

Week 19 RBC

February 10, 2016

Question 2 - Derivations

First, we were asked to derive three equations characterizing the steady state based on the following assumptions: The depreciation rate is zero, i.e. $\delta = 0$, utility is logarithmic, i.e. $u(c) = \log(c)$, the production function is Cobb-Douglas, i.e. $Af(K, h) = AK^\alpha h^{1-\alpha}$, and, not explicitly stated but implicitly assumed in the solutions, the subjective discount rate is equal to the interest rate, so the discount factor is $\beta = 1/1 + r$.

To derive these equations, start from the equilibrium conditions derived in seminar last week. Also, note that in steady state we can drop time subscripts. Making use of the above assumptions and rearranging, the three equilibrium conditions are as follows.

The condition for the intertemporal consumption-savings decision becomes

$$\begin{aligned}u'(c) &= \beta(1 + Af_K(K, h) - \delta)u'(c) \\1 &= \beta(1 + Af_K(K, h) - \delta) \\ \frac{1}{\beta} - 1 + \delta &= Af_K(K, h) \\1 + r - 1 &= \alpha AK^{\alpha-1} h^{1-\alpha} \\K &= \left(\frac{\alpha A}{r}\right)^{\frac{1}{1-\alpha}} h\end{aligned}\tag{1}$$

The condition for labour supply becomes

$$\begin{aligned}\frac{v'(1-h)}{u'(c)} &= Af_h(K, h) \\cv'(1-h) &= (1-\alpha)AK^\alpha h^{-\alpha}\end{aligned}\tag{2}$$

The capital accumulation equation becomes

$$\begin{aligned}K &= (1-\delta)K + Af(K, h) - c \\c &= Af(K, h) - \delta K \\c &= AK^\alpha h^{1-\alpha}\end{aligned}\tag{3}$$

Second, we were asked to obtain an implicit expression for steady state hours worked in terms of parameters and exogenous variables only. To do so, first plug (3) into (2), then plug (1) into (2), and rearrange

$$\begin{aligned} [AK^\alpha h^{1-\alpha}] v'(1-h) &= (1-\alpha)AK^\alpha h^{-\alpha} \\ \left[A \left(\frac{\alpha A}{r} \right)^{\frac{\alpha}{1-\alpha}} h \right] v'(1-h) &= (1-\alpha)A \left(\frac{\alpha A}{r} \right)^{\frac{\alpha}{1-\alpha}} \\ hv'(1-h) &= (1-\alpha) \end{aligned}$$

Note that technology does not enter steady state hours worked, the significance of which we discussed in the seminar.

Why does technology not enter steady state hours worked? The following was not required for the seminar, but it was mentioned in the solutions (which were based on problem sets in previous years when this question was included) because it shows more clearly how the result - that technology does not enter steady state hours worked - relates to the assumption of logarithmic utility. It just repeats the previous derivations but relaxes the assumption of logarithmic utility. As a result, the derivations are a bit more tedious. Here, I show the derivations just for the sake of completeness and so you can see how I got the end result which I mentioned in the seminar. Please feel free to ignore this part.

To proceed, assume the more general utility function $u(c_t) = \frac{1}{1-\sigma} c_t^{1-\sigma} + v(1-h_t)$. Also, note that $\lim_{\sigma \rightarrow 1} \frac{1}{1-\sigma} c_t^{1-\sigma} = \log(c_t)$. That is, the utility function now assumed nests logarithmic utility as a special case for $\sigma = 1$. Again we want to obtain an implicit expression for steady state hours worked in terms of parameters and exogenous, but now relaxing the assumption of logarithmic utility and zero depreciation (in my understanding, the crucial assumption is logarithmic utility, and zero or nonzero depreciation does not affect the substantial result of technology cancelling out. I just allow for nonzero depreciation to be in line with what is mentioned in the solutions).

Based on our assumptions, the three equilibrium conditions become

$$\begin{aligned} u'(c) &= \beta(1 + Af_K(K, h) - \delta)u'(c) \\ 1 &= \frac{1}{1+r}(1 + AK^\alpha h^{1-\alpha} - \delta) \\ K &= \left(\frac{\alpha A}{r + \delta} \right)^{\frac{1}{1-\alpha}} h \end{aligned} \tag{4}$$

$$\begin{aligned} \frac{v'(1-h)}{u'(c)} &= (1-\alpha)Af_h(K, h) \\ \frac{v'(1-h)}{c^{-\sigma}} &= (1-\alpha)A \left(\frac{K}{h} \right)^\alpha \end{aligned} \tag{5}$$

$$\begin{aligned}
K &= (1 - \delta)K + Af(K, h) - c \\
c &= AK^\alpha h^{1-\alpha} - \delta K \\
c &= (AK^{\alpha-1} h^{1-\alpha} - \delta) K
\end{aligned} \tag{6}$$

To simplify, substitute for K in (6) using (4).

$$\begin{aligned}
c &= \left(A \left(\left(\frac{\alpha A}{r + \delta} \right)^{\frac{1}{1-\alpha}} h \right)^{-(1-\alpha)} h^{1-\alpha} - \delta \right) \left(\frac{\alpha A}{r + \delta} \right)^{\frac{1}{1-\alpha}} h \\
c &= \left(\frac{r + \delta}{\alpha} - \delta \right) \left(\frac{\alpha A}{r + \delta} \right)^{\frac{1}{1-\alpha}} h
\end{aligned} \tag{7}$$

Next, plug (7) into (5), substitute for K using (4) and rearrange to get

$$\begin{aligned}
c^\sigma v'(1-h) &= (1-\alpha)A \left(\frac{K}{h} \right)^\alpha \\
\left[\left(\frac{r + \delta}{\alpha} - \delta \right) \left(\frac{\alpha A}{r + \delta} \right)^{\frac{1}{1-\alpha}} h \right]^\sigma v'(1-h) &= (1-\alpha)A \left(\frac{K}{h} \right)^\alpha \\
\left[\left(\frac{r + \delta}{\alpha} - \delta \right) \left(\frac{\alpha A}{r + \delta} \right)^{\frac{1}{1-\alpha}} h \right]^\sigma v'(1-h) &= (1-\alpha)A \left(\frac{\alpha A}{r + \delta} \right)^{\frac{\alpha}{1-\alpha}} \\
\left[\left(\frac{r + \delta}{\alpha} - \delta \right) \left(\frac{\alpha}{r + \delta} \right)^{\frac{1}{1-\alpha}} \right]^\sigma h^\sigma v'(1-h) &= (1-\alpha) \left(\frac{\alpha}{r + \delta} \right)^{\frac{\alpha}{1-\alpha}} A^{1 + \frac{\alpha}{1-\alpha} - \frac{\sigma}{1-\alpha}} \\
h^\sigma v'(1-h) &= (1-\alpha)\gamma A^{\frac{1-\sigma}{1-\alpha}}
\end{aligned}$$

where $\gamma \equiv \left(\frac{\alpha}{r + \delta} \right)^{\frac{\alpha}{1-\alpha}} \left[\left(\frac{r + \delta}{\alpha} - \delta \right) \left(\frac{\alpha}{r + \delta} \right)^{\frac{1}{1-\alpha}} \right]^{-\sigma}$ is a bunch of parameters not of direct interest. The main result is that for $\sigma = 1$ we get $A^0 = 1$, so we can see here that technology cancels out because we choose $\sigma = 1$ (which is the implicit assumption when we use logarithmic utility). The reason for choosing $\sigma = 1$ was discussed in the seminar.