Personalization, Discrimination and Information Revelation

Sinem Hidir, Nikhil Vellodi*

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Abstract

We study a game in which a buyer has private valuations over a seller's product range, and can communicate these through costless messaging. The seller's preferred outcome is a fully revealing, efficient equilibrium that secures him the entire surplus whilst matching each buyer type to their preferred product. The buyer's preferred outcome is partially revealing; he releases only enough information to guarantee trade. This outcome is inefficient - almost all types purchase a sub-optimal product. Such a characterization rests on a natural trade-off faced by the buyer - reveal his type in order to secure his preferred good, or hide his type in order to secure a better price. We use our results to speak directly to the debate regarding *product steering* versus *price discrimination* in online retail.

JEL Classification: D82, L11, L12. **Keywords**: cheap-talk, bargaining, price discrimination, product differentiation.

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1 Introduction

We study a simple game between a privately informed buyer and a seller, in which the buyer communicates his preferences over the seller's product line. The buyer faces a fundamental trade-off: should he reveal what he likes, in order to get his preferred product, or hide what he likes, in order to get the best price?

Such deliberations are central to the debate regarding consumer privacy in online retail. Consider a consumer searching via an online retail site. The consumer naturally benefits from supplying the site with finer information, in order that product recommendations be of greater relevance - a practice often referred to as *product steering*. However, finer information allows the site to potentially engage in *price discrimination*, securing the gains from the supply of information for itself. A recent White House Report on personalized pricing describes the problem explicitly[1]:

"Much of what companies learn... is used to design products and services that deliver more value to the individual consumer. At the same time, if sellers can accurately predict what a customer is willing to pay, they may set prices so as to capture much of the value in a given transaction"

Price discrimination is of first-order concern for consumers in such settings; [Hannak et al. (2014)] find that 9 out the 10 e-commerce sites they studied engaged in some form of both steering and price discrimination based on consumer-specific information, with prices differing in some instances by hundreds of dollars. As the White House report continues to argue, *"Whilst there are substantial concerns about differential pricing,... whether [it] helps or harms the average consumer depends on how and where it is used."*

[1]See https://obamawhitehouse.archives.gov/sites/default/files/whitehouse_files/docs/Big_Data_Report_Nonembargo_v2.pdf

Such issues arise in many other contexts, ranging from market trading stands, to worker-firm relationships. Consider an employee who knows that her abilities are particularly suited to a specific role in her organization. She might want to signal this, and yet worry the firm might leverage this information to capture the gains for itself via an increased workload. Finally, the problem might also arise in intermediated trades - a real estate agent, who typically takes commission from the price of the transaction, might have strong incentives to release the buyer's private information to the seller, thus allowing for a higher traded price.

A few features are essential for such a trade-off to exist. First, the seller must lack commitment power. If the seller can commit to prices in advance, the buyer need not fear the prospect of price discrimination.[2] Second, there must be a degree of horizontal differentiation in preferences. If it is commonly known that one product is preferred to all the others, then the seller will offer that product for sure, thus obviating the role of communication.

We combine these features in a simple bargaining game with cheap-talk. A buyer faces the seller of multiple, heterogeneous goods, over which the buyer has private valuations. They engage in the following bargaining game. First, the buyer sends a costless signal of their valuation type to the seller. The seller then makes a take-it-or-leave-it price offer for a good from his stock. Finally, the buyer may accept or reject this offer.

As is often the case in games with cheap-talk, our model admits a large multiplicity of equilibria. We begin by examining each player's preferred equilibrium outcome. For the seller, the most informative equilibrium is optimal, as it allows the greatest scope for price discrimination. Indeed, in the benchmark model, a fully separating equilibrium exists that transfers the entire surplus to him — Proposition 1. Each type

[2]See [Roesler and Szentes (2017)] for a similar assumption on seller's lack of commitment.

is offered their preferred good at precisely their willingness-to-pay. Furthermore, since the matching between types and goods is perfect, this allocation also maximizes overall welfare.

Characterizing the buyer's preferred outcome — Theorem 6 — constitutes the main result of the paper. Whilst the construction is subtle, the equilibrium itself admits a simple description; it is the *least informative equilibrium that guarantees trade*, a result that relies on the buyer's mixed incentives to pool. The intuition is as follows. The buyer's gain from pooling comes through price discounts; if types pool, the seller's optimal price offer decreases, as he attempts to trade with a broader set of types. If these discounts are sufficiently large, this gain outweighs the buyer's loss from being offered a sub-optimal good. However, if the seller's price offer is not discounted enough, the buyer's loss from being mis-matched outweighs the price gain, resulting in a breakdown in trade. In the language of online retail, the buyer seeks to balance the *gains from product steering* against the *losses from price discrimination*. Our analysis thus contributes to the literature on consumer privacy by formally analysing this trade-off. This simple characterization extends to richer menus of offers, as discussed in Section 4.1 and more general preferences, as discussed in Appendix B.

Whilst sender-optimality has been used as a reasonable equilibrium selection criterion throughout the cheap-talk literature,[3] we appeal to Farrell's early notion of "neologism-proofness"to help discipline our predictions further. We show that the set of neologism-proof equilibria involve only partial information revelation. This proposition is unusual in that in most applications, refinements generally serve to support greater separation, not less.[4] We characterize the refined set fully — Theorem

[3]See [Blume, Kim and Sobel (1993)] for an extensive discussion, as well as [Lipnowski and Ravid (2017)] more recently.

[4]Farrell gives an example of a 2×2 game in which the unique neologism-proof equilibrium is babbling, not separating (Example 2). Not only does our analysis offer a natural economic setting in which such forces might prevail, but it also shows how babbling may itself not be neologism-proof.

10 — proving that the ex-ante buyer-optimal equilibrium is always contained in it, and compute bounds on both buyer welfare, as well as on how informative such equilibria are.

The model also delivers several comparative static results. For instance, as goods become less substitutable, more information is revealed; there is little point buying a worthless good, even at a discount. We also show that the buyer's share of the surplus in his preferred equilibrium varies non-monotonically. Two countervailing forces are at play here. As the buyer's loss from product mismatch increases, information provision increases in a manner that does not perfectly offset this loss.

The paper is structured as follows. Section 2 introduces the benchmark model. Section 3 characterizes both the seller- and buyer-optimal equilibria, discussing some comparative static and welfare implications. The benchmark model is extended by considering richer menus of offers in Section 4.1, whilst Section 4.2 is devoted to characterizing the set of neologism-proof equilibria. Section 5 outlines the contribution of our paper to various literatures, whilst Section 6 concludes with some possible extensions. All proofs are relegated to the Appendix, which also extends the main results to a broader set of preferences, as well as discussing alternative refinement concepts.

2 Model

A buyer and a seller interact. The seller has a commonly known stock of indivisible, heterogeneous goods, each indexed by $V = [0, 1]$. The buyer's privately known valuation for a good v is captured by some θ drawn from an ambient set Θ, which we also set as $[0, 1]$. A buyer of type θ has willingness-to-pay for good v given by $u(v, \theta)$, where $u \in \mathcal{C}^2(V \times \Theta)$. The seller's prior over Θ is given by some continuous distribution function F. The seller values all goods at 0; that is, we focus on the sub-game following any production or procurement required by the seller. To make the central point as cleanly as possible, we employ the following specification in the main part of the paper: $u(v,\theta) = \bar{u} - a(v-\theta)^2$, where $\bar{u}, a > 0, V = [0,1]$ and $F \sim \mathcal{U}[0,1]$.

2.1 Game

The players play the following game, designed to capture the main point as simply as possible. First, the buyer sends a message $m \in \mathcal{M}$ to the seller, where $\mathcal{M}$ is some set large enough to reveal θ. The seller then makes an offer of some good v to the buyer at price p, which the buyer either accepts or rejects. We focus on *Perfect Bayesian equilibria in pure strategies*. Given the simplicity of the game, and the familiarity of the solution concept, we relegate formal definitions to the Appendix.

2.2 Discussion of Model

We take a moment to discuss the modelling assumptions made above. The assumption that the buyer is privately informed of his type before communicating with the seller seems natural in most applications. We view our model as a reduced form description of consumer search via online platforms, in which communication is tantamount to performing a search query. As such, it seems natural to assume the consumer knows his own taste before searching for products. This assumption casts the model as one of cheap-talk, rather than persuasion alá [Kamenica and Gentzkow (2011)], and as such, gives rise to a multiplicity of equilibria. Far from viewing this as problematic, we see this as providing an opportunity to thoroughly understand the conflict between the seller and buyer by comparing their preferred outcomes. Furthermore, the buyer's preferred equilibrium, as identified in Section 3.3 comes with a particularly simple intuition.

The assumption that the seller can only offer one good from his range might be motivated on the grounds of limited inattention on the part of the buyer (see [Ravid (2017)]) or costly search through alternatives (see [Eliaz and Spiegler (2015)] for an application to platforms). Nevertheless, the results of Section 4.1 demonstrate that with all but the most complex menus available to the seller, the main results still go through. The main takeaway is that more complex menus transfer surplus to the seller by forcing types to sort through their acceptance strategies. This last assumption also ties to the seller's lack of commitment. We argue that the motivating debate surrounding consumer privacy and price discrimination hinges on the seller being able to set prices *after* learning a consumer's preferences.

We assume a standard form of horizontal preferences. In Appendix B, we substantially generalize this preference structure , imposing a few standard assumptions such as single-crossing, single-peakedness and strict concavity. The main results of the paper, namely that the seller prefers the most informative outcome whilst the buyer prefers the least informative outcome that guarantees trade, remain robustly in tact.

3 Results

3.1 Seller Optimality: Full Revelation

Before proceeding to characterize the equilibrium set, I state and prove the existence of a fully revealing equilibrium. Such an equilibrium is unusual in models in which Θ is one-dimensional. Indeed, in the leading example of [Crawford and Sobel (1982)], full separation is not possible. In [Crawford and Sobel (1982)], preference misalignment was captured by a single, one-dimensional *bias* term. Here, the misalignment is over two dimensions. The buyer and seller are *fully aligned* over the optimal choice of good - both would like to bargain after settling on the buyer's ideal good, as this maximizes joint surplus. However, they are *fully misaligned* over the optimal price. It is this multi-dimensional aspect of alignment that permits a separating equilibrium to exist.

Proposition 1. *A fully revealing equilibrium exists. Furthermore, it maximizes total welfare, whilst transferring the entire surplus to the seller.*

Each type perfectly transmits their tastes to the seller, who then offers them their ideal good, but at the maximal price $\bar{u}$. Clearly, each type has no incentive to deviate, as they would be paying the same price for an inferior good. The seller achieves his first-best payoff. Whilst full separation is not robust to more general preferences, the essential feature of this equilibrium remains in tact, namely that the seller's preferred equilibrium is as revealing as possible.

3.2 A Complete Characterization

As this is a costless signalling model, we should expect multiple equilibria to exist. This is indeed the case. Each equilibrium falls into one of three categories: partitional, semi-separating, and fully separating.

Definition 2. For $n \in \mathbb{N}$, and an interval $I = [\theta, \theta'] \subset \Theta$, we say an equilibrium σ is **n-partitional on *I*** if there exist numbers $\theta = a_0 < a_1 < \cdots < a_n = \theta'$ such that $\mu_{i+1} := \mu(.|\theta'')$ is uniform on $[a_i, a_{i+1}]$ for $\theta'' \in [a_i, a_{i+1}]$. σ is **n-partitional** if it is n-partitional on Θ. Given an interval I, an equilibrium σ is **separating on *I*** if for all $\theta \in I$, $m(\theta)$ is invertible. We say an equilibrium σ is **semi-separating** if for $n, m \in \mathbb{N}$, there exists a partition $I_1, \ldots, I_n, J_1, \ldots, J_m$ of Θ such that σ is separating on $I_1, \ldots, I_n$ and partitional on $J_1, \ldots, J_n$.

Proposition 3. *1. For every $n \in \mathbb{N}$, there exists an n-partitional equilibrium.*

2. All equilibria either partitional, semi-separating or fully separating equilibrium.

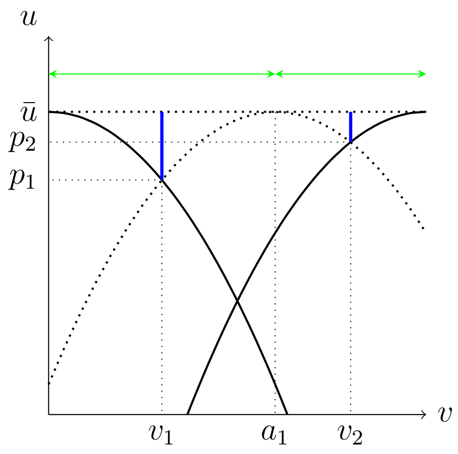


Figure 1: A 2-partitional equilibrium

3. *In a partitional equilibrium, it must be that each boundary type receives* 0*.*

Part 3 of Proposition 3 will be useful in the sequel, particularly in the construction of so-called self-signalling sets, and is a special case of the usual boundary indifference condition in partitional equilibria. If it were not true, the seller would effectively be leaving rents on the table. Figure 1 shows an example of an equilibrium, in which types pool into two distinct intervals, with the seller's optimal offer ensuring that all types trade; note that only the boundary types in this example are indifferent between accepting and rejecting.

3.3 Buyer Optimality: The Least Separating Clearing Equilibrium

We saw in Section 3.1 how a fully separating equilibrium exists and is trivially the seller's preferred outcome, securing him the entire surplus. Let's now turn to the buyer's *ex-ante* preferred equilibrium. It stands to reason then that some pooling will benefit the buyer, but how so? With full separation, the seller is able to set maximal prices. If some types pool, the seller is forced to offer a good that he knows to be inferior for the majority of these types, and so will optimally reduce prices in order to ensure trade, transferring surplus to the buyer. Of course, there are limits to this argument. Beyond a point, the seller's loss from reducing prices outweighs the gain from ensuring trade. The subsequent loss in trade harms both the seller and the buyer. To summarize, some pooling of types benefits the buyer through price discounts, but too much pooling harms the buyer through a reduction in trade. Our discussion naturally motivates the notion of a *clearing equilibrium*, an outcome in which trade is guaranteed.

Definition 4. An interval $I \subset \Theta$ is **clearing** if, faced with this set of types, the seller's optimal offer results in all types accepting, that is, $x^*(\theta, \mathrm{br}(I)) = 1$ for all $\theta \in I$. An interval is residual if it is not clearing. A partitional equilibrium σ is **clearing** if each member of its partition is clearing, and is residual if it is not clearing.

Lemma 1. *$I = [\theta, \theta']$ is clearing if and only if $|\theta' - \theta| \leqslant \Lambda(a, \bar{u})$, where $\Lambda(a, \bar{u}) = 2\sqrt{\frac{\bar{u}}{3a}}$*

Lemma 1 identifies a number — a *critical width* of the game — such that all partitional equilibria that contain only intervals smaller than this width are clearing, and all partitional equilibria with at least one interval larger than this width are residual. Armed with this number, we can provide a full description of clearing equilibria. Enough intervals are needed in the partition so that the width of each interval is below the critical width. In the benchmark model, the critical width is easy to calculate - it is the size of the interval served by the seller's static monopoly price, which is given by $p_M = \frac{2\bar{u}}{3}$.

Proposition 5. *Let $N^* = \lceil \frac{1}{\Lambda(a,\bar{u})} \rceil$, where $\lceil x \rceil$ maps the number x to the smallest following integer. Then for all $n \geqslant N^*$, there exists a clearing n-partitional equilibrium, and for $n < N^*$, no clearing equilibrium exists.*

There are many ways of identifying the lower bound N^* in Proposition 5. Our approach is constructive, based on the following steps. Start with a babbling equilibrium. If the width of the entire interval $[0, 1]$ exceeds the critical width $\Lambda(a, \bar{u})$, then this babbling equilibrium is clearing. Otherwise, it is residual. Now form the babbling equilibrium with the largest residual set possible, and create a new equilibrium by having this residual set form their own interval and the seller best respond. If this best response is itself residual, repeat the previous steps to keep "filling in the holes". Such an approach has the benefit of not only proving that N^* exists, but it also constructs a special clearing equilibrium. This equilibrium, which we will refer to as the **least separating clearing (LSC) equilibrium**, is essentially the least informative equilibrium, subject to it being clearing. The equilibrium is essentially unique, i.e. the players' ex-ante expected utilities are identical across any such equilibrium.

The preceding results formalize the sense in which too much pooling is harmful through a reduction in trade. Casting back to our earlier discussion, how can we capture the sense in which the buyer *prefers* pooling due to the subsequent price discounts? The following result does just that. Formally, it states that if two intervals can pool together and still be clearing, then almost all types in these intervals strictly prefer this outcome, as the seller lowers his price sufficiently, without reducing trade.

Lemma 2. *Take a clearing n-equilibrium σ with boundary types $0 = a_0 < \cdots < a_n = 1$. If $|a_{i+2} - a_i| \leqslant \Lambda(a, \bar{u})$ for some $i \in \{0, \ldots, n-2\}$, then all types $\theta \in (a_i, a_{i+2})$ strictly prefer the equilibrium σ' defined by the boundary types $a_0, \ldots, a_i, a_{i+2}, \ldots, a_n$, whilst all other types are indifferent between σ' and σ.*

Combining Lemma 2 and Proposition 5, we might postulate that the buyer's preferred equilibrium exhibits maximal pooling subject to ensuring trade. This intuition turns out to be correct, and describes precisely the LSC equilibrium identified in Proposition 5. We formalize this in the following theorem, which constitutes the main result of the paper.

Theorem 6. *The LSC equilibrium is the unique ex-ante buyer-optimal equilibrium.*

It should be noted that for $a \leqslant \frac{4\bar{u}}{3}, N^* = 1$, and hence the LSC equilibrium is babbling. That is, if goods are sufficiently substitutable, the buyer is willing to fully concede on the choice of good in favor of a price discount. On the other hand, as the buyer becomes less willing to be mis-matched, he will communicate his preferences more accurately. After all, if a buyer wants only one particular item in a shop, he can't help but reveal this. We formalize this logic with the following comparative static result, the proof of which is direct from the expression for N^* given in Proposition 5.

Corollary 7. *N^* is (weakly) increasing in a.*

We conclude this section by performing a second comparative static exercise; how does *ex ante* buyer welfare in the LSC equilibrium vary as goods become less substitutable? The answer is, perhaps surprisingly, non-monotonically.

Proposition 8. *Let $W_{LSC}(a)$ denote the ex ante buyer payoff under the LSC equilibrium for a given value of a. Then*

$$W_{LSC}(a) = \frac{a}{6}\left[(N^* - 1)\Lambda(a,\bar{u})^3 + \big(1 - (N^* - 1)\Lambda(a,\bar{u})\big)^3\right]$$

where $\Lambda(a,\bar{u}) = 2\sqrt{\frac{\bar{u}}{3a}}$ and $N^* = \lceil \frac{1}{2}\sqrt{\frac{3a}{\bar{u}}} \rceil$. In particular, $W_{LSC}(a)$ is continuous and non-monotone in a.

One might expect that as the buyer becomes more choosy, he is forced to reveal his preferences more precisely whilst facing higher prices. This intuition is only partially correct. In the LSC equilibrium, the buyer reveals precisely enough information to force the seller to charge the *same* (monopoly) price as often as possible. Thus, whilst the buyer's loss from product mismatch increases, increased information provision offsets this loss. This is best understood through Figure 3.3. In a loose sense then, the buyer is better of being selective, but not too much so.

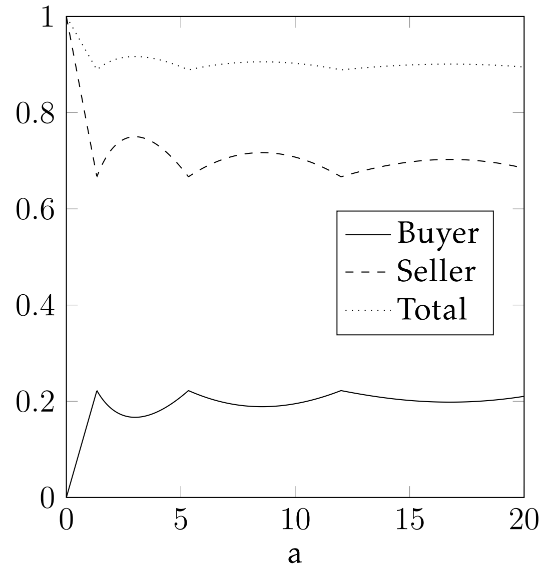


Figure 2: Buyer, seller *ex ante* welfare in the LSC equilibrium ($\bar{u} = 1$)

4 Discussion

4.1 Menus

Having the seller make offers with only one good greatly simplifies the analysis, allowing the key economic forces of the model to come through cleanly. This modelling assumption might be motivated on the grounds of rational inattention on the part of the buyer (see [Ravid (2017)]) or costly search through alternatives (see [Eliaz and Spiegler (2015)] for an application to platforms). In many applications however, sellers can certainly offer a selection of goods before the final choice is made.

Returning to the benchmark model, suppose the seller can offer a *complete* menu of take-it-or-leave-it offers $\{(v, p)\}_{v \in V}$. Then, for any belief, an optimal menu for the seller is simply $\{(v, \bar{u})\}_{v \in V}$. All types accept the offer pertaining to their ideal good, and receive 0, much as in the fully separating equilibrium of the benchmark model. The buyer's communication strategy becomes completely irrelevant. Such a situation would best model competitive retailers, e.g. supermarkets, who clearly have the ability to commit to a menu of prices on all goods in their stock. Indeed, buyers rarely hesitate when telling a supermarket assistant what they are looking for.

A more moderate approach would be to ask what happens if the seller can offer bundles with *finitely* many goods, i.e. that there are more buyer types than there are different products on offer. As with the complete menu case, offering multiple goods greatly helps the seller, allowing him to set higher prices with the knowledge that types will be forced to sort themselves more selectively through their acceptance rules. Hence, richer menus can be viewed as playing an analogous role to more precise communication, and so we might expect the buyer's incentive to reveal information to be attenuated. Focussing again on the LSC equilibrium, the following result summarizes this intuition formally.

Proposition 9. *Suppose the seller can offer menus of k goods. Then*

- *The LSC equilibrium is clearing and N_k^*-partitional, with $N_k^* = \lceil \frac{1}{2k}\sqrt{\frac{3a}{\bar{u}}} \rceil$*
- *The LSC equilibrium is the ex-ante buyer optimal equilibrium.*
- *The seller optimal equilibrium is still fully separating.*

The key economic forces from the single-good benchmark remain in tact in this setting, since N_k^* remains finite and, for $a \geqslant \frac{4k^2\bar{u}}{3}$, greater than 1, and so the LSC equilibrium remains partially informative. Furthermore, N_k^* is decreasing in k, showing how as menus become richer, the buyer reveals less information.

4.2 Neologism-Proof Equilibria

Section 3.3 showed how the buyer's preferred equilibrium carefully balances the gains from pooling that come through price discounts with the losses from pooling that come through product mis-match and the subsequent reduction in trade. In this section, we will show how a well-known refinement concept introduced specifically for cheap-talk games - the “neologism-proofness”concept of[Farrell (1993)] - operates through very similar forces. Although being one of the earliest and best-known refinement concepts for cheap-talk games, neologism-proofness typically suffers from non-existence as well as stability issues.[5] In the current setting, not only do these equilibria always exist and demonstrate a form of stability (see Appendix C), but they admit a simple characterization (Theorem 10). In particular, they are all clearing and involve significant pooling, as was true for the LSC equilibrium.

Neologism-proofness starts with a putative equilibrium, and asks,“what would the seller do in response to facing some set of types G, and would said types prefer this response to the equilibrium?”If the answer leaves types in G better of, and if types not in G do not stand to gain by joining G, then G is referred to as a self-signalling set. An equilibrium σ is neologism-proof if no such set exists. This is formalized in Appendix A.

Towards a complete description, note that in any equilibrium which admits some zone of separation, all types in that zone receive a zero payoff. Any interval subset G of that zone can serve as a profitable, self-signalling set so long as the seller serves that set entirely with their subsequent pricing decision. Lemma 1 guarantees that such a G exists, as long as its width is below the critical width $\Lambda(a, \bar{u})$. Thus, if an equilibrium is either semi-separating, fully-separating or residual, then it is not neologism-proof.

Theorem 10 provides the key features of NP equilibria. Firstly, it gives bounds on interval lengths. The first bound comes directly from Lemma 1, whereas the second comes from Proposition 2; that result stated that almost all types in adjacent intervals that could pool together and still form a clearing interval would strictly prefer such a situation. Hence, viewed as a deviating set G, this would constitute a self-signalling set.

[5] For example, the leading example of [Crawford and Sobel (1982)] has no neologism-proof equilibria.

Combining these two bounds yields a tight bound on the number of intervals.

Theorem 10. *An equilibrium is neologism-proof if and only if* $|a_{i+1} - a_i| \leqslant \Lambda(a, \bar{u})$ *for all* $i = 0, \ldots, n-1$ *and* $|a_{i+2} - a_i| > \Lambda(a, \bar{u})$ *for all* $i = 0, \ldots, n-2$. *In particular, if an n-partitional equilibrium is neologism-proof, then* $n \in \{N^*, \ldots, 2N^* - 1\}$. *The LSC equilibrium is neologism-proof.*

5 Literature and Contribution

Our results contribute to the ongoing debate regarding consumer privacy in online retail. [Acquisti, Taylor and Wagman (2016)] and [Varian (1997)] provide extensive surveys on the privacy literature and [Fudenberg and Villas-Boas (2006)] provide a survey on models with behavioral price discrimination where firms price discriminate using past purchasing behavior.[6] [Taylor (2004)] considers the sale of consumer information and finds that when consumers are strategic, firms cannot benefit from targeted pricing. [Calzolari and Pavan (2006)] consider the sharing of consumer information between two principals who sequentially contract with the same agent and give conditions under which commitment to privacy is optimal.[7]. [Acquisti and Varian (2005)] and [Conitzer, Taylor and Wagman (2012)] consider a monopolist who sells a good over two periods where customers are strategic and may choose to maintain anonymity at a cost. In contrast to these settings, we allow for direct, unconstrained communication by the buyer regarding his preferences, rather than simply through equilibrium accept/reject decisions. This richness allows us to go beyond both full privacy (babbling) and full information sharing (perfect separation). As such, we are able to characterize

[6][Villas-Boas (1999)] and [Fudenberg and Tirole (2000)] consider behavior- based price discrimination in duopoly where customers have the choice between remaining loyal to a firm and switching to a competitor.

[7]Some other examples of behavior based price discrimination include [Gehrig and Stenbacka (2007)], [Chen and Zhang (2009)], [Jing (2011)] , [Shy and Stenbacka (2016)].

sharply the extent to which the buyer actively seeks to provide personal information. Furthermore, our paper also extends the discussions on privacy by allowing for product choice.

That the good offered is chosen conditional on buyer information connects our analysis to the literature on targeted advertising. [Shen and Villas-Boas (2018)] consider behavior-based advertising by a monopolist where purchase decisions determine future advertising decisions. [Bergemann and Bonatti (2015)] consider the problem of a data provider who sells consumer information to firms which then tailor their advertisements to the individual match value. [Corniere and Nijs (2016)] consider an online platform that auctions an advertising slot to competing firms. In their setup, consumer information is taken as freely available to the platform and disclosure improves the match between the advertiser and consumer while also increasing the prices. They abstract from the consumer's strategic incentives to reveal information, and allow the seller to commit to prices. As such, the characterization we offer in Theorem 5 is qualitatively distinct.

[Roesler and Szentes (2017)] consider a buyer who acquires information regarding her valuation for a single good, after which a seller posts a take-it-or-leave-it price offer. They define the buyer optimal signal structure, showing that it ensures trade whilst minimizing the seller's profit. Their characterization also exhibits minimal informativeness subject to ensuring trade. However, in their setting, their is a single good over which preferences are vertical, and as such *product steering* - and thus the trade-off at the heart of our analysis - is absent.

In concurrent work, [Ichihashi (2017)] studies information disclosure by consumers to a multi-product firm. He studies a persuasion model and focusses on welfare comparisons between discriminatory and non-discriminatory pricing. Whilst his characterisation of the consumer optimal disclosure policy allows for two products and binary valuations, our richer preference structure affords a broader characterization of the degree of information revelation.

On the theoretical front, that the set of neologism-proof equilibria are neither fully separating nor babbling for a wide range of parameters, even when both exist, is a novel result. Other cheap talk models have delivered fully, or almost-fully separating equilibria. These typically involve multiple dimensions, either within the type space or number of players (e.g. [Battaglini (2002)], [Chakraborty and Harbaugh (2010)], [Krishna and Morgan (2000)]). Recently, [Li, Rantakari and Yang (2016)] exhibit cheap-talk equilibria that are not Pareto-rankable in a setting with many competing senders. A literature also exists on cheap-talk in bargaining games: see [Farrell and Gibbons (1989)], [Matthews and Postlewaite (1989)], [Gardete (2013)], [Lim (2014)]. See [Kim and Kircher (2015)] for an application of cheap-talk to trading in larger markets. The focus in these papers is on showing how cheap-talk can improve equilibrium payoffs. For example, the unique neologism-proof equilibrium in [Farrell and Gibbons (1989)] and [Kim and Kircher (2015)] is separating, which is never the case in our model.

The tension driving incentives to reveal information bears some resemblance to the so-called *ratchet effect* identified in the early literature on dynamic adverse selection (see [Hart and Tirole (1988)], [Laffont and Tirole (1988)]). [Ickes and Samuelson (1987)] show how switching between skill-independent jobs can break the link between current and future incentives, thereby reducing the ratchet effect. Both their model and results are far removed from the current setting. In particular, in their setting, types are independent across jobs, rather than perfectly correlated, whilst the agent's ability to signal comes solely in the form of costly effort, rather than cheap talk. [Coles, Kushnir and Niederle (2013)] model signalling in matching markets in which preferences have both horizontal and vertical differentiation.

6 Conclusion

This paper presented a model of trade with costless communication in which a buyer trades off revealing his preferences in order to get the good he most desires against hiding his preferences to protect himself against price discrimination. Our main result is a characterization of the buyer's preferred equilibrium - the buyer reveals as little information as possible regarding his tastes, to ensure low prices, whilst providing enough information to ensure the seller offers relevant products. These conflicting objectives map directly into the ongoing debate regarding consumer privacy, in the form of two well-known practices - *product steering* and *price discrimination*.

Whilst the simplicity of the model allows a transparent investigation of the key economic forces, it paves the way for a number of extensions. For instance, incorporating multiple rounds of bargaining into the game might produce a more realistic extensive form. We posit that introducing multiple trading rounds would transfer surplus to the seller, whilst reducing the information revealed by the buyer via his cheap-talk messaging.[8] The intuition here is that multiple offers for seller works much like a menu of offers - rather than types sorting simultaneously through the menu, they would sort *dynamically* through the seller's sequential offers. We leave a formal analysis of this and other extensions for future work.

References

[Acquisti, Taylor and Wagman (2016)] Acquisti, A., Taylor, C., Wagman, L., 2016, "The Economics of Privacy", *Journal of Economic Literature*, 54(2), 442-492.

[Acquisti and Varian (2005)] Acquisti, A., Varian, H.R., 2005, "Conditioning Prices on Purchase History", *Marketing Science*, 24(3), 367-381.

[8] This is at odds with the literature surrounding the Coase conjecture (see [Gul, Sonnenschein and Wilson (1986)],[Fudenberg, Levine and Tirole (1987)]). There, as the number of offers become infinite, and period lengths shrink, the uninformed party is reduced to marginal cost

[Battaglini (2002)] Battaglini, M., 2002, "Multiple Referrals and Multidimensional Cheap Talk ", *Econometrica*, 70(4): 1379-1401.

[Bergemann and Bonatti (2015)] Bergemann, D., Bonatti, A., 2015, "Selling Cookies", *American Economic Journal: Microeconomics*, 7(3), 259–294

[Blume and Sobel (1995)] Blume, A., Sobel, J., 1995, "Communication-Proof Equilibria in Cheap-Talk Games ", *Journal of Economic Theory*, 65: 359-382.

[Blume, Kim and Sobel (1993)] Blume, A., Kim, Y-G., Sobel, J., 1993, "Evolutionary Stability in Games with Communication ", *Games and Economic Behaviour*, 5: 547-575.

[Calzolari and Pavan (2006)] Calzolari, G.,Pavan, A., 2006, "On the optimality of privacy in sequential contracting", *Journal of Economic Theory*, 130, 168-204.

[Chakraborty and Harbaugh (2010)] Chakraborty, A., Harbaugh, R., 2010, "Persuasion by Cheap Talk", *American Economic Review*, vol 100(5), pp 2361-2382.

[Chen, Kartik and Sobel (2008)] Chen, Y., Kartik, N., Sobel, J., 2008, "Selecting Cheap-Talk Equilibria", *Econometrica*, vol 76(1), pp 117-136.

[Chen and Zhang (2009)] Chen, Y.Z., Zhang, J., 2009, "Dynamic Targeted Pricing with Strategic Customers", *International Journal of Industrial Organization*, 27(1), 43-50.

[Coles, Kushnir and Niederle (2013)] Coles, P., Kushnir, A., Niederle, M., 2013, "Preference Signalling in Matching Markets", *American Economic Journal: Microeconomics*, vol 5(2), pp 99-134.

[Conitzer, Taylor and Wagman (2012)] Conitzer, V., Taylor, C., Wagman, L., 2012, "Hide and Seek: Costly Consumer Privacy in a Market with Repeat Purchases", *Marketing Science*, 31(2), 277–292.

[Corniere and Nijs (2016)] de Corniere, A., de Nijs, R., 2016, "Online Advertising and Privacy", *RAND Journal of Economics*, Vol. 47, No.1, 48–72.

[Crawford and Sobel (1982)] Crawford, V., Sobel, J., 1982, "Strategic Information Transmission", *Econometrica*, vol 60(6), pp 1431-1451.

[Eliaz and Spiegler (2015)] Eliaz, K., Spiegler, R., 2015, "Search Design and Broad Matching", *American Economic Review*, vol 105, pp 563-586.

[Farrell (1993)] Farrell, J., 1993 "Meaning and Credibility in Cheap-Talk Games", *Games and Economic Behaviour*, (5): 514-531.

[Farrell and Gibbons (1989)] Farrell, J., Gibbons, R., 1989 "Cheap Talk Can Matter In Bargaining", *Journal of Economic Theory*, (48): 221-237.

[Fudenberg, Levine and Tirole (1987)] Fudenberg, D., Levine, D.K., Tirole, J., 1987 "Incomplete Information Bargaining with Outside Opportunities", *Quarterly Journal of Economics*, 102 (1): 37–50.

[Fudenberg and Tirole (2000)] Fudenberg, D., Tirole, J.,2000, "Customer poaching and brand switching", *RAND Journal*, 31(4), 634-657.

[Fudenberg and Villas-Boas (2006)] Fudenberg, D., Villas-Boas M.J., 2007, "Behavior-Based Price Discrimination and Customer Recognition", *Economics and Information Systems, Volume 1. Oxford: Elsevier Science*.

[Gardete (2013)] Gardete, P.M., 2013, "Cheap-talk advertising and misrepresentation in vertically differentiated markets", *Marketing Science*, 2013, 32 (4), 609–621.

[Gehrig and Stenbacka (2007)] Gehrig, T., Stenbacka, R., 2007, "Information Sharing and Lending Market Competition with Switching Costs and Poaching", *European Economic Review*, 51(1), 77-99.

[Graham and Knuth (1994)] Graham, R.L., Knuth, D., 1994, "Concrete Mathematics", *Reading Ma.: Addison-Wesley*.

[Gul and Sonnenschein (1988)] Gul, F., Sonnenschein, H., 1988, "On Delay in Bargaining with One-Sided Uncertainty", *Econometrica*, 56 (3): 601-611.

[Gul, Sonnenschein and Wilson (1986)] Gul, F., Sonnenschein, H., Wilson, C., 1986, "Foundations of Dynamic Monopoly and the Coase Conjecture", *Journal of Economic Theory*, 39 (1): 155–90.

[Hannak et al. (2014)] Hannak, A., Soeller, G., Lazer, D., Mislove, A., Wilson, C., 2014, "Measuring Price Discrimination and Steering on E-commerce Web Sites", *IMC '14 Proceedings of the 2014 Conference on Internet Measurement Conference*, 305-318.

[Hart and Tirole (1988)] Hart, O.D., Tirole, J., 1988, "Contract Renegotiation and Coasian Dynamics", *Review of Economic Studies*, 509-540.

[Ichihashi (2017)] Ichihashi, S., 2017, "Online Privacy and Information Disclosure by Consumers ", Working Paper.

[Ickes and Samuelson (1987)] Ickes, B.W., Samuelson, L., 1987, "Job Transfers and Incentives in Complex Organizations: Thwarting the Ratchet Effect", *RAND Journal of Economics*, Vol. 18, No. 2 (Summer, 1987), pp. 275-286.

[Jing (2011)] Jing, B., 2011, "Pricing Experience Goods: The Effects of Customer Recognition and Commitment", *Journal of Economics and Management Strategy*, 20(2), 451-473.

[Kamenica and Gentzkow (2011)] Kamenica, E., Gentzkow, M., 2011, "Bayesian Persuasion", *American Economic Review*, Vol. 101, 2590–2615.

[Kartik (2009)] Kartik, N., 2009, "Strategic Communication with Lying Costs", *Review of Economic Studies*, Vol. 76, 1359-1395.

[Kim and Kircher (2015)] Kim, K., Kircher, P., 2015, "Efficient Competition through Cheap Talk: The Case of Competing Auctions", *Econometrica*, Vol. 83(5), 1849-1875.

[Krishna and Morgan (2000)] Krishna, V, Morgan, J., 2000, "A Model of Expertise", *Quarterly Journal of Economics*, Vol. 116, 747-775.

[Laffont and Tirole (1988)] Laffont, J-J., Tirole, J., 1988, "The Dynamics of Incentive Contracts", *Econometrica*, Vol. 56, No. 5 (September, 1988), 1153-1175.

[Lipnowski and Ravid (2017)] Lipnowski, E., Ravid, D., 2017, "Cheap Talk with Transparent Motives", *Working Paper*, 2017.

[Li, Rantakari and Yang (2016)] Li, Z., Rantakari, H., Yang, H., 2016, "Competitive cheap talk", *Games and Economic Behavior*, 96: 65-89.

[Lim (2014)] Lim, W., 2014, "Communication in Bargaining over Decision Rights", *Games and Economic Behavior*, 85: 159-179.

[Matthews et al. (1990)] Matthews, S., Okuno-Fujiwara, M., Postlewaite, A., 1990, "Refining Cheap-Talk Equilibria", *Journal of Economic Theory*, (55), pp. 247-273.

[Matthews and Postlewaite (1989)] Matthews, S.A., A. Postlewaite, 1989, “Pre-play communication in two-person sealed-bid double auctions”, *Journal of Economic Theory*, 1989, 48 (1), 238–263.

[Mussa and Rosen (1978)] Mussa, M., Rosen, S., 1978, “Monopoly and Product Quality”, *Journal of Economic Theory*, vol 18 (1978), pp. 301-317.

[Ravid (2017)] Ravid, D., 2017, “Bargaining with Rational Inattention”, *Working Paper*, 2017.

[Roesler and Szentes (2017)] Roesler, A.K, Szentes, B., 2017, “Buyer-Optimal Learning and Monopoly Pricing”, *American Economic Review*, vol 107(7),2072-2080.

[Shen and Villas-Boas (2018)] Shen, Q., Villas-Boas, M.J., 2018, “Behavior-Based Advertising ”, *Management Science*, 1-18.

[Shy and Stenbacka (2016)] Shy, O., Stenbacka, R., 2015, “Customer Privacy and Competition”, *Journal of Economics and Management Strategy*, vol 25(3), 539-562.

[Sobel (2009)] Sobel, J, 2009, “Signalling Games”, *Encyclopedia of Complexity and System Science*, R Meyer, ed, 2009.

[Taylor (2004)] Taylor, C. R., 2004, “Consumer Privacy and the Market for Customer Information”, *RAND Journal of Economics*, vol 35(4), 631-650.

[Varian (1997)] Varian, H.R., 1997, “Economic Aspects of Personal Privacy”, in Privacy and Self-Regulation in the Information Age U. S. Department of Commerce, Washington, DC.

[Villas-Boas (1999)] Villas-Boas, J.M., 1999, “Dynamic competition with customer recognition”, *RAND Journal of Economics*, vol 30(4),604-631.

A Formalities

A.1 Strategies and Equilibrium

Let $m : \theta \to \mathcal{M}$ be the profile of messages sent by the buyer. Let $S = V \times \mathbb{R}_+$ be the space of offers - a good and a price, and let $s : \mathcal{M} \to S$ the seller's choice of offer, conditional on having received message m. Finally, let $x : [0,1] \times V \times \mathbb{R}_+ \to \{0,1\}$ be the acceptance strategies for each buyer type (where 0 indicates a rejection), given an offer (v,p).

Type θ's utility from accepting an offer (v,p) is $u(v,\theta) - p$, and hence his expected utility, given a strategy profile $\sigma = (m,s,x)$ is $\mathbb{E}_\sigma(u(v,\theta) - p)$, whilst the seller's expected profit is simply $\mathbb{E}_\sigma(px)$. Let $U(v,p,\theta) = u(v,\theta) - p$. In the event of a rejection, payoffs to both agents are zero.

Definition 11. An equilibrium σ is a profile of message strategies m, offer strategies s and acceptance strategies x such that for each $\theta \in [0,1]$, $m(\theta)$ solves

$$\max_{m \in \mathcal{M}} U(v(m), p(m), \theta) \tag{B1}$$

given m, $s(m) = \big(v(m), p(m)\big)$ solves

$$\max_{v,p} \int px(\theta, s) d\hat{\mu}(\theta|m) \tag{S}$$

where $\hat{\mu}(.|\theta)$ is derived from m by Bayes' rule, and given $s(m)$, $x(s)$ solves

$$\max_{x \in \{0,1\}} x\big(U(v(m), p(m), \theta)\big) \tag{B2}$$

A.2 Neologism-Proofness

For some $G \subset \Theta$, let

$$\text{br}(G) = \arg\max_{v,p} \int_G p\mathbb{I}_{u(v,\theta) \geqslant p} d\theta$$

That is, br(G) is the seller's best-response to the belief that he faces types G, given that they behave optimally at the accept/reject stage that follows and under the presumption that they indeed are in G.

Definition 12. Let $G \subset \Theta$. For all $\theta \in G$, we define

$$\underline{U}(\theta|G) = \min_{(v,p) \in \text{br}(G)} \mathbb{E}_{\text{br}(G)}(U(v,p,\theta))$$

$$\bar{U}(\theta|G) = \max_{(v,p) \in \text{br}(G)} \mathbb{E}_{\text{br}(G)}(U(v,p,\theta))$$

Given an equilibrium σ, a set $G \subset \Theta$ is **self-signalling** if

$$\underline{U}(\theta|G) \overset{a.s.}{>} \mathbb{E}_\sigma(U(v,p,\theta)) \ \ \forall\theta \in G$$
$$\bar{U}(\theta|G) \leqslant \mathbb{E}_\sigma(U(v,p,\theta)) \ \ \forall\theta \in \Theta \setminus G$$

σ is **neologism-proof** if no self-signalling set exists relative to it.

A few technical remarks are needed at this point. Firstly, if the seller's best response to G is not unique, we use the convention introduced by [Matthews et al. (1990)] - a deviating type assesses his worst-case deviation against the putative equilibrium, whereas a non-deviating type assesses his best-case deviation. I abuse terminology and maintain the moniker "neologism-proofness". This is not to undermine the contribution of [Matthews et al. (1990)], but simply because my adapted definition seems closer in essence to [Farrell (1993)]. Secondly, I slightly adapt Farrell's original definition for finite type spaces to the current setting. I impose that at most a measure 0 of types in G are indifferent between the deviation and the equilibrium payoffs. Whilst this convention seems the most natural, how one takes a stand on this point turns out to be important for the existence of self-signalling sets. Were I to impose that the deviation be strict *for all* types in G, then all equilibria would be neologism-proof, by Proposition 3.

B Generalizing the Buyer's Preferences

Horizontal differentiation is one of the two canonical models of consumer preferences over multiple products. The other - *vertical differentiation* - posits that all consumers have the same ranking over the product line. Of course, in reality, a combination of the two seems the most appropriate description. Many brands offer premium and budget lines for their products, with all consumers ranking the former higher, whereas within a given line, consumers might have subjective preferences across brands, e.g. a preference for Apple over Samsung based on compatibility concerns.[9]

For a start, take the standard model of vertical preferences, given by $u(v,\theta) = v\theta$ (see for instance [Mussa and Rosen (1978)]). In this setting, the conclusion of the analysis changes dramatically; no communication occurs in equilibrium. To see this, note that for a given offer (v,p), types $\theta \geqslant \frac{p}{v}$ accept, and types $\theta < \frac{p}{v}$ reject. For a given belief μ, the seller's profit from an offer (v,p) is then $\Pi(v,p|\mu) = \int_{\frac{p}{v}}^{1} p d\mu(\theta)$. Clearly then, offering $v = 1$ weakly dominates. Suppose the equilibrium was 2-partitional. Then the seller offers v for both intervals, at prices p_0, p_1. If $p_0 < p_1$, then types in μ_1 would strictly benefit by deviating and pooling with μ_0, and vice versa if $p_1 < p_0$. A similar argument holds for any equilibrium with some separation.

[9]See [Coles, Kushnir and Niederle (2013)] for preferences that combine vertical and horizontal differentiation.

Plainly put, when all types agree on the ranking of goods, they want the lowest price possible, knowing that the seller will offer the best good, leaving no room for effective communication.

Remark 13. *With $u(v, \theta) = v\theta$, no communication occurs in equilibrium.*

That monotonicity in preferences leads to no communication is a well-known result in games with cheap-talk; Proposition 13 is simply an expression of this[10]. A more natural exercise would be to allow for a more general $u(v, \theta)$, satisfying minimal conditions that combine both horizontal and vertical differentiation. To this end, we impose the following restrictions:

Assumption 14. (A1) $\frac{\partial u}{\partial v}|_{v=\theta} = 0$ (A2) $\frac{\partial u}{\partial \theta}|_{v=\theta} > 0$ (A3) $\frac{\partial^2 u}{\partial v^2} < 0$ (A4) $\frac{\partial^2 u}{\partial \theta \partial v} \geqslant 0$ (A5) $u(\theta, \theta) > 0$

Properties (A1) and (A3) ensure that u is single-peaked and strictly concave in v. (A2) captures the idea that, although it is not true that all types share the same ranking, it is the case that higher v products entail potentially greater joint surplus. (A4) is a standard sufficient condition for single-crossing, and (A5) ensures gains from trade in every good.

How then do such restrictions affect the results from Sections 2 and 4.2? The following result demonstrates the fragility of the fully separating equilibrium to even the smallest vertical perturbation. By (A2) the gain for type θ to deviate marginally to the left is first-order, whereas by (A1) the loss is second-order. This fragility provides yet more support for the efficacy of neologism-proofness as a refining concept in this setting.

Specifically, in a fully separating equilibrium, it must be that $s(v, p|\theta) = \big(\theta, u(\theta, \theta)\big)$ - by (A1) and (A3), $v = \theta$ uniquely maximizes $u(v, \theta)$. Suppose type θ considers a deviation to $\theta - \epsilon$, for some small $\epsilon > 0$. His payoff from accepting is then $u(\theta - \epsilon, \theta) - u(\theta - \epsilon, \theta - \epsilon)$. But for small ϵ,

$$u(\theta - \epsilon, \theta) - u(\theta - \epsilon, \theta - \epsilon) \approx \epsilon u_\theta(\theta, \theta) > 0$$

by (A2).[11]

Proposition 15. *If $u(v, \theta)$ satisfies Assumption 14, then a fully separating equilibrium cannot exist. Furthermore, there exists an M^* such that all partitional equilibria have at most M^* intervals.*

Proof. The first part is immediate from the preceding paragraph. For the remainder, we proceed with some propositions.

First, for $\theta \in [0, 1]$, let $b(\theta) \in [0, 1] \setminus \{\theta\}$ solve $u(b, b) = u(b, \theta)$ (if no solution to the equation exists in $[0, 1]$, set $b(\theta) = 0$). Then the function $b : [0, 1] \to [0, 1]$ is well-defined, and furthermore $b(\theta) \in [0, \theta)$ by (A2).

Define $\lambda : [0, 1] \to \mathbb{R}_+$ by $\lambda(x) = x - b(x)$.

[10]See [Sobel (2009)] for a detailed discussion.

[11]It should be noted that assumption (A2) is not necessary for the non-existence of full separation. All that is required is that there exists *a single* type θ for whom (A2) holds.

Proposition 16. *Given a partitional equilibrium σ, if $a_{i+1} - a_i < \lambda(a_{i+1})$, then $U(v_i, p_i, a_{i+1}) > 0$ and $U(v_i, p_i, a_i) = 0$.*

Proof. The last equality follows from the usual arguments forcing the lowest type IC to bind. Towards a contradiction, suppose $U(v_i, p_i, a_{i+1}) = 0$, i.e. $p_i = u(v_i, a_{i+1})$. Then there exists $v_m \in [a_i, a_{i+1})$ such that $x(\theta, v_i, p_i) = 0 \forall \theta \in [a_i, v_m], x(\theta, v_i, p_i) = 1 \forall \theta \in [v_m, a_{i+1}]$, i.e. v_m forms a cut-off type. Then the seller's profit is given by

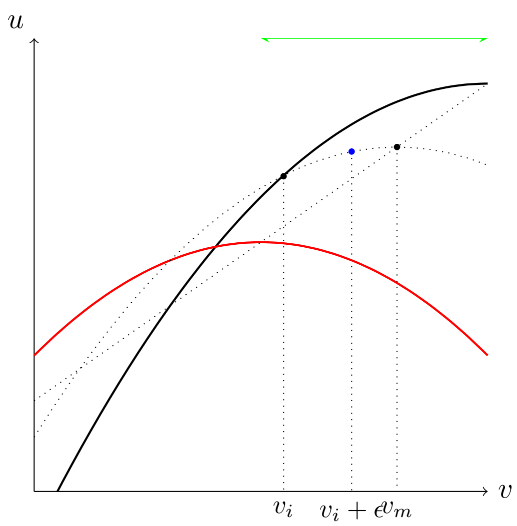
$$\Pi(v_i, p_i) = \frac{p_i(a_{i+1} - v_m)}{a_{i+1} - a_i} = \frac{u(v_i, a_{i+1})(a_{i+1} - v_m)}{a_{i+1} - a_i}$$

Consider instead the offer $(v', p') = (v_i + \epsilon, u(v_i + \epsilon, v_m)$, for some small $\epsilon > 0$. By definition, v_m solves $(u(v_i, a_{i+1}) = u(v_i, v_m)$. Hence, for sufficiently small ϵ, $v' < v_m$, thus the acceptance set is the same as the original offer (v_i, p_i). The seller's profit becomes

$$\begin{aligned}
\Pi((v', p') &= \frac{u(v_i + \epsilon, v_m)(a_{i+1} - v_m)}{a_{i+1} - a_i} \\
&\approx \frac{\big(u(v_i, v_m) + \epsilon u_v(v_i, v_m)\big)(a_{i+1} - v_m)}{a_{i+1} - a_i} \\
&= \frac{u(v_i, v_m)(a_{i+1} - v_m)}{a_{i+1} - a_i} + \frac{\epsilon u_v(v_i, v_m)(a_{i+1} - v_m)}{a_{i+1} - a_i} \\
&> \Pi(v_i, p_i)
\end{aligned}$$

by (A2). Hence, (v', p') constitutes a profitable deviation for the seller.

Set $\lambda(f) = \inf_{\theta \in \Theta} (\lambda(\theta)$. This is clearly well-defined, since $\lambda([0,1]) \subset [0,1]$.



□

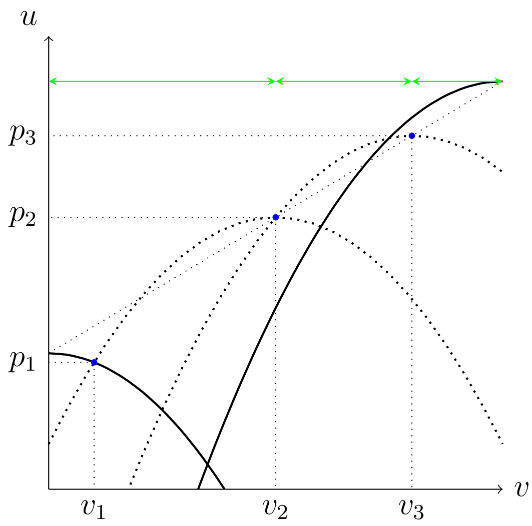


Figure 3: An M* equilibrium ($u(v,\theta) = 3 + \frac{3}{5}\theta - \frac{1}{5}(v-\theta)^2$)

Finally, to prove the proposition, suppose no such M^* exists. Then for large enough M, there must exist two consecutive intervals $[a_i, a_{i+1}], [a_{i+1}, a_{i+2}]$ such that $a_{i+j} - a_{i+j-1} < \lambda(f)\ \forall j = 1, 2$. By Proposition 23, type a_{i+1} receives positive surplus by pooling with $[a_i, a_{i+1}]$, but zero surplus by pooling with $[a_{i+1}, a_{i+2}]$, a contradiction. □

Much like in [Crawford and Sobel (1982)], M^* can be determined as the solution to a recursive difference equation. In this case, if a_{i+1} is a boundary type, then the solution to the difference equation

$$u(a_i, a_i) = u(a_i, a_{i+1}), \quad a_{i+1} \neq a_i$$

forms a lower bound for how close a_i can be to a_{i+1} in equilibrium. Figure 3 shows this construction graphically.

Whilst such a result reveals a technical connection to [Crawford and Sobel (1982)], the economic forces are quite different. In [Crawford and Sobel (1982)], whilst equilibria cannot be *ex-post* Pareto-ranked, it is the case that both sender and receiver *ex-ante* expected payoffs are increasing in the degree of equilibrium separation. Indeed, applications tend to use this as a criterion for selecting the most informative equilibrium, whilst many refinement concepts were proposed solely to select this outcome (see [Chen, Kartik and Sobel (2008)] for a detailed discussion). This is not the case here. Certainly, all residual equilibria are *ex-ante* dominated. However, within the class of clearing equilibria, as the degree of separation increases, the expected surplus transfers from the buyer to the seller.

Finally, does the “refining down”feature of Theorem 10 hold in this general setting? Consider the case when $u(v, \theta) = a\theta - (v - \theta)^2$.[12] As $a \to \infty$, no solution to the equation $u(a_i, a_i) = u(a_i, a_{i+1})$ exists, preferences become fully vertical, and hence no communication occurs. It is vacuously false then that the most informative equilibria are refined away. This limit clearly embodies a convergence towards Remark 13. We can, however, recover the result if the vertical perturbation is sufficiently small. To formalize this, consider the function $f(\theta) = u(\theta, \theta)$ that describes the locus of local maxima of the function $u(v, .)$, and suppose $a > 0$ is such that[13]

$$a = \max_{\theta \in [0,1]} f'(\theta)$$

In the benchmark case, $f(\theta) = \bar{u}$ and $a = 0$. As a becomes small, the locus of maxima flattens, converging to the horizontal benchmark. As such, the first-order gain that prevents full separation starts to shrink, thus allowing finer partitions to emerge in equilibrium. Denoting this dependence as $M^*(a)$, we obtain the following generalizations of Theorems 10 and 6.

Proposition 17. *1. Suppose $u(v, \theta)$ satisfies Assumption 14, and $a > 0$ is defined as above. Then as $a \to 0$, $M^*(a) \to \infty$. Furthermore, there exists δ such that for all $a < \delta$, any $M^*(a)$-equilibrium is not NP.*

2. All LSC equilibria are NP. The ex-ante buyer optimal equilibrium is an LSC equilibrium. In particular, the NP set is non-empty.

Proof. For sufficiently small a, (A5) allows us to restrict attention to clearing equilibria. Using the definitions of the functions $b(\theta)$ and $\lambda(\theta)$ as in the proof of Proposition 15, we proceed in steps.

Proposition 18. *Take a boundary type a_i. As $a \to 0$, $b(a_i) \to a_i$.*

Proof. Let $a_{i-1} = b(a_i)$. Applying the Mean Value Theorem to $f(\theta)$ on the closed interval $[a_{i-1}, a_i]$, there exists $y \in (a_{i-1}, a_i)$ such that $f(a_i) - f(a_{i-1}) = f'(y)(a_i - a_{i-1})$. Hence

$$\begin{aligned}
|u(a_i, a_{i-1}) - u(a_{i-1}, a_{i-1})| &< |u(a_i, a_i) - u(a_{i-1}, a_{i-1})| \\
&= |f(a_i) - f(a_{i-1})| \\
&= |f'(y)||a_i - a_{i-1}| \\
&\leqslant \max_{\theta \in [0,1]} f'(\theta)|a_i - a_{i-1}| \\
&= a|a_i - a_{i-1}| \\
&\leqslant a \\
&\to 0
\end{aligned}$$

[12]It is easily verified that this function satisfies Assumption 14.

[13]Note that since f' is continuous on a compact set, the max here is well-defined.

as $a \to 0$. The first inequality follows from (A1). The result follows from continuity of $u(.,\theta)$. $\square$

In particular, the Mean value bound constructed in the proof of Proposition 24 says that the minimum interval width in an M^*-equilibrium scales with a, and hence $M^* \to \infty$ as $a \to 0$.

To prove the remainder of the proposition, take an M^*-equilibrium, and a sequence of boundary types a_i, a_{i+1}, a_{i+2}. We will show that the set $G = [a_i, a_{i+2}]$ forms a self-signalling set. Single-crossing (A4) ensures that types outside $[a_i, a_{i+2}]$ prefer the equilibrium, given it and G are clearing. By (A3), it is sufficient consider local IC constraints around the boundary types a_i, a_{i+2}.

First consider type $a_i + \epsilon$, for some small $\epsilon > 0$. In equilibrium, this type receives payoff $u(a_i, a_i + \epsilon) - u(a_i, a_i)$. Denote this $U_\sigma(a_i + \epsilon)$. Under the self-signalling deviation G, this type receives $u(v, a_i + \epsilon) - u(v, a_i)$, for some v; by (A4), we know that $a_i < v < a_{i+2}$, so let $\delta_1, \delta_2 > 0$ be such that $v = a_i + \delta_2 = a_{i+2} - \delta_2$. Then

$$
\begin{aligned}
u(v, a_i + \epsilon) - u(v, a_i) &= u(a_i + \delta_1, a_i + \epsilon) - u(a_i + \delta_1, a_i) \\
&\approx u(a_i, a_i + \epsilon) + \delta_1 u_v(a_i, a_i + \epsilon) - \big[u(a_i, a_i) + \delta_1 u_v(a_i, a_i)\big] \\
&= U_\sigma(a_i + \epsilon) + \delta_1 \big[\underbrace{u_v(a_i, a_i + \epsilon)}_{>0} - \underbrace{u_v(a_i, a_i)}_{=0}\big] \\
&> U_\sigma(a_i + \epsilon)
\end{aligned}
$$

Note that Proposition 24 validates the first-order approximation. Similarly, the payoff for type $a_{i+2} - \epsilon$ in equilibrium is $u(a_{i+1}, a_{i+2} - \epsilon) - u(a_{i+1}, a_{i+1}) = U_\sigma(a_{i+2} - \epsilon)$, whilst under G,

$$
\begin{aligned}
u(v, a_{i+2} + \epsilon) - u(v, a_{i+2}) &= u(a_{i+2} - \delta_2, a_{i+2} - \epsilon) - u(a_{i+2} - \delta_2, a_{i+2}) \\
&\approx u(a_{i+2}, a_{i+2} - \epsilon) - \delta_2 u_v(a_{i+2}, a_{i+2} - \epsilon) - \big[u(a_{i+2}, a_{i+2}) - \delta_2 u_v(a_{i+2}, a_{i+2})\big] \\
&\geqslant U_\sigma(a_{i+2} - \epsilon) - \delta_2 \big[\underbrace{u_v(a_{i+2}, a_{i+2} - \epsilon)}_{<0} - \underbrace{u_v(a_{i+2}, a_{i+2})}_{=0}\big] \\
&> U_\sigma(a_{i+2} - \epsilon)
\end{aligned}
$$

where the third inequality holds by (A4), since $u(a_{i+2}, a_{i+2} - \epsilon) - u(a_{i+2}, a_{i+2}) > u(a_{i+1}, a_{i+2} - \epsilon) - u(a_{i+1}, a_{i+1}) = U_\sigma(a_{i+2} - \epsilon)$. Hence only boundary types are left indifferent under the deviation G, and so G is self-signalling.

The proof of the second part of the proposition is immediate from the arguments in the preceding proof, which showed that the same necessary and sufficient condition for neologism-proofness - that no two intervals must be able to pool and face a clearing offer - still holds. $\square$

Proposition 16 maintains the robust "refining down"feature of Theorem 10. Were I to impose further restrictions on the second derivative $\frac{\partial^2 u}{\partial v^2}$, the "refining up"feature might also be recovered. The second part demonstrates how the identity of the buyer's preferred outcome is also robust to these generalized preferences. Note that, in general, there might be many LSC equilibria.

C Alternative Refinements

In this section, we briefly discuss alternative refinements purpose-built for cheap talk games. First, consider the *no incentive to separate* concept, recently introduced by [Chen, Kartik and Sobel (2008)]. An equilibrium σ satisfies *no incentive to separate* (NITS) if $\mathbb{E}_\sigma(U(v,p,0)) \geqslant \mathbb{E}_{\text{br}(\{0\})}(U(v,p,0))$. That is, the lowest buyer type weakly prefers σ to revealing his type (if he could), and the seller best-responding to this revelation. It is immediately obvious that, in the current setting, such a refinement has no power. For if type 0 reveals himself truthfully, the seller offers good 0 at price $\bar{u}$, which is accepted. Since type 0 gets 0 in this deviation, it is clear that any incentive-compatible strategy profile satisfies NITS.

The concept of neologism-proofness is known to lack certain stability properties. Here, we employ the notion of *communication-proofness* as in [Blume and Sobel (1995)] - a concept invented specifically to study stability in the context of cheap-talk games - to determine whether in this context, NP equilibria are stable. In the bechmark model, an equilibrium is communication-proof if and only if it is clearing. Hence, all NP equilibria are communication-proof. In the language of [Blume and Sobel (1995)], the clearing equilibria are "good", and hence cannot be de-stabilized by other clearing equilibria. The NP set is thus a subset of the stable set with respect to this concept. Intuitively, any residual equilibrium is de-stabilized by one in which the set of types not served are served in intervals, much as in the proof of Proposition 5. Clearing equilibria cannot be ex-post Pareto-ranked; simply consider two clearing equilibria with different boundary types.

D Proofs

D.1 Proof of Proposition 1

To prove that such an equilibrium exists is straightforward. In such an equilibrium, $m(\theta)$ is injective on Θ, and hence we let θ_m be the unique member of Θ such that $m(\theta_m) = m$. Then define the strategies $s(m) = (\theta_m, \bar{u})$ and $x(\theta_m, s(m)) = 1$, for all $\theta_m \in [0,1]$. The seller achieves his maximum profit, subject to each type's IC constraint. Each type θ receives $\bar{u} - a(\theta - \theta)^2 - \bar{u} = 0$ in equilibrium. Since the signals are fully revealing, it is sufficient to consider a deviation for type θ to some message $\theta' \neq \theta$ instead. Subsequent acceptance of the seller's equilibrium offer would then yield a payoff of $\bar{u} - a(\theta' - \theta)^2 - \bar{u} < 0$, so such a deviation cannot be profitable.

To show that this allocation maximizes welfare, consider the social planner's ex ante problem. This can be viewed as an assignment problem. That is, the solution takes the form of a joint distribution $\pi : V \times [0,1] \to [0,1]$ that solves

$$\max_{\pi \in \Pi} \int u(v,\theta) d\pi$$

where Π is the set of all such distributions.[14] But

$$\max_{\pi \in \Pi} \int u(v,\theta) d\pi = \max_{\pi \in \Pi} \int \bar{u} - a(v-\theta)^2 d\pi = \min_{\pi \in \Pi} \int a(v-\theta)^2 d\pi$$

This is clearly achieved when $\pi(v,\theta) = \mathbb{I}_{v=\theta}$.

D.2 Proof of Proposition 3

Take $n \in \mathbb{N}$. Pick an arbitrary sequence $0 = \theta_0 < \theta_1 < \cdots < \theta_{n-1} = 1$, and set $\mu_{i+1} = \mu(.|\theta)$ uniform on $[\theta_i, \theta_{i+1}]$ for $\theta \in [\theta_i, \theta_{i+1}]$. Henceforth, let $[a,b]$ also denote the uniform distribution on the interval $[a,b]$. Consider the seller's problem, conditional on the belief $[\theta_i, \theta_{i+1}]$. If he offers good v at price p, his expected profit is

$$\Pi(v,p|\mu_{i+1}) = p \int \mathbb{I}_{u(v,\theta) \geqslant p} d\theta$$

We have that $u(v,\theta) \geqslant p$ holds for types such that

$$\max\{\theta_i, v - \left(\frac{\bar{u}-p}{a}\right)^{\frac{1}{2}}\} \leqslant \min\{\theta_{i+1}, v + \left(\frac{\bar{u}-p}{a}\right)^{\frac{1}{2}}\}$$

and hence the seller's expected profit is given by

$$\Pi(v,p|\mu_{i+1}) = p \int_{g_l(v,p)}^{g_u(v,p)} d\theta = p\big(g_u(v,p) - g_l(v,p)\big)$$

where $g_l(v,p) = \max\{\theta_i, v - \left(\frac{\bar{u}-p}{a}\right)^{\frac{1}{2}}\}$, and $g_u(v,p) = \min\{\theta_{i+1}, v + \left(\frac{\bar{u}-p}{a}\right)^{\frac{1}{2}}\}$. Simple analysis shows that the function $\Pi(v,p|\mu)$ is either weakly or strictly concave in v, depending on $\theta_{i+1} - \theta_i$, and strictly concave in p. Hence, p^* is unique in equilibrium, whereas V^* may either be a singleton or a closed interval.

[14] The max is well-defined by the Weierstrass extreme value theorem, since u is continuous on a compact space.

Suppose (v^*, p^*) constitute the seller's equilibrium strategy. We claim that

$$p^*_{i+1} \geq \max\{u(v^*_{i+1}, \theta_i), u(v^*_{i+1}, \theta_{i+1}), 0\}$$

To see this, suppose not, and wlog suppose $p^*_{i+1} < u(v^*_{i+1}, \theta_i)$. By the single-crossing property, there exists a type θ_m such that $p^*_{i+1} = u(v^*_{i+1}, \theta_m)$ and $p^*_{i+1} > u(v^*_{i+1}, \theta)$ for all $\theta \in (\theta_m, \theta_{i+1}]$. Hence, we have that $g_l(v^*_{i+1}, p^*_{i+1}) = \theta_i, g_u(v^*_{i+1}, p^*_{i+1}) = \theta_m$, and so equilibrium profits are

$$\Pi^* = p^*_{i+1}(\theta_m - \theta_i)$$

Now consider the profile $(v^*_{i+1} + \epsilon, p^*_{i+1})$. By continuity, we can find a sufficiently small ϵ such that $p^*_{i+1} < u(v^*_{i+1} + \epsilon, \theta_i)$ and $p^*_{i+1} < u(v^*_{i+1} + \epsilon, \theta_m)$. Hence by continuity again, there exists a type $\theta_n > \theta_m$ such that $p^*_{i+1} = u(v^*_{i+1} + \epsilon, \theta_n)$. Profits are now given by

$$\Pi(v^*_{i+1} + \epsilon, p^*_{i+1}) = p^*_{i+1}(\theta_n - \theta_i) > p^*_{i+1}(\theta_m - \theta_i) = \Pi^*$$

a contradiction.

We construct the strategy s. On any interval $[\theta_i, \theta_{i+1}]$, we have that $\Pi(v, 0) = \Pi(v, \bar{u}) = 0$,

By the claim, we can restrict attention to $p^*_{i+1} \geq \max\{u(v^*_{i+1}, \theta_i), u(v^*_{i+1}, \theta_{i+1})\}$. On this range, $\Pi(., p)$ is strictly concave and hence there exists a unique maximal $p^*(v) \in \arg\max \Pi(v, p)$. Next, fix p. Since $\Pi(v, p^*(v))$ is linear in v, it may be that several v solve $\arg\max_v \Pi(v, p^*(v))$. Pick one such v^*, and now set $s(\theta_i, \theta_{i+1}) = (v^*_{i+1}, p^*_{i+1}(v^*))$.

Finally, we check that no types have an incentive to deviate. Consider type $\theta_m \in [\theta_i, \theta_{i+1}]$, for some $i \in \{0, \ldots, n\}$. If θ_m deviates to the group $[\theta_{i+1}, \theta_{i+2}]$, he faces an offer (v^*_{i+2}, p^*_{i+2}) such that $u(v^*_{i+2}, \theta_m) < p^*_{i+2}$. This follows from the earlier claim, which established that $u(v^*_{i+1}, \theta_m) \leqslant p^*_{i+2}$, and hence the statement follows by continuity. The second part of the proposition holds, since non-partitional equilibria are ruled out in the usual way by the single-crossing and pure-strategy assumptions.

D.3 Proof of Lemma 1

Without loss, we take $v^* = \frac{\theta+\theta'}{2}$. By part 3 of Proposition 3, if p doesn't bind, then the FOC is sufficient and p solves $\frac{\partial \Pi(p)}{\partial p} = 0$, where $\Pi(p) = 2p\left(\frac{\bar{u}-p}{a}\right)^{\frac{1}{2}}$. Hence

$$\begin{aligned}
\Pi'(p) = 0 \Rightarrow (\bar{u} - p)^{\frac{1}{2}} &= \frac{1}{2}p(\bar{u} - p)^{-\frac{1}{2}} \\
(\bar{u} - p) &= \frac{1}{2}p \\
p &= \frac{2\bar{u}}{3}
\end{aligned}$$

If p does bind, the p satisfies $p = u(\frac{\theta+\theta'}{2}, \theta - \frac{\theta+\theta'}{2}) = u(\frac{\theta+\theta'}{2}, \theta' - \frac{\theta+\theta'}{2})$. By continuity of $\Pi(p)$, it must be that the value $\Lambda(a, \bar{u})$ solves

$$\bar{u} - a\big(\frac{\Lambda(a,\bar{u})}{2}\big)^2 = \frac{2\bar{u}}{3}$$
$$\Lambda(a,\bar{u}) = 2\sqrt{\frac{\bar{u}}{3a}}$$

D.4 Proof of Proposition 5

For a strategy profile σ, let

$$\Theta_1(\sigma) = \{\theta \in \Theta | x(\theta, s) = 1\}$$
$$\Theta_0(\sigma) = \{\theta \in \Theta | x(\theta, s) = 0\}$$

Call $\Theta_1(\sigma)$ the acceptance set, and $\Theta_0(\sigma)$ the rejection set. The steps constituting the construction are as follows.

Step 1 Take the profile σ in which $\mu(.|\theta)$ is uniform over $\mathcal{M}$, and $s(m) = (v^*, p^*)$ solves the seller's problem S. Then σ is a babbling equilibrium. If σ is clearing, we are done. To see this, note that we can form an n-partitional equilibrium from σ as follows. Take types $[0, \epsilon]$ for some $\epsilon > 0$, and form $n-1$ subdivisions. By Lemma 1, we have that as $\epsilon \to 0$, for each $i \in 0, \ldots, n-1$, the seller's best response is clearing.

Step 2 If σ is not clearing, then by arguments in Proposition 3, there exists a closed, convex set $V_0^* = [v_0^l, v_0^r]$ of goods such that $s(m) = (v^*, p^*)$ form part of a babbling equilibrium for all $v^* \in V_0^*$. Wlog, take $v_0^* = v_0^r$, and construct the babbling equilibrium σ_0 with $s_0(m) = (v_0^r, p_0^*)$, where $p_0^* = p^* = \frac{2\bar{u}}{3}$, since σ was residual. Then σ_0 is residual, with $\Theta_0(\sigma_0)$ and $\Theta_1(\sigma_0)$ convex and partitioning Θ.

Step 3 Consider the seller's best response to the belief $\Theta_0(\sigma_0)$. Again, by Proposition 3, the price p^* is uniquely defined, whilst v^* is defined up to a convex set $V_1^* = [v_1^l, v_1^r]$. If v^* is unique, then form the strategy profile σ_1 by setting $\mu_1 = \Theta_0(\sigma_0), \mu_2 = \Theta_1(\sigma_0), (v_1^*, p_1^*) = (v_0^r, \frac{2\bar{u}}{3}), (v_2^*, p_2^*) = (v^*, p^*)$. By construction, σ_1 is a clearing 2-partitional equilibrium, and the same argument as in step 1 completes the proof.

Step 4 If v^* is not unique, then take $v_1^* = v_1^r$ and $p_1^* = \frac{2\bar{u}}{3}$, and construct the strategy profile σ_1 such that $\mu_1 = \Theta_0(\sigma_0), \mu_2 = \Theta_1(\sigma_0), (v_1^*, p_1^*) = (v_0^r, \frac{2\bar{u}}{3}), (v_2^*, p_2^*) = (v_1^r, \frac{2\bar{u}}{3})$. Again, $\Theta_0(\sigma_1)$ and $\Theta_1(\sigma_1)$ are convex and partition Θ. Repeat steps 3 and 4.

Lemma 1 ensures this process ends after finitely many iterations, since for some finite N^*, the maximal width of any interval induced by this algorithm will be smaller than $\Lambda(a, \bar{u})$. To calculate N^* explicitly, note that the set $\Theta_1(\sigma_0)$ constructed by step 2 has width $\Lambda(a, \bar{u})$, since it is minimally clearing. Hence, the equilibrium constructed by the algorithm has a clear structure - $N^* - 1$ intervals are of width $\Lambda(a, \bar{u})$, and the boundary interval has width strictly less than $\Lambda(a, \bar{u})$. Hence,

$$N^* = \lceil \frac{1}{\Lambda(a, \bar{u})} \rceil = \lceil \frac{1}{2}\sqrt{\frac{3a}{\bar{u}}} \rceil$$

D.5 Proof of Proposition 2

Proof. It suffices to show that almost all types in $[a_i, a_{i+2}]$ strictly gain from the deviation when $[a_i, a_{i+2}]$ is clearing. By hypothesis, under σ, $[a_i, a_{i+1}]$ and $[a_{i+1}, a_{i+2}]$ are both clearing, and hence $v_i = \frac{a_i + a_{i+1}}{2}, p_i = \bar{u} - a\big(\frac{a_{i+1} - a_i}{2}\big)^2$ and $v_{i+1} = \frac{a_{i+1} + a_{i+2}}{2}, p_i = \bar{u} - a\big(\frac{a_{i+2} - a_{i+1}}{2}\big)^2$. Furthermore, if $[a_i, a_{i+2}]$ is clearing, then the seller's unique best response is $v' = \frac{a_i + a_{i+2}}{2}, p_i = \bar{u} - a\big(\frac{a_{i+2} - a_i}{2}\big)^2$.

First, take $\theta \in (a_i, a_{i+1})$. Under σ, type θ's payoff is

$$\bar{u} - a\Big(\frac{a_i + a_{i+1}}{2} - \theta\Big)^2 - \bar{u} + a\Big(\frac{a_{i+1} - a_i}{2}\Big)^2 = a\Big[\Big(\frac{a_{i+1} - a_i}{2}\Big)^2 - \Big(\frac{a_i + a_{i+1}}{2} - \theta\Big)^2\Big] \tag{1}$$

whereas under $br[a_i, a_{i+2}]$, θ receives

$$a\Big[\Big(\frac{a_{i+2} - a_i}{2}\Big)^2 - \Big(\frac{a_i + a_{i+2}}{2} - \theta\Big)^2\Big] \tag{2}$$

Subtracting 1 from 2, we have type $\theta' s$ gain from deviating

$$a\Big[\Big(\frac{a_{i+2} - a_i}{2}\Big)^2 - \Big(\frac{a_{i+1} - a_i}{2}\Big)^2\Big] - a\Big[\Big(\frac{a_{i+2} + a_i}{2} - \theta\Big)^2 - \Big(\frac{a_{i+1} + a_i}{2} - \theta\Big)^2\Big] \tag{2}$$

The first term is strictly positive since $a_i < a_{i+1} < a_{i+2}$. Define the function f as

$$f(\theta) = a\Big[\Big(\frac{a_{i+2} + a_i}{2} - \theta\Big)^2 - \Big(\frac{a_{i+1} + a_i}{2} - \theta\Big)^2\Big]$$

Then

$$f'(\theta) = 2\Big(\frac{a_i + a_{i+1}}{2} - \theta\Big) - 2\Big(\frac{a_i + a_{i+2}}{2} - \theta\Big) = \frac{a_{i+1} - a_{i+2}}{2} < 0$$

Since f is linear it must be that $\sup_{\theta \in [a_i, a_{i+1}]} f(\theta) = a_i$. Hence f is maximized at a_i, and so

$$\begin{aligned}
&a\Big[\Big(\frac{a_{i+2} - a_i}{2}\Big)^2 - \Big(\frac{a_{i+1} - a_i}{2}\Big)^2\Big] - a\Big[\Big(\frac{a_{i+2} + a_i}{2} - \theta\Big)^2 - \Big(\frac{a_{i+1} + a_i}{2} - \theta\Big)^2\Big] \\
&> a\Big[\Big(\frac{a_{i+2} - a_i}{2}\Big)^2 - \Big(\frac{a_{i+1} - a_i}{2}\Big)^2\Big] - a\Big[\Big(\frac{a_{i+2} - a_i}{2}\Big)^2 - \Big(\frac{a_{i+1} - a_i}{2}\Big)^2\Big] \\
&\geqslant 0
\end{aligned}$$

That is, all types $\theta \in (a_i, a_{i+1})$ strictly benefit from the deviation to $[a_i, a_{i+2}]$, with type a_i indifferent. By a symmetric argument, all types $\theta \in (a_{i+1}, a_{i+2})$ strictly gain, with type a_{i+2} indifferent. That is, $[a_i, a_{i+2}]$ is a self-signalling set.

□

D.6 Proof of Theorem 6

Let $\mathcal{E}$ denote the equilibrium set. Let $W : \mathcal{E} \to \mathbb{R}$ map equilibria to their ex-ante buyer welfare. That is, for an equilibrium σ, let

$$W(\sigma) = \int_{\Theta} \mathbb{E}_{\sigma}(U(v, p, \theta))d\theta$$

Proposition 19. *All residual equilibria are ex-ante buyer welfare dominated in $\mathcal{E}$.*

Proof. Take a residual equilibrium σ, with clearing set G. By the arguments made in the proof of Proposition 5, the profile σ' defined as $\sigma'|_{\Theta_1(\sigma)} = \sigma, \sigma'|_{\Theta_0(\sigma)} = br(G)$ constitutes an equilibrium and leaves the buyer strictly better off ex ante. □

Note that in the set of clearing equilibria, each σ is uniquely defined by the boundary types $0 = a_0, \ldots, a_n = 1$, since the seller best-responses are uniquely defined. Let $M(\sigma)$ denote the set of intervals defined by σ. That is $M(\sigma) = (M_1, \ldots, M_n) = ([a_0, a_1], \ldots, [a_{n-1}, a_n])$. I will abuse notation and refer to denote the widths of these intervals by M_i as well.

Proposition 20. *If $\sigma = (a_1, \ldots, a_{n-1})$ is a clearing equilibrium with associated intervals $(M_1, \ldots, M_n)$, then*

$$W(\sigma) = \frac{a}{6} \sum_{i=1}^{n} M_i^3$$

Furthermore, W is convex in $(M_1, \ldots, M_n)$.

Proof. Take a clearing n-equilibrium σ. Then

$$\begin{aligned}
W(\sigma) &= \sum_{i=1}^{n} \int_{a_{i-1}}^{a_i} U(v,p,\theta) d\theta \\
&= \sum_{i=1}^{n} \int_{a_{i-1}}^{a_i} \bar{u} - a\big(\frac{a_i + a_{i-1}}{2} - \theta\big)^2 - \bar{u} + a\big(\frac{a_i - a_{i-1}}{2}\big)^2 d\theta \\
&= \sum_{i=1}^{n} a \int_{a_{i-1}}^{a_i} \big(\frac{a_i - a_{i-1}}{2}\big)^2 - \big(\frac{a_i + a_{i-1}}{2} - \theta\big)^2 d\theta \\
&= a \sum_{i=1}^{n} \big[\big(\frac{a_i - a_{i-1}}{2}\big)^2 \theta - \frac{1}{3}\big(\frac{a_i + a_{i-1}}{2} - \theta\big)^3\big]_{a_{i-1}}^{a_i} \\
&= \frac{a}{6} \sum_{i=1}^{n} (a_i - a_{i-1})^3 \\
&= \frac{a}{6} \sum_{i=1}^{n} M_i^3
\end{aligned}$$

W is clearly then invariant over equilibria with identical intervals, regardless of their order. Since $(M_1, \ldots, M_n)$ are such that $M_i \in [0,1]$ for all i, and $\sum_{i=1}^{n} M_i = 1$, we may consider W as a function $W : \Delta_n \to \mathbb{R}_+$, where Δ_n is the n-dimensional simplex. Since Δ_n is convex, a sufficient condition for W to be a convex function is for its Hessian

$$\mathbf{D}^2 W = \frac{a}{6} \begin{pmatrix} 6M_1 & 0 & \ldots & 0 \\ 0 & 6M_2 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & 6M_n \end{pmatrix}$$

to be positive semi-definite. But $|\mathbf{D}^2 W - x\mathbf{I}| = \frac{a}{6} \prod_{i=1}^{n} (6M_i - x)$, hence $\mathbf{D}^2 W$ has all positive eigenvalues and hence the condition is met. □

To prove the proposition, note that by the proof of Proposition 5, any equilibrium σ that is not LSC is such that there exist at least two intervals $M_i, M_j \in M(\sigma)$ such that $M_i \leqslant M_j < \Lambda(a, \bar{u})$, where the first inequality is without loss. Construct a profile σ' such that $(M_1, \ldots, M_i, \ldots, M_j, \ldots, M_n) = (M_1, \ldots, M_i - \epsilon, \ldots, M_j + \epsilon, \ldots, M_n')$. For sufficiently small ϵ, σ' is a clearing n-equilibrium. Since W is convex in $(M_1, \ldots, M_n)$, it must be that $W(\sigma') > W(\sigma)$.

D.7 Proof of Proposition 8

In the LSC equilibrium, $N^* - 1$ intervals have width $\Lambda(a, \bar{u})$, with a residual interval having width $1 - (N^* - 1)\Lambda(a, \bar{u})$. Hence

$$\begin{aligned}
W_{LSC}(a) &= \frac{a}{6}\sum_{i=1}^{n} M_i^3 \\
&= \frac{a}{6}\Big[\sum_{i=1}^{N^*-1}\Lambda(a,\bar{u})^3 + (1-(N^*-1)\Lambda(a,\bar{u}))^3\Big] \\
&= \frac{a}{6}\big[(N^*-1)\Lambda(a,\bar{u})^3 + (1-(N^*-1)\Lambda(a,\bar{u}))^3\big]
\end{aligned}$$

To prove non-monotonicity, we first argue that if a is such that $1/\Lambda(a, \bar{u}) \in \mathbb{N}$, then $W_{LSC}(a) = \frac{2\bar{u}}{9}$. At such values of a, all intervals have width $\Lambda(a, \bar{u})$, and the seller charges a price $\frac{2\bar{u}}{3}$. Let $1/\Lambda(a, \bar{u}) = N$. Then $\frac{1}{2}\sqrt{\frac{3a}{\bar{u}}} = N$ and so $a = \frac{4N^2\bar{u}}{3}$. Hence

$$\begin{aligned}
W_{LSC}(a) &= W_{LSC}\big(\frac{4N^2\bar{u}}{3}\big) \\
&= \sum_{i=1}^{N}\Big[\int_0^{\frac{1}{N}} \bar{u} - \frac{4N^2\bar{u}}{3}\big(\frac{1}{2N}-\theta\big)^2 d\theta\Big] - \frac{2\bar{u}}{3} \\
&= N\Big[\frac{\bar{u}}{N} - \frac{2a}{3}\big(\frac{1}{8N^3}\big)\Big] - \frac{2\bar{u}}{3} \\
&= \frac{2\bar{u}}{9}
\end{aligned}$$

We now prove that, over regions where N^* is constant, $W_{LSC}(a)$ is strictly convex. Clearly, in these regions, $W_{LSC}(a)$ is differentiable, and hence it suffices to show that $\frac{\partial^2 W_{LSC}}{\partial a^2} > 0$. Define $W_1(x) = (N^* - 1)x^3 + (1 - (N^* - 1)x)^3$. Then

$$\begin{aligned}
W_{LSC}(a) &= \frac{a}{3}W_1(\Lambda(a)) \\
W'_{LSC}(a) &= W'_1(\Lambda(a))\big(\Lambda'(a)\frac{a}{3}\big) + W_1(\Lambda(a)) \\
W''_{LSC}(a) &= \Lambda'(a)\frac{a}{3}W''(\Lambda(a))\Lambda'(a) + W'_1(\Lambda(a))\big(\Lambda''(a)\frac{a}{3} + \frac{\Lambda'(a)}{3}\big) + W'_1(\Lambda(a))\Lambda'(a) \\
&= \frac{\Lambda'(a)^2 W''_1(\Gamma(a))a}{3} + W_1(\Lambda(a))\big(\Lambda''(a)\frac{a}{3} + \frac{4}{3}\Lambda'(a)\big)
\end{aligned}$$

By direct calculation, it suffices to show that $\Lambda''(a)\frac{a}{3} + \frac{4}{3}\Lambda'(a) < 0$. But

$$\begin{aligned}
\Lambda''(a)\frac{a}{3} + \frac{4}{3}\Lambda'(a) &= \big(\frac{3}{\bar{u}}\big)^2\big(\frac{3a}{\bar{u}}\big)^{\frac{-5}{2}}\frac{a}{3} - \frac{4}{3}\frac{3}{\bar{u}}\big(\frac{3a}{\bar{u}}\big)^{-\frac{3}{2}} \\
&= \frac{a^{-\frac{3}{2}}}{u^{-\frac{1}{2}}}\big(3^{-\frac{5}{2}} - 4(3^{-\frac{3}{2}})\big) \\
&< 0
\end{aligned}$$

as required.

D.8 Proof of Proposition 9

To prove the first part of the proposition, note that as before, the seller is unwilling to charge a price of lower than $\frac{2\bar{u}}{3}$, and hence we can derive the critical width as $\Lambda_k(a, \bar{u}) = 2k\sqrt{\frac{\bar{u}}{3a}}$. The remainder of the construction is identical to that in the proof of Proposition 5.

For the second part, it is again without loss to focus on clearing equilibria. Let $0 = a_0 < \cdots < a_n = 1$ denote the boundary types on pooling intervals as before. Let $(v_{i,j}, p_{i,j}), i = 1, \ldots, n, j = 1, \ldots, k$ denote the seller's offer and $b_{i,j}, i = 1, \ldots, n, j = 1, \ldots, k$ be types indifferent between $(v_{i,j}, p_{i,j})$ and $(v_{i,j+1}, p_{i,j+1})$, so that $b_{i,k} = b_{i+1,1} = a_{i+1}$. Further, let $M_{i,j} = b_{i+1,j} - b_{i,j}$ be the widths of intervals accepting the same option within a menu, and $M_i = a_{i+1} - a_i$ the widths of pooling types. As before, in equilibrium it must be that each $b_{i,j}$ receives 0 surplus, and hence the seller's expected profit from a given menu can be computed as

$$\begin{aligned}
\Pi(v_{i,j}, p_{i,j}) &= \sum_j M_{i,j}(\bar{u} - a(\frac{M_{i,j}}{2})^2) \\
&= \bar{u}M_i - \frac{a}{4}\sum_j M_{i,j}^3
\end{aligned}$$

Refer to a menu $(v_{i,j}, p_{i,j})$ as *balanced* if $M_{i,j} = M_{i,j+1}$ for all $j = 1, \ldots, k$. That is, a menu is balanced if the seller's offer partitions each pooling interval into equal groups. We now prove that the seller's optimal offer is balanced. Suppose not, i.e there exists an i and r, s such that $M_{i,r} \neq M_{i,s}$. Without loss, and since the equilibrium is clearing, we take $M_{i,r} < M_{i,s} \leqslant \Lambda_k(a, \bar{u})$. Consider a new menu $(v'_{i,j}, p'_{i,j})$ with $M'_{i,r} = M_{i,r} + \epsilon$, $M'_{i,s} = M_{i,s} - \epsilon$ for small $\epsilon > 0$, and all remaining $M_{i,j}$ unchanged. For small enough ϵ, such a menu is still clearing and hence is incentive compatible, whilst the above formula implies that the new menu yields greater profit to the seller.

Finally, using this result, the buyer's ex-ante welfare for a given clearing equilibrium σ can be computed as

$$W(\sigma) = \frac{a}{6}\sum_{i=1}^{n} k(\frac{M_i}{k})^3$$
$$= \frac{a}{6k^2}\sum_{i=1}^{n} M_i^3$$

Notice that this expression is identical to that in the proof of Proposition 19 scaled by $\frac{1}{k^2}$, and hence the result follows by the same argument.

D.9 Proof of Theorem 10

We prove the result with a sequence of claims. The main idea is to find necessary and sufficient conditions for a set G to be self-signalling.

Proposition 21. *Take a clearing n-equilibrium σ with boundary types $0 = a_0 < \cdots < a_n = 1$. If $G \subset \Theta$ is self-signalling wrt σ, then $G = [a_j, a_{k+1}]$, for some $j \in \{1, \ldots, n\}$, $k > j$.*

Proof. Clearly $G = [a_i, a_{i+1}]$ cannot be self-signalling, since σ is an equilibrium. By single-crossing, we may then restrict our attention to interval sets $G = [b, c], b < c$. Suppose b is not a boundary type, i.e. $b \in (a_i, a_{i+1})$ for some i. For $\epsilon > 0$ and define

$$\mathcal{O}_\epsilon(b) = \{\theta \in [a_i, a_{i+1}] || \theta - b| < \epsilon\}$$
$$U(\mathcal{O}_\epsilon^-(b); \sigma) = \inf_{\theta \in \mathcal{O}_\epsilon(b)} \mathbb{E}_\sigma(U(v, p, \theta))$$
$$U(\mathcal{O}_\epsilon^+(b); \sigma) = \sup_{\theta \in \mathcal{O}_\epsilon(b)} \mathbb{E}_\sigma(U(v, p, \theta))$$

For sufficiently small ϵ, we have that $\mathcal{O}_\epsilon(b) \subset (a_i, a_{i+1})$ and $U(\mathcal{O}_\epsilon^-(b); \sigma) = \delta(\epsilon) > 0$. Fix such an ϵ. By Proposition 3, we have that $U(b; br(G)) = 0$, and hence by continuity of U, there exists $\epsilon^* > 0$ such that $U(\mathcal{O}_{\epsilon^*}^+(b); br(G)) < \delta(\epsilon)$. That is, the positive measure of types $\mathcal{O}_{\epsilon^*}(b)$ are strictly worse off from the deviation. An identical argument shows that c must also be a boundary type. □

Proposition 22. *σ is NP if and only if σ is clearing and for all $G = [a_i, a_{i+2}]$, G is residual. Hence, if σ is babbling, then σ is NP if and only if it is clearing.*

Proof. For the first part, note that the $\Rightarrow$ implication follows directly from the contrapositive statement of Proposition 2. For the $\Leftarrow$ implication, note first that if $[a_i, a_{i+2}]$ is residual, then by Lemma 1 so too is $[a_i, a_{i+k}]$ for all $k > 2$. Now take any residual G. We claim this cannot be a self-signalling set. To this end, fix an offer $(v, p) \in br(G)$. Since G is residual, there exists an open set $\mathcal{O}(v,p) = \{\theta \in G | E_{br(G)}(U(v,p,\theta)) = 0\}$, and hence $\mathcal{O}(v,p)$ cannot strictly gain from the deviation. Since $(v,p) \in br(G)$ was arbitrary, G cannot be self-signalling. The implication then follows from 20. $\square$

Theorem 23. *An equilibrium is NP if and only if it is clearing and* $[a_i, a_{i+2}] > \Lambda(a,\bar{u})$ *for all consecutive boundary types* a_i, a_{i+1}, a_{i+2}. *In particular, if an n-partitional equilibrium is NP, then* $n \in \{N^*, \dots, 2N^* - 1\}$.

Proof. The bounds are now immediate from Proposition 21. For the last part, a simple counting argument shows that the upper bound on the number of intervals is given by $\lfloor \frac{1-\Lambda(a,\bar{u})}{\Lambda(a,\bar{u})/2} \rfloor + 2$. Applying Hermite's Identity (see [Graham and Knuth (1994)]), we have that

$$
\begin{aligned}
\lfloor \frac{1-\Lambda(a,\bar{u})}{\Lambda(a,\bar{u})/2} \rfloor + 2 &= \lfloor \frac{2}{\Lambda(a,\bar{u})} \rfloor \\
&= \lfloor \frac{1}{\Lambda(a,\bar{u})} \rfloor + \lfloor \frac{2}{\Lambda(a,\bar{u})} + \frac{1}{2} \rfloor \\
&= N^* - 1 + N^* \\
&= 2N^* - 1
\end{aligned}
$$

$\square$