Week 6 - Amendment

November 13, 2015

Question 1 - Derivation of k^*/y^*

We want to find an expression for the steady state capital to income ratio. It is equivalent to look for the ratio in levels or in terms of effective workers

$$\frac{K^*}{Y^*} = \frac{\frac{K^*}{EN}}{\frac{Y^*}{EN}} = \frac{k^*}{y^*}$$

First, we derive the steady state value of capital per effective worker, k^*. Note that the $'*'$ denotes steady state values. Start from the steady state condition for the capital stock in the long run. The value of capital per effective worker which solves this condition is k^*.

$$\frac{MPK}{1+\mu} = \bar{r} + \delta$$

Find an expression for MPK using the aggregate production function $Y = K^\alpha (EN)^{1-\alpha}$.

$$\begin{aligned} MPK = \frac{\partial Y}{\partial K} &= \alpha K^{\alpha-1}(EN)^{1-\alpha} \\ &= \alpha \frac{K^{\alpha-1}}{(EN)^{-(1-\alpha)}} \\ &= \alpha \left(\frac{K}{EN}\right)^{\alpha-1} = \alpha k^{\alpha-1} \end{aligned}$$

Plug the expression for MPK back into optimality condition for capital and solve for k.

$$\begin{aligned} \frac{\alpha k^{\alpha-1}}{1+\mu} &= \bar{r} + \delta \\ k^{\alpha-1} &= \frac{(\bar{r}+\delta)(1+\mu)}{\alpha} \end{aligned}$$

$$k = \left[\frac{(\bar{r}+\delta)(1+\mu)}{\alpha}\right]^{\frac{1}{\alpha-1}}$$

$$k^* = \left[\frac{\alpha}{(\bar{r}+\delta)(1+\mu)}\right]^{\frac{1}{1-\alpha}}$$

Second, we derive the steady state value for output per effective worker, y^*. Start from the aggregate production function and divide it by EN to get output per effective worker.

$$\begin{aligned} \frac{Y}{EN} \equiv y &= \frac{K^\alpha (EN)^{1-\alpha}}{EN} = K^\alpha (EN)^{-\alpha} \\ &= \left(\frac{K}{EN}\right)^\alpha = k^\alpha \end{aligned}$$

Steady state output is the aggregate production function evaluated at steady state capital. Thus, plug in k^* to get steady state output per worker.

$$\begin{aligned} y^* &= k^{*\alpha} \\ &= \left[\frac{\alpha}{(\bar{r}+\delta)(1+\mu)}\right]^{\frac{\alpha}{1-\alpha}} \end{aligned}$$

Third, compute the ratio k^*/y^*. The long run capital to income ratio in the Ramsey model is

$$\begin{aligned} \frac{k^*}{y^*} = \frac{K^*/EN}{Y^*/EN} = \frac{K^*}{Y^*} &= \left[\frac{\alpha}{(\bar{r}+\delta)(1+\mu)}\right]^{\frac{1}{1-\alpha}} \left[\frac{\alpha}{(\bar{r}+\delta)(1+\mu)}\right]^{\frac{-\alpha}{1-\alpha}} \\ &= \left[\frac{\alpha}{(\bar{r}+\delta)(1+\mu)}\right]^{\frac{1-\alpha}{1-\alpha}} \\ &= \frac{\alpha}{(\bar{r}+\delta)(1+\mu)} \end{aligned}$$

Question 3 - Increase in μ

In class we discussed changes in the depreciation rate and the rate of impatience. Now, we want to know how the steady state value of capital per effective worker changes for an increase in the price mark-up μ. The graph depicts the following condition

$$\frac{\alpha k^{\alpha-1}}{1+\mu} - \delta = \bar{r}$$

The downward sloping curve denoted LHS depits the left-hand side of this condition. This LHS can also be interpreted as the net gain from investing in the capital stock. The y-axis plots this gain in terms of r as a function of k.

First, consider an increase in μ for a given value of k. For a given k, a higher μ decreases the value of the LHS:

$$\frac{\alpha k^{\alpha-1}}{(1+\mu)\uparrow} \downarrow -\delta$$

Thus, the curve depicting the LHS shifts downwards, i.e. for a given level of k, the net gain from investing in the capital stock is lower.

Second, consider the slope of the LHS curve:

$$\text{Slope} = \frac{\partial LHS}{\partial k} = \frac{\alpha\left(\alpha-1\right)}{1+\mu}k^{\alpha-2} = -\frac{\alpha\left(1-\alpha\right)}{1+\mu}k^{\alpha-2}$$

where the last equality is just to point out that the slope is negative because $0<\alpha<1$. Now, consider an increase in μ:

$$-\frac{\alpha\left(1-\alpha\right)}{(1+\mu)\uparrow}k^{\alpha-2}\downarrow$$

Hence, a higher price mark-up decreases the slope of the LHS, the LHS curve becomes flatter.

There are thus two changes of the LHS curve in response to a higher mark-up: a downward shift and an anti-clockwise rotation making the curve flatter. Both act to decrease the steady state capital stock per effective worker for a given long-term real interest rate: The LHS decreases for a given k, the RHS with $\bar{r}$ stays unchanged, so the LHS has to increase to equal $\bar{r}$ again. This is achieved at a lower capital stock per effective worker, where the MPK and thus the return on investment are higher.

How can we explain this result intuitively? A higher mark-up decreases the net gain from investing in the capital stock because a) it increases the costs of capital (the model assumes there is only one type of good, which can either be consumed or invested to be turned into capital, thus the price for consumption and capital goods is the same, i.e. $P=(1+\mu)MC$), and b) it decreases the marginal value product of capital (recall that the MVPK=MPK*MR was shown to equal $MPK/(1+\mu)$ in week 4's seminar). To see b), recall from week 3's seminar that a higher mark-up indicates a steeper demand curve and thus a steeper marginal revenue curve. Thus, for a given k and thus a given level of production, a steeper demand curve implies firms will have lower revenues.

Hence, for a given k the net gain from investing in the capital stock will be lower. Firms find it optimal to invest in the capital stock until this net gain equals the costs of borrowing as given by $\bar{r}$ (recall that the model assumes firms borrow money to invest in one unit of capital and repay the loan next period). At the initial k, the cost of borrowing are higher than the net gain of investing in capital. So firms will acquire a lower capital stock in steady state, at a higher marginal product of capital and thus a higher net gain from investing, such that the net gain is again equal to the costs of investing in the capital stock.