the shape of average demand. However, once the average demand and variance pair are observed, there are sharp data restrictions. As we had spoken about in the introduction, this highlights SMD theory's fragility in the face of variance data.

Before we move onto the next section, we give a simple example where utility maximization is contradicted.

Example 1 (Test of Demand). Suppose there are 2 goods, Let

$$\mu(p, I) = (\frac{I}{2p_1}, \frac{I}{2p_2})$$

and

$$\sigma^2(I, p) = \frac{1}{4}I^2 \begin{pmatrix} \frac{1}{p_1^2} & \frac{-1}{p_1 p_2} \\ \frac{-1}{p_1 p_2} & \frac{1}{p_2^2} \end{pmatrix} + \big(\mathcal{G}(p)\big)$$

Where $\mathcal{G}(p)$ is an arbitrary function of price.

Claim. The above mean variance pair is not jointly rationalizable.

Proof. Let us conduct our analysis; we see that

$$\hat{S} = \mathbf{D}_p\mu + \frac{\partial \mu}{\partial I}\mu^\intercal$$

$$= I \begin{pmatrix} -\frac{1}{2p_1^2} & 0 \\ 0 & -\frac{1}{2p_2^2} \end{pmatrix} + I \begin{pmatrix} \frac{1}{4p_1^2} & \frac{1}{4p_1 p_2} \\ \frac{1}{4p_1 p_2} & \frac{1}{4p_2^2} \end{pmatrix} = I \begin{pmatrix} -\frac{1}{4p_1^2} & \frac{1}{4p_1 p_2} \\ \frac{1}{4p_1 p_2} & -\frac{1}{4p_1^2} \end{pmatrix}$$

Further, we can compute the variance correction.

$$\frac{1}{2}\frac{\partial}{\partial I}\sigma^2(p, I) = \frac{1}{4}I \begin{pmatrix} \frac{1}{p_1^2} & \frac{-1}{p_1 p_2} \\ \frac{-1}{p_1 p_2} & \frac{1}{p_2^2} \end{pmatrix}$$

which gives us the "true" average substitution.

$$S = I \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Notice that the diagonal elements are non-negative, which contradicts utility maximization. Also, the rank condition is violated. This result is particularly interesting because the average demand is simply a Cobb-Douglas. However, given the specific form of average demand, the variance cannot arise from individual utility maximization. □

In the next section, we talk of conditions under which the variance correction disappears.

5 Normative Representative Consumers

One much studied problem in the economics of aggregation is the "exact aggregation" of consumer demands. It poses the following question.

When can a rational preference relation generate the demand of a population? Furthermore, when is this preference welfare relevant?.

This problem has been studied in Antonelli (1971) Gorman (1953) and Nataf (1954). (Gorman, 1953) showed that for this to be true for arbitrary distributions of income, income effects must be constant across the population, and independent of income.

Now, suppose an analyst observes average demand as a function of prices, all that he can check is if the average demand satisfies the Slutsky equation. Nevertheless, this agent may not, in general, be welfare-relevant for society. The 2 papers Jerison (1996, 1994) refer to this as a "positive" representative consumer, as opposed to the above case of "normative" representative consumer who is welfare relevant. They also show that the conditions for the existence of a positive normative consumer are much weaker than the "Gormann Conditions". Dow and da Costa Werlang (1988) show that this agent may be Pareto inconsistent, preferring situations in which each agent of society is worse off. Noticing these issues, constructing a test to see how "normative" a representative consumer is may be helpful; we construct such a test.

We first show a link between the Gorman conditions and our decomposition.

Proposition 5.1. *Let a group of consumers satisfy the "Gorman conditions", namely income effects are identical and independent of income. The "variance correction" $\frac{\partial}{\partial I}\sigma^2(p, I)$ is always zero.*

Proof. Suppose all individuals have identical income effects, that is to say, for each pair of agents

$$\forall i, j \quad \frac{dx_i}{dI} = \frac{dx_j}{dI} = k$$

Now let us see how a small change in income affects the variance of demand. By definition,

$$\sigma^2(p, I) = \frac{1}{n}\sum_{i=1}^{n}[(x_i(p, I) - \mu(p, I))(x_i(p, I) - \mu(p, I))^\intercal]$$

Now let us try to compute

$$x_i(p, I + \Delta I) - \mu(p, I + \Delta I) = [x_i(p, I) - \mu(p, I)] - \Delta I \left[\frac{dx_i}{dI} - \frac{d\mu}{dI}\right] \tag{6}$$

But we now must remember, that in fact

$$\mu(p, I) = \frac{1}{n}\sum_{i=1}^{n} x_i(p, I)$$

$$\implies \frac{d\mu}{dI} = \frac{1}{n}\sum_{i=1}^{n} \frac{dx_i}{dI}$$

now let invoke the Gorman condition, and replace every $\frac{dx_i}{dI}$ by a constant k. This will give us,

$$\frac{d\mu}{dI} = \frac{1}{n}\sum_{i=1}^{n} k = k$$

plugging this back into 6, we get

$$x_i(p, I + \Delta I) - \mu(p, I + \Delta I) = [x_i(p, I) - \mu(p, I)] - \Delta I [k - k]$$

$$\implies x_i(p, I + \Delta I) - \mu(p, I + \Delta I) = [x_i(p, I) - \mu(p, I)]$$

now plugging this back into the equation to determine σ^2, we get

$$\sigma^2(p, I + \Delta I) = (x_i(p, I + \Delta I) - \mu(p, I + \Delta I))(x_i(p, I + \Delta I) - \mu(p, I + \Delta I))^{\intercal}$$

but for reasons we just showed above,

$$(x_i(p, I + \Delta I) - \mu(p, I + \Delta I))(x_i(p, I + \Delta I) - \mu(p, I + \Delta I))$$

$$= (x_i(p, I) - \mu(p, I))(x_i(p, I) - \mu(p, I))$$

but the right hand side is just $\sigma^2(p, I)$ so it must be that $\sigma^2(p, I + \Delta I) = \sigma^2(p, I)$ which means that

$$\frac{d\sigma^2}{dI} = 0$$

□

Now we prove the general theorem we had alluded to above. Which allows us to test for aggregation conditions given mean-variance data. However, for this, we give two conditions that we think a "reasonable" representative agent would satisfy **at a point**.

1. *The Slutsky Condition*: $\mathbf{D}_p\mu(p,I) - \frac{\partial\mu}{\partial I}\mu(p,I)^\intercal$ is symmetric and negative semi-definite.

2. *Normativity*: $\frac{\partial\mu}{\partial I}\mu(p,I)^\intercal = \frac{1}{n}\sum_{i=1}^{n}\frac{\partial x}{\partial I}x(p,I)^\intercal$

The first condition states that the mean demand behaves like an individual maximizing utility. The second condition requires the individual to approximate average welfare measures for society.

Theorem 5.1. *The variance correction term is zero if and only if the mean demand satisfies both the slutsky condition and normativity.*

Proof. The If part:-

Suppose the Slutsky condition and normativity were to be satisfied this means that $\mathbf{D}_p\mu(p,I)+\frac{\partial\mu}{\partial I}\mu(p,I)^\intercal = \hat{S}$ is symmetric, in particular, plugging this into equation 4 we get,

$$S = \frac{1}{2}\left\{\mathbf{D}_p\left[\mu(p,I)\right] + \mathbf{D}_p\left[\mu(p,I)\right]^\intercal + \frac{\partial}{\partial I}[\mu(p,I)][\mu(p,I)]^\intercal\right\} + VC$$

$$= \frac{1}{2}\left[[\mathbf{D}_p\mu(p,I) + \frac{\partial\mu}{\partial I}\mu(p,I)^\intercal] + [\mathbf{D}_p\mu(p,I) + \frac{\partial\mu}{\partial I}\mu(p,I)^\intercal]^T\right] + VC$$

$$= \frac{1}{2}\left[\hat{S} + \hat{S}^\intercal\right] + VC$$

$$= \hat{S} + VC$$

now by normativity $\hat{S} = S$ which means that VC must be zero.

Only if part, Suppose that the variance correction is 0, this means that

$$\frac{d}{dI}\sigma^2(p,I) = \frac{d}{dI}\left\{\frac{1}{n}\sum_{i=1}^{n}(x_i x_i)^\intercal - (\mu_i)(\mu_i)^\intercal\right\} = 0$$

Simplifying, we get,

$$\frac{\partial\mu}{\partial I}\mu^\intercal = \frac{1}{n}\sum_{i=1}^{n}\frac{\partial x_i}{\partial I}(x_i)^\intercal$$

This is the same as the normativity condition. Now we can differentiate demand for every individual i, giving us,

$$\mathbf{D}_p x_i - \frac{\partial x_i}{\partial I}(x_i) = S_i$$

which is symmetric and NSD, adding across individuals and dividing across individuals, we get that.

$$\mathbf{D}_p\mu - \frac{1}{n}\sum_{i=1}^{n}\frac{\partial x_i}{\partial I}(x_i) = \frac{1}{n}\sum_{i=1}^{n} S_i$$

plugging in the above equation, we get,

$$\mathbf{D}_p\mu - \frac{\partial \mu}{\partial I}\mu^\intercal = \frac{1}{n}\sum_{i=1}^{n} S_i$$

where the RHS is symmetric NSD, which proves that the mean demand satisfies the Slutsky equation.

□

Remark. Our result makes intuitive sense if one buys our method of decomposition. If society were indeed exactly aggregatable to a representative agent, it must be that the only effect which influences substitution is the "symmetrized demand". Thus the variance correction must disappear.

Remark. The conditions we have highlighted are weaker than those that Gorman found because we restrict ourselves to constant income distributions in society.

Remark (**The Law of Demand**). What Hildenbrand (1983) refers to as the "Law of Demand" as follows.

$$(p_1 - p_2)^\intercal(\mu(p_1, I) - \mu(p_2, I)) \leq 0$$

He shows that if $\frac{\partial \sigma^2}{\partial I}$ is positive semidefinite then $\mathcal{D}_p\mu(p, I)$ is NSD. This implies that the law of demand holds. However, this does not imply the Slutsky condition, which would mean that $\mathcal{D}_p\mu(p, I) + \frac{\partial \mu(p,I)}{\partial I}\mu^\intercal$ is symmetric and NSD.
So it does not guarantee even a "positive" representative consumer.

Remark (**Measure of Normativity**). We can think of the term $\frac{\partial \sigma^2}{\partial I}$ as a "distance" from normativity or how badly the representative agents mimics average welfare..

6 Second Order Welfare Computation

The above analysis leads to the last natural application of our results we discuss before moving to cross-sectional data.

Definition 4 (Average Equivalent Variation)**.** Suppose there is an increase of prices from p_1 to p_2, the mean equivalent variation, AEV$(p1, p2)$, is the minimum average transfer to be given to the population to make them at least as well off as they were earlier.

$$AEV(p_1, p_2) = \frac{1}{n}\sum_{i=1}^{n}[e_i(p_1, V_i(p_1, I)) - e_i(p_2, V_i(p_1, I))] = I - \frac{1}{n}e_i(p_2, V_i(p_1, I))$$

Where $V_i(p, I)$ represents the *indirect utility function* defined as

$$V(p, I) = u_i(x_i(p, I))$$

and $e_i(p, u)$ is the *expenditure function* of the ith individual, defined as

$$e_i(p, u) = \inf\{e \in \mathbb{R} | V_i(p, I) \geq u\}$$

There is a long tradition in economics that tries to estimate Equivalent Variation (EV) empirically from consumer data under price changes. The existing approaches to compute this measure, such as Vartia (1983)Hausman (1981)Hausman and Newey (1995), attempt both computational and analytic methods, trying to solve a differential equation for the expenditure function through direct use of Shepherd's Lemma.

$$\frac{\partial e(p, u)}{\partial p_i} = -h_i(p, u)$$

Define $S(t, I) = I - e(p(t), u_0)$, a direct application of Shepherd's lemma gives us

$$\frac{\partial S}{\partial t} = -\frac{x(p(t), I - S(t, I))}{\partial t}$$

This technique was first used by Hausman (1981) who solved the problem for analytically several well studied demand systems. Vartia (1983) gave computational approaches to solve the above problem and Hausman and Newey (1995) used non-parametric estimation to compute the metric in situations where the analyst faces unobserved heterogeneity in the population.

The problem with these approaches is that they are non-linear and, as a result, do not aggregate well from individual to aggregate observations. The only case where this can is possible is the case where income effects are constant across the population as observed by in a general sense by Gorman (1953), the partial dual of this result was shown by Hausman and Newey (2016), where they showed non-identification in the case where income effects vary across the population.

This literature has attempted, in several ways, to try to compute average equivalent from mean demands. The most related to our approach is Schlee (2007), which gives assumptions under which equivalent variation of mean demand acts as bound for average equivalent variation. Hausman and Newey (2016) uses an approach which leverages bounds on income effects to give both upper and lower bounds to AEV. (Blundell, Browning, and Crawford, 2003) shows that if the idiosyncratic preference component in the demand equation is multiplicatively separable in preference type and the price-income pair, the EV of mean demand is a first order approximation to AEV. We differ from the above approaches by using approximations of compensating incomes; instead, we directly approximate compensated demands.

Theorem 6.1. *Let (μ, σ^2) be a mean variance demand pair,*

$$AEV(p_1, p_2) \approx (p_1 - p_2) \cdot \mu(p_1, I) + \frac{1}{4}(p_1 - p_2)^\intercal \left[\hat{S} + \hat{S}^\intercal + \frac{\partial}{\partial I}\sigma^2\right] (p_1 - p_2)$$

Proof. By shepherds lemma

$$\frac{\partial e_i(p, u)}{\partial p} = h_i(p, u) \tag{7}$$

Further, by definition of the substitution matrix is:-

$$S_i(p, I) = \mathbf{D}_p h_i(p, u)$$

this gives us 2 approximations:

1. Let $\mathcal{P}(t) : [0, 1] \to \mathbf{R}^l$ be a path from p_1 to p_2 indexed by t. Summing equation 7 across society, and integrating, we get,

$$\frac{1}{n}\sum_{i=1}^{n}[e_i(p_1, u) - e_i(p_2, u)] = \int_0^1 \left\{\frac{1}{n}\sum_{i=1}^{n} h_i(\mathcal{P}(t), V_i(p_1, I))\right\} \frac{d\mathcal{P}}{dt} \tag{8}$$

Remark. If we assumed that the compensated demand equals the demand, we could use this equation to compute consumer surplus.

2. We also have for equation 4

$$\frac{1}{2}\left\{\hat{S} + \hat{S}^T + \frac{\partial \sigma^2}{\partial I}\right\} = S = \sum_{i=1}^{n} \mathbf{D}_p h_i(p, u)$$

Using this, we can generate a first degree approximation for the average Hicksian demands.

$$\frac{1}{n}\sum_{i=1}^{n} h_i(p + \Delta p, V_i(p, I)) = \frac{1}{n}\sum_{i=1}^{n}[h_i(p, V_i(p, I)) + \mathbf{D}_p h_i(p, u)\Delta p]$$

$$= \frac{1}{n}\sum_{i=1}^{n} h_i(p, V_i(p,I)) + \frac{1}{n}\left\{\sum_{i=1}^{n} \mathbf{D}_p h_i(p,u)\right\}\Delta p$$

$$\mu(p,I) + S\Delta p = \mu(p,I) + \frac{1}{2}\left\{\hat{S} + \hat{S}^T + \frac{\partial \sigma^2}{\partial I}\right\}\Delta p \tag{9}$$

We now try to combine the above 2 equations to get a second order approximation for the average equivalent variation. In order to do this, We choose the linear path defined by,

$$\mathcal{P}(t) = p_1 + t(p_2 - p_1)$$

We can plug this back into expression 8 to get,

$$AEV(p_1, P_2) = \int_0^1 (p_2 - p_1)^\intercal \cdot \left\{\frac{1}{n}\sum_{i=1}^{n} h_i(p_1 + t(p_2 - p_1), V_i(p_1, I))\right\} dt$$

Now we can use our second approximation, equation 9 into the right hand side; this gives us:-

$$\int_0^1 (p_2 - p_1)^\intercal \cdot \left\{\mu(p_1, I) + \frac{t}{2}\left\{\hat{S}(p_1, I) + \hat{S}(p_1, I)^T + \frac{\partial \sigma^2(p_1, I)}{\partial I}\right\}(p_2 - p_1)\right\} dt$$

$$AEV \approx (p_2 - p_1)^\intercal \mu(p_1, I) + \frac{1}{4}(p_1 - p_2)^\intercal \left[\hat{S} + \hat{S}^\intercal + \frac{\partial}{\partial I}\sigma^2\right](p_1 - p_2) \tag{10}$$

Which is what was to be shown □

Remark. Denote $(p_2 - p_1)$ as Δp If society does obey the slutsky condition, then $\hat{S}$ is symmetric, which means that:-

$$AEV \approx \Delta p^\intercal \mu(p_1, I) + \frac{1}{2}\Delta p^\intercal \left[\hat{S}\right]\Delta p + \frac{1}{4}\Delta p^\intercal \left[\frac{\partial}{\partial I}\sigma^2\right]\Delta p$$

Notice that $\Delta p^\intercal \mu(p_1, I) + \frac{1}{2}\Delta p^\intercal \left[\hat{S}\right]\Delta p$ is the welfare computed by treating mean demand as representative for society. Again, this clarifies how the "variance correction" differentiates normative consumers from positive consumers.

Remark. Also, notice that the *local* performance of mean demand at approximating welfare depends only on the magnitude of the derivative of variance *with respect to* income, but, *in the direction of the price change*. This is different in flavor yet somewhat similar to the income effect "Bounds" approach leading to upper and lower bounds on exact surplus (Hausman and Newey, 2016, Theorem 3).

We give an approximation but use not just bounds but also some information on the distribution of income effects that the variance of consumption provides.

7 Cross Sectional Data

Now we go through the final main results in the paper; one could argue that observing the variance of demand is asking a little too much from the data, and though we do have access to a lot of cross-sectional data, i.e., quantile distributions of demand for particular goods, joint distributions of consumption are hard to compute. To this criticism, we have 2 reposts.

1. Because of the nature of our tests and results, which are independent of the true magnitude, as highlighted in remark 4, our methods may be attractive even if more micro data is available if data suffers a large additively separable measurement error.

2. The cross sectional data gives us access to the variance of specific goods and their changes with respect to price; this by itself gives us some observable restrictions and lets us compute welfare approximations when only 1 price is changing. Furthermore, the nature of the variance lets us construct bounds on the possibilities of the σ^2 matrix, which then lets us construct bounds for the measures we would like to compute.

Equation 4 is a matrix equation; if there are l goods, it can be thought of as $l \times l$ linear equations.

Observing the cross-sectional variance is "as if" we observe only l of these equations, namely the ones corresponding to the diagonal elements of the matrix. This can be rewritten as follows for the l^{th} good.

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\frac{\partial h_i^l(p,u)}{\partial p_l} &= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{\partial x_i^l(p,I)}{\partial p_l}\right] + \mu(p,I)\frac{\partial \mu(p,I)}{\partial I} + \frac{\partial \sigma_{ll}^2}{\partial I} \\ &= \frac{\partial \mu^l(p,u)}{\partial p_l} + \mu(p,I)\frac{\partial \mu(p,I)}{\partial I} + \frac{\partial \sigma_{ll}^2}{\partial I}\end{aligned}$$

So this already gives us an observable restriction on cross sectional data, namely, for every good,

$$\frac{\partial \mu^l(p,I)}{\partial p_l} + \mu(p,I)\frac{\partial \mu(p,I)}{\partial I} + \frac{\partial \sigma_{ll}^2}{\partial I} \leq 0$$

Welfare Computation

Single Price Changes

Now suppose the price of only one good changes; WLOG we can assume that it is the first good which changes. By equation 10, we get

$$AEV(p_1^1, p_2^1) \approx (p_1^1 - p_2^1)\mu^1(p,I) + (p_1^1 - p_2^1)\frac{1}{4}\left[\mu(p,I)\frac{\partial \mu(p,I)}{\partial I} + \frac{\partial \sigma_{ll}^2}{\partial I}\right](p_1^1 - p_2^1)$$

$$= (p_1^1 - p_2^1)\mu^1(p,I) + \frac{(p_1^1 - p_2^1)^2}{4}\left[2\mu(p,I)\frac{\partial\mu(p,I)}{\partial I} + \frac{\partial\sigma_{ll}^2}{\partial I}\right]$$

$$= \underbrace{(p_1^1 - p_2^1)\mu^1(p,I) + \frac{(p_1^1 - p_2^1)^2}{2}\mu(p,I)\frac{\partial\mu(p,I)}{\partial I}}_{\text{Mean Demand Term}} + \underbrace{\frac{(p_1^1 - p_2^1)^2}{4}\frac{\partial\sigma_{ll}^2}{\partial I}}_{\text{Variance Correction}}$$

This result is akin to Hausman (1981), who showed that computation of EV for price changes of one good only requires knowledge of income effects for that specific good.

Multiple Price Changes

If we face multiple price changes, then there are 2 methods in which changes in average welfare can be approximated.

1. TThe first way is to vary the prices of goods to generate an approximate path to be able to compute welfare.
2. TThe second approach uses cross-sectional data to find bounds to find bounds for the σ^2 matrix; this can then give bounds on welfare approximations and even generate testable restrictions.

Bounding $\frac{\partial}{\partial I}\sigma^2$ When not all Entries are Observed

In order to generate bounds on the matrix, if we only observe the diagonal elements, realize the following must be the case.

1. The Matrix must be symmetric
$$\sigma_{ij}^2 = \sigma_{ji}^2$$
2. It must lie in the kernel of price
$$p \cdot \frac{\partial}{\partial I}\sigma^2 = \frac{\partial}{\partial I}\sigma^2 \cdot p = 0$$
3. It must satisfy the Cauchy Schwarz inequality
$$[\sigma_{ij}^2] \leq \sqrt{\sigma_{ii}^2\sigma_{jj}^2}$$

These 3 equations ensure that the parameters we require have bounded support, even if we only observe the diagonal elements.

8 Conclusion

We have argued that when mean demand is observable, the variance of consumption is a parameter that is of interest. It enables the analyst to place testable restrictions on data that they observe and conduct counterfactual welfare analysis arising from unobserved situations in the data. We hope to have shown that at least the second moment and perhaps the other higher moments of population demand carry much important information.

References

J. Andreu. Rationalization of market demand on
nite domains. *Journal of Economic Theory*, 28:201–204, 1983.

G. Antonelli. On the mathematical theory of political economy. In J. Chipman, L. Hurwicz, M. Richter, and H. Sonnenschein, editors, *Preference, Utility and Demand*. Harcourt Brace Jovanavich Ltd, 1971.

Kenneth J Arrow. Economic theory and the hypothesis of rationality. In *Utility and Probability*, pages 25–37. Springer, 1990.

Richard W. Blundell, Martin Browning, and Ian A. Crawford. Nonparametric Engel Curves and Revealed Preference. *Econometrica*, 71(1): 205–240, January 2003. URL https://ideas.repec.org/a/ecm/emetrp/v71y2003i1p205-240.html.

D. Brown and R. Matzkin. Testable restrictions on the equilibrium manifold. *Econometrica*, 64:1249–1262, 1996.

P. A. Chiappori and I. Ekeland. Aggregation and Market Demand: An Exterior Differential Calculus Viewpoint. *Econometrica*, 67(6):1435–1458, November 1999a. URL https://ideas.repec.org/a/ecm/emetrp/v67y1999i6p1435-1458.html.

P.-A. Chiappori and I. Ekeland. Aggregation and market demand: an exterior differential calculus viewpoint. *Econometrica*, 67:1435–1457, 1999b.

P.-A. Chiappori and I. Ekeland. The microeconomics of group behavior: General characterization. Unpublished manuscript, 2002.

P.-A. Chiappori and I. Ekeland. Individual excess demand. *Journal of Mathematical Economics*, 40:41–57, 2004.

P.-A. Chiappori, I. Ekeland, F. Kubler, and H. Polemarchakis. Testable implications of general equilibrium theory: a differentiable approach. *Journal of Mathematical Economics*, 40:105–119, 2004. URL http://www.polemarchakis.org/a64-tge.pdf.

G. Debreu. Excess demand functions. *Journal of Mathematical Economics*, 1:000–000, 1974.

Holger Dette, Stefan Hoderlein, and Natalie Neumeyer. Testing multivariate economic restrictions using quantiles: The example of Slutsky negative semidefiniteness. *Journal of Econometrics*, 191(1):129–144, 2016.

doi: 10.1016/j.jeconom.2015.07. URL https://ideas.repec.org/a/eee/econom/v191y2016i1p129-144.html.

W. E. Diewert. Generalized slutsky conditions for aggregate consumer demand functions. *Journal of Economic Theory*, 15(2):353–362, August 1977. URL https://ideas.repec.org/a/eee/jetheo/v15y1977i2p353-362.html.

James Dow and Sérgio Ribeiro da Costa Werlang. The consistency of welfare judgments with a representative consumer. *Journal of Economic Theory*, 44(2):269–280, 1988.

Andrew Foster and Jinyong Hahn. A consistent semiparametric estimation of the consumer surplus distribution. *Economics Letters*, 69(3):245–251, December 2000. URL https://ideas.repec.org/a/eee/ecolet/v69y2000i3p245-251.html.

J. D. Geanakoplos and H. Polemarchakis. On the disaggregation of excess demand functions. *Econometrica*, 48:217–229, 1980. URL http://www.polemarchakis.org/a13-ded.pdf.

W. M. Gorman. Community preference fields. *Econometrica*, 21:63–80, 1953.

Jean-Michel Grandmont. Distributions of preferences and the" law of demand". *Econometrica: Journal of the Econometric Society*, pages 155–161, 1987a.

J.M. Grandmont. Stabilizing competitive business cycles. *Journal of Economic Theory*, 40:57–76, 1987b.

J.M. Grandmont. Transformation of the commodity space, behavioral heterogeneity and the aggregation problem. *Journal of Economic Theory*, 57: 1–35, 1992.

Birgit Grodal and Werner Hildenbrand. Cross-section engel curves, expenditure distributions, and the "law of demand". In *Aggregation, Consumption and Trade*, pages 37–53. Springer, 1992.

Wolfgang Härdle, Werner Hildenbrand, and Michael Jerison. Empirical evidence on the law of demand. *Econometrica: Journal of the Econometric Society*, pages 1525–1549, 1991.

Jerry A Hausman. Exact consumer's surplus and deadweight loss. *The American Economic Review*, 71(4):662–676, 1981.

Jerry A Hausman and Whitney K Newey. Nonparametric estimation of exact consumers surplus and deadweight loss. *Econometrica: Journal of the Econometric Society*, pages 1445–1476, 1995.

Jerry A Hausman and Whitney K Newey. Individual heterogeneity and average welfare. *Econometrica*, 84(3):1225–1248, 2016.

Jerry A. Hausman and Whitney K. Newey. Nonparametric Welfare Analysis. *Annual Review of Economics*, 9(1):521–546, September 2017. doi: 10.1146/annurev-economics. URL https://ideas.repec.org/a/anr/reveco/v9y2017p521-546.html.

W. Hildenbrand. On the law of demand. *Econometrica*, 51:997–1019, 1983.

W. Hildenbrand. *Market Demand*. Princeton University Press, 1994.

Stefan Hoderlein and Jörg Stoye. Revealed Preferences in a Heterogeneous Population. *The Review of Economics and Statistics*, 96(2): 197–213, May 2014. URL https://ideas.repec.org/a/tpr/restat/v96y2014i2p197-213.html.

Stefan Hoderlein and Jörg Stoye. Testing stochastic rationality and predicting stochastic demand: the case of two goods. *Economic Theory Bulletin*, 3(2):313–328, October 2015. doi: 10.1007/s40505-014-0061-5. URL https://ideas.repec.org/a/spr/etbull/v3y2015i2d10.1007_s40505-014-0061-5.html.

M. Jerison. Nonrepresentative representative consumers. Unpublished manuscript, 1996.

Michael Jerison. Social welfare and the unrepresentative representative consumer. *DP, SUNY*, 1984.

Michael Jerison. Optimal income distribution rules and representative consumers. *The Review of Economic Studies*, 61(4):739–771, 1994.

F. Kubler. Observable restrictions on general equilibrium with financial markets. *Journal of Economic Theory*, 110:137–153, 2003.

Arthur Lewbel. Demand systems with and without errors. *American Economic Review*, 91(3):611–618, 2001.

R. Mantel. On the characterization of aggregate excess demand. *Journal of Economic Theory*, 7:348–353, 1974.

R. Mantel. Homothetic preferences and community excess demand functions. *Journal of Economic Theory*, 12:197–201, 1976.

Rolf R. Mantel. Implications of Microeconomic Theory for Community Excess Demand Functions. Cowles Foundation Discussion Papers 409, Cowles Foundation for Research in Economics, Yale University, 1975. URL https://ideas.repec.org/p/cwl/cwldpp/409.html.

F Marhuenda. Distribution of Income and Aggregation of Demand. *Econometrica*, 63(3):647–666, May 1995. URL https://ideas.repec.org/a/ecm/emetrp/v63y1995i3p647-66.html.

Daniel McFadden. Revealed stochastic preference: a synthesis. *Economic Theory*, 26(2):245–264, August 2005. doi: 10.1007/s00199-004-0495-3. URL https://ideas.repec.org/a/spr/joecth/v26y2005i2p245-264.html.

Daniel McFadden and Marcel K Richter. Stochastic rationality and revealed stochastic preference. In John S Chipman, Daniel McFadden, and Marcel K Richter, editors, *Preferences, Uncertainty, and Rationality*, pages 161–186. Westview Press, 1991.

André Nataf. *Sur des questions d'agrégation en économétrie*. PhD thesis, Sc. math., Paris, 1954.

John K.-H. Quah. The Law of Demand when Income Is Price Dependent. *Econometrica*, 65(6):1421–1442, November 1997. URL https://ideas.repec.org/a/ecm/emetrp/v65y1997i6p1421-1442.html.

S. Abu Turab Rizvi. The Sonnenschein-Mantel-Debreu Results after Thirty Years. *History of Political Economy*, 38(5):228–245, Supplemen 2006. URL https://ideas.repec.org/a/hop/hopeec/v38y2006i5p228-245.html.

Edward E Schlee. Measuring consumer welfare with mean demands. *International Economic Review*, 48(3):869–899, 2007.

E. Slutsky. Sulla teoria del bilancio del consumatore. *Giornale degli Economisti*, 51:1–26, 1915.

H. Sonnenschein. Do walras' identity and continuity characterize the class of community excess demand functions? *Journal of Economic Theory*, 6: 345–354., 1973a.

Hugo Sonnenschein. Market Excess Demand Functions. *Econometrica*, 40 (3):549–563, May 1972. URL https://ideas.repec.org/a/ecm/emetrp/v40y1972i3p549-63.html.

Hugo Sonnenschein. The Utility Hypothesis and Market Demand Theory. Discussion Papers 51, Northwestern University, Center for Mathematical Studies in Economics and Management Science, September 1973b. URL https://ideas.repec.org/p/nwu/cmsems/51.html.

Yrjö O Vartia. Efficient methods of measuring welfare change and compensated income in terms of ordinary demand functions. *Econometrica: Journal of the Econometric Society*, pages 79–98, 1983.

Eduardo Zambrano and Timothy J. Vogelsang. A Simple Test of the Law of Demand for the United States. *Econometrica*, 68(4):1013–1022, July 2000. URL https://ideas.repec.org/a/ecm/emetrp/v68y2000i4p1013-1022.html.