The Gentzen rules for propositional logic ($\mathcal{G}_{\mathbf{PL}}$)

$\wedge$ rules

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi}\,\wedge I \qquad \frac{\psi \quad \varphi}{\varphi \wedge \psi}\,\wedge I \qquad \frac{\varphi \wedge \psi}{\varphi}\,\wedge E \qquad \frac{\varphi \wedge \psi}{\psi}\,\wedge E$$

$\vee$ rules

$$\frac{\varphi}{\varphi \vee \psi}\,\vee I \qquad \frac{\psi}{\varphi \vee \psi}\,\vee I \qquad \frac{\varphi \vee \psi \quad \begin{matrix}[\varphi]\\ \vdots \\ \chi\end{matrix} \quad \begin{matrix}[\psi]\\ \vdots \\ \chi\end{matrix}}{\chi}\,\vee E$$

Note: $\vee E$ discharges *both* the hypotheses φ and ψ.

$\rightarrow$ rules

$$\frac{\begin{matrix}[\varphi]\\ \vdots \\ \psi\end{matrix}}{\varphi \rightarrow \psi} \rightarrow I \qquad \frac{\varphi \rightarrow \psi \quad \varphi}{\psi} \rightarrow E$$

Note: Although in the form given, $\rightarrow I$ discharges the hypothesis φ, it also admits the form $\frac{[\psi]}{\varphi \rightarrow \psi} \rightarrow I$ in which ψ is *not* discharged.

$\neg$ rules

$$\frac{\begin{matrix}[\varphi]\\ \vdots \\ \bot\end{matrix}}{\neg\varphi}\,\neg I \qquad \frac{\varphi \quad \neg\varphi}{\bot}\,\neg E$$

$\bot$ rules

$$\frac{\bot}{\varphi}\,\bot E \qquad \frac{\begin{matrix}[\neg\varphi]\\ \vdots \\ \bot\end{matrix}}{\varphi}\,\text{RAA}$$

Note: RAA discharges the hypothesis $[\varphi]$.

$\leftrightarrow$ rules

$$\frac{\begin{matrix}[\varphi]\\ \vdots \\ \psi\end{matrix} \quad \begin{matrix}[\psi]\\ \vdots \\ \varphi\end{matrix}}{\varphi \leftrightarrow \psi} \leftrightarrow I \qquad \frac{\varphi \leftrightarrow \psi \quad \varphi}{\psi} \leftrightarrow E \qquad \frac{\psi \leftrightarrow \psi \quad \varphi}{\varphi} \leftrightarrow E$$

Note: $\leftrightarrow I$ discharges both the hypotheses φ and ψ.

The Gentzen rules for first-order logic ($\mathcal{G}_{\mathbf{FOL}}$)

$\mathcal{G}_{\text{FOL}}$ consists of the propositional rules of $\mathcal{G}_{\text{PL}}$ together with the following first-order rules:

$\forall$ rules

$$\dfrac{\begin{matrix}[\Gamma]\\ \vdots \\ \varphi(x)\end{matrix}}{\forall x\varphi(x)}\ \forall I \quad \text{where } x \notin \text{FV}(\Gamma)$$

$$\dfrac{\forall x\varphi(x)}{\varphi(x)[t/x]}\ \forall E \quad \text{where } t \text{ is free for } x \text{ in } \varphi(x)$$

$\exists$ rules

$$\dfrac{\varphi(t)}{\exists x\varphi(x)}\ \exists I \qquad \text{where } t \text{ is free for } x \text{ in } \varphi(x)$$

$$\dfrac{\exists x\varphi(x) \qquad \begin{matrix}[\varphi(x)]\\ \vdots \\ \psi\end{matrix}}{\psi}\ \exists E$$

if $x \notin \text{FV}(\psi)$ and x is also not free in any hypothesis on which the subderivation of ψ depends other than $\varphi(x)$

Identity axioms

$$\dfrac{}{x = x}\ RI_1 \qquad \dfrac{x = y}{y = x}\ RI_2 \qquad \dfrac{x = y \quad y = z}{x = z}\ RI_2$$

$$\dfrac{x_1 = y_1, \ldots, x_n = y_n}{t(x_1, \ldots, x_n) = t(y_1, \ldots, y_n)}\ RI_4$$

$$\dfrac{x_1 = y_1, \ldots, x_n = y_n \quad \varphi(x_1, \ldots, x_n)}{\varphi(y_1, \ldots, y_n)}\ RI_5$$

In $RI_1, \ldots, RI_5$ it is assumed that the variables $y_1, \ldots, y_n$ are free for $x_1, \ldots, x_n$ in φ.