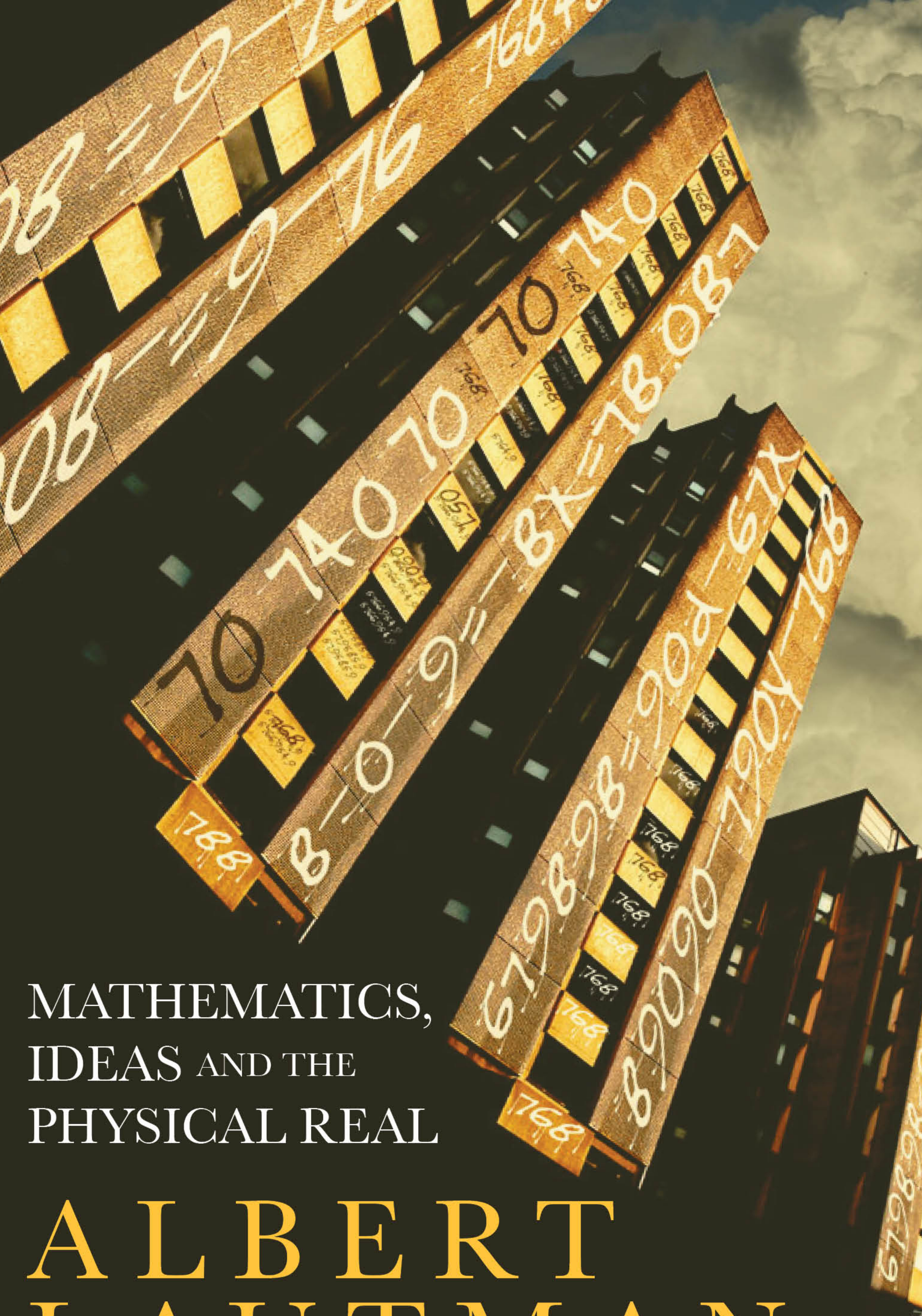

MATHEMATICS, IDEAS AND THE PHYSICAL REAL

ALBERT LAUTMAN

MATHEMATICS, IDEAS AND THE PHYSICAL REAL

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Albert Lautman

Translated by Simon B. Duffy

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Translator's Note

The collected work of Albert Lautman was first published in 1977 as *Essai sur l'unité des mathématiques et divers écrits* by Union Générale d'Éditions. It was not until 2006 that the re-edited collection upon which this translation is based appeared from Librairie Philosophique J. Vrin. In addition to permission to include all of the texts by Lautman that appeared in the 2006 volume, Fernando Zalamea has given permission to include his introduction to Lautman's work that also appeared in it. This Zalamea article will provide the main introduction to the work of Lautman in the current volume. The secondary bibliography of reviews and philosophical commentaries on Lautman's work that was compiled by Zalamea for the 2006 volume has been updated and expanded for inclusion in the current volume.

The four main essays published by Lautman are included in the chronological order of their appearance. The first two that appeared in 1938 were for the Doctorat D'Etat. 'Essay on the unity of the mathematical sciences in their present development' (1938a) was Lautman's secondary thesis, and 'Essay on the notions of structure and existence in mathematics. I. The schemas of structure. II. The schemas of genesis' (1938b) was his principal thesis. Lautman often refers to these two essays in his other work in this way, that is, as his 'principal thesis' and his 'secondary thesis'. Whenever he does so, I have included the citations as above.

I have made a few typographic corrections to the equations with reference to the original publications. Where English translations of cited or referenced material are available, I have provided citations from the English edition and/or given page references to the English edition in square brackets after the reference to the original language edition, except where otherwise indicated.

TRANSLATOR'S NOTE

In the French translation of Heidegger's *The Essence of Reason* (1969 [1929; 1938]) used by Lautman, *Dasein* is translated as 'human reality'. I have retained this French usage in all quotations from Heidegger 1969, and indicated this in the text with square brackets around [human reality]. I have also altered the Malik translation to render Heidegger's distinction between *Sein* – as 'being', the gerund of 'to be' – and *Seiend* (singular) or *Seiendes* (plural) – as 'an entity' and 'entities'. References in the text are to the English translation edition (Heidegger 1969) followed by page numbers in square brackets to the French translation.

I have translated the term *dominer*, which Lautman uses to describe the nature of the relation that exists between dialectical ideas on the one hand, and mathematical theories on the other, with the term 'to govern'. I have translated the French *sensible* as 'sensible' in English, as in sensibility in contrast to understanding, in keeping with the common English translation of the Kantian term *sinnlich*.

Lautman refers to combined subdomains of mathematics, such as algebraic topology, differential geometry, algebraic geometry and analytic number theory as either *les mathématiques mixtes*, and simply *les mixtes* or *un mixte*, which I have translated, respectively, as 'mixed mathematics' and the 'mixes' or a 'mix'.

Additional translator's notes on particular points have been included in the Notes following each section. These are indicated by the following: —Tr.

I would like to thank Daniel W. Smith, who, in consultation with Continuum, suggested that I undertake this project, and Sandra L. Field, for her support and encouragement throughout.

Simon B. Duffy

Acknowledgements, 2006 Edition

After the new study by Fernando Zalamea and the re-publication of the preface written in 1977 by Jean Dieudonné, the chronological order has been retained. In this new edition of the collection it seemed justified to provide the view that the two articles, ‘Essay on the unity of the mathematical sciences in their present development’ (secondary thesis) and ‘Essay on the notions of structure and existence in mathematics’ (principal thesis), introduce, among other things, the ideas and the positions formulated by the author after these were first published. Similarly, with the latest articles, his conviction of a profound affinity between mathematical structures and the exigency of the external real revealed in the formalization of physical theories is seen to take an increasingly important place in the course of his short career.

A secondary bibliography listing reviews and philosophical commentaries of the work is also presented. Fernando Zalamea established the second bibliography and here authorized its use. I thank him warmly.

The bibliography of all cited references has been established for this edition, thanks to the resources of the Library of Mathematics at the École Normale Supérieure. Permission was granted to refrain from reproducing the original bibliographies, which remain rather imperfect, since, at that time, philosophers other than historians of philosophy had little respect for the rules which have since become the required standards.

Finally, I would like to express the gratitude of my brother and myself to the Librairie Philosophique Vrin and Madame Arnaud, its director, and to Jean-François Courtine for having welcomed these old and difficult texts.

Jacques Lautman

Introduction

by Jacques Lautman

The present book, with preface by Jean Dieudonne from the 1977 edition, preceded by a recent study by Fernando Zalamea, reproduces the entire collection, plus a few additions, of the texts of Albert Lautman which had been reunited in 1977 under the title *L'Unité des sciences mathematiques et autres écrits* in one volume of the series 10/18, prepared by Maurice Loi. The title, taken from the supplementary thesis published in 1938 by Hermann, indicates one of the many directions of his work, but it was a poor guide to the hidden metaphysical ambition of the author, which is central, though it remained underdeveloped. Since Parmenides, the great philosophers have scrutinized the very complex relations between the opposites: the finite and the infinite, the continuous and the discrete, the open and the closed, the local and the global, the same and the other, and movement and immobility. All are pairs of concepts used to describe the situations, or processes, of nature. The simplest example is chirality (the impossibility of superimposing two dissymmetrical objects in the same orientation) and the importance of dissymmetry is well known in the discovery of crystallography by Pasteur.

Albert Lautman understood the fundamental rupture between mathematics up until Augustin-Louis Cauchy and modern mathematics that arises with Évariste Galois and Niels Henrik Abel, and progressed rapidly with Bernhard Riemann, Georg Cantor and then David Hilbert to create a multitude of developments in which opposites interpenetrate, create inclusive links, open new domains to the creative imagination and, in the process, produce quantities of mathematical entities whose identity is established, more or less clearly, at various levels. Essential are the mixes that appear, unexpectedly or secretly, and that make new improbable passages possible. Mathematics has its particular development but Max Planck and Albert Einstein have brought to the fore how physics and cosmology require, *ex post*, the creations of Riemann and Henri Poincaré, from which comes Albert Lautman's conviction that dialectical pairs dominate the physical real in the same way as the functioning of the mind, and that mathematics is the most accomplished modality of the development of the possibilities of operational connections between opposites. In his lectures, he loved to cite Malebranche: 'The study of mathematics is the purest application of the mind to God'.[1] Lautman's last texts, written in 1943–1944, clearly attest to a shift in his work towards physics.

This work on mathematical philosophy, short and very focused, is indebted to the entry by Jean Petitot in the *Enciclopedia Einaudi* (Petitot 1982) for not being completely ignored, even by the specialists. Despite the two theses defended in December 1937 in front of Leon Brunschvicg and the mathematician Elie Cartan, both great figures, having been immediately subject to a critical review by Jean Cavaillès in the *Revue de Metaphysique et Morale* (1938a) and, in 1940, in the *Journal of Symbolic Logic*, a penetrating review, highlighting the unresolved difficulties concerning the status of the existence of mathematical objects, by Paul Bernays (1940), a colleague of Hilbert. The break in scientific exchanges because of the war and the death of the author at 36 years of age in 1944 explains in part the forgetfulness that was only interrupted from time to time, with the 1977 edition by 10/18 (Lautman 1977), and in 1987 by Petitot in the *Revue d'histoire des sciences*.[2]

Three other reasons, each situated at very different levels relative to one another, are worth mentioning: the uncertainty about a receptive audience, the silence of the logicians and Cavaillès's shadow. These texts, whose argument relies heavily on the precise analysis of a number of mathematical creations during the years 1860 to 1943 are difficult reading for ordinary philosophers. Mathematicians follow more easily; however, they are normally devoted to mathematics. It is, in general, only late in life that some, usually only the great figures, interest themselves in the ultimate questions, and, as will be found in the preface to the 1977 edition by Jean Dieudonné, pivotal to the Bourbaki group at its creation and for a long time thereafter.

Detached from examples of mathematics that are the subject matter on which work that is technically philosophical is based, the metaphysical contribution, in the strict sense, which carries the meaning of the work, has been puzzling: suggestive but too underdeveloped not to be ambiguous, since, even though both the realist and idealist conceptions of science could retrieve it, it would remain absolutely intolerable to the nominalist. The absence of interest from mathematical logicians and the frank hostility of philosophical logicians, with the exception of Ferdinand Gonseth,[3] are better appreciated when it is understood that the first two articles published by Albert Lautman comprise frontal attacks against the heritage of the *Principia Mathematica* of Bertrand Russell and Alfred North Whitehead, and also against Rudolf Carnap and the Vienna Circle.

The few pages from 1939 in which he expressed his interest in *Being and Time* (1962 [1927]) have retained much less attention than that his rationalist reading is far removed from the Martin Heidegger of the French after the war[4], if not from Heidegger himself. Gilles Deleuze (1994 [1968], chapter 4) and, more recently, Alain Badiou (2005 [1998]) appear to be the only ones to have made explicit a strictly philosophical use of his work, outside of a strictly epistemological context or the history of the discipline.

It is hard to write that the image, quite rightly great, of Jean Cavaillès cast a shadow over the destiny of the writings of Albert Lautman. Their trajectories, philosophical as much as during war time, were both too parallel and too unequal in visibility for it to have been otherwise. Cavaillès, the elder by a few years, was a young substitute professor at the Sorbonne in 1941. He had a place of national importance in the Resistance, and with the combination of these two qualities, he has become, post-mortem, the symbol of a resistant university, in part mythical. For those not looking carefully, Albert Lautman, whose name has often been quoted next to that of Cavaillès, could only have been a disciple, at best a double. This is not how Lautman and Cavaillés saw themselves, as evidenced by their public exchange, organized by Leon Brunschvicg before the *Societe française de philosophie* on 4 February 1939 (Cavaillès and Lautman 1946), and also those few remaining letters from Cavaillès to Lautman (see Benis-Sinaceur 1987). Their friendship, the identity of their ethico–political commitment and their work community in a domain in which those who might follow them were scarce did not exclude a fairly large difference in angle of attack on mathematics and the task of the philosopher.

The life of Albert Lautman had been deeply inflected twice. In autumn 1923, in the elementary mathematics class at Lycée Condorcet, he met Jacques Herbrand[5] – in the words of his contemporaries, a quite exceptional person – whose influence was absolutely crucial on his orientation towards mathematical philosophy. Less than 15 years later, the announcement of the triumph of Nazism will transform the philosopher, for whom the defence of freedom and universal values are a necessity that take precedence over all, in wartime.

Lautman was born in Paris in 1908. His father, Sami Lautman, a Jew of the Austro–Hungarian Empire, had been excluded from the competitive examinations to train at the hospitals in Vienna, under the *numerus clausus*. Lüger, elected Mayor of Vienna in 1886, is the one who put in place the first anti-Semitic provisions of contemporary times. So Sami Lautman arrived in Paris in 1891 and had to start by obtaining a baccalaureate before resuming his medical studies *ab initio*. In 1914, he joined the French Army, but because he was newly naturalized, he could not be an officer and, therefore, could not be classed a doctor. He was a warrant officer stretcher-bearer and, of course, served as a doctor. Critically wounded, he received the *Croix de Chevalier* (Knight's Cross) of the *Legion d'Honneur*. Albert's mother, née Lajeunesse, is from a family of both Avignonians (Papal Jews) and Alsacians who were settled in Paris for several generations.

In 1926, Albert entered the École Normale Supérieure via the Letters examination and met Jacques Herbrand again, who had entered in 1925, first in the Science examination, and who, to join his friend, went to the literature student rooms. 'Loving philosophy with a passion, but not seeking in it a rule of life because the practical problem was not interesting' (Chevalley and Lautman 1931), Herbrand was the individual and daily instructor for Lautman in mathematics. Together, they became friends and intellectual companions with Claude Chevalley, then with Charles Ehresmann, two of the five future founders of the Bourbaki group.[6] In February 1928, Lautman was taken by Celestin Bougie, deputy director of the school, to Franco–German meetings in Davos, where he met his future wife, then a student of George Davy in Dijon. He returned convinced that mathematical creation is done in Germany and made sure to spend a semester in Berlin in 1929. In 1929, an event of micro-history is indicative of his politics and morality. A majority of student primary school teachers from Quimper had decided to refuse to take military service for reserve officers and were threatened with exclusion from their École Normale Primaire by the Rector of Rennes. In Rue d'Ulm,[7] a petition of support circulated, borne mainly by former *khagneux*[8] at the Lycée Henri IV converted to pacifism by Alain,[9] with Simone Weil at the head.[10] Albert Lautman signed, but, unlike several other signatories, he did not refuse military service, not wanting the signatories to be excluded or to make them fail. He promoted freedom of choice, but was elitist and considered that Noblesse oblige. Qualified in philosophy in 1930, he completed his military service with fellow mathematicians in the 401st Field Artillery at Metz.

In spring 1931, the Institute of Western Languages in Osaka wrote to the Director of the École Normale Supérieure, seeking to recruit a young teacher of that school for two years to teach literature and French philosophy. The offer tempted Jean-Paul Sartre and Lautman, who both decided to apply. Lautman sent his letter via the USSR and the Trans-Siberian railway, and it arrived before that of Sartre, which was delivered by ship from Marseilles. Missing, without doubt, other selection criteria, the Japanese responded positively to the application that arrived first. On his return from Osaka two years later, welcomed without excessive warmth by the Inspector General of Philosophy (unique at the time) Dominique Parodi, who, three years later would open the columns of the *Revue de Metaphysique et Morale*, he was assigned to the Lycee de Vesoul. He was there only one year and earned a one-year scholarship to the *Caisse Nationale des Sciences*,[11] the forerunner of the CNRS.[12] In October 1935, he was allocated to the boys school at Chartres, which allowed him to attend seminars at the *Institut Henri Poincaré*, notably that of Gaston Julia, as evidenced by many references in his theses. At the invitation of Célestin Bouglé, each year he gave a small series of lessons to the *aggregatifs*[13] of the Rue d'Ulm. He frequented the Sunday mornings of Leon Brunschvicg, who received students, colleagues and also members of important radical parties, being the Deputy Secretary of State in the government of Léon Blum. At the end of 1937, he defended his theses.[14]

Politically, Lautman was on the left, but he kept his distance from the Communists because he did not admit that the end justifies the means. He had contacts with German colleagues, a number of whom he saw pass through France on their way to the United States, and not all of them were Jews. He was convinced, from the beginning of 1938, that war was inevitable, and that the sooner the better. This is why he signed up voluntarily for fairly heavy training for reserve officers, with training periods in Suippes and Mourmelon. This earned him, at mobilization, the assignment of commander of an anti-aircraft artillery battery, and he was quickly promoted to 'temporary' captain. His battery had some success: four planes of the Luftwaffe shot down, three hit. As captain he received the Military Cross. Posted in mid-June 1940 to cover the re-embarkation of British troops, he was taken prisoner and sent to Oflag IV D at Hoyerswerda, Silesia. A first hazardous attempt to escape failed before he had even crossed the last barbed wire fence. Hence, he was sent to the lockup, where attempted escapees were reunited. He took an active part in the university of the camp, which had notable lectures on major topics.[15]

While incarcerated, he made the acquaintance of Jacques Louis, a Saint-Cyrian[16] and captain in a regiment of Tunisian *goumiers*,[17] also committed to escape for good. The child of a northern city, Louis found himself separated from his parents in 1914, during the first German offensive. Taken in by British soldiers, he spent the war with an English family and was miraculously found by his parents in 1919. An adventurer but also an organizer, he soon realized that Lautman the intellectual, who still didn't know how to turn an empty tin can into a pickaxe, a shovel or a candle-holder – he'd eventually learn – presented as a critical asset: he spoke fluent German and could buy tickets for rail and bread without being noticed. Louis assembled a group of 28, who spent nine months digging an 80-meter tunnel. On 18 October 1941, they set off. Sixteen arrived safely, after about ten days; the most delicate passage was that between Alsace–Lorraine, already regarded as under the control of the Reich, and then France. The railway workers of the SNCF[18] were known intermediaries and the handcar workers at Forbach did a great service, helping the escaped soldiers to freedom.

Demobilized, Lautman was immediately dismissed as a Jew. After a stay in Aix for a few months, he moved to Toulouse, where he found Louis, who devoted himself to recruiting for the Secret Army. Lautman quickly became one of those responsible for the Haute-Garonne staff headquarters (see Latapie 1984); however, he also accepted the responsibility for organising passages to Spain for the O'Leary network[19] (see Belot 1998, 106–10), the smugglers were Spanish Republicans in exile. For two years, at the rate of about 12 to 20 per month, Anglo-Saxon airmen, Resistance fighters on the run, hounded Jews and young people anxious to join the Free French forces had crossed the Pyrenees, at the risk then of a few months incarceration in Miranda prison, before being able to make it to Algiers. In April 1944, there was a setback: a summer route over the mountains had been chosen, yet it snowed late. Tracks were found and the group was stopped. However, within the scope of his activities in the Secret Army, when, in the spring of 1943, the message was received announcing that the Normandy landing would not take place soon, Lautman understood, like many others, that he must organize the material life and also the activity of these young people who had become illegal by refusing the STO (*Service du Travail Obligatoire*, or forced labour, in Germany) and were in the process of swelling the ranks of the Maquis.[20] He was particularly interested in the Corps-Franc near Toulouse[21] when, sensing that the landing was imminent, he decided to join them from 17 May 1944, to ensure their training. But on Monday, 15 May 1944, when he arrived at the destination of the rendezvous with one of his smugglers, he and the smuggler were arrested by the German police of the Occupation. Although never proven, the hypothesis of treason, or at least that there was a weak link in the network, remains the most likely explanation of his arrest. Lautman was part of the convoy for deportation, which left Toulouse on 9 July but was turned back and ended up at Bordeaux (see Nitti 1944). Fifty detainees whose cases were the most serious were condemned to death. They were taken on 29 July to the execution posts of Camp de Souges. The required squad of French mobile police did not appear; it was the same the next day. On 1 August, a squad of German non-commissioned officers completed the task.

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Albert Lautman and the Creative Dialectic of Modern Mathematics

by Fernando Zalamea*

It is possible today to observe in hindsight the epistemological landscape of the twentieth century, and the work of Albert Lautman in mathematical philosophy appears as a profound turning point, opening to a true understanding of creativity in mathematics and its relation with the real. Little understood in its time or even today, Lautman's work explores the difficult but exciting intersection where modern mathematics, advanced mathematical invention, the structural or unitary relations of mathematical knowledge and, finally, the metaphysical and dialectical tensions underlying mathematical activity converge. Well beyond other better-known names in philosophy of mathematics – who are focused above all on questions concerning the logical problem of foundations, important but fragmentary studies in the vast panorama of modern mathematics – Lautman broaches the emergence of inventiveness in the very broad spectrum of the development of the mathematical *real*. Group theory, differential geometry, algebraic topology, differential equations, functional analysis, functions of complex variables and number fields are some of the domains of his

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preferred examples. He detects in them methods of construction, structuration and unification of modern mathematics that he connects to a precise Platonic interpretation in which powerful pairs of ideas serve to organize the edifice of effective mathematics.

In what follows, an interpretation of the work of Lautman will be offered that freely uses some mathematical advances of the second half of the twentieth century, since, in our view, interest in Lautman consists primarily in what it has to say to us *today*. His ideas, his method, his wagers are now more striking than ever. After the particular limits encountered by analytic philosophy, after the reductionist linguistic dissections of knowledge, after the zigzags of postmodernism, a critical return to the big questions of the history of philosophy and toward a complex appreciation of reality brings the work of Lautman into close proximity with contemporary inquiry.[1] We will proceed in three stages, according to a triple *back-and-forth* between various levels of the concrete and the abstract: plurality of effective mathematics and unity of structural methods (Sections 1 and 2); 'mixed' mathematics, Lautmanian 'notions' and 'ideas' (Sections 3, 4: the adjective *Lautmanian* deserves to enter into usage); Platonic dialectic and the dialectic of the mathematical theory of categories (sections 5, 6). The aim is to pay homage in this way to Lautman's very method, which permits rising to the most pure and universal through a dialogical back-and-forth between complementary notions, following a scale in which the complex and the concrete are gradually liberated to attain the simple and the abstract.

1. EFFECTIVE MATHEMATICS

With the term 'effective mathematics',[2] Lautman tackles the theories, structures and constructions conceived in the very activity of the mathematician. The term refers to the structure of mathematical knowledge, and what is effective refers to the concrete *action* of the mathematician to gradually build the mathematical edifice, that such action is constructivist or existential.[3] The mathematical – beyond its ideal set theoretical reconstruction – develops along a hierarchy of real configurations of rather diverse complexity, in which the concepts and examples are connected through structural processes of liberation and saturation, resulting in mathematical creations like mixes between opposite polarities. Lautman detects some *specific* features of advanced mathematics that are not given in elementary mathematics:[4] a) the *complex hierarchisation* of various theories, irreducible to systems of *intermediate* deduction; b) the *richness* of the models, irreducible to linguistic manipulation; c) the *unity* of structural methods and of conceptual polarites, beyond the effective multiplicity of models; d) the *dynamics* of the creative activity, in a permanent back-and-forth between freedom and saturation, open to the Platonic division and the Platonic dialectic; e) the *mathematically demonstrable relation* between what is multiple on a given level and what is singular on another, through a sophisticated lattice of mixed ascents and descents.

The reduction of all mathematical theorems to 'tautologies', equivalent to each other from the point of view of the base Zermelo-Fraenkel axioms, has *levelled* (conveniently for many) the broken terrain of mathematics. Lautman protested against this 'resignation' and his work deserves to be understood as an effort to describe a mathematical topography that is much more complex and eventful. Attentive to the ascents and descents in Galois theory, in class field theory, or in the construction of universal covering surfaces, Lautman shows not only how the process of saturation of an imperfect structure gives rise to mathematical creativity, but also how these processes combine gradually through scales of construction and very precise inverse correspondences: between intermediate fields and subgroups of the Galois group, between class fields and groups of ideals, between covering surfaces and subgroups of Poincaré's fundamental group. In these cases, mathematical creativity is responding to a hierarchization of knowledge, in which the multiple *intermediate levels* of correspondence between structures are much more important than a purely logical alternation between particularization and generalization. According to Lautman, mathematical creation emerges with the division and definition of differences (hence the influence of Lautman on Deleuze)[5], coupled with the utilization of mixes which permit the liberation of simple notions. It happens to be a process that is opposed to the arbitrary search for generalizations, in which 'a whole conception of mathematical intelligence' is put in play:

> It is, in effect, extremely important for the philosopher to prevent the analysis of ideas and the research of notions that are the most simple and separable from each other, from appearing as research of the broadest kind.

In fact, a closer look at the gradations and the back-and-forth of concepts allows the *emergence* of mathematical thought to be observed. The hatching and the genesis of mathematical structures, that the analytic set-theoretical approach hides, can be studied with greater insight by the Lautmanian approach. When he detects that the metrization of topologies is not always possible, or that the Hilbert space serves as a 'remarkable synthesis of continuous and discontinuous notions', when he discovers the precise moment when algebra and analysis are related in the work of Hilbert on integral equations, when he recalls the 'affinity of the underlying dialectical structure' to the Weil conjectures, when he shows the emergence of Herbrand's domains as gradual mixes which allow the construction of proofs of consistency, when he marvels at lattices that appear throughout mathematics, Lautman is always on the lookout for creative movements in mathematics: movements in which a problem, a concept or a construction is transformed through the sheaf of partial solutions to the problem, the definitional delimitations of the concept, and the saturations and divisions of the construction. Mathematics, a space of thought that is always alive and in constant evolution, can in this way appear as truly *iconic* in the work of Lautman, one of whose principal merits (as Bernays remarks) consists precisely in the capacity to think the mathematical world *faithfully*.

At the base of his work, when he contemplates the dynamic spectrum of the technique of mathematics, Lautman discovers a number of tectonic shifts in the new mathematics that will shake it *right up to the late twentieth century*: 'the primacy of geometric synthesis over that of the "numerical" analysis' (see the combinatorial geometries of Boris Zilber, Poizat 2000), a full perception of the intrinsic richness of arithmetic systems: 'we may be led to believe that it is wrong to claim to consider arithmetic as fundamentally more simple than analysis' (see the *reverse mathematics* of Friedman and Simpson, Simpson 1999), a linkage of properties of reflection and properties of closure in the back-and-forth between the local and the global (see the theorems of representation in Peter Freyd's theory of allegories, Freyd et al. 1990). The agile receptivity of Lautman allows the opening up of a number of major currents in the mathematics of his time, that still have much to teach us.

2. STRUCTURE AND UNITY

According Lautman, mathematical theories are constructed following 'a whole series of precisions, limitations, exceptions', through which the Ideas and concepts acquire their effective life. In fact, 'restrictions and delimitations [. . .] should not be conceived as an impoverishment, but rather as an enrichment of knowledge, due to the increased precision and certainty provided.' Nevertheless, the *exact*, concrete and differentiated impetus consigned in this way has as a constant complement an inverse alternating impetus, oriented towards integrality and unity. In the science of the 1930s, the ineluctable emergence of the notion of mathematical structure gave Lautman a supple instrument to order the diversity. Still nascent when he was writing, and not yet defined in all their generality (the labour of Bourbaki), mathematical structures included a wide network that underlies the work of the German school (Hilbert, Emmy Noether, Emil Artin), studied closely by Lautman during his final years at the École Normale, with Herbrand, Chevalley and Ehresmann. He was capable of grasping the emergence of various levels of mathematical structure: structured *concrete* objects (groups, algebras, Hilbert spaces, etc.), *intermediate* structural correspondences (duality theorems, Galois correspondences, conformal representations, etc.), *generic* structures (lattices, etc.). Going even further, he began to see 'structural schemas' in which – at an 'upper' logical level: more simple and universal – a number of *free* oppositions are assembled (local/global, intrinsic/extrinsic, discrete/continuous, etc.) throughout collections of possible structures.[6]

Lautman emphasises a synthetic perception, ready to determine the value of complex networks of mathematical interaction, beyond a stifling search for 'primary' notions. The unity of mathematics is expressed, not in a common base to rebuild the whole, but in the convergence of its methods and in the passage of ideas between its various networks: logical, arithmetic, algebraic, analytic, topological, geometric, etc. The penetration of the methods of algebra in analysis, analysis subordinated to topology, the ubiquitous appearance of the geometric idea of domain, the agreement of the complex variable in arithmetic – these are all examples studied in detail by Lautman, in which, in the local fragment, the global unity of mathematics is reflected. It happens to be a *real* unity, at the interior of the synthetic universe of effective mathematics, that *disappears* when the plurality of mathematical knowledge is reduced to its fictional analytic reconstruction. After all, the set theoretical presentation provides convenient layers of relative consistency, but, in practice, it is increasingly evident that mathematics develops far from its so-called fundamentals.[7] An epistemological inversion shows how, contrary to what one might think in the first instance, a practical observation of diversity can then reinstate the multiplicity in the unity. In fact, a full awareness of diversity does not lead to a lack of connectivity, but returns to the unity, whether in Charles Peirce's pragmatism, in Walter Benjamin's montage, in the figurative relay of Pierre Francastel, or in the difference of Deleuze. The metaphor of 'rising again' (*remonter*) – present in several of Lautman's references: return, ascend, sources, etc. – exposes once again the integral behind the differential. Through logical gradations in which the structure and dynamics are in dialogue, the ascent towards Ideas is one of the preferred Lautmanian displacements. There is perhaps no more audacious and magnificent an ascension than his penultimate text, when he assures us – and then convinces us in the demonstrative context of statistical mechanics – that 'the materials of which the universe is formed are not so much the atoms and molecules of the physical theory as these great pairs of ideal opposites such as the Same and the Other, the Symmetrical and Dissymmetrical, related to one another according to the laws of a harmonious mixture.' Similarly, in a mathematical sheaf, the sections are complemented with projections, in a back-and-forth that unites reality and knowledge, 'the conceptual analysis necessarily succeeds in projecting, as an anticipation of the concept, the concrete notions in which it is realized or historicized'.

3. MIXES

One of the characteristic traits of modern mathematics is its ongoing effort to put an end to impermeable enclosures, its constant *transport* of examples, theories and ideas between different subdomains of mathematics, with the consequent transformation of objects when they are read from multiple variable contexts. If one of the strengths of mathematics has always been its ability to realize the passages between the possible (models), the necessary (theorems) and the current (applications), this facility to control and mould the mediations becomes a veritable plastic arsenal throughout the twentieth century. Mixed mathematics are legion. Among the most studied by Lautman, one finds algebraic topology, differential geometry, algebraic geometry and analytic number theory. The connections of the noun (central subdiscipline) and the adjective (infiltrating subdiscipline) only signal very weakly the real osmosis of modem mathematical thinking. The rigid and outdated delimitations disappear, and flexible frontiers arise in a new classification (MSC 2000), which, rather than a Porphyrian tree, seems more like a liquid surface in which the information flows between mobile nuclei of knowledge.[8]

Albert Lautman is the *only* modern philosopher of mathematics to have remarked upon and studied, on the one hand, the emergence of mixed mathematics in action, and on the other, the 'ideas' and 'notions' that allow an understanding of the increasing potential of the mixes. Given the central importance of mixed structures in contemporary mathematics, it is natural to emphasize the immense value that Lautmanian philosophy of mathematics can (and should) acquire today. The failure of a supposed mathematical tautology (purist invention of analytic philosophy) and the opening towards a *contaminated* mathematics, much closer to reality, are the order of the day. Contamination by physics and metaphysics is indispensable for mathematical creativity.

Clearly opposed to the efforts of purification advanced in the foundations of arithmetic, Lautman exalts the richness of 'transcendent' analytical methods in number theory, and explains *why* mediations and mixes tend to be required in the creative act in mathematics. In fact, the primary role of mixes consists in their joint *reflective action*: a reflective back-and-forth between partial properties, which are located in a neighbourhood mediate between extrema and which act as a precise *relay*[9] in the transmission of information. Whether in a given structure (Hilbert space), a collection of structures (Herbrand's ascending domains) or a family of functions (Montel's normal families), mixes, on the one hand, imitate the structure of underlying domains, and on the other hand, act as partial blocks to construct the higher domains. Without the desired form of contamination, without these premeditated alloys, contemporary mathematics would simply be incomprehensible. A technical success like the proof of Fermat's theorem (1994),[10] the symbol of the full deployment of twentieth-century mathematical invention, is only possible as a final effort *after* a very complex back-and-forth of mixes: a problem on elliptic curves and modular forms resolved through deep connections between algebraic geometry and complex variables, developed around ζ-functions, their Galois representations and their deformations and associated rings. The implicit scope of Lautman's work, registered in such a deployment of mixes, opens on surprising perspectives that appear to exceed the present philosophy of mathematics. The 'mixes' appear in Lautman's first known writing, when he brilliantly describes the construction of Herbrand's 'domains' and shows how 'the Hilbertians knew to interpose an intermediate schematic, that of individuals and domains considered not so much for themselves as for the infinite consequences that allow finite calculations to operate through them'. Comparing this intermediate schematisation with Russell's hierarchy of types, Lautman indicates that 'we are in the presence, in each case, of a structure whose elements are neither entirely arbitrary, nor really constructed, but composed as a *mixed form* that derives its fertility from its dual nature' (emphasis added). Understanding at the outset the significance of mixed forms of logic in the 1930s, when logic was viewed on the contrary as a pure form, demonstrates well the independence and the acuity of the young philosopher. When it has today become evident that it is precisely these mixed forms of logic that led to the real explosion in mathematical logic (happily infiltrated by algebraic, topological and geometric methods), and when it has became clear that a logical system should not be understood *a priori*, but rather as closely related to a collection of mathematical structures *a posteriori*, it is all the more remarkable that these contemporary *mathematical facts* are able to be found with such fidelity in the profound philosophical insights of Lautman.

In his article on the 'method of division', Lautman connects the reference to 'mixes' to the Platonic tradition (*Sophist*, *Philebus*, Plato 1997), emphasises their dynamic interest, and situates mathematical creativity in a dialectic of liberation and composition. With terms that are not found in his work, but that summarize his position on better known terrain, his work shows how mathematics, on the one hand, divides the contents of a concept through definitions (syntax) and derivations (grammar), thereby releasing simple components, and how, on the other hand, it constructs intermediate relays through models (semantic) and transport theory (pragmatics), thus rekindling the existence of simple filaments and allowing for their reorganization into new concepts. When one of these mixes succeeded in uniting a clear simplicity and a high powered reflector – as in Riemann surfaces or in Hilbert spaces, admired and exemplarily studied by Lautman in his two theses – the mathematical creativity reaches perhaps its greatest heights.

4. NOTIONS AND IDEAS

In Lautman's theses, the whole dynamics of his philosophy is driven by an alternating contrast between complementary concepts – global/local, whole/part, extrinsic/intrinsic, continuous/discrete, etc. – but it is in his *New Research on the Dialectical Structure of Mathematics* that Lautman introduced the terms that govern these dialectical connections. He defines a *notion* as one of the poles of a conceptual tension, and an *idea* as a partial resolution of this polarity. Thus, the concepts of finitude, infinitude, localization, globalization, computation, modeling, continuity or discontinuity are 'notions' (examples given by Lautman), and the propositions that express state that infinity is obtained as the non-finite (cardinal skeleton), the global as the patching together of the local (compactness), the modelizable as the realisation of the deductively calculable (set-theoretical semantics), or the continuous as the completion of the discrete (Cantorian right) are some 'ideas' (examples given by us).

The interest in notions and ideas is threefold: they allow the *filtering* (liberation) of some unnecessary ornaments and in this way allow the mathematical framework to become clear; the *unification* of various constructions through a 'higher' dialectic level; and the *opening* of the spectrum of mathematics to alternatives. Whether by filtering or unifying the mathematical landscape – duality theorems in algebraic topology and in the 'general theory of structures' (i.e. lattices) (Lautman's examples) – whether by opening it to a new framework of possibilities – non-standard ideas that resolve in *another way* the oppositions between notions: the infinite as unbounded (Robinson), the discrete as a sequence of demarcations on a primitive continuous (L. E. J. Brouwer), the deductively calculable as a system of coordinates for the modelizable (Per Lindström) (examples proposed by us) – in all cases, the Lautmanian notions and ideas cover the universe of mathematics *transversely*, and explain the amplitude of this universe as much as its surprising harmonic agreement between the one and the many.

For Lautman, notions and ideas are situated at a 'higher' level, in which the intellect can imagine the *possibility of a problematic* independently of mathematics, but whose real meaning can only be obtained when it is embodied in effective mathematics. In the tension between a universal (or generic) problematic and its partial concrete (or effective) resolutions, according Lautman, much of the structural and unitary back-and-forth of mathematics is to be found. As we will see later, it is a very precise question of the paradigm promoted by the mathematical theory of categories.

Lautman is aware that he seems to introduce a delicate *a priori* in the philosophy of mathematics, but he explains it as a simple 'exigency of problems, anterior to the discovery of their solutions': 'in a purely relative sense, and in relation to mathematics, [this *a priori*] is exclusively the possibility of experiencing the concern of a mode of connection between two ideas and to describe this concern phenomenologically'. In fact, the anteriority of the problem should be understood solely at a purely conceptual level, since the elements of a solution are often given first in practice and incorporated only *after* in a problem (which does not prevent, in a conceptual rearrangement, the problem in the end from preceding the solution). So:

> Mathematical philosophy, as we conceive it, therefore consists not so much in retrieving a logical problem of classical metaphysics within a mathematical theory, than in grasping the structure of this theory globally in order to identify the logical problem that happens to be both defined and resolved by the very existence of this theory. (Lautman 1938b)

It is distressing to observe that the huge effort made by Lautman to grasp globally a few *mathematical* theories of his time, and extract valuable *ideas* from them, is no longer repeated in philosophy.[11]

In parallel to his strategy to apprehend the global structure of a theory before defining its logical status, Lautman situates the place of logic at the very interior of mathematics, as a discipline that does not precede mathematics, but should instead be situated at the *same level* as the other mathematical theories. He prefigures – as the later Peirce – our conception of logic as it arises from model theory, in which a 'logic' is not only determined, but even *defined* (à la Lindström) by an adequate collection of structures. According to Lautman, 'for logic to exist, a mathematics is necessary', and it is in the back-and-forth of *mixed* logical schemas with their effective realizations that the strength of mathematical thinking lies.

The reciprocal enrichment between effective Mathematics and the Dialectic (Lautman's capitals) is reflected in an ascent and descent between Lautmanian *notions* and *ideas*, on the one hand, and *mixes*, on the other. In fact, if from the mixes one ascends, notions and ideas are 'liberated' which allows situating these mixes in a more ample dialectic; and if, conversely, from the notions one descends, new mixes are constructed to clarify and incarnate the content of the ideas in play. One of the most salient features of Lautman's work is to have shown in detail how these processes of ascent and descent must be inextricably linked in the philosophy of Mathematics *in extenso*, in the same way that they are in a Galois correspondence *in nuce*.

5. PLATONISM

As Jean Cavaillès remarks in the first sentence of his report on Lautman's theses, his work constitutes 'a new attempt to define the inherent reality of mathematical theories: the most recent works are used and the result invokes Plato' (1938a, 9). In effect, according Lautman, mathematical reality – far from being reduced to the observation of natural objects, independent of the human eye – extends to encompass an entire hierarchical lattice of *ideal* constructions, which cease to be arbitrary when they are integrated generically in stable scientific networks. Mathematical constructions then become *general reals*, and mathematical reality is organised around a complex imbrication of levels, situated below a schematic echelon formed by ideas:

> We do not understand by Ideas the models whose mathematical entities would merely be copies, but in the true Platonic sense of the term, the structural schemas according to which the effective theories are organized. (p. 199)

The invoked Platonism is, therefore, far from a simplistic idealism that he denounces. Following Léon Robin, Julius Stenzel, and Oscar Becker,[12] Lautman reads a dynamic Plato, far from pure contemplation, and much closer to a full becoming of qualitative distinctions, which allows the incarnation of the schematic tensions of ideas in effective mathematical theories.

Even if Lautman's first references to a rereading of Platonism can be found in the conclusion of his main thesis (1938b), from his first articles he establishes a dialogue with Plato, to underline a 'participation of the sensible in the intelligible' – which serves to support understanding the osmosis of mathematical activity, its processes of transport and its creative mixes – and to begin to explore, with striking examples in modern mathematics, 'the Platonic distinction between the Same and the Other that is found in the unity of Being'. This dialogue with Plato takes shape in *New research on the dialectical structure of mathematics*, when he indicates that 'the Ideas of this Dialectic are certainly transcendent (in the usual sense) with respect to mathematics', Lautman is not thinking of any temporal, methodological or even logical priority, he assumes no *a priori*, but he suggests simply the possibility of the existence of a connection of the 'why' with the 'how', where the question can come to be situated before the answer. What is proper and specific to mathematical activity is then to *transit* between a world of ideas of possible, free relations, and a world much richer in determinations, full of precisions and delimitations. The back-and-forth is inevitable, and it opens up new possibilities for a Platonism better adapted to the complexities of mathematical reality: a structural and dynamic Platonism, not reified, immobile or eternal.

In this Platonic *manifold* – according to Lautman, closer to the true Plato – mathematics and physics converge. Whether in the forms of mathematical discovery that neighbor physical dissociations, or, in a conception of mathematical theories as middle terms between ideas and experience:

$$\frac{\text{Ideas}}{\text{Theories}} \equiv \frac{\text{Theories}}{\text{Experience}}$$

or, in 'the hypothesis of a similar importance of the dissymmetrical symmetry in the sensible universe and the anti-symmetric duality in the mathematical world', the structural base that, with powerful arguments, renews support for the classical correlation:

$$\frac{\text{Physics}}{\text{Sensible world}} \equiv \frac{\text{Mathematics}}{\text{Ideal world}}$$

Lautman always manages to compare forms of physical knowledge and those of mathematical knowledge *by projecting them onto situational structural networks* whose relative coordinates can be put in perfect correspondence, even if the objects studied in each network are quite different. Thus, a *reticular commensurability* of physics and mathematics acquires the right of existence, and is clearly situated in opposition to its reduced (and too proclaimed) grammatical incommensurability.[13]

6. CATEGORY THEORY

A first idea of the objectives and methods of the mathematical theory of categories can be obtained when the ontological content of an already quoted sentence of Heidegger's is eliminated: 'renewing the relation between structures each time to a unity, so that from this unity the whole history of an entity can be followed up to its concreteness'. In fact, the dialectics of the one and the multiple, of the structural and the concrete, reached one of its most fortunate expressions in category theory, because an object defined through the universal properties in abstract categories – *one* – becomes in turn a *multiple* throughout the plurality of concrete categories in which it is 'incarnated'. The technical power of the theory resides in the permanent back-and-forth of functors, natural transformations and adjunctions, which serve to constitute a very ductile network of exchanges and blockages in mathematics. It is remarkable that Lautman's conceptions are able to take shape fully (that is, theorematically, with their corresponding 'procession of precisions') through the fundamental concepts of category theory.[14]

Lautman often describes the conceptual resources implied in certain techniques of category theory: functors in algebraic topology (the description of Poincaré's and James Alexander's duality theorems), functors representable in manifolds (description of the ascent to a universal covering surface and the hierarchy of intermediate isomorphisms connected to subgroups of the fundamental group), logical adjunctions (the description of an *inversion* between Kurt Gödel's completeness theorem and Herbrand's theorem), free allegories (the description of a 'structure is like a first drawing of the temporal form of sensible phenomena'). Basically, when he maintains that it is necessary to accept 'the legitimacy of a theory of abstract structures, independent of the objects connected together by these structures', Lautman is very close to a mathematical theory oriented towards the structural relations *beyond objects*: the mathematical theory of categories.

The Lautmanian language, with its idiosyncratic 'notions', 'ideas' and dialectical hierarchies, acquires much delimited technical support in category theory. The 'notions' may be defined through universal categorical constructions (diagrams, limits, free objects), the 'ideas' through the elevation of classes of free objects to pairs of adjoint functors, the dialectical hierarchies through scales of levels in the natural transformations. Thus, for example, the Yoneda lemma[15] shows technically the *inevitable* presence of ideal constructions in any full consideration of mathematical reality – one of Lautman's strong positions – when the lemma proves that *any* small category can be submerged in a category of functors, in which, in addition to the representable functors that form a copy of the initial category, other ideal functors (presheafs) also appear that complement the universe. It is a matter of a *forced* apparition of the ideal at same moment as grasping the real, a permanent and penetrating osmosis in all forms of mathematical creativity. The *irreducible* back-and-forth between ideality and reality (very much present elsewhere in Hilbert's article 'On the Infinite', 1926, known to Lautman) is one of the major strengths of modern mathematics.

Most of the structural schemas and schemas of genesis studied by Lautman in his main thesis (1938b) can be explained and, above all, extended, through the aide of category theory. For example, the study of the duality of the local and the global is extended to a complex duality of functorial localizations and global reintegrations (Freyd's representation theorem); the duality of the extrinsic point of view and of the intrinsic point of view is extended to the internal logic of an elementary topos (F. William Lawvere's geometric logic); the logical schemas of Galois's theory are extended to a general theory of residuality (Galois's abstract connections, à la George Janelidze). When we see how, according to Lautman, the affinities of logical structure allow different mathematical theories to be brought together, because of the fact that they each provide an outline of different solutions to one and the same dialectical problem; how one can talk about the participation of mathematical theories distinct from a common dialectic that governs them; how the dialectic is purely problematic, outline of the schemas; or how the indetermination of the dialectic at the same time ensures its exteriority (See p. 126–130), it is impossible not to situate these ideas in the environment of category theory. Whether in the back-and-forth between abstract categories ('common dialectic') and concrete categories ('distinct mathematical theories'), in free objects ('indetermination of the dialectic') whose external applicability on the whole mathematical spectrum is precisely the consequence of their indetermination, or in diagrams, sketches and limits that allow the grand schemas of mathematical exchanges to be to outlined.

A lot of the apparent distance between modern mathematics, classical philosophy and contemporary thought can be reduced. In this sense – one of the senses of Lautman's militant work, open at once to Plato and Hilbert, to the ancient dialectic and to the advanced mathematics of his time – we suggest in the following table some translations between Lautman's thought, category theory and the *trimodal* thought of universals according to Avicenna:

Lautman	*Categories*	*Avicenna*
notions and ideas	universal constructions (abstract categories)	*universals ante multitudinem*
effective mathematics	given structures	universals *in multiplicitate*
mixes	kinds of structures (concrete categories)	universals *post multiplicitatem*

The universals *ante multitudinem* of Avicenna (mixed products in the Neoplatonic understanding of Aristotle's categories) are intrinsic concepts, without referentials; the universals *in multiplicitate* are collections of individuals, or are referentials; the universals *post multiplicitatem* are forms conceived by the understanding beyond collections.[16] Between Avicenna and the mathematical theory of categories – simultaneously near and yet so far – the work of Lautman is situated in a fascinating mixed terrain, which serves as a bridge between some of the major ideas of philosophy and some of the great achievements of contemporary mathematics. Full of ardor and exigency,[17] Lautman's work opens up original perspectives for a vigilant, engaged mathematical philosophy, by taking hold of both the real activity of the discipline and the major problems – still alive despite their alleged death – in the history of philosophy.

Preface to the 1977 Edition

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Contemporary philosophers who are interested in mathematics are in most cases occupied with its origins, its relations with logic, or its problems with foundations; quite a natural attitude for that matter, since these are the questions that themselves call for philosophical reflection. Few are those who seek to get an idea of the broad tendencies of the mathematics of their time, and of what guides more or less consciously present mathematicians in their work.

Albert Lautman, on the contrary, seems to have always been fascinated by these questions. Like Jean Cavaillès, he made the effort to become initiated in the basic mathematical techniques, which enabled him to find out about the latest research without risking being drowned in a flood of abstract notions that are difficult to assimilate for a layman. Also, in contact with his comrades and friends Jacques Herbrand and Claude Chevalley (two of the most original minds of the century), he had acquired of mathematics in the years 1920–1930 views far more extensive and precise than had most mathematicians of his generation, often narrowly focused. I can vouch for what concerns me personally.

It is these views that he presents in his two theses, and with 40 years of distance, one can't but be struck by their prophetic bearing. Because from the title of these works, one finds, as if highlighted, the two key ideas–forces that have dominated all subsequent developments: the concept of mathematical structure, and the profound sense of the essential unity underlying the apparent multiplicity of the diverse mathematical disciplines.

The word 'structure' is one of those that has been the most overused during the last decades, but for mathematicians, it has acquired a very precise meaning. In 1935 this meaning had not yet been completely explicit, but many mathematicians were quite aware of the reality that he recovers, notably among those who were inspired by Hilbert's ideas on the axiomatic conception of mathematics. The essential point in this conception is that a mathematical theory is concerned primarily with the *relations* between the objects that it considers, rather than the nature of these objects. For example, in group theory, it is most often secondary to know that the elements of the group are numbers, functions or points of a space. What is important is to know whether the group is commutative, or finite, or simple, etc. This view has so permeated the development of mathematics since 1940 that it has become quite banal, but this was not yet the case at the time when Lautman was writing, and he repeatedly insisted on it, as for example when he stressed the fundamental identity of structure between the Hilbert space, composed of functions, and the usual Euclidean space. Even more remarkable is the long passage devoted to what is now called the notion of universal covering (*revêtement*) of a manifold (at the time it was referred to as a 'universal manifold of covering (*recouvrement*)'). The 'ascent towards the absolute' that he discerned, and in which he saw a general tendency, took in effect, through the language of categories, a form applicable to all parts of mathematics: it is the notion of 'representable functor' that today plays a significant role both in the discovery and in the structuration of a theory.

The theme of the unity of mathematics seems more central again in the thought of Lautman. We know that, since antiquity, philosophers have been pleased to highlight the opposition of points of view and of methods in all domains of intellectual activity. Many mathematicians have long been impressed by these antagonistic pairs that philosophers had taught them to discern in their own science:

Finite versus Infinite
Discrete versus Continuous
Local versus Global
Algebra versus Analysis
Commutative versus Non commutative, etc.

A significant portion of Lautman's theses are devoted to examination of these oppositions, on which his position is similar to that of Hilbert, in whom one finds the strongly expressed conviction that they only happen to be superficial appearances masking much more profound relationships. The entire development of mathematics since 1940 has only confirmed the soundness of this position. It has thus been well recognized that these supposed oppositions are actually poles *of tension* within a same structure, and that it is from these tensions that the most remarkable progress follows.

As regards the 'local–global' antagonism, Lautman had known to appeal to the work of E. Cartan, the value of which very few appreciated before 1935, and whose central place in all mathematics is now universally recognized. But this fertile polarity now extends far beyond its initial geometrico–topological framework. One of the grand axes of the work of Algebra and Number Theory, since Hensel, Krull, Hasse and Chevalley, has been to analyze the problems by 'localizing' them to begin with, the notion of prime ideal (or of 'valuation') replacing the 'points' of geometry; while in the opposite sense, the technique of sheafs and their cohomology, created by J. Leray, provided adequate instruments in the most diverse theories to 'globalize' the local results and measure, in some way, the obstructions to this globalization.

If this evolution was possible, it is because it overcame at the same time an opposition that seemed much more radical (since it goes back to Greek mathematics, and has caused much ink to flow over the centuries): that of the 'discrete' and the 'continuous'. The time has passed when the ideas of approximation and completion seemed foreign to Algebra for which they have become particularly valuable tools. And the extraordinary intuition of A. Weil, foreseeing the possibility of structures recovered from Topology (later discovered by A. Grothendieck) at the very heart of Number theory, the science of the 'discreet' par excellence, has opened vast horizons. A whole new science was created, Homological Algebra, which borrows its methods from algebraic Topology to enrich itself, for example, group theory or 'abstract' rings, and inversely allows Topology to be considered as a peculiar application of an essentially 'combinatorial' theory: 'simplicial' Algebra.

The no less venerable opposition of the finite and the infinite has suffered the same fate. To give one striking example, one of the most important advances in finite group theory has been the discovery, by Chevalley, of a method allowing a whole new family of finite simple groups to be deduced from simple complex Lie groups (formerly called 'continuous groups'). To illustrate the interpenetration of the 'commutative' and the 'noncommutative', Lautman again had recourse to the work of E. Cartan, making Grassmann's exterior algebra (noncommutative) the essential tool in Differential Geometry and Pfaffian systems theory. He could also have mentioned, from that era, Brauer's group theory, which plays a central role in the theory of commutative fields, despite having the noncommutative algebras constructed on these fields as its object. But today there are many other examples of this type: in the theory of spherical functions (a generalization of Laplace's 'spherical harmonics', which essentially relates to the theory of certain noncommutative Lie groups), the key result is the fact (discovered by Gelfand) that a certain algebra of functions is commutative; while in the inverse sense, it was found that the theory of commutative algebraic groups is closely related to the structure of a certain noncommutative ring.

As for the old classification of mathematics into Arithmetic, Algebra, Geometry and Analysis, it has become as out-of-date as the divisions of the 'animal kingdom' by the early naturalists into species grouped according to fortuitous and superficial resemblances. Modern mathematical objects appear as centers where surprising combinations of many diverse structures come to converge. A typical modern mathematical paper[1] will unfold as follows: The aim is to demonstrate that a certain group, occurring in number theory (and, more precisely, in the arithmetic theory of roots of unity) is finite. One begins by interpreting this group using homological Algebra, which reduces the theorem to be proved to a result concerning the cohomology of certain discrete subgroups of a Lie group; and finally this last result is obtained by calling upon E. Cartan's theory of symmetric spaces and Hodge's theory of harmonic forms. It started with 'Arithmetic', and then passed via 'Algebra' to end up ultimately in 'Geometry' and 'Analysis'! One could give many other analogous examples, showing conclusively that the old conceptions can only be intolerable fetters in the understanding of mathematics today.

This shows that Lautman had foreseen this extraordinary development of mathematics, which fate prevented him from participating in. He had filled it with enthusiasm, as much for the unparalleled harvest of new theories and solutions to old problems,[2] as for the eminently aesthetic character that the central parts of this vast edifice now offer (to those who, like Lautman, seek to understand them). It is hoped that the new edition of his works gives rise among philosophers to young emulators, capable like him to appreciate and interpret one of the most amazing monuments of the human spirit.

Considerations on Mathematical Logic*

It is impossible to speak of mathematical logic without first of all mentioning certain recent work. There is a broad historical and critical exposé of the question in the magnificant Treatise of Formal Logic by Professor Jorgen Jürgensen (Jürgensen 1931). Moreover, the problems posed by the philosophy of mathematics have been the subject of a special philosophy of science congress held at Koenigsberg in 1930; the report of the congress sessions has been published in Volume II of the new German review of philosophy Erkenntnis;[1] and finally Arnold Reymond, Professor at the University of Lausanne, has had a book appear last year entitled: Les principes de la logique et la critique contemporaine (1933), which is the most comprehensive and most precise of books on mathematical logic, the only one that is also written in French.

It is known that the interest of mathematical logic is the reunification of two independent series of research. Logicians along with Boole and Schröder had wanted to share with logical deduction the benefits of the algebraic calculus. They had reached a calculus of classes, propositions and relations, and thus opened the 'calculus ratiocinator' that Leibniz had required, a domain that was extending itself to understand the mathematics. Independently of this work, the mathematicians were led to investigate the logical foundations of mathematics whose certainty seemed shaken by the discovery of the famous paradoxes of set theory. But set theory proved

* This text, easily datable from 1933, appears in the 10/18 edition of 1977 without reference to an original publication, of which neither Fernando Zalamea nor I have succeeded in finding a trace (Jacques Lautman).

so successful for the study of number theory that it was able to keep the benefit of transfinite calculus while introducing numbers and sets in a fairly rigorous way so that the contradictions, like the set of all sets, are eliminated in advance. It then went on to try to deduce all of mathematics from a small number of logical notions and primitive propositions. And, in addition, as the paradoxes raised by Cantorian set theory were in nature more grammatical than mathematical, it undertook to transcribe all mathematical arguments in a symbolic language, the idea of which also goes back to Leibniz and to his 'lingua characteristica' project. This was the work first of Frege, then of Peano and above all of Russell and Whitehead in *Principia Mathematica*. Mathematics and logic were in this way conflated in a general theory of deduction: logicism.

The philosophical problems posed by the existence of a theory of deduction are twofold. It is a matter of knowing, on the one hand, whether the initial claims relative to the deduction of mathematics from logical notions and primitive propositions have been justified, that is, that there has been no appeal to a principle other than the primitive principles, which would thereby be found to be necessary and sufficient. It is then a matter of assessing the validity of the same theory, that is, to obtain the certainty that it can be applied without ever encountering a contradiction. We will see that independent of any preconceived metaphysics and for technical reasons of calculation, the answer to these two questions is possible in Russell's theory only at the cost of asserting a certain reality of the outside world. A metaphysical structure of the world is entailed by the demands of the theory, and in such a way that the stages of the deduction are always compatible with ordinary experience.

Let us examine first of all the logical construction of mathematics. The frame of this construction is constituted by the famous theory of types. Russell, having noted that the antinomies of analysis and set theory had been made possible by the consideration of sets whose elements were defined through the consideration of the set itself or through functions that could take as arguments values defined by the totality of possible argument values, wanted to make such definitions illegitimate. To do so, he defined the hierarchy of individuals as type 0, properties of individuals as type 1, properties of properties as type 2, and so on, in such a way that elements of a class or the arguments of a function are always of the type inferior to the class or function in question. The theory of types thus eliminates the antinomies like the set of all sets that contain itself as an element, but it does not exclude the possibility of vicious circles within the same type of functions or properties. When the attribution of a property to an individual is based on the totality of properties of that individual, the consideration of that totality is illegitimate. Take, for example, the theory of cardinal numbers. This theory is based on primitive notions of '0' and the 'recurrence of n to n + 1'. A whole number n + 1 is a number that has all the recurrent properties of n starting from 0. However, the property of being a whole number defined by recurrence is also among the properties of the number n. The notion of whole number would therefore be illegitimately implicated in the elements of its definition (see Carnap 1931). That's why Russell was led to establish a new ramification at the interior of the set of properties of the same type. The properties involved in the definition by recurrence will be of a certain order and the property of being a cardinal number will be a property of the same type and higher order. The notion of cardinal number was thereby saved, however a very large number of theorems of the theory of real numbers, in which the consideration of all real numbers irrespective of order played a part, were rendered void. Their preservation was only possible through the introduction of the 'axiom of reducibility' which asserts that there exists for all predicative functions of any order, a function equivalent to the same type and of order 1. The introduction of this axiom is the recognition that mathematics does not form a set of tautological propositions. It is indeed impossible to indicate a procedure for constructing the function of order 1 equivalent to a function of any order. Russell and Whitehead rely only on the realist certainty that certain beings possess certain properties even though we could not attribute these to them by only appealing to rigorous logical operations. Ramsey, a disciple of Russell who unfortunately died in 1930, considered for example the property of being the tallest man in the room. The attribution of this property to a specific man requires consideration of all the men in the room, and it may be that it is impossible to find the man in question even though we are confident that there is a man who has this property. The axiom of reducibility is the translation into symbolic language of sometimes impossible operations by the strict application of the theory of ramified types, but only by the necessity with which we possess an empirical intuition. This axiom of existence was not the only theory of deduction in the *Principia Mathematica*. In the same way as we defined whole numbers by the fact that they are different from each of their predecessors, we had been obliged by this to admit an axiom of infinity, posing the existence of an infinity of objects. If the number of objects in the universe were limited to 10 for example, 10 + 1, 10 + 2, etc., would be identical to the null class and would therefore all be equal to one another, unlike the general property of whole numbers. The notion of cardinal number is inseparable from the real existence of things to count; it relies on the existence of classes. There exist for example dualities necessary to the definition of a law of correspondence between these dualities, which is the cardinal number 2. It is through this constant appeal to experience in the choice of primary notions and in the introduction of different logical operations that Russell and Whitehead are certain to have eliminated in advance the contradictions and paradoxes. One doesn't say that the King of France is bald, not that he isn't, but that he doesn't exist. Suppose we wanted to introduce the use of the article 'the' (in a sentence like: W. Scott is the author of Waverley, for example), we will only introduce beings defined by 'the' with all necessary precautions to be in agreement with experience. If Waverley exists and if there is only one man who wrote Waverley, that man is Sir Walter Scott. Whereas, and we shall see later with the example of Hilbert's theory, a deductive theory is generally obliged to give proofs of consistency for it to be admitted in all rigor. Russell's theory linking mathematical existence to sensible reality, escapes into the intention of its author at this requirement. It is nevertheless true that the admission of the axiom of infinity, the axiom of choice (taken from set theory), the axiom of reducibility, lack logical justification. Also the followers of Russell are trying hard to eliminate the axioms of logicism. It also seems to them that this point of support in experience that is at the base of the *Principia* harms the purely symbolic character of mathematics. It seems nevertheless that the theory of types is indispensable to all rational construction and mathematical philosophy can retain as the fundamental contribution of logicism this notion of an order common to the generation of mathematical functions and to the description of any logical properties.

There is one category of mathematician for whom the most skilful efforts of rigorous deduction seem to be unreal exercises, it is those who, as a result of Brouwer and Weyl, see in mathematical objects the product of an activity of the mind exercising itself freely and independently of the possibilities of any logical transcription. The only law to which this activity is subject also governs all operational activity whether manual or intellectual. It is the law of only being possible in a finite and discontinuous number of steps. The intuitionists thus prohibit the consideration of elements distinguished by certain properties within infinite sets. Classical logic, like logicism, admitted that the rule of the excluded middle could be applied to the disjunction formed by a universal proposition and its contradictory particular as in the following reasoning: either all elements of this set have such-and-such a property, or there is one that doesn't. The Brouwerians assert that it makes no sense to deny a universal proposition as long as the object which is the manifest contradiction of it has not been effectively constructed. In a very large number of cases, the mind would therefore not be entitled to conclude from the absence of certain properties, the assertion of the contrary property. The intuitionist is related by this to the phenomenologist disciples of Husserl, Heidegger and Oscar Becker (see Heyting 1931). There exists for them a positivity of non-being that is not conceived as a simple negation, but as the object of a sui generis intention of thought. The concern to see that the reasoning never goes beyond these actually effected operations leads the intuitionists to only attribute to mathematical notions a provisional always revisable existence, at the mercy of the first particular determination that will come to change the meaning of any outlined edifice. The definition of mathematical entities is secured the moment in which the mind stops in the series of operations undertaken. While the mathematician has thereby perhaps gained in certainty, mathematics has been diminished in scope.

It is nevertheless possible to reconstruct mathematics by avoiding the antinomies while retaining Cantor's results, but without relying on the realism of the notion of class like the logicists. This is the result of the work of Hilbert and his followers, principally von Neumann in Berlin and Herbrand in France, in the elaboration of the axiomatic method. Mathematics is conceived in a set of signs devoid of any meaning and of which the calculation is as follows: Take a certain number of letters, some of which are always called constants, others variables, the others properties or functions, functions of functions, etc. Also take a certain number of signs corresponding to the logical operations of disjunction and negation. Then write certain sets of letters and signs that constitute the primitive axioms. Take also a certain number of rules of passage that allow certain sets of letters in the axioms to be replaced by others, and say that a proposition is demonstrated in the system of axioms when a process exists that allows it to be transformed by the use of rules of passage into an identity derived from the axioms.[2] The result is thus a symbolism somewhat similar to that of Russell, but whose meaning is profoundly different. The characteristic of such a mathematics is in effect to give the definitions in understanding and no longer in extension to avoid the vicious circles in the definitions. The whole number is no longer Russell's inductive number, but the simple property common to a set of signs, to be one of the arguments in the axioms that define the use of the letter Z (first letter of the German word Zahl). Several groups of axioms can be distinguished, each corresponding to an extension of the domain defined by the set of preceding groups. Analysis is thus reconstructed with the axioms of arithmetic, the set of axioms defining the total induction, those that introduce the notion of function, and those that define the expressions: all and some, that is, the axioms of set theory. The strictly mathematical operations being each time finite in number and bearing upon signs devoid of meaning, it is necessary to involve other arguments than mathematical reasoning to guarantee the consistency of the system of axioms and of their transfinite consequences. The Hilbertians' intention would also be to prove that the systems of axioms introduced are at complete determination, that is, that it is possible to recognize in a finite number of steps if a given proposition is or is not demonstrable in the system of axioms. Unlike mathematical reasoning, the proofs of consistency and research on general solvability (or complete determination) are concerned therefore with a real object, namely the formalist mathematical theory. Hilbert gave the name of metamathematics to this study of mathematical theories.

Up until Hilbert, there were four methods of proving consistency (see Poirier 1932, 150). The first is to develop the theory and not to find contradictions, but this method does not provide the certainty that one will never be found. The second is to see clearly via intuition the simplicity and compatibility of the primitive axioms; this is a little the certainty of Russell at the start of the *Principia*. The third is to find an empirical interpretation that justifies the invention of such an entity, for example a physical interpretation: the existence of diagonals justifies that of the irrationals; a problem of mechanics has a solution if it corresponds to a physical phenomenon. This assimilation of consistency with the construction carried out is furthermore the sole definition of compatibility admitted by the intuitionists. The fourth is to reduce one system of axioms to another, which only postpones but does not solve the problem. Hilbert and his followers have in truth invented a new method that I will try to characterize according to the work of Herbrand (Herbrand 1930; 1931).[3] Consider all the elementary propositions of a theory. Attribute to each of them a logical value, that is, that each has a sign T (true) or F (false) associated with it; then set the rules of attribution of the signs T and F to the propositions formed with the signs of disjunction and negation, then reduce all propositions of the theory to a conjunctive or disjunctive canonical form. If this canonical form contains n different elementary propositions, one can, after 2n tries, find out if the canonical form has or doesn't have the logical value T, whatever the logical values given to elementary propositions. Suppose that variables or functions have not yet been introduced, there is still only the calculus of logical propositions and it is proved that in this theory, we can determine whether a given proposition is or is not demonstrable. There is therefore, in this very simple case, consistency and complete determination (Entscheidung). Let us now consider expressions containing variables. It is possible to reduce these expressions to a canonical form by involving the notion of domain. A domain is a set of n letters such that if certain of them denote variables, others will serve by convention of 'value' the functions having as arguments the values that replace the variables and so on until the exhaustion of all individuals and functions of the expression. For a proposition containing a finite number of variables and functions, it is possible to consider all expressions obtained by substituting the variables with letters in any way and the functions by their 'value'. The logical value of each of the expressions obtained can be calculated for all cases and if this logical value is always T, the proposition is said to be true in the considered domain of order n. In these circumstances, if it is possible to construct for all h a domain of order such that all the hypotheses of the theory y are true, then the theory is not contradictory. Herbrand died before he could apply his conception of consistency to a theory broader than ordinary arithmetic. Further research seems to show that it does not apply to analysis. We will mention here in any case only the essential idea guiding Herbrand. It is impossible to carry out all the calculations implied by a theory, because obviously they are infinite in number, and the intuitionists are right to say that by proceeding in this way there would never be certainty of not finding contradictions. But it is possible to replace, as concerns the study of logical value, the consideration of an infinity of particular values with a letter or a function 'of choice' so that the results obtained in the finite domain of these values of choice have transfinitely valid consequences for all particular mathematical entities whose values are symbolized by this value of choice. This is to take up in another form Hilbert's former logical function which Hilbert himself gave a vivid interpretation with the example of Aristide: if Aristide the unbribable has also let himself be bribed, it is certain that a fortiori all men will be bribable (Hilbert 1923; 1936, 183 [1996, 1141]). The conclusions valid for an individual of choice are also valid for an infinite class of individuals of which he is the representative. Between the intuitionists' demands of construction and the pure introduction of notions by axioms, the Hilbertians knew to interpose an intermediate schematic, that of individuals and domains considered not so much for themselves as for the infinite consequences that allow finite calculations to operate through them. There remains a strong analogy between the domains of mathematics and Russell's hierarchy of types and orders. We are in the presence, in each case, of a structure whose elements are neither entirely arbitrary, nor really constructed, but composed as a mixed form that derives its fertility from its dual nature.

None of these three theories, no more that of the logicists than that of the intuitionists or the formalists, has yet been presented in a manner deemed satisfactory by their authors themselves. But for the philosopher, more than anyone else, attempted and roughly outlined theories are just as fruitful as definitive results.

Mathematics and Reality

PRESENTATION TO THE INTERNATIONAL CONGRESS
ON SCIENTIFIC PHILOSOPHY, PARIS 1935*

The logicians of the Vienna Circle claim that the formal study of scientific language should be the sole object of the philosophy of the sciences. This is a thesis that is difficult to accept for those philosophers who consider establishing a coherent theory of the relations of logic and the real as their essential task. There is a physical real and the miracle that is to be explained is that there is a need of the most developed mathematical theories to interpret it. There is also a mathematical real and it is similarly an object of admiration to see domains resisting exploration until they can be tackled with new methods. It is in this way that analysis was introduced into arithmetic or topology into the theory of functions. A philosophy of the sciences that isn't entirely concerned with the study of this solidarity between domains of reality and methods of investigation would be singularly devoid of interest. The philosopher is not in effect a mathematician by nature. If the logico–mathematical rigor can seduce him, it is certainly not because it allows a system of tautological propositions to be established, but because it sheds excellent light on the connection between the rules and their domain. It even brings up the curious fact that what is for the logicist an obstacle to be eliminated becomes for the philosopher the highest

* Lautman 1936.

object of his interest. These are all the 'material' or 'realist' implications that logicism is obligated to admit: they are Russell's well-known axioms, the axiom of infinity and the axiom of reducibility. It is, particularly in Wittgenstein, the assertion that to all true propositions there corresponds an event in the world, which entails a whole procession of restrictions and precautions for logic. In particular all propositions relating to the set of propositions, all logical syntax, as defined by Carnap, is impossible since this would require being able to consider correlatively the world as a totality, which is illegitimate.

The logicists of the Vienna Circle always assert their full agreement with Hilbert's school. Nothing is however more debatable. The logicism school, that follows from Russell, tries hard to find the atomic constituents of every mathematical proposition. The operations of arithmetic are defined from the primary notions of element and class, and the concepts of analysis are defined by extension from arithmetic. The notion of number therefore plays a crucial role here, and this role is augmented again by the arithmetization of logic that follows from the work of Gödel and Carnap. This primacy of the notion of number seems however not to be confirmed by the development of modern mathematics. Poincaré had already indicated in respect of the theories of dimension that the arithmetization of mathematics does not always correspond to the true nature of things. Hermann Weyl, in the introduction to his book *The Theory of Groups and Quantum Mechanics* (1928), established a distinction that seems fundamental to us and that will be taken into account by all future philosophy of mathematics. He distinguishes two currents in mathematics. One from India and the Arabs highlights the notion of number and leads to the theory of functions of a complex variable. The other is the Greek point of view which claims that each domain carries with it a characteristic system of numbers. It is the primacy of the geometric idea of domain over that of whole number.

The axiomatic of Hilbert and his students, far from claiming to reduce all of mathematics to being only a promotion of arithmetic, aims on the contrary to identify for each domain studied a system of axioms such that from the reunion of the conditions implicated by the axioms arises both a domain and valid operations in this domain. It is in this way that group theory, the theory of ideals, and of systems of hypercomplex numbers, etc., are constituted axiomatically in modern algebra.

The consideration of a purely formal mathematics must therefore give way to the dualism of a topological structure and of the functional properties in relation to that structure. The formalist presentation of similarly axiomatized theories is only a question of greater rigor. The object studied is not the set of propositions derived from axioms, but the organized, structured, complete entities, having an anatomy and physiology of their own. As an example, the Hilbert space defined by axioms that give it a structure appropriate to the resolution of integral equations. The point of view that prevails here is that of the synthesis of necessary conditions and not that of the analysis of primitive notions.

This same synthesis of the domain and the operation is found in physics from a slightly different point of view. Carnap sometimes seems to consider the relation between mathematics and physics as that between form and matter. Mathematics would provide the coordinate system in which the physical data is inscribed. This conception seems hardly defendable since modern physics, far from maintaining the distinction between geometric form and physical matter, unites on the contrary spatio–temporal data and material data in the common framework of a mode of synthetic representation of phenomena; whether through the tensorial representation of the theory of relativity or by the Hamiltonian equations of mechanics. There is thus for each system a simultaneous and reciprocal determination of the container and the contents. This is once more a determination unique to each domain at the interior of which a distinction between matter and form no longer subsists. Carnap seems, it is true, to also have another theory of the relation between mathematics and physics, much more in accordance with his logicism tendency. He regards physics, not as the science of real facts, but as a language in which experimentally verifiable statements are expressed. This language is subject to the rules of syntax, of a mathematical nature, when uniformly valid throughout their domain of definition, of a physical nature, when their determination varies with experience. There is here once more that assertion of the universality of mathematical laws as opposed to the variation of physical data. It seems to us that this conception does not account for the fact that this variation of physical data only makes sense in relation to the prior choice of magnitudes that are likely to vary, and this choice is physical. Carnap's example in his book *The Logical Syntax of Language* (1934, 131 [1937, 178]) is characteristic. When the components of the metrical fundamental tensor are constants, he argues, it is a mathematical law; when they vary, they obey a law of physics. The real philosophical problem is rather to know how a differential geometry can become a theory of gravitation. This accord between geometry and physics is proof of the intelligibility of the universe. It results from the clarification by the mind of a way of structuring the universe in profound harmony with the nature of this universe. It is conceivable that this penetration of the real by human intelligence has no meaning for certain extreme formalists. They would in effect rather see in the pretensions of the mind to know nature an approach drawn from the studies of Levy-Bruhl (1931). Understanding for them would be a mystical belief analogous to the participation of the subject in the object in the primitive soul. The term participation has nevertheless in philosophy another more noble origin, and Brunschvicg has rightly denounced the confusion of the two meanings. The participation of the sensible in the intelligible in Plato permits the identification, behind the changing appearances, of the intelligible relations of ideas. If the first contacts with the sensible are only sensations and emotions, the constitution of mathematical physics gives us access to the real through the knowledge of the structure with which it is endowed. It is even impossible to talk about the real independently of the modes of thought by which it can be apprehended, and far from denigrating mathematics to being only a language indifferent to the reality that it would disparage, the philosopher is commited to it as though in an attitude of meditation in which the secrets of nature are bound to appear to him. There is therefore no reason to maintain the distinction made by the Vienna Circle between rational knowledge and intuitive experience, between *Erkennen* and *Erleben*. In wanting to suppress the connections between thought and reality, as in refusing to give to science the value of a spiritual experience, the risk is to have only a shadow of science, and to push the mind in search of the real back towards the violent attitudes in which reason has no part. This is a resignation that the philosophy of science must not accept.

International Congress of the Philosophy of Science: Paris, 15–23 September 1935*

The eighth International Congress of Philosophy held in Prague in September 1934 was preceded by a sort of 'small Congress' of the Philosophy of Science in which the Vienna Circle in particular took part, and whose work Cavaillès summed up in an article that was presented here in January 1935. With the intention of marking the autonomy of the philosophy of science with respect to general philosophy, this 'Preparatory Conference' had agreed to hold regular Congresses of the philosophy of science, and a committee, including Carnap, Frank, Jörgensen, Lukasiewicz, Morris, Neurath, Reichenbach, Schlick and Rougier, were responsible for preparing the first of these Congresses whose meetings were held in Paris from 15 to 23 September 1935. Despite some resistance, Rougier, who took almost all the organizational work upon himself alone, was concerned that all trends in contemporary philosophy of science would be presented and discussed, also it was agreed that the papers would focus on general problems over technical questions being in principle reserved for future Congresses.

The popularity of the neo-positivist Vienna Circle in Central Europe and America, and above all the influence of the work and personality of Carnap naturally lead the speakers to be divided into two classes: there were those who placed themselves on the terrain of the Vienna Circle, and the others. The first studied the same problems and spoke the same language, their adversaries presented isolated theses in relation to personal conceptions of science or philosophy, less likely to be condensed into formulas and erected into a common doctrine.

* Lautman 1936b.

The presentation by Carnap was concerned with the relations between science and philosophy as understood by the Vienna Circle, the most complete presentation of which lies in the important work of Carnap: *The Logical Syntax of Language* (1934). Following Wittgenstein, Carnap considers an experimental science not as the study of a certain domain of reality, but as a coherent set of propositions that involve certain words and certain attributes corresponding to objects of experience and their observable properties. These are the statements of physics, psychology, sociology, etc. These statements must always correspond to a determined experience, the Viennese denote them by the term 'protocols' to give them the character of simple accounts of experiences. The protocols form the only synthetic propositions of science, and neopositivism is essentially an empiricism. To obtain from these 'protocol–statements' other statements, it is necessary to submit the protocol to a purely formal 'calculus' of logic and mathematics. A logicism is therefore going to be grafted onto the empiricism, and thus neopositivism realises the synthesis of Mach's phenomenalism and Russell's logicism. The translation of experimental statements into a formalized scientific language is thus the essential condition for scientific reasoning. The scientific language understands then, regardless of the science in question, two kinds of signs: descriptive signs corresponding to the objects and empirical properties, and formal signs borrowed from logic and mathematics. The only problems that arise in such a formalism are problems relative to these signs, independent of the meaning of the signs: what are the rules that allow the recognition that an assemblage of signs constitutes a proposition of the science studied? (This is the problem of formal determinations, *Formbestimmungen*). What are the rules that allow the deduction, from certain admissible premises, to other propositions? (This is the problem of formal transformations, *Umformungbestimmungen*). All of these rules constitute what Carnap calls, by analogy with the grammar that is the study of the syntax of ordinary language, the syntax of scientific language. There is up to this point nothing that is characteristic of the Viennese logicism, since these two syntactic problems have already been quite clearly formulated by Herbrand in the preface to his thesis on proof theory (Herbrand 1930).

What characterizes the logicist neopositivism of Wittgenstein and Carnap, on the contrary, is the reduction of philosophy to the syntactic study of scientific statements. The role of philosophy is thus a role of clarification of the propositions involved in what is generally called the theory of knowledge. Some of these propositions are concerned with questions that fall within the sciences proper, as, for example, all the propositions relative to space and time. There are also other propositions that are concerned with the logical relations between concepts or scientific propositions. The role of the 'logic of science' is to submit these propositions to the critique from the syntactic point of view. Some will be amenable to a correct formulation, others, which might have a meaning in common language, cannot be correctly formulated in scientific language in the least and will thus be eliminated as concerned with pseudo-problems. Rougier has, in this direction, shown how many philosophical problems of Aristotelianism had been made possible by the logical scandal of the dual meaning of the verb *to be* in Greek, which serves at the same time to link the attribute to the subject and to assert the substantial existence of this same subject.

Cavaillès has indicated in his article the principal problem that Carnap tackled in his book: can the rules of syntax be formulated in a formal language governed by this syntax? Carnap resolved the question in the affirmative for a simple enough language no. 1, only employing the signs of logical quantification ('there is an x such that' . . . and 'for all x') in the case of finite collections of objects. The rules of syntax are intended to establish the conditions under which properties can be attributed to a proposition that comes from the possibilities of its admission in the deductive system being studied, such as, for example, the following properties: to be demonstrable, to be refutable, to be compatible (meaning several propositions), to be complete (meaning a system of axioms). Carnap shows how these properties, which result from logical connections existing between all the propositions of a system, can still be expressed formally as the properties of one or several propositions taken in isolation, and he can do so through a process of arithmetization due to Gödel. By making the numbers selected in a certain way correspond to all the signs involved in a proposition, a characteristic number of all the propositions can be identified such that the syntactic properties of the proposition are related to arithmetic properties of this number. The determination of the rules of syntax of language no. 1 is thus reduced to the determination of the arithmetic statements expressible integrally by means of the signs of this same language. Carnap also defines a language no. 2 in which he introduces the operators (for all x) where the general variable can take an infinity of values. In this language no. 2, Carnap appeals to a much broader mode of consecution of the propositions than proof in a finite number of steps, namely, to sequential determinations (*Folgebestimmungen*) capable of including an infinite number of steps, and that can thus give correct definitions in this system of syntactic properties analogous to those whose formal conditions of attribution are set moreover in the simpler system of language no. 1.

It is not certain that all the notions introduced by Carnap are already formulated in his eyes in a definitive way. The syntactic study of scientific language is still in its infancy and Carnap's *Logical Syntax* (1934) is presented instead in the form of logical tests rather than a dogmatic treatise. Carnap was concerned, indeed, in a remarkable effort of intellectual sincerity, to mark the points where he is in disagreement with the other currents of contemporary logicism. For Wittgenstein, there cannot be syntactic studies, since the properties of propositions, like their compatibility, for example, appear intuitively in the examination of the propositions themselves, and cannot be formulated. This prohibition of all reflexive consideration of the statements of science comes from the author of the *Tractatus Logico-Philosophicus* (Wittgenstein 1922), from his realist attitude. Wittgenstein considered, in effect, that all propositions must correspond to a real situation in the world of facts, so that a proposition true for all sentences of the language, as would be a rule of syntax, would have the sense of a proposition relative to the totality of the universe. But the consideration of this totality is illegitimate, because it cannot be given in the experience of any situation. That is why the validity of scientific language cannot be discussed; this validity is manifested in the act by which we feel that our words correspond to things. The Vienna Circle has abandoned the realistic point of view of Russell and Wittgenstein, so that the restrictions that are going to be imposed on logicism no longer arise from consideration of real, but solely from the necessities of the formal calculus. The most famous restriction in this sense is that which was discovered by Gödel. Gödel established that we can never demonstrate, in a formal theory containing arithmetic, the consistency of this theory. He indicates, in effect, for all formalisms containing arithmetic, a process that leads to the effective construction of unprovable propositions, that is, propositions that it is equally contradictory to suppose either their truth or the truth of their negation.

Now, the following proposition: 'The theory studied is not contradictory' is justifiably an unprovable proposition in such a formalism. There is a limit here to the symbolism that arises from symbolism itself, just as in quantum mechanics, Heisenberg's uncertainty relations can be proved from the formal properties of mathematical operators corresponding to the physical magnitudes studied.

Carnap's syntax is also different from Hilbert's metamathematics, although it deals with the same problems. For the Hilbertians, metamathematics cannot be part of the formalized mathematics, and this is explained by Herbrand as follows: a metamathematical reasoning is necessarily discursive and relies by the same token upon a recurrence in the finite whose validity comes from being able, as Descartes claimed, to make a revision of reasoning so rapid that it becomes intuitive (Herbrand 1930). This metamathematical recurrence is thus the necessary condition for the proof of the consistency of reasoning by ordinary recurrence in arithmetic. But by the same token it belongs to another language, of a higher type, as defined by Russell's hierarchy of types. There were at least two Hilbertian mathematicians at the Congress, Bernays and Chevalley, but it seems they were not concerned to defend the point of view of mathematical formalism in front of the logicians of the Vienna Circle. Bernays, who, with Hilbert, was the author of *Grundlagen der Mathematik* (Hilbert and Bernays 1934; 1939), preferred, to the delight of philosophers and the surprise of logicists, to show that there were things in metaphysics other than the famous pseudo-problems, and Chevalley, who resumed Hilbertian class field theory from new foundations in algebra, tried to rediscover in mathematical thinking the effort of the human person in order to insert everything that had ceased to be about life and real needs into the automatisms.

Carnap's opposition to logic is only manifested on the technical terrain of logical syntax with the Polish metalogicians and Tarski. Tarski seems to be more concerned than Carnap with effective proofs in metamathematics. Carnap does not reason, in effect, on the true axioms of mathematically determined theories, but on schematic models of possible systems of axioms. He can thus define a predicate as the 'provable' predicate and attribute it to a given proposition, which assures a logical sense to the following proposition: 'Such a proposition is provable', but there is still here no outline, however vague, of the effective proof of the proposition in question. It can therefore only be said that the fate of Mathematics as engaged in syntactic research is detached from real mathematical facts. Tarski seems to follow the lead of Herbrand in trying to define metamathematical or syntactical notions in a manner that is in conjunction with effective proofs of consistency, of the independence of notions with respect to each other, of compatibility, etc. Abandoning the pure point of view of comprehension which is that of Carnap, he reintroduced into metamathematics the consideration in extension of domains of individuals whose construction is necessary in order to study the propositions. Let us, for example, define the property for a system of axioms to be 'complete' (*vollständig*), that is, to be such that a new axiom cannot be added to it without introducing contradictions in the consequences. Tarski has shown, in his Prague lecture (Tarski 1935), that any definition is purely nominal if it doesn't tie the research of the '*Vollständigkeit*' [completeness] to the consideration of the 'interpretations' of this system of axioms, that is, of the classes of individuals likely to present among themselves these relations implicitly defined by the axioms. A system of axioms being said to be 'monotransformable' if all its interpretations are isomorphic, Tarski proves that any system of monotransformable axioms is 'complete' as defined by '*Vollständigkeit*'. The converse of this theorem is yet to be proved, but the example helps to understand the thought of Tarski. He seeks to attribute the predicate 'complete' to a system of axioms only if certain mathematical relations can actually be found between the individuals of the domains attached to this system of axioms. This notion is therefore not part of mathematics, because it is of a type superior to the propositions that play a part in the system studied. The proposition: 'this system of axioms is complete', or, what amounts to the same, 'this x is complete', in effect only makes sense in a 'metalogic' in which the variable subject x, as well as the properties likely to be attached to it, are of a type superior to variables and properties that are related to the individuals of the domains attached to the propositions of the system studied. Tarski not only makes Hilbertian notions of metamathematics enter into his metalogic, he tried to make a 'semantic' of it, or a general theory of correspondence between signs and things signified, and began the study of notions such as that of truth or of definition that question the very essence of formalism and its philosophical value. In an orthodox formalism, the object of the calculus is the graphic sign independent of any reference to a reality designated by the sign. To assert the truth of a proposition signifies uniquely that the proposition is provable in the formalism. If one adopts then, like Wittgenstein and Carnap, a tautological conception of logico-mathematical formalism, 'true' is conflated with 'analytic' and 'false' with 'contradictory' (the problem of the verification or refutation of protocol statements by experience is not posed within the formalism itself). Tarski, on the contrary, restores to formalism its nature of language serving to express a reality and tries to give in this formalism a definition of truth that ensures the correspondence between the results of the calculus and the real. Consider a proposition P obtained in the formal language. This proposition is of a certain type determined by the type of variables involved in it. As previously, let *x* be a new variable of a higher kind introduced in metalogic to designate a proposition of the formal calculus. In these conditions, one has the right to write '*x* is true if P is true'. This is tantamount to saying, as Carnap notes with some irony that doesn't at all exclude the sympathy, that 'it is true that snow is white' if we already have this proposition: 'snow is white'. Tarski's semantics is no less a very serious effort to justify the name of the science of significations and has the great merit of showing that within the formalism there exists, thanks to Russell's wonderful theory of types, a means of exiting the pure calculus and regaining contact with physics.

This necessity of adapting the formalism to the physics of physicists, Carnap cannot escape from, and his book reveals in many places how he undertook to solve the question of the relation between logic and reality. The problem is posed from the initial definition of descriptive signs when Carnap says simply that they correspond to the objects and to the properties involved in the protocols. This definition seems to imply that Carnap conceives a protocol statement as necessarily attributing a property to an object. Gonseth has rightly noted that the task of logic would be greatly facilitated if it found itself facing a set constituted by judgments of attribution or of existence that would require nothing further than to be codified, whereas in fact the least description of experience implies that the mind has to impose on things the order of a certain relation. Carnap tries nevertheless to avoid the objection by giving a purely formal criteria to the distinction of signs into logical signs and descriptive signs: let, for example, a mathematical sign play a part in a physical theory like the components $g \vee v$ of the metrical fundamental tensor of the theory of relativity. When the value of these components is determined by a general law, as in homogeneous spaces of constant curvature, they are mathematical signs. When these components vary in a non-homogeneous space with the distribution of matter in this space, they are physical signs.

The basic problem that arises in full amplitude with this definition of descriptive signs is that of the introduction of new signs within a formal theory. It is a problem of logic that is encountered both in mathematics and physics and that can receive two solutions: either the new signs can be defined by an effective combination of previously established signs, as in any reconstruction of mathematical logic analogous to the *Principia Mathematica* (1910) of Russell and Whitehead; or the sign can be implicitly defined by a new axiom, as in Hilbert's axiomatic. As concerns numbers, the problem was handled in masterly fashion in front of the Congress by Padoa. Padoa, taking up the problem of the introduction into mathematics of negative numbers, fractions, and real and imaginary numbers, showed that these categories of numbers cannot be considered as extensions of the notion of whole number. Let us, for example, define the fractional numbers by the sign A/B. If the exact division of A by B is possible, the sign B only defines a whole number C that is already known, if it is not possible, the sign A/B only defines the existence of a fractional number if we admit in advance that only whole numbers exist. Padoa, recalling his work on the axiomatic definition of numbers, argued that it is necessary to abandon the logicist point of view, which only considers the precondition of the set of whole numbers, and presupposes in advance the existence of a 'maximal' set within which the different categories of numbers are defined by the different groups of axioms.

The opposition of methods is no less strong in physics. The Vienna Circle originally only admitted signs, in fact descriptive signs, connectable to an object of experience or definable from the same signs. This is the attitude of Carnap in *The Logical Structure of the World* (1928). This integral positivism can be generalized into a system of philosophy exactly like Comtism, and it is physicalism in its primitive rigor. Under the influence of Karl Popper (1935), Carnap admitted that the laws of physics were not protocol statements, that there are, in effect, notions that play a part which are neither more nor less relatable to any experience. Thus, for example, the vector of an electric field or a magnetic field in Maxwell's theory should be considered as defined implicitly by Maxwell's equations and is not an object of experience. It is mostly English-speaking philosophers who are attached to the Congress to define the meaning of the notions involved in the laws of physics independently of all experience. The problem was furthermore treated differently by Benjamin (Chicago) and Braithwaite (Cambridge).

Benjamin defined with great discernment the range of the two methods. In the so-called method of construction, the aim is to deduce the unknown (to which corresponds what Benjamin calls 'suppositional' symbols) solely from known elements, as if it was implicated by them. The hypothetical (or hypothetico–deductive) method on the contrary gives new signs axiomatically and tries to make with them the antecedent relations of implication whose consequents would be experimentally verifiable. The use of both methods is absolutely necessary and Benjamin reintroduced certain indispensible metaphysical hypotheses in the philosophy of science to legitimize the hypothetical method, as, for example, the hypothesis that the known and the unknown are part of a same nature. Braithwaite, trying to define the meaning of a word, like the word *electron*, also admits, under Ramsey's influence, the impossibility of defining the new terms of physics by means of the terms already known. Such an attitude would, in effect, render the passage from an old theory to a new theory very difficult, which is most often anything but a simple generalization of the former theory. These are ideas that have been presented in France by Bachelard in his latest book, *The New Scientific Spirit* (1934). Bachelard spoke out against the easy conception that claims to see in the new mechanics a generalization of the old mechanics, although his discovery corresponds to a veritable 'mutation' of the scientific mind. Braithwaite, like Benjamin, is also not satisfied to ask of the axioms that introduce non protocol signs to permit the deduction of experimental consequences, he gives these abstract symbols the meaning of an ideal reality analogous to the reality of a fairy tale. There is here an attitude that is not without analogy with Husserl's phenomenology, in which the real is characterized by a disposition of the mind to accept it as such.

These metaphysical attitudes that are found in certain theorists of knowledge seem to be necessary to prevent the philosophy of science from ending up as a radical nominalism toward which the Vienna Circle tends, as did the scholasticism of Occam in the past. The word scholasticism was mentioned by Enriques with mildly critical intent; the memory of Occam had been recalled with the very clear intention of being linked to the Vienna Circle by Morris, of Chicago, who had already contributed to the Congress in Prague to establish by his general theory of signs (*semiotic*) a rapprochement between American pragmatism and the logicist nominalism of Vienna. When one considers in effect that a statement is meaningful only in the language in which the signs involved in the statement are defined, even though one admits that these statements can be rendered false by a specific experience translated into another statement of the language, but one does not admit that a correctly verified statement provides any knowledge whatsoever of the reel. One adopts the radical conventionalism of Le Roy against which Poincaré spoke out so strongly in the final pages of *The Value of Science* (1905) and which seems nevertheless to have been taken up by Ajdukiewicz. A philosophical development was necessary and it was Schlick who provided one in a presentation sent to the Congress which he unfortunately could not attend. Schlick established a crucial distinction between a physical law and the statement of the law. Such a statement has meaning only in a particular language, it can be true in one system and false in another. But something subsists beyond the diversity of expression; it is the truth of the law itself. This truth seems to lie for Schlick in the invariance of the relation that expresses the law with respect to its different possible methods of verification. It is seized by an act of intellectual intuition beyond the discourse, in a moment of contact between the mind and reality.

Similar consequences were already discernible in Schlick's article on the foundations of knowledge published in the journal *Erkenntnis* in 1934 (Schlick 1934). There Schlick established an equally crucial distinction between the protocol statements that describe an experiment in a scientific language, and the expected intuitive observation of facts, at the moment when the experience that they constitute is witnessed. The joy in which consciousness is then enfolded is the very guarantee for consciousness of contact with reality. But these moments of contact imply a difficult effort of intellectual tension, the mind falls back as soon as it has raised itself up to the level of things and must then follow its course again into pure logic to further recognize the real. The definitive ban that the Viennese logicians had believed to pronounce for all reference to an unthinkable world to which the statements of science would apply is in any case removed, and Schlick finds again, in the activity of the intelligence, a mode of intuitive knowledge in which the statements assert their own truth value. Schlick distances himself, perhaps intentionally, from this school founded by him, but surely draws closer to the position of Brunschvicg or Enriques.

Enriques has already had the opportunity to introduce to the French philosophical public his conception of the philosophy of science, in particular at the time of the discussion that was held on his ideas at the Societé française de Philosophie (Enriques 1934). In his communication to the Congress he showed how the history of science was an instrument at least as necessary for the study of scientific truth as logicist formalism. It is known that Enriques rejects the purely phenomenalist attitude of Mach and stresses the importance of *a priori* rational demands in the progress of science. The true object of the philosophy of science lies for him much less in the study of the constitution of the statements of science than in the fertile role of hypotheses about the nature or the simplicity of things. It is in this regard regretable that a discussion about phenomenalism in contemporary physics had not been established before the Congress. It is known, in effect, that it is in specific reference to the ideas of Mach that Heisenberg, renouncing any representative hypothesis in atomic physics, oriented quantum mechanics in the direction of a calculation based solely on measurable magnitudes. The philosophical problems posed by theoretical physics were only discussed at the Congress in their relation with the calculation of probabilities, and this thanks to the presentations by Reichenbach.

Cavaillès has already given detailed guidance, in his article last year (Cavaillès 1935), on the junction that Reichenbach has made in his masterful treaty on probabilities (Reichenbach 1935) between the calculation of probabilities and logic with an infinity of values. Cavaillès pointed out that, in addition to classical logic or logic of two values (true and false), three valued logics, simultaneously developed by Post and Lukasiewicz, already existed (by adding an intermediate value corresponding to the probable), but they don't give the certainty of finding the mathematical rules of total and compound probabilities. Reichenbach defines the probability of a proposition asserting a fact as a frequency attached to this probability when it is considered within a series of propositions asserting or denying that fact. A logical value between 0 and 1 is attributed to this proposition; and the determination of the logical value of the sum, product, equivalence and implication of propositions is made by the same rules as the calculation of probabilities.

Reichenbach was able to establish on this basis a purely logical theory of induction, in the ordinary sense of the anticipation of a statement in physics. The hypothesis of induction comes back in effect to a wager, in a series of frequencies, on the limit towards which it is believed these frequencies tend. A proposition relative to this measure of the probability of the limit is neither true nor false, it is equally a probable proposition, whose probability is measured according to the rules of combination of elementary probabilities. Each observation adding a new term to the series of frequencies comes to change and correct the successive wagers, that is, the hypotheses made about the limit of the sequence, and whether there really is a limit. It is necessary that from a certain point these hypotheses are verified. Induction thus refers to probabilities of probabilities, which, as Reichenbach demonstrates, converge much faster than the first order probabilities in the case of the existence of a limit of these probabilities. From the moment we admit that it is better to know rather than not know, we are thus engaged in this successive formulation of wagers each correcting the other. The only way to get a true statement related to the future is just to continue to wager. Thus all pragmatist or finalist hypotheses on the basis of induction are eliminated, and the result is obtained by the sole means of a rich enough logic.

Probability theory to date provides the clearest case in which one can see a logic obtained from Russell fruitfully contribute to the formulation or resolution of a mathematical problem. However there is in the development of contemporary mathematical philosophy another current that is often conflated with logicism and which is no less profoundly different in its methods and its goals: it is the axiomatic gained from Hilbert. Instead of trying to recompose all mathematical statements from the same set of primitive logical notions, as in the *Principia Mathematica* (1910), the axiomatic in its recent development is rather an inverse effort to characterize mathematical theories in their irreducible specificity with respect to one another.

Chevalley has already shown, in his article on style in mathematics (1935), how the axiomatic definition of notions such as 'limit', by their characteristic properties, respond to an attitude quite different from that seen in the generalization of constructive methods in analysis, the tool par excellence of rigor in mathematics. When we consider the systems of axioms that are the basis of modern theories of arithmetic, algebra, functional calculus: axioms of group theory, ideals, Hilbert spaces, hypercomplex numbers, etc., we envisage simultaneously a domain and operations carried out in this domain. So what matters in the establishment of a system of axioms is not the logical elements of which the statements are composed, but the structural concern to appropriate exactly the chosen axioms to the domain that we want to define by its properties. The solidarity which is thus manifested between the domain and the possible operations on this domain places the connection between abstract operations and concrete domains at the forefront of mathematical research.

It is mainly the Swiss mathematicians who have made every effort to describe this obvious connection in their communications to the Congress. Swiss Romandy mathematicians were in fact in the habit of meeting philosophers in the Philosophical Society of Switzerland Romandy chaired by Arnold Reymond, Professor at the University of Lausanne. Their relative conceptions of the philosophy of mathematics provide extremely precise suggestions to the proccupations of the philosophers. Gonseth, in his lecture on logic considered as a science of any object, resumed the ideas he had already presented in his lecture on the law in Mathematics that appeared in the publication of the Centre de Synthèse (Gonseth 1934). He considered logical axioms as the last stage of a process of progressive abstraction and objectification from concrete experience to physical objects. At each level of abstraction, the mind studies more and more general properties. Logic is merely the last chapter of physics, which studies the properties of concrete objects that come from the simple fact of existence, as, for example, the following properties: two things being given, either they are simultaneously present or simultaneously absent, or else there is one without the other. The problems of logic are thus attached not only to a real mathematics, but also a real physics. Juvet, focusing on problems of group theory, showed how the study of the structure of a group allowed the formal point of view and the practical point of view to be joined in all branches of mathematics. The abstract structure of a group of transformations, for example, reflects the characteristic properties of space in which the transformations of the algebraically defined group operate. In his book *La Structure des Nouvelles Theories Physiques* (1933), Juvet gives a magnificent comparison that allows one to understanding what profound harmony can exist between a schematic structure and a material realization:

> Placed at a great distance from a window, the eye can first distinguish two axes of symmetry. On approaching, it recognizes in each quarter of the structure two new symmetries; certain motifs are repeated five times around a center. From closer yet again, more subtle ornamentations are seen in these motifs. It is the same with physical reality and the mind that examines it. The symmetries of phenomena, their alternations observe certain invariants at a given scale. The description that we give actually preserves these invariants and mimics these symmetries and these alternations in a game that reflects the structure of a group. Can it not be said that, in its way, physical reality at this scale mimics the structure of the group, or as Plato said, participates in this group? (Juvet 1933, 173)

The reference to Plato is particularly significant and beneficial. To study only the signs, we can in effect come to believe that science deals only with those signs and excludes any consideration of a reality that the symbolism would have as its function to describe. The rational idea that the mind penetrates the becoming of things by knowledge of the mathematical connections in which they participate appears to some to be as obscure as the mystical beliefs in the participation of the subject in the object for the primitives spoken of by Levy-Bruhl (1931). Philosophers are then entitled to ask themselves whether or not the philosophy of science lacks the essential mission of every philosophy when it ceases to search for methods that give man access to the real. Placed in front of a purely tautological conception of mathematics, the philosopher should stop linking the discovery of truth in science to the spiritual progress of a consciousness in search of a real to know and dominate. Scientific philosophy, by its formalism, would thus have contributed to the rejection of philosophy as belonging to the exclusive cult of irrational attitudes. One may however wish for the philosophy of science a higher ambition.

On the Reality Inherent to Mathematical Theories

PRESENTATION TO THE NINTH INTERNATIONAL CONGRESS OF PHILOSOPHY*

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SUMMARY – I try to show that the reality inherent to mathematical theories comes to them from what is incarnated in their own movement as the schema of connections that between them support certain abstract ideas that are dominating with respect to mathematics. I show in particular how the problem of the relations of essence and existence in effective mathematics receives quite a different solution to those of intuitionism and formalism.

Mathematical philosophy tends often actually to be mistaken for the study of different logical formalisms. This attitude generally entails as a consequence the assertion of the tautological character of mathematics. The mathematical edifices that appear to the philosopher so hard to explore, so rich in results and so harmonious in their structures contain in fact no more reality than is contained in the principle of identity. We claim to

* Lautman 1937a.

show how it is possible for the philosopher to disregard such flawed conceptions and find within mathematics a reality that fully satisfies the expectation they have of it.

This work conveys all the more to the philosopher that it is the reality inherent to mathematics, as all reality, in which the mind encounters an objectivity that is imposed on it. It is necessary to know to relate the modalities of the spiritual experience to the intrinsic nature of this reality in which it can be apprehended. The reality of mathematics is not made in the act of the intellect that creates or understands, but it is in this act that it appears to us and it cannot be fully characterized independently of the mathematics that is it's indispensible support. In other words, we think that the proper movement of a mathematical theory lays out the schema of connections that support certain abstract ideas that are dominating with respect to mathematics. The problem of connections that these ideas are likely to support can arise outside of any mathematics, but the effectuation of these connections is immediately mathematical theory. Mathematical logic does not enjoy in this respect any special privilege. It is only one theory among others and the problems that it raises or that it solves are found almost identically elsewhere.

We will show on a precise point how we think we can justify this presentation of things, and to do this let us study the problem of the relation of essence and existence. This problem, which is connected furthermore to the problem of the finite and the infinite, is essentially philosophical. Classical metaphysics, with the dialectical means of which it is disposed, has always tried to carry out the passage, for a single entity, from the essence of this entity to its existence. This problem is met with again in mathematics in the discussions concerning the transfinite and the axiom of choice, and the intuitionist or formalist mathematicians have in general placed the debate on the terrain of traditional philosophy.

For supporters of the actual infinite, the non-contradictory definition of a mathematical entity entails its existence. For the nominalists, there is only effectively constructed existence. It seems that what these two attitudes have in common is that they still conceive the problem of the relation of essence and existence as arising with regard to a same entity. If we now abandon the idea that a schema of solution for such a problem could even be conceivable independently of mathematics and try on the contrary to derive from the movement of mathematical theories the framework that underlies them, we arrive at very different conclusions. When the passage from essence to existence is possible, it has always taken place from one kind of entity to another kind of entity, and likewise in logic and in the rest of mathematics.

The point of view of 'essence' in logic and that of the non-contradictory structure of a system of axioms is the structural point of view, or '*beweistheoretisch*', that Bernays opposes to the extensive point of view, or '*mengentheoretisch*'. The extensive point of view is that of the existence of interpretations of a system of axioms, of domains of individuals that realize it, and almost all the metamathematical proofs try to establish a link between the structural properties of the propositions of a system and the existence of domains in which these propositions are verified. The passage from essence to existence thus results from the structure or essence of the system of axioms being apt to give rise to interpretations of the system. We'll find analogous schemas of genesis in the most different mathematical theories.

The structure of a Riemann surface is expressed by its genus. The genus p makes it possible to know the maximum number of $2p$ closed curves that can be drawn on this surface without dividing it into two separate regions. Now that number $2p$, which is thus connected to the 'canonical cutting' surface, is immediately interpretable in terms of existence for entities other than the base surface, since it also measures the maximum number of elements of a real base of Abelian integrals, everywhere finite, definable on this surface. The overall structure of a group is reflected in the number of 'classes of elements' of the group, and this number, in the case of a finite group, is equally that of the irreducible and non-equivalent representations definable on the space of the group. The structure of an algebraic field is manifested, like that of a group, by a decomposition into 'classes' of elements of the field, and the properties of this structural number depends on the existence in the base field k of a number such that $k\sqrt{a}$ is a quadratic class field over k. Again, let an operator be defined over the Hilbert space. The bringing the eigenvectors and eigenvalues of this operator to the fore results in a structural decomposition of the Hilbert space into eigenspaces, each defined by the eigenvectors of the operator in question. In all these purely mathematical examples, we always see a mode of structuration of a basic domain interpretable in terms of existence for certain new entities, functions, transformations, numbers, that the structure of the domain thus appears to preform. The problem of the geneses in which the passage between essence and existence is carried out is perhaps formulable abstractly but it is only in the proper movement of mathematical theories that the distinctions necessary for its solution thus take shape.

It would likewise be possible to extract from the mathematical theories the schema of connections that support other logical, or more precisely dialectical, ideas: those of whole and part, complete and incomplete, intrinsic and extrinsic, existence and choice. We only wish to indicate here the Platonic conclusion that these researches seem to us to impose: the reality inherent to mathematical theories comes to them from their participation in an ideal reality that is dominating with respect to mathematics, but that is only knowable through it.

These ideas are quite distinct from pure arrangements of signs, but they have no less need of them as mathematical matter that lends them a field in which the layout of their connections can be provided.

The Axiomatic and the Method of Division*

The development of abstract theories in modern mathematics and the research into axiomatic definitions are often accompanied by a restoration of the idea of generalization. Axioms are considered both, in comprehension, as a system of conceivable conditions independently of the mathematical entities that they realize, and, in extension, as defining the most extended class of entities likely to realize them. This point of view of extension is sometimes so well allied with the classification into genera and species that the possibility of fitting the least extensive types in the most extensive is met with again even in the constitution of 'abstract' theories. Thus, for example, in his book *Les espaces abstraits* (1928), Frechet envisages the axiomatic constitution of abstract spaces from the point of view of successive generalizations of the notion of space. He first defines the D spaces or distances, that is, the spaces in which to any pair of points a number can be attached that is called the distance between these two points, and which satisfies the axioms of distance. He then considered L spaces or spaces in which a convergence of sequences of elements can be defined without it being necessary to define a distance in advance. Any space D is a space L, but the reciprocal is not true; there are L spaces which are not D spaces. Frechet then considers a more general category, V spaces, whose definition does not even appeal to the notion of convergence and relies solely on the notions of neighborhood and point of accumulation. It is possible to show that the class of L spaces is entirely contained in the class of V spaces.

* Lautman 1937b.

These examples are sufficient to show that the axiomatic study of abstract spaces is able to be interpreted as a generalization. We shall see however that it is possible to give to axiomatic thought a completely different bearing the idea of which moreover is equally found in the work of Frechet.

Consider again the importance of the idea of generality in the conceptions of Bouligand relative to causality in mathematics. Bouligand calls a proof 'causal' if the proof of a relation between hypotheses and a conclusion is such that any reduction carried out in the statement of the hypotheses is likely to compromise the conclusion (*Cf.* Bouligand 1934; 1935, 175). This relation can then be 'realized' in an extensive enough domain of mathematical facts that constitute the domain of causality of the relation in question. The idea of group then necessarily introduces the fact that the different realizations of the same relation obey the laws of composition of the elements of a group. With respect to its domain of causality, the causal statement thus plays the role of an invariant with respect to a group. Bouligand expressly links the concern of causality to the revision of the initial notions implicated by the axiomatic method and to the search for greater generality. The link between these three ideas, for example, is asserted in this sentence from 'Geometry and Causality':

> The search for the most general conditions of validity of a determined statement, if it is ready to reveal its causal proof, doesn't succeed without a constant reworking of the implemented notions. (Bouligand 1935)

Thus, for example, the classic statement of Pythagoras's Theorem is too restrictive and obscures the causal fact that the area of any figure constructed on the hypotenuse is the sum of the areas of figures similar to the first and constructed on the other two sides. These areas can be of squares but also of any figures whatsoever. It can be seen that the consideration of the most general statement is destined to shed light on a necessary connection that proves insufficient in particular cases. The search for generality, in Bouligand, is therefore in no way due to a concern with generalization, it presents itself rather as a consequence of the search for the necessary connection and we would like to further study the true logical nature of this approach.

Whether it's a matter of the constitution of abstract notions, or the search for necessary connections, it seems to us that mathematical discovery does not at all consist in subsuming the particular under the general but in carrying out the dissociations comparable to those that condition the progress of physical knowledge. Physical experimental discovery very often results from being able to carry out, within a phenomenon, a dissociation by which the complexity of facts, which had beforehand seemed to be simple, is revealed. This experimental dissociation is often preceded or followed by a theoretical dissociation established within the system of notions corresponding to the experience. In atomic physics, it is the dissociation of spectral lines into doublets or multiplets, the dissociation of molecules in ortho and para molecules, the discovery of the positive electron, de Broglie's hypothesis on the complementarity of two half-photons that constitute the photon. It is the same in the theory of relativity: the plurality of time or the duality of masses. When it takes place in theoretical physics, this dissociation can sometimes result from an activity totally abstracted from the critical postulates implicitly admitted in the common notions. The dissociation of the unicity of time results in a critique of the notion of coincidence just as the criticism of the notion of measurement can cause the dissociation of the point of view of the observed and of the observing. It thus seems that the concrete experiences present themselves to the intelligence as resulting from the exceptional encounter of certain notions whose separation can be carried out abstractly. This separation does not in the least have the character of a generalization since the consideration of other particular facts of an order of higher precision is substituted for a particular fact of a certain order of precision.

If one tries to characterize the role of the method of division in mathematics, one finds first of all two fairly simple and well known cases: 1) when two properties are wrongly identified, the discovery of one case in which one is realized without the other shows their difference (Weierstrass's discovery: continuous functions without derivatives; Borel's discovery: functions of a complex variable that are 'monogenic', in Cauchy's sense, without being analytic, in Weierstrass's sense); 2) another form of dissociation is that which, by an appropriate treatment, establishes the differences between certain elements having use of a common property, the singular points of an analytic function are those in the neighborhood of which the function cannot be expanded into a series but a dissociation into essential singular points and non-essential singular points (or poles) is introduced immediately when instead of studying the series expansion of $f(z)$ in the neighborhood of these points, the development of

$$\frac{1}{f(z)}$$

is studied. A similar development exists for the poles and not for the essential singular points. We now wish to describe, in a few examples of axiomatized notions, a third form of dissociation the philosophical importance of which seems to us to be considerable because it shows, in mathematics at least, the close connection of critical reflection and effective creation.

1. EQUALITY

In the *Grundlagen der Mathematik* (1939), by Hilbert and Bernays, arithmetic equality is defined by two axioms:

$$a = a$$
$$a = b \rightarrow [A(a) \rightarrow (b)].$$

The first axiom establishes the reflexivity of equality and it can easily be proven that transitivity and symmetry are implicated by the second. The second axiom states that for two numbers to be equal, it is necessary that any arithmetic property that applies to one applies to the other. It is necessary, in sum, that two equal numbers are indiscernible at all possible points of view, as regards at least the properties defined by the signs of predicates A introduced in the theory in question. Conversely, if two numbers are discernable in comprehension, they measure classes of distinct elements in extension. Thus we see that the notion of indiscernibility is closely linked to the axioms that define the number of elements of a set. Bernays introduced, in addition, these axioms of countability immediately after the axioms of equality by indiscernibility. Here, for example, is the axiom that defines the fact that a domain only contains one individual: $(x)(y)(x = y)$, there is only one individual in the given domain if given any object x and any object y, we have $x = y$. Similarly, the formula $(x)(y)(z)(x = y \vee y = z \vee x = z)$ means that there are at most two individuals in the domain considered, and the formula $(\exists x)(\exists y)(x \neq y)$ (there is an x and y such that x is different from y) means that there are at least two of them. If we then agree to arrange the elements between which there is a relation of equality into the same class, we see that there is in any set of numbers as many elements as classes of equal individuals. It is perfectly possible to dissociate the number of classes of elements of a domain, from the number of elements of this domain by no longer envisaging the relation of equality (=), but an equivalence relation ($\equiv$) which is reflexive, symmetric, transitive like equality, and amenable to being defined in many ways. Suppose, for example, that the domain in question was any group. Two elements of the group a and a' are said to be equivalent, or belong to the same class, if b is any fixed element, such that $a' = bab^{-1}$. The result for the elements of a same class is no longer a complete indiscernability, but a certain partial indiscernibility. If a number $\chi(a)$, such that $\chi(ab) = \chi(a)\chi(b)$, is made to correspond, in effect, with any element a of a group, then the functions χ (which are the characters of the group) take the same value, for all the elements of a same class: $\chi(a) = \chi(bab^{-1})$. The consequence of this partial character of the indiscernibility is that the formulas of countability corresponding to a relation of equivalence can no longer measure the number of elements of the group, but only the number of their classes. A formula asserting that there is at most n nonequivalent elements in a group would signify that there are at most n classes of equivalent elements, without any indication of the number of elements of the group. This shows how the study of equivalence relations allows the dissociation of the point of view of countability of classes from that of the countability of individuals, which are conflated in the equality of arithmetic.

2. MULTIPLICATION

Consider a domain of individuals x, y, whose nature isn't specified for the moment, and write the equation $y = ax$, in which a is any whole number. If x equally traverses the domain of whole numbers we have the ordinary multiplication of arithmetic and y is equally a whole number. Multiplication can even be defined when the elements a and the elements x traverse different domains. For example, consider the case of a vector space. We know that in such a space, multiplying a vector x by a number a has a meaning and gives, as a result, a new vector $y = ax$. The whole numbers and vectors constitute respectively distinct domains which obey distinct laws: the vectors form a module, that is, a domain in which the sum of two elements is uniquely defined, and the whole numbers form a ring, that is, a domain in which the sum and the product of two elements are defined. The following axioms are obtained, the first of which concerns the addition of vectors, the second and third the addition and the multiplication of numerical multipliers

$$a(x + y) = ax + ay$$
$$(a + b)x = ax + bx$$
$$a(bx) = (ab)x.$$

The ordinary notion of multiplication is seen here as split into two distinct ideas: 1) the idea that we can make the elements of a domain of *operators* act on the elements of a basic domain to recover other elements of the basic domain; 2) the idea that this action of operators on the elements of a domain is, in some cases, reducible to the formation of arithmetic products. In the case of numbers that multiply vectors, the formation of products is still found, because the vector *x* is defined by the coordinates $(x_2 \; . \; . \; . \; x_n)$ the vector *ax* is defined by coordinates $(ax_2 \ldots ax_n)$, but the operators acting on a domain to restore an element of the domain can be considered without the formation of arithmetic products playing a part in any way. These operators are not always composed according to the axioms defined above, but they nevertheless constitute a domain of elements as characterized as the elements of the basic domain on which they act.

If the basic domain is constituted by the points of a space, transformations of a group can be defined as operators acting on these points, which are composed among themselves following the law of group composition. The transformation S acting on the point p, transforms it into a point p´ = S_p and the transformation T transforms the point p´ = S_p into a point p″ = TS_p such that there is a transformation R of the group directly transforming the point p into p″ = R_p. If the basic domain is constituted by the functions y = $(fx_1 \; . \; . \; . \; fx_n)$ the differential operators

$$\frac{\partial}{\partial x_\mu}$$

can be considered to be acting to the left on functions *y* to restore another function:

$$\frac{\partial}{\partial x_\nu} y = \frac{\partial y}{\partial x_\nu}.$$

These operators are likely to be composed following the known laws of the differential calculus:

$$\frac{\partial}{\partial x_\mu}\left(\frac{\partial y}{\partial x_\nu}\right) = \frac{\partial 2y}{\partial x_\mu \partial x_\nu}.$$

This shows how the axioms defining the action of operators on a basic domain result from what has been dissociated in ordinary multiplication, the operational function, which is independent of the specific nature of the envisaged operators, from the formation of products, which is linked to the particular nature of arithmetic operators. This duality between the action and the nature of the operators is particularly significant in the study of unity.

3. UNITY

When writing $x1 = x$ (or $1x = x$), the term 1 plays a dual role: it is the unit element of the domain of operators acting on the domain of x, and it is also the identical operator that transforms the elements x of the basic domain. Here's what these distinctions mean: suppose that the domain of operations is a ring of numbers. The unit element 1 of this ring is such that for every element a of the ring, there would be $a1 = a$ and $1a = a$. This is an internal definition that only envisages the nature of the unit element within the domain to which it belongs. If, furthermore, this unit element acting on the element x of the basic domain transforms this element, in addition to the fact of being a unit element of the ring of operators, it is the identical operator, and this concerns not so much its own nature as its outward actions. This distinction of the specific nature of certain elements and the action they exercise on the elements of other domains, is for us essential. We shall see later that it permits the method of division in mathematics to be characterized as a whole.

4. MEASURE AND THE INTEGRAL

In the classical theory of the integral, the expression

$$\int_b^a f(x)\,dx$$

is defined by means of the sum of an infinite number of products. Each product represents the multiplication of an ordinate by the length dx of an infinitesimally small increase of the abscissa. The lengths dx therefore play a dual role: being defined as measures of magnitude, they are practically conflated with those geometric segments to which they are connected; and secondly, their function in the determination of the integral is to be the numbers that play a part as the factors in a product of numbers. Here again the Lebesgue generalization can be interpreted in terms of dissociation. The fact of being given in a product as factors, like the contribution to a basic domain, is not in the least connected to the geometric magnitude of this domain. Rather, *dx* can be considered to be simply a number attached to this domain, satisfying certain determined conditions.

This is necessarily so when the basic domain is no longer a segment but a family E of sets of points of any kind. The determination of numbers attached to these sets can be made as follows (*Cf.* Possel 1935): let us call *set function* a function that makes a number μ correspond to each set of the family E whose value is real, finite or equal to $+\infty$, but never to $-\infty$. This function is said to be *completely additive* if, E_1, E_2 . . . being disjoint sets of E whose sum comes to E:

1) $\mu(E_1) + \mu(E_2)$ has a finite sum or is equal to $+\infty$, independent of the order of the terms;
2) this sum of values is equal to the value of the sum of the sets.

It proves that only those set functions that are never negative can be considered among the completely additive set functions. Such a function is called a *measure* if:

3) any set E of E such that $\mu(E) = +\infty$ can be covered by a countable infinity of sets for which μ is finite;
4) the family of sets E contain every intersection and every reunion of a finite or countable number of sets of E;
5) when $\mu E = 0$, any portion E′ of E belongs to E, and we have: $\mu E' = 0$.

Here's how such a measure is suitable for the formation of the Lebesgue integral: let a point function $f(p)$, defined over a set of points E, take the non-negative values (and if necessary $+\infty$) and let μ be a measure defined in E. It is then possible, in certain conditions the statement of which is irrelevant here, to define on the set E an integral

$$\int_E f(p)\,d\mu,$$

through the formation of Lebesgue sums, each of which contains an infinity of products of type $l_i.\mu E_i$ in which l_i measures an interval (in ordinates) of the set of real numbers and in which μE_i is the measure attached to a subset the fundamental set E. The measure nicely plays here a role analogous to that of the length of infinitely small increments of the abscissa in the classical theory of the integral. It brings the contribution of space as defined by the function to an infinity of products. But this new conception of the integral could only be constituted by the work of Lebesgue and his successors through the dissociation of the idea of measure attached by convention to a space, from the consideration of *magnitudes* that were originally conflated with the numbers that measured them.

5. THE ABSOLUTE VALUE

Consider a number field K, that is, a set of numbers such that it is always possible to form the sum, product and quotient of any two numbers. This field is said to be *ordered* if, for any element a, the property of *being positive* (> 0) can be defined by the following axioms (*Cf.* van der Waerden 1931, ch. 10):

(I) 1) For any element a of k, either $a > 0, a = 0$, or $-a > 0$;
2) if $a > 0$ and $b > 0$, then $a + b > 0$ and $ab > 0$;
3) if $-a > 0$, then a is negative.

This having been posed, we call the *absolute value* $|a|$ of an element a of an ordered field that of the two terms a and $-a$ that are not negative. The absolute values satisfy the following axioms

(II)
$$|ab| = |a| \,.\, |b|$$
$$|a + b| \leq |a| + |b|.$$

The notion of absolute value, being linked to the definition of the positivity of the elements of a field, is thus closely linked to the *order* of the elements of the field. It is also equally linked to another quite different property, the *closure* of the field question. This is what should be understood by that: a field is said to be closed (or complete) if every sequence of terms $a_1, a_2, \ldots,$ such that from a certain rank the *absolute value* of the difference between two terms is less than any given positive number ε, converges in absolute value towards a limit a. It proves that given an ordered field K, an extension field Ω of K which is both ordered and closed can always be constructed. When K is the field of rational numbers, Ω is the field of real numbers.

The two notions of order and closure are thus closely associated in the notion of absolute value, but it has been possible to dissociate them by noting that the construction of the closure Ω of K doesn't use the order of elements a of K but the order of absolute values $|a|$. In the case of ordered fields, we make an element a correspond to an absolute value $|a|$ by choosing, of the two terms a and $-a$, the one that is not negative, because by virtue of axioms (I), there is always one. But what matters in the construction of closure is not that the 'absolute value' of a is $|a|$, but rather that this absolute value obeys axioms (II). There is therefore a 'generalization' of the notion of absolute value, when, to each element a of a field still *not amenable to order*, an '*evaluation*' (*Bewertung*) $\varphi(a)$ can be made to correspond that satisfies the axioms analogous to axioms (II):

(III) 1) $\varphi(a)$ is an element of an ordered field
2) $\varphi(a) > 0$ for $a \neq 0, \varphi(o) = 0$
3) $\varphi(a).\varphi(b) = \varphi(ab)$
4) $\varphi(a+b) \leq \varphi(a) + \varphi(b)$

It can be shown by emphasising this time the properties of the evaluation, that given a valued field K, we can always construct an extension field Ω whose evaluation extends that of K and such that any fundamental sequence y is convergent (in the sense of the evaluation defined by axioms III).

The absolute values (II) do indeed constitute a particular case of evaluations (III), but the passage from axioms (II) to axioms (III) consisted essentially in detaching the properties defined in (II) from their too close connection with the properties of order defined in (I).

The few examples that have just been analyzed will allow us to be more specific about the philosophical importance of the activity of dissociation in mathematics. We have seen in all the cases certain notions of elementary arithmetic and algebra, which seemed simple and primitive, envelop a plurality of logical or mathematical notions, delicate to specify but in all cases clearly distinguishable from one another. It is in this way that arithmetic equality is the only equivalence relation such that the countability of the individuals of a set is conflated with the countability of classes of equivalent individuals as defined by this relation. Likewise, the idea of multiplication contains both the formation of arithmetic products and the action of operators on a domain of elements distinct from these operators. The idea of unity can be considered either from the point of view of the unit element of a ring of numbers, or from the point of view of the operator identical to a domain of operators. The length of a segment is connected to the magnitude that it measures but it is only a number attached by convention to this magnitude. Finally, the absolute value of classical algebra envelopes both the idea of order and the construction of the closure of a number field. The passage from notions said to be 'elementary' to abstract notions doesn't present itself as a subsumption of the particular under the general but as the division or analysis of a 'mix' which tends to release the simple notions with which this mix participates. It is therefore not Aristotelian logic, that of genera and species, that plays a part here, but the Platonic method of division, as taught in the *Sophist* and the *Philebus*, for which the unity of Being is a unit of composition and a starting point for the search for principles that are united in the Ideas.

Another rapprochement with the Platonic dialectic is necessary. We have seen that distinct notions revealed by the dissociation of a mathematical idea are related most often, on the one hand, to the intrinsic nature of certain entities, and on the other, to the action of these entities on other entities. Let us take a few examples. The fact that the unequal elements of a set of numbers are discernible, from a certain point of view, is a property concerning the nature of these numbers. The fact that the relation of equality, like any relation of equivalence, determines within the set in question a division into classes (each of which only contains one element), relates to the structure what the relation of equality imposes on the set. The fact that numbers can be multiplied with each other relates to the intrinsic properties of the ring of whole numbers, whereas, on the other hand, the fact that the multiplication of a number by a vector gives a vector is linked to the fact that a vector space admits the action of numerical operations on it. Likewise, finally, the positivity of a number is rightly an intrinsic notion, but the closure of a field that can be carried out with the notion of absolute value or evaluation relates rightly to the constructive fertility of the notion of evaluation. The axiomatic definitions 'by abstraction' of equivalence, measure, operators, evaluation, etc., thus characterize not a 'type' in extension but the possibilities of structuration, integration, operations, of closure conceived in a dynamic and organizational way. The distinction that is thereby established within a same notion between the intrinsic properties of an entity or notion and its possibilities of action seems to be similar to the Platonic distinction between the Same and the Other that is found in the unity of Being. The Same would be that by which a notion is intrinsic, the Other that by which it can enter in relation with other notions and act on them.

We said at the beginning of this study that certain mathematicians like Frechet or Bouligand sometimes associate the effort of axiomatic abstraction and the idea of generalization. Generalization is however, with them too, only the consequence of more essential preoccupations. With Bouligand, as we have seen, it is the search for the necessary connection, with Frechet, a concern for analysis that often relegates the point of view of generalization to second place, as in these lines intended for philosophers, where he writes:

> It is extremely curious to see a notion such as that of distance which appears, primarily, to be a primary notion, an irreducible notion, able to be dissociated into notions of nature very different from each other. (Frechet 1928, 158)

It is, in effect, extremely important for the philosopher to prevent the analysis of Ideas and the search for notions that are the most simple and separable from each other, from appearing like the search for the most extended types. A whole conception of mathematical intelligence, issuing from Platonism and Cartesianism is, in effect, at stake in this distinction.

BOOK I

Essay on the Unity of the Mathematical Sciences in their Current Development

Introduction: Two Kinds of Mathematics

In 1928, in the preface to *The Theory of Groups and Quantum Mechanics*, Hermann Weyl wrote:

> There exists, in my opinion, a plainly discernible parallelism between the more recent developments of mathematics and physics. Occidental mathematics has in past centuries broken away from the Greek view and followed a course which seems to have originated in India and which has been transmitted, with additions, to us by the Arabs; in it the concept of number appears as logically prior to the concepts of geometry. The result of this has been that we have applied this systematically developed number concept to all branches, irrespective of whether it is most appropriate for these particular applications. But the present trend in mathematics is clearly in the direction of a return to the Greek standpoint; we now look upon each branch of mathematics as determining its own characteristic domain of magnitudes. The algebraist of the present day considers the continuum of real or complex numbers as merely one 'field' [or 'antisymmetric field' among an infinity of others that have the right to the same consideration. (The property of being antisymmetric results from not conserving the commutative law of multiplication).] The recent axiomatic foundation of projective geometry may be considered as the geometric counterpart of this view. This newer mathematics, including the modern theory of groups and 'abstract algebra', is clearly motivated by a spirit different from that of 'classical mathematics', which found its highest expression in the theory of functions of a complex variable. The continuum of real numbers has retained its ancient prerogative

> in physics for the expression of physical measurements, but it can justly be maintained that the essence of the new Heisenberg-Schrödinger-Dirac quantum mechanics is to be found in the fact that there is associated with each physical system a set of magnitudes, constituting a non-commutative algebra in the technical mathematical sense, the elements of which are the physical magnitudes themselves. (Weyl 1928, vi [1931, p. viii]) (Comments inserted by Lautman)

This text asserts in the clearest way the existence of an essential division in contemporary mathematics. It would be necessary in effect to distinguish 'classic' mathematics, which starting from the notion of whole number leads to analysis, from 'modern' mathematics, which, opposed to the mathematics of number, asserts on the contrary the primacy of the notion of domain with respect to numbers attached to this domain.

There is no doubt about the existence of a new mathematics, animated by a very different spirit to that of the mathematics of last century. Analysis of the nineteenth century, as its name suggests, analyses the infinitesimally small. In the theory of functions, it is the convergence of series, the passage to the limit, continuity, derivation and integration. In the theory of differential equations, it is the search for local integration in the neighborhood of the origin. It is the differential geometry that emerges from Gauss's *General Investigations of Curved Surfaces* (1827) and Monge's *Applications de l'Analyse à la Géométrie* (1807): the theory of contacts, curvature, envelopes, etc. However, in spite of the radiance with which the analysis of the infinitely small shined throughout the course of the nineteenth century, it did not stop the emergence of ideas whose development would lead to the synthetic mathematics of the modern epoch. Some of these new theories, such as set theory, are due to the very difficulties of analysis and the need to find a way to overcome them. Others, such as group theory, have their origins in algebraic problems completely foreign to analysis. We will not stop at these questions of origin, because our goal here is solely to characterize in their common resemblances the diverse theories that, in opposition to analysis, have as their object the study of the global structure of a 'whole'.

Let us envisage in the first instance topology. Currently, two kinds of topology are sometimes distinguished: the topology based on the methods of the set theory of points; and combinatorial topology, or algebraic topology, which characterizes the properties of invariant figures arithmetically or algebraically by a continuous transformation. The global character of these two theories is obvious. They generalize in effect intuitive considerations of synthetic geometry like the fact for a surface to be opened or closed, to have holes or to cover itself with places. The considerable development in contemporary topology derives from it being currently impossible to conceive a theory of analysis that is not based on a prior topological study of the domain of definition of the envisaged functions. Consider, for example, the expression $y = f(x)$. This expression asserts the existence of a selective correspondence between a domain of the x plane and a domain of the y plane. The function f which ensures this correspondence can be perfectly determined uniquely by the topology of the two domains in question, without needing to know the algebraic or transcendental expression,[1] which gives the value of y from that of x. This necessity to upport analysis with topology derives from Riemann as regards the functions of a complex variable, and the topology in question is that which has since been called combinatorial topology. As for the theory of functions of real variables, since the work of Cantor, Borel and Lebesgue, it is intimately linked to the topology of point sets and the global 'measure' of the domains they represent. The distinction established by Weyl between classical mathematics and modern geometrical mathematics would therefore not tend to oppose the methods of the theory of functions and those of topology. They are the same, and nothing more is needed to be convinced of this than to read the book by Weyl, *The Concept of a Riemann Surface* (1913), that is often cited in our principal thesis (Lautman 1938b).

By opposing the methods of modern mathematics to those of classical mathematics, Weyl thought, and the indications that he gives prove it, of the theories of modern algebra as they are found presented for example in the two volumes of the work of van der Waerden (1930; 1931). We would like briefly to emphasize the differences that separate the spirit of classical arithmetic and analysis from the spirit of modern arithmetic and algebra: group theory, rings, fields, ideals, hypercomplex systems, etc. Classical mathematics is constructivist, first of all as regards the definition of the operations of analysis beginning with the operations of elementary arithmetic, but also and above all as regards the individual generation of real or complex numbers from whole numbers. Modern algebra is on the contrary axiomatic, and the result is that by giving the axioms which the elements of a group or field obey, the often infinite totality of elements of the group or field are also given, at the same time. These elements are in general no longer amenable to being individually constructed from the arithmetic of whole numbers. They can in fact be of any kind, in addition to numbers there are also vectors, operators, transformations, matrices etc.

This axiomatization of modern algebraic theories brings with it a number of consequences that for Weyl seem essential: the first relates to the priority of the global structure of a domain with respect to numbers attached to this domain. This can be understood in two different but complementary ways. The domain described by Weyl can be the first 'domain of magnitudes' (or in the narrow sense of the word: algebra) that constitutes the studied magnitudes themselves, whose individual properties are then governed by the laws of organization of the 'whole' of which they are the elements. What in effect characterizes an algebra are not the elements of which it is the set, but the fact that, whatever elements are considered, they maintain relations among themselves which affirm their membership of this algebra.

The priority of the notion of domain can then be asserted with respect to the notion of number, by considering the appropriation of a system of numbers or magnitudes of a domain previously defined in a geometrical, topological or even physical way. We have already alluded above to the connections that unite the topology of certain domains to the existence of functions defined on these domains. The connection of topology and algebra is infinitely closer because the algebraic invariants attached to a domain are not only defined on it, but serve moreover to explore and recognize it.

There is another aspect in the text of Weyl that distinguishes modern algebra from classical mathematics, that which is relative to the non-commutativity of multiplication. The non-commutative algebra, in which a product *ab* is different to a product *ba*, also differs as radically from ordinary algebra as the algebra of logic in which $a \times a = a$. The algebra of logic does not however constitute, in all truth, a new mathematics because its field of application is too narrow, while the non-commutative algebras have an increasingly appreciable importance, both in mathematics and physics.

Finally, a fourth characteristic of the theories of modern algebra can be described, which, though not formulated in Weyl's text, is nonetheless an extension of the previous characteristics: in contrast to the analysis of the continuous and the infinite, algebraic structures have a clearly finite and discontinuous aspect. Whatever the infinity of elements that constitute a group, a body, an algebra (in the narrow sense of the word), the methods of modern algebra consist most often in imposing on these elements a division into classes of equivalent elements, and in thus substituting for an infinite set, the consideration of a number of classes which, in application, is most often finite.

The methods and the fundamental conceptions of modern algebra being thus distinguished from those of analysis, an interpretation of the meaning of this duality, which appears to be deeply installed within contemporary mathematics, can be sought. Is it an essential duality between theories irreducible to one another, or is it rather a duality of methods that could one day be reconciled? The text of Weyl quote above does not contain a clear answer to this question; however, he suggests the idea that there would be more of an opposition between the methods of the theory of functions of a complex variable and algebra. On the other hand, Hilbert has repeatedly expressed his firm belief in the possibility of finding a conjoint method of elaboration of algebra and analysis: *eine methodisch einheitliche Gestaltung von Algebra und Analysis* (1935, 57). We propose to show in the pages that follow how modern mathematics is engaged in the process of this unification of algebra and analysis, and that it is so by the penetration of increasingly sophisticated structural and finitist methods of the algebra in the domain of analysis and the continuous. In sum, the conflict between the methods of algebra and analysis are resolved in favor of algebra. Weyl's distinction between two kinds of mathematics thus seems only to correspond to the historical conditions of the development of mathematics, and leaves intact the unity of mathematics and the unity of intelligence.

In the four chapters of this essay we try to show how, in the modern theories of analysis, aspects which seem to characterize modern algebra can be found. The algebraic idea of dependence of magnitude with respect to the domain to which it belongs has led to the study of the influence of the 'dimensional' structure of a set on the mode of individual decomposition of its elements, and to consider the importance of dimensional decompositions in the theory of functions (Chapter 1). The priority of the topology of certain domains over the numbers attached to these domains is reflected in the role of non-Euclidean metrics in the theory of analytic functions (Chapter 2). We will then study (Chapter 3) the non-commutative algebra which plays a part in the theory of equivalence of differential equations. Finally we show how the consideration of finite and discontinuous algebraic structures can be used to determine the existence of functions of a continuous variable (Chapter 4).

CHAPTER 1

The Structure of a Domain of Magnitudes and the Decomposition of Its Elements: Dimensional Considerations in Analysis

The decomposition of a mathematical entity can be envisaged from two very different points of view: sometimes the decomposition brings to light the particular properties of an entity within the set to which it belongs, and it is characterized by the specificity of its very structure; sometimes on the contrary the decomposition of the entities of the same set is done according to a plan common to them all and thus reflects not only their particular properties, but also their belonging to the same set whose global structure is reflected in that of its elements.

The type of decomposition of the first kind is the arithmetic decomposition of a whole number into a product of equal or unequal prime factors. This decomposition is possible in elementary arithmetic in only one way and every number is thus uniquely characterized by the factors that are proper to it.

Take as a type of decomposition of the second kind the algebraic-geometric decomposition of a vector $\vec{X}$ in an n-dimensional vector space R. Let a system of n linearly independent vectors $\vec{e_1} \ldots \vec{e_n}$ be in this space, that is, such that the equation:

$$a_1\vec{e_1} \ldots a_n\vec{e_n} = 0$$

is only solvable if all the coefficients $\vec{X}$ cancel each other out. The space being n-dimensional, these vectors form a basic system or fundamental system for the set of vectors $\vec{X}$ of the space. This means that any vector $\vec{X}$ can be represented by a decomposition of the following type:

$$\vec{X} = x_1\vec{e_1} + \ldots x_n\vec{e_n}$$

These decompositions differ clearly from the decompositions of the first kind. The coefficients $x_1 \ldots x_n$ characterize uniquely, it is true, the vector $\vec{X}$ in the adopted basic system but the fact that any vector can be decomposed into n components reflects less an individual property of these vectors than their membership in an n-dimensional space. These are the global dimensional properties of this space that thereby impose a uniform mode of decomposition to each of its elements, and this influence of the structure of the set on the nature of the elements is manifested in an even more striking way when the space R can be represented as a direct sum of subspaces independent of one another:

$$R = R_1 \oplus R_2 \ldots \oplus R_n$$

Each R_i then contains all the vectors $x_i\vec{e_i}$ of the same index, so that any vector X is thus obliged, by its affiliation with R, to have a component in each of the n subspaces of the decomposition of R.

The decompositions of a second kind present yet another essential aspect. The n base vectors of such a decomposition, $e_1 \ldots e_n$ are, in the same way as X, vectors of space R, whereas the factors of a decomposition of the first kind are still, as we shall see, of a simpler nature than the entities that they characterize. The vectors of space R therefore present these multiple forms of solidarity, and a decomposition into components reflects not only the dimensional structure of the global space, but also the choice of the basic system adopted to support and organise the relations that unite all the elements of the space to a fundamental set of n.

It is possible to find in algebra the two types of decomposition that have just been distinguished. The theorem that has long been called the fundamental theorem of algebra is a theorem of 'proper' decomposition. It states that any polynomial in x of degree m is equal to the product of a constant by m equal or unequal factors of the form $(x - \alpha)$:

$$f(x) = C(x - \alpha_i)(x - \alpha_2).(x - \alpha_m).$$

To this decomposition, which brings to light the unique roots of a polynomial, we would oppose all 'imposed' decompositions in the modern theories of algebra. Consider for example a hypercomplex system. Such a system O is defined by a set $b_1 \ldots b_n$ of basic elements and a commutative field P whose elements act as multipliers on the basic elements. This gives the following decomposition for O

(I) $$O = b_1 P + \ldots\ldots b_n P;$$

(P denotes the set of numbers of field P), and for each a of O

(II) $$a = b_1 \lambda_1 + \ldots\ldots b_n \lambda_n = \lambda_1 b_1 + \ldots\ldots \lambda_n b_n)(\lambda_i \varepsilon P).$$

The law of individual decomposition (II) inscribed thus in each element a of O the structural properties (I) of the system to which these elements belong.

The two kinds of algebraic decomposition are in addition not totally foreign to one another. The 'proper' decomposition of a polynomial into a product of factors is closely connected to the global nature of the field in which this decomposition is attempted. The polynomial $x^2 - 2$ has no root in the field k of rational numbers and is only decomposable into a field which contains at least the field $k(\sqrt{2})$.[1] The fundamental theorem of algebra relative to the complete decomposition into m factors of any polynomial of degree m is basically only the translation of a global property of the field of complex numbers: that of being algebraically closed. Such a dependency is even found with respect to the base field (*Abhängigkeit vom Grundkörper*) in arithmetic decompositions. The number 21 is decomposable into a unique product of prime factors only in the field of rational numbers, in which 21 = 3 x 7. On the contrary, it can be decomposed in a second way in the field $k\sqrt{-5}$, in which there is, in addition to 21 = 3 x 7, the following decomposition: $21 = (1 + 2\sqrt{-5})(1 - 2\sqrt{-5})$. The proper decompositions are in a certain way connected to the basic domain as 'imposed' decompositions.

One essential difference nevertheless remains. The proper decompositions still study mathematical entities in isolation, independently of all those that have the benefit of analogous properties, while the second kind of decompositions impose on all elements of a set the same structure that results from the dimensional structure of the set to which they belong, and thus establishing a close dependency between the totality of entities studied and a certain number of them taken as the basic system. The dependency with respect to the base field that sometimes presents the decompositions of the first kind is thus very different from the interdependence of the elements in the global system of contemporary algebra, and that brings to light the decompositions of the second kind.

Before researching the respective importance of these two modes of decomposition of a mathematical entity in analysis, we would like to show how the linear interdependence of elements of a set is in close relation to the theory of systems of linear algebraic equations.

Consider first of all the homogeneous system (without second member)

$$\text{(III)} \qquad \begin{array}{l} a_{11}x_1 + \ldots\ldots a_{1n}x_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + \ldots\ldots a_{mn}x_n = 0 \end{array}$$

Let r be the rank of the system, that is, the degree of the largest non-zero determinant formed with the coefficients of system (III). A solution of the system is any vector X with coordinates $x_1 \ldots x_n$ that satisfies the equations (III), and it can be demonstrated that for $r < n$, there is a maximum number of linearly independent $n - r$ solutions forming a fundamental system of solutions. Any solution $\vec{X}$ of the system (III) is a linear combination of fundamental solutions,

$$\vec{X} = c_1\vec{X}_1 + \ldots\ldots c_{n-r}\vec{X}_{n-r}$$

and the system (III) thus defines a vector space to $n - r$ dimensions. We will not discuss here the case of a non-homogenous system, that is, whose second members are not zero as in (III); but simply indicate that in the study of these systems, the dimensional idea of the composition of a general solution from any particular solution, and of a fundamental system of solutions from the corresponding homogeneous system is sometimes found.

This junction which thus appears between the theory of decompositions of the second kind and the resolution of linear algebraic equations is in our view very important, because it is through the consideration of such systems of equations that the substitution of the synthetic point of view of dimensional decompositions for the individual point of view of proper decompositions occurs very often in analysis.

The domain of decompositions of the second kind is primarily formed by the approximate representation of functions belonging to different functional spaces. It is known that the trigonometric functions 1, *cos x, cos* 2*x*, . . . ; *sin x, sin* 2*x* . . . , form in the space of continuous and differentiable functions $f(x)$, for $< -\pi \leq x \leq \pi$ with $f(-\pi) = f(\pi)$ a system of basic functions, such that for any function $f(x)$ of the space, there is the Fourier series expansion:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{n=\infty}(a_n \cos nx + b_n \sin nx) :$$

the a_n and b_n satisfying certain determined conditions.

We have also recalled elsewhere the analogous properties of the space of functions of summable squares. This space is constituted by the set of functions $f(s)$ defined on a measurable space E, such that

$$\int_E f(s)\,d\mu$$

exists, and that

$$\int_E |f(s)|^2 d\mu$$

has a finite value.[2] Such a space is isomorphic to the Hilbert space (*Cf.* Lautman 1938b, ch. 5); it is thus a vector space to an infinity of dimensions and there exists in this space an infinity of basic systems $\varphi_1(s), \varphi_2(s)\ldots$ each formed by an infinity of orthonormal functions[3] such that every function of the space is decomposable in series:

$$f(s) = x_1\varphi_1(s) + x_2\varphi_2(s) + \ldots\ldots$$

This result has played a large role in the history of the problems studied here, because it is by the extension of these algebraic methods of decomposition of the second kind to unknown functions of integral equations or partial differential equations that Hilbert conceived a possibility for the conjoint elaboration of algebra and analysis. The unification of the mathematical disciplines must thus be carried out by the penetration of the dimensional and finitist methods of algebraic origin into the domain of analysis. The decompositions of the second kind being connected in algebra to the study of systems of a finite number of equations, of a finite number of variables, Hilbert developed the theory of systems of an infinity of linear equations to an infinity of variables to find the theorems of linear combination analogous to those of the finite, and applicable to the theory of integral equations. Here is a passage in which Hilbert expresses with admirable clarity the hope of unifying algebra and analysis by a similar generalization of algebraic methods:

> The theory of forms to an infinity of variables 'is' a new domain, and to some extent intermediate between algebra and analysis, which by its methods is based on algebra, but which belongs to analysis by the transcendent nature of its results. (Hilbert 1909, 62; 1935, 59)

We will now briefly show how Hilbert carries out the reduction of a problem of analysis to an algebraic problem.

Let the linear integral equation be:

$$\text{(IV)} \qquad f(s) = \varphi(s) - \lambda \int_0^1 k(s,t)\varphi(t)\,dt$$

in which (s) is the function sought after and in which $f(s)$ and $k(s, t)$ are given. Hilbert first constructed a system, of arbitrary orthonormal continuous functions $\omega_p(s)(0 \leq s \leq 1)$, that satisfy in addition, relative to an arbitrary continuous function $u(s)$, the following conditions of completion:

$$\int_0^1 u(s)^2 ds = \left[\int_0^1 u(s)\omega_1(s)ds\right]^2 + \left[\int_0^1 u(s)\omega_2(s)ds\right]^2 + \ldots +$$

Such a system forms a basic system in the space of functions $u(s)$. Hilbert then introduces the following expressions:

$$x_p = \int_0^1 \varphi(s)\omega_p(s)ds,\ f_p = \int_0^1 f(s)\omega_p(s)ds\ ,$$

$$k_q = \int_0^1 k(s,t)\omega_p dt\ ,$$

$$k_{pq} = \int_0^1 \int_0^1 k(s,t)\omega_p(s)\omega_p(t)dsdt$$

which are such that $\sum x_p^2, \sum f_p^2, \sum k_q(s)^2, \sum k_{pq}(s)^2$ converge.

In these conditions, from the integral equation (IV), the system following from an infinity of linear algebraic equations, an infinity of unknowns is deduced

(V)
$$(1-\lambda k_{11})x_1 - \lambda k_{12}x_2 \ldots\ldots = f_1$$
$$-\lambda k_{21}x_1 - (1-\lambda k_{22})x_2 \ldots = f_1$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The resolution of the integral equation (IV), a problem of analysis, is therefore reduced to the resolution of the system of algebraic equations (V).[4]

Let $x_1, x_2 \ldots$ be a set of values forming a solution to this algebraic system; a continuous solution of the integral equation (IV) can be obtained by forming the uniformly convergent series:

$$\varphi(s) = f(s) + \lambda \sum_q k_q(s) x_q$$

and it is easy to show that this solution $\varphi(s)$ admits a decomposition into Fourier series,

(VI)
$$\varphi(s) = \sum_{p=1}^{\infty} x_p \omega_p(s)$$

the basic system being the system of adopted $\omega p(s)$ and the coefficients x_p obeying the relations:

$$x_p = \int_0^1 \varphi(s)\omega_p(s)\,ds$$

The expression (VI) correctly represents a development of the function 'of the second kind'; $\varphi(s)$ is sought after, and it is this algebraic-looking result that Hilbert had in mind in the constitution of his theory.

We will now try to determine the role of decompositions of the first kind in the theory of analytic functions and see how in this domain too, the dimensional considerations issuing from algebra could give new meaning to the results of proper decomposition theorems of arithmetic inspiration.

The proper decomposition of a polynomial is the one that emphasizes its roots. The analytic function that most resembles a polynomial is the integral function, which only has singularities at the infinite, and it is possible to single out the zeros of such a function by a decomposition into products analogous to the decomposition of a polynomial into factors of the first degree. Let $a_0, a_1, \ldots,$ be an infinite sequence of points different to 0. The product:

$$\prod = \prod_{n=0}^{n=\infty} \left\{ \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \dots \frac{1}{k_n}\left(\frac{z}{a_n}\right)^{k_n}} \right\}$$

under certain conditions represents an integral function which admits the points $a_0, a_1, \dots a_n \dots$ as zeros, and any integral function that also admits these points as zeros is represented by the expression:

$$G(z) = e^{H(z)}.\prod$$

in which $H(z)$ is an arbitrary integral function.

Now envisage the meromorphic functions, that is, the functions that admit, in the finite, a finite number of poles $a_i \dots a_n$, in the neighborhood of which their expansion into series has the form:

$$f(z) = \frac{A_1}{(z-a_i)^m} + \frac{A_2}{(z-a_i)^{m-1}} + \dots \frac{A_m}{(z-a_i)} + G(z); (a_i = a_1 \dots a_n),$$

$G(z)$ being an integral function.

The set of fractional terms of this development constitutes the principal part

$$g_i\left(\frac{1}{z-a_i}\right)$$

of $f(z)$ corresponding to the pole a_i.

This having been posed, it is possible to give in advance the poles and principal parts of a meromorphic function and thus reconstruct this function from partial fractions (the different principal parts), each admitting no more than one pole:

$$\text{(VII)} \qquad f(z) = g_1\left(\frac{1}{z-a_1}\right) + g_2\left(\frac{1}{z-a_2}\right) + \dots g_p\left(\frac{1}{z-a_n}\right) + G(z)$$

The decomposition of an integral function into products, and the decomposition of a meromorphic function into partial fractions are essentially decompositions of the first kind, because they reveal the very existence of an infinity of functions admitting given points as zeros or poles, but they reveal nothing about the global structure of this infinity. They are attached exclusively in effect to the individual properties of the functions studied and no at all to the organization of the class that these functions could constitute. We will now see the dimensional considerations reappear at the heart of these problems, and the algebraic solidarity between elements of the same set be established once again.

The Riemann-Roch theorem proposes to study the structure of the set of functions that admit as poles on a Riemann surface n given points $a_1 \ldots a_n$, which are uniform everywhere on the surface (*Cf.* Osgood 1907, part 2).

By virtue of the theorem of decomposition into partial fractions, the functions studied can be put in the following form:

$$F(z) = A_1 Y_{a_1}(z) + A_2 Y_{a_2}(z) \ \ldots + A_n Y_{a_n}(z) + C \qquad \text{(VIII)}$$

in which $Y_{a_i}(z)$ are Abelian integrals of the second kind which each only admit one pole, the point a_i. This decomposition is absolutely equivalent to (VII). The functions $F(z)$ are presented in the form of a sum of functions, from a certain point of view simpler than the function $F(z)$ since they each have only one pole. In any case these functions are not themselves parts of the set of functions $F(z)$. The decomposition obtained is therefore a decomposition of the first kind, and it is the determination of the coefficients A_i that will introduce into this problem of analysis a dimensional point of view of algebraic inspiration.

It is known that (*Cf.* Lautman, 1938b, ch. 3) a multiply connected Riemann surface can be rendered simply connected by a 'canonical cutting'. Here is briefly what is meant by this: the Riemann surface can be brought about by continuous deformation under the form of a disk with 2 sides of pierced with p holes; p being the genus of surface. We can then trace on this surface a maximal system of $2p$ closed curves that are not reducible to a point by continuous deformation and such that none of these curves divide the surface into two separate regions. By cutting the surface according to these $2p$ curves, a simply connected surface is obtained. When the variable passes from one side of one of these cuts to the other, the abelian integrals are subject to a jump which is manifested by the existence of a discontinuity in the values of the function. The abelian integrals of the second kind can be demonstrated to only have discontinuities different to 0 for the p half of the $2p$ retrosections traced on the surface.

Let $Q_1(a_1), Q_2(a_1) \ldots Q_p(a_1)$ be the p discontinuities of the integral $Y_{a_1}(x)$ of the formula (VIII); likewise, let $Q_1(a_1), Q_1(a_2) \ldots Q_1(a_n)$ be the discontinuities of the functions $Y_{a_2}(z) \ldots Y_{a_n}(z)$. So that the function $F(z)$ is uniform on the p cuts where each of these components is subject to a discontinuity, it is necessary that there are the following p relations:

$$A_1Q_1(a_1) + A_2Q_1(a_2) + \ldots\ldots A_nQ_1(a_n) = 0$$
$$A_1Q_2(a_1) + A_2Q_2(a_2) + \ldots\ldots A_nQ_2(a_n) = 0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$A_1Q_p(a_1) + A_2Q_p(a_2) + \ldots\ldots A_nQ_p(a_n) = 0$$

There is therefore a system of algebraic equations to n columns and p rows. To simplify, let p be the rank of this system. We know that $n - p$ coefficients $A_1 \ldots A_{n-p}$ can be chosen arbitrarily and the p others are expressed in functions of Q_i and the $n - p$ arbitrary coefficients. The result of that which precedes is that the most general function that admits the points $a_i \ldots a_n$ as poles depends on $n - p$ similar functions, that is, we have

$$\text{(IX)} \qquad F(z) = A_1F_1(z) + \ldots\ldots A_{n-p}F_{n-p}(z) + C$$

This decomposition is eminently a decomposition of the type which we have called the second kind. The functions $F_1(z) \ldots F_{n-p}(z)$ belong to the set of functions $F(z)$ sought after. They form in that space an $(n - p)$-dimensional basic system and all the others let themselves be obtained by the linear combination of these basic functions.[5]

The comparison of decomposition (IX) and decomposition (VIII) is extremely suggestive. It shows us how the same mathematical entities, in this case the functions $F(z)$, can be decomposed in two distinct ways: the first decomposition characterizes their individual properties, the second the links that they support between them. These are indeed two modes of thought of different inspiration, and whose reunion is the work of Riemann.

The examples that have been given thus permit us to understand that if there are different ways of thinking in mathematics, it is very unlikely that these differences in method correspond to differences of domain. The duality of the types of decomposition which have been stressed throughout this chapter is a certain fact that asserts itself for any observer, but this duality of methods does not end up constituting two different mathematics, that which would be a promotion of the arithmetic of whole numbers, and that which would be an extension of algebra. The same entities are able to be studied in both ways and it is this encounter of methods that gives rise to the profound unity of mathematics.[6]

CHAPTER 2

The Domain and Numbers: Non-Euclidean Metrics in the Theory of Analytic Functions

We have shown, in the first chapter of our principal thesis (Lautman 1938b), the differences between the synthetic conception and the infinitesimal conception of geometry. The Riemannian definition of a space by the formula that gives the distance between two infinitely near points is a purely local conception, the entire space is constructed by the step by step juxtaposition of infinitesimal neighborhoods, whose assemblage is not, without new conventions, amenable to collective characterization. On the contrary, the definition of the properties of a space by those of the group of transformations whose properties in question are invariants, is a global definition which envisages the possibilities of group action on the totality of points of the space. It is also often possible to define an invariant metric in 'global' spaces with respect to the group that operates on the space, but this metric can be very different from the Riemannian metric.[1] These in effect only appeal to notions defined in the neighborhood of each point of origin considered, while the formula for an invariant metric with respect to a group, contains in general terms referring to properties of the whole space. What is seen to appear here is a new aspect of the opposition of the two mathematics. In the local conception, the expression of numbers which form the metric of the surface are given prior to the constitution of the surface since the surface results precisely from the bringing together of neighborhoods defined by the value of the distance between two infinitely near points. In the synthetic conception, space preexists the metric, since the geometry of the entire space influences the determination of the metric. Numbers are then posterior to the domain and it is particularly significant in this regard that Weyl alludes to projective geometry in the text that these pages are intended to comment on.

Projective geometry has in effect provided the first example of a geometry in which a number is attached to any pair of points that is defined independently of any reference to Euclidean distance and that satisfies nonetheless certain formal conditions which allow it to be considered abstractly as the 'distance' between two points on this space. Chief among these conditions is the additivity of distance: let $d(a,b)$ be the number d attached to the pair of points (a,b); it is necessary that

$$d(a,b) + d(b,c) \geq d(a,c)$$

The formal axioms that define the notion of distance can be satisfied by an infinity of different systems of numbers, the object of abstract geometry is to choose one that ensures the invariance of the metric with respect to the overall transformations that are envisaged on the surface. In projective geometry, for example, the distance between two points is defined by the anharmonic ratio attached to the 'play' of the four following points: the two given points and the intersections of the projective line passing through these two points with the conic which is globally transformed in itself by a sub-group of the group of projective transformations. It is well known how Klein operated the union of notions of projective metric, due to Cayley and non-Euclidean geometries: Riemann's geometry corresponds to the case in which the 'absolute', that is, the projectively invariant conic, is imaginary; Lobatchewsky's geometry corresponds to the case of a real conic; and Euclid's geometry to the case in which the conic degenerates into two imaginary points. From the point of view that interests us here, the non-Euclidean geometries and projective geometry are conflated with one another: what is seen in effect is the appropriation of a system of numbers by a space that results from the prior consideration of the structural properties of the space, which thus receives a system of numbers on the space made specifically for it.

What is proposed for the remainder of this chapter is the rediscovery of the part played by non-Euclidean metrics in the theory of analytic functions, and the fact that it is of great philosophical importance. Weyl wanted in effect to arrange all of analysis in the mathematics that issues from the notion of whole number, in order to oppose it to the modern mathematics of domains. It is certain that even the applications of analysis to geometry, such as it is conceived in France by the school of Monge, and in Germany by that of Gauss, are only geometric interpretations of analytical results obtained independently of any reference to geometry.[2] This concern to eliminate geometric intuitions from analysis is moreover in the spirit of Cartesianism, for which synthetic geometry is imaginative work and remains well short of the infinite possibilities of calculation carried out uniquely by the understanding.

This difference of planes between a purely intellectual analysis and a synthetic geometry that's more sensible has in our day almost completely vanished, because topology and group theory have allowed the clarification of synthetic methods in geometry, which are just as intellectual as methods of analysis. A reversal has thus been made in the relations of analysis and geometry, and analysis has once again, not been 'restricted to the consideration of figures' but more or less subordinated to a prior topological study of the domain of definition of the functions envisaged. This new aspect of the theory of analytic functions is related to the development of the theory of the Riemann surface of an analytic function. We will only envisage here the consequences relative to the metric.

When Poincaré used a non-Euclidean metric in a problem of analysis, the first it seems, he thought he was witnessing the accidental and almost inexplicable encounter between two orders of thought totally foreign to one another. Non-Euclidean geometry had seemed up until then to be 'a simple mind game that was only of interest to the philosopher, without being of any use to the mathematician',[3] and it found itself to be essential in the uniformization theory of algebraic functions. It is necessary to insist for a moment on the conditions that brought about the connection of the old and the new mathematics.

We have already had occasion to define the problem of uniformization (*Cf.* Lautman 1938b, ch. 3). Given an analytic function $\zeta = f(z)$, this function in general on the plane of the complex variable z, possesses points of ramification in the neighborhood of which it can take several distinct values. To uniformize the function $\zeta = f(z)$ is to find a complex variable t such that $\zeta = \varphi(t)$ and $z = \psi(t)$ are two uniform functions on the complex plane t. The resolution of this problem requires not only the construction of the Riemann surface F corresponding to the function $\zeta = f(z)$, but the construction of the universal covering surface $\overline{F}$ of the surface F. It is possible in effect to establish a conformal correspondence between this simple convex surface $\overline{F}$ and the totality or a portion of the plane of a complex variable t. From the existence of this conformal correspondence, the existence on the plane t of the sought after uniform functions can then be easily deduced. There exist in this respect three types of universal covering surfaces: those whose conformal representation can be made on the open complex plane; those whose conformal representation can only be made on the plane completed by the point at infinity and are topologically equivalent to the complex sphere; and finally those whose representation can be made on the unit circle. Let us now place ourselves in the latter case, by envisaging for greater simplicity only a correspondence between interior points of the surface and points interior to the circle. The surface $\overline{F}$ admits a discontinuous group of internal transformations, that is, a point P_o of this surface can be made to correspond to as many points P_1, P_2 . . . as there are on the Riemann surface of the given function, and as many classes of closed paths, irreducible to one another by continuous deformation, and issuing from the point p_o corresponding to P_o. To the group of internal transformations of the surface $\overline{F}$ there corresponds the discontinuous group of linear transformations at the interior of the unit circle, and it turns out that the invariant metric by this group coincides with the metric of Lobatchewsky's geometry at the interior of the circle $|z| < 1$. By agreeing arbitrarily to call the non-Euclidean distance between two points on the surface $\overline{F}$ the non-Euclidean distance between corresponding points at the interior of the unit circle, Poincaré thus obtained on the surface $\overline{F}$ a 'hyperbolic' metric connected to the topological fact that $\overline{F}$ was representable on the unit circle.

Let ds be the Euclidean length of an arc of the conformal representation on the unit circle, r the distance from this arc to the center of the circle, and $d\sigma$ the hyperbolic (or non-Euclidean) length of the arc, then

$$d\sigma = \frac{ds}{1 - r^2}$$

In the case in which the surface $\overline{F}$ is of the elliptical type and therefore representable on the complex sphere, the spherical distance between two points can be called the distance between corresponding points on the sphere,

$$d\sigma = \frac{ds}{1 + r^2}$$

and this distance is invariant with respect to the group of rotations of the sphere.

The Euclidean length of an arc is evidently unique, since the metric is not defined with respect to special properties of the surface, but the non-Euclidean lengths being numbers attached by convention to an arc, it is perfectly possible to confer several different non-Euclidean metrics to an arc. Consider for example the simply connected surfaces representable on the unit circle. Several different metrics can be conferred to them at the same time; in particular a hyperbolic metric

$$d\sigma = \frac{ds}{1 - r^2}$$

invariant under the group of transformations internal to the circle, and a spherical metric

$$d\sigma = \frac{ds}{1 + r^2} \ ,$$

that receives the unit circle when envisaged as a portion of the sphere. The choice of metric is then controlled by the nature of the invariants that are sought to be obtained on the surface.

We will now see, following the book of Nevanlinna (1936), how such considerations have currently made a name for themselves in a central problem of the theory of analytic functions, that of Picard's exceptional values. Consider a meromorphic function at the interior of any circle of radius r, $r < R < \infty$, with its center the point at the origin o of the complex plane. It is a matter of studying the 'affinity' of the function $w(z)$ for the value a of this function. Nevanlinna introduced two expressions for this, of which one, $N(r,a)$ is related to the number of z values interior to the circle r of center o for which $w(z)$ effectively takes the value a; and the other, $m(r,a)$, measures as it were the average intensity with which the values of $w(z)$ gather around a on the circumference $|z| = r$. There exists then a remarkable swing between the frequency with which the function $w(z)$ reaches any value a and the average of these deviations with respect to a. Nevanlinna has in fact proved the following theorem:

$$m(r,a) + N(r,a) = T(r) + \text{a bounded quantity;}$$

in which $T(r)$ is a constant relative to the circle $|z| = r$.

If in the circle $|z| = r$, $w(z)$ only rarely takes the value a, the function of frequency $N(r,a)$ has a very low value which may be zero in the case in which a is an exceptional Picard value. On the other hand, the function $w(z)$ deviates very little in average from the value a on the circle $|z| = r$, the function $m(r,a)$ is all the greater, and the sum $m(r,a) + N(r,a)$ thus reaches the constant value $T(r)$. This theorem is of great important since it shows that in the study of exceptional values, it does not suffice to seek the values that a function never reaches, which is what was done previously, but it is necessary to also consider how the function approaches them.

Consider for example the exponential function ε^z which allows as exceptional values $a = 0$ and $a = \infty$; then, by calling $O(1)$ a quantity which remains bounded (of the order of 1):

$$N(r,a) = 0, m(r,a) = \frac{r}{\pi} \text{ for } a = 0 \text{ and } a = \infty$$

$$N(r,a) = \frac{r}{\pi} + O(\mathbf{1}), m(r,a) = O(\mathbf{1})$$

for an ordinary value of a, so that in all cases:

$$m(r,a) + N(r,a) = \frac{r}{\pi} + \text{potentially } O(1),$$

$\frac{r}{\pi}$ is the characteristic function $T(r)$ attached to the function e^z. For a fixed value $|z| = r$, $T(r)$ is an invariant and, through the notion of a non-Euclidean metric on a Riemann surface, it has been possible for Ahlfors (1929) and Shimizu (1929) to give a geometric interpretation of this invariant.[4] Consider the complex plane of the variable z and the complex plane of variable w whose points represent the values of the meromorphic function $w(z)$ for $|z| < R \leq \infty$. Let F be the area on the plane w corresponding to the circle $|z| < r$ of the plane z. Let us introduce a spherical metric on the plane w. This metric associates a 'spherical' value to the area F, that of the stereographic projection of F on the unit sphere. Let $A(r)$ be the spherical value (divided by π) of this area F, which on the plane w corresponds to the circle $|z| = r$ of the plane of the complex variable z. $A(r)$ is thus an invariant attached to the circle of radius r, and there is between Nevanlinna's invariant $T(r)$ and the invariant $A(r)$ of the geometric theory, the relations:

$$\frac{dT(r)}{d\log r} = A(r) \text{ or again } \int_0^r \frac{A(t)}{t} dt$$

The invariant of Nevanlinna's analytic theory therefore gets its whole meaning by the introduction on the envisaged Riemann surface of a global metric specially appropriate for the topology of this surface: a spherical metric.

A geometric interpretation of certain other theorems of the theory of meromorphic functions can also be arrived at by introducing this time a Lobatchewsky–Poincaré hyperbolic metric (*Cf.* Nevanlinna 1936, 242 [1970, 248]), and all the theorems of the theory can be retrieved by envisaging, as did Ahlfors, any metric satisfying the general axioms mentioned at the beginning of this chapter. In all cases, what results from these examples, through the definition of the metric on a Riemann surface, is that the theory of analytic functions is very much acquainted with the ideas of the new mathematics, which coordinates in each domain a suitable system of numbers, selected with regard to the structure of this domain, and thereby establishes to a certain extent the primacy of geometric synthesis over 'numerical' analysis.

CHAPTER 3
The Algebra of Non-Commutative Magnitudes: Pfaffian Forms and the Theory of Differential Equations

One aspect of the new mathematics that we propose to examine here is relative to the non-commutativity of multiplication in certain of the most important modern algebraic theories. Let a and b be two magnitudes of a theory. This theory is non-commutative if it has $ab \neq ba$. This is a property that profoundly distinguishes these theories from ordinary arithmetic and algebra and whose richness is considerable, since it contains the key to uncertainty relations in quantum mechanics.

Schrödinger's mechanics associates in effect an abstract expression called an 'operator' to any physical magnitude, and two magnitudes A and B are simultaneously measurable only if the operators corresponding to them are permutable, that is, if the multiplication of these operators is commutative. Now, if the operator Q_k is made to correspond[1] to a coordinate q_k, in which the operator Q_k signifies multiplication by q_k, and the operator

$$P_k = -\frac{h}{2\pi i}\frac{\partial}{\partial q_k}$$

is made to correspond to the quantity of movement according to the coordinate q_k, then it is found that these operators are not permutable:

$$-\frac{h}{2\pi i}\frac{\partial}{\partial q_k}.q_k = -\frac{h}{2\pi i}q_k\frac{\partial}{\partial q_k} - \frac{h}{2\pi i}$$

$$\text{or again: } Q_k P_k - P_k Q_k = \frac{h}{2\pi i}$$

What follows from these equations is that a coordinate and the corresponding quantity of momentum are not simultaneously measurable and the indeterminacy that affects the system of the two measures is itself measured by the value of a difference of the type $ab - ba$.

The operator theory of quantum mechanics, as Von Neumann has shown (1935), is only a special case of the general theory of rings[2] of operators in Hilbert space, and the problems related to the commutativity or non-commutativity of two elements in the study of the structure of these rings play the same role as in the study of rings of algebraic numbers. This structure is in effect characterized by the possibilities of dimensional decomposition of the ring directly into components, analogous to the global decompositions that was described in Chapter 1, and the point of view of the commutativity of multiplication plays a part by the fact that the global decomposition of the ring, in certain cases, is interdependent with a global decomposition of the center of the ring. The latter is formed by the set of elements a that 'commute' with all the others, and the connection between the decomposition of the ring and that of its center is as follows: if the ring O is decomposed directly into sub-rings (having certain specific properties)[3]

$$O = o_1 + \ldots\ldots o_n$$

the center Z of the ring is decomposed directly into the centers of different sub-rings of the decomposition of O

$$Z = Z_1 + \ldots\ldots Z_n$$

It was shown in Chapter 1 how the theories of modern algebra can be characterized by the importance played by the theorem of dimensional decomposition, and the close relation that unites the existence of these decompositions to the distinction between commutative and non-commutative multiplication can now be seen.

In the rest of this chapter we intend to show how, thanks to the work of Cartan and his predecessors, the mode of thought that is essential to modern algebra, which is the calculus of non-commutative magnitudes, has penetrated contemporary analysis. We will first briefly define the calculus of Pfaffian forms and then envisage their application to analysis.

Given two series of differentials[4]

(I) $$dx_1, dx_2 \ldots\ldots dx_n$$

(II) $$\delta x_1, \delta x_2 \ldots\ldots \delta x_n$$

let us say, following Grassmann's notation of exterior products,

$$[dx_i, dx_j] = dx_i \delta x_j - \delta x_i dx_j$$

Now let two Pfaffian forms be:

$$\omega(d) = a_1 dx_1 + a_2 dx_2 + \ldots a_n dx_n$$
$$\varpi(d) = b_1 dx_1 + b_2 dx_2 + \ldots b_n dx_n$$

the exterior product of these two forms, and by $[\omega, \varpi]$ the bilinear expression with respect to the two series of variables (I) and (II) is noted:

$$[\omega, \varpi] = \omega(d)\varpi(\delta) - \omega(\delta)\varpi(d) = \sum a_i b_i [dx_i, dx_j]$$

Such symbolic multiplication is essentially non-commutative and we have:

$$[\omega, \varpi] = -[\varpi, \omega]$$

Cartan also established a 'symbolic derivation' of Pfaffian forms:

Let $\omega(d) = a_1 dx_1 + \ldots a_n dx_n$ be a form to n variables; by defining the symbolic derivative or exterior derivative of $\omega(d)$ by $d\omega$, we have:

(III) $$d\omega = d\omega(\delta) - \delta\omega(d) = \sum_i^n da_j \delta x_j - \delta a_j d_j$$

$$= \sum_{(fk)} \left(\frac{\partial a_k}{\partial x_j} - \frac{\partial a_j}{\partial x_k} \right) [dx_j, dx_k]$$

$$= \sum a_{jk} [dx_j, dx_k]$$

The exterior products $\sum a_j b_j [dx_i, \delta x_j]$ and the exterior derivatives $\sum a_{jk} [dx_i, \delta x_k]$ are differential forms of degree 2. Differential forms of any degree can even be defined by successive multiplications or derivations, and the set of these differential forms, univocally satisfying a law of addition and a law of multiplication, constitutes a ring comparable to operator rings and the rings of algebraic numbers which were spoken of above.

We will now see the immediate connection that Cartan established between the algebra of Pfaffian forms, as just defined, and the analytic theory of differential equations.[5]

Consider the second order differential equation

$$\frac{d^2y}{dx^2} = F\left(x, y, \frac{dy}{dx}\right)$$

Let us pose

$$\omega_1 = dy - y'dx$$
$$\omega_2 = dy' - F(x, y, y')dx$$

Two Pfaffian forms are obtained to 3 variables *dy*, *dy´* and *dx*; and the differential equation is reduced to the following Pfaffian system

$$\omega_1 = 0$$
$$\omega_2 = 0$$

This example allows us to understand how Cartan was able, in general, to reduce the study of systems of differential equations and partial differential equations, to the study of corresponding algebraic-geometric Pfaffian systems. This is a huge domain in which the methods of algebra are again seen to penetrate the domain of analysis. We will try to give an idea of this by first envisaging equivalence problems.

Consider two systems of Pfaffian forms $\omega_1 \dots \omega_n$, $\varpi_1 \dots \varpi_n$, the first constructed with the variables $x_1 \dots x_n$ and their differentials, the second with the variables $\overline{x}_1 \dots \overline{x}_n$ and their differentials. The systems are said to be equivalent if they are transformable into one another by analytical transformations making the variables pass from x_i to $\overline{x}_i = \varphi_i(x_1 \dots x_n)$, the φ_i being analytic functions. This problem is doubly a problem of analysis: first of all by the character of the functions φ_i whose existence is sought to be demonstrated, and secondly because in the applications, the Pfaffian forms envisaged always correspond to the systems of differential equations or partial differential equations. Cartan's methods then allow this problem of analysis to be substituted with a problem of algebra: for the two Pfaffian systems $\omega_1 \dots \omega_n$, $\varpi_1 \dots \varpi_n$ to be equivalent as defined by analysis, it is necessary that in two corresponding points $(x, \overline{x})$ both systems of forms, considered as algebraic forms in dx_i and $\overline{dx}_i$, are transformable into one another by a linear substitution, that is, that we have:

(IV) $$\varpi_i = a_i\omega_i + \ldots a_{in}\omega_n$$

The existence of φ_i functions is connected to the conditions of solvability of the algebraic system (IV). It is even possible to substitute for the forms ω_i and ϖ_i the forms $\omega_i^* = \varpi_i^*$ such that the problem (IV) is reduced to the search for the conditions of compatibility of equations

(V) $$\varpi_i^* = \omega_i^*$$

In conformity with the general theory of Pfaffian systems, it is necessary to add to equations (V) the equations between the exterior derivatives:

$$d\varpi_i^* = d\omega_i^*$$

or again by virtue of (III)

(VI) $$\sum c_{jk}^i[\omega_j^*, \omega_k^*] = \sum \overline{c}_{jk}^i[\varpi_j^*, \varpi_k^*]$$

We see that the coefficients c_{jk}^i must be invariant. Let us now place ourselves in the most important particular case, that in which these invariants are constants. It will be demonstrated that the constancy of these algebraic coefficients is a necessary and sufficient condition for the existence of the sought after analytic transformations. There is still more: the transformations φ_i form a finite continuous group to n parameters and the c_{jk}^i are the algebraic structural constants of this group as defined by Lie and Cartan. The problems relative to the equivalent of differential equations or of differential forms, problems of analysis, are thus resolved by the 3rd fundamental theorem of the algebraic theory of Lie groups: given a system of constants c_{jk}^i satisfying certain determined algebraic relations, there is a finite continuous group of transformations admitting the c_{jk}^i as structural constants and leaving the Pfaffian forms, whose exterior derivatives have these same constants as coefficients, invariant.

In a simple example given by Cartan, it will be shown how the existence of continuous transformations can result from the algebraic conditions of compatibility of two Pfaffian systems: that is to find the conditions of applicability of two surfaces defined by ds^2. The ds^2 of each surface is defined as a quadratic differential form of coordinates x_1, x_2 for the first, $\overline{x}_1, \overline{x}_2$ for the second, the surfaces are applicable to one another with conservation of the metric, if there are specific biunivocal and bicontinuous correspondences between x_i and $\overline{x}_i$. It is possible to find the conditions of existence of these continuous correspondences by considering the Pfaffian forms attached to the ds^2 surfaces.

The calculus of the exterior derivative of one of these forms gives in effect:

$$d\omega_3^* = -K[\omega_1^*, \omega_2^*]$$

The quantity K playing the role of structural coefficients of formula (VI) requires that the surfaces are applicable, that this quantity, formed from the coefficients of the ds^2 of each surface, is the same for both surfaces. This is the total curvature as defined by differential geometry. This theory thus gives, in a purely algebraic way, a classic result of the applications of analysis to geometry.

The role of the non-commutative calculus of Pfaffian forms in analysis is therefore considerable. It allows, in many cases, a substitution of algebra for analysis to be operated as follows: the theory of differential equations reduces to that of Pfaffian forms; which merges with the theory of continuous Lie groups, and the latter has become, thanks to Cartan, an algebraic theory. The fertility of these algebraic methods is even manifested in the problems that have always seemed to be the center of analysis: those that relate to the integration of differential equations or partial differential equations. The infinitesimal transformations which constitute a Lie group can in effect, in some cases,[6] be defined as solutions to a system of partial differential equations giving the transformed variables based on primitive variables. It is however not directly by the integration of these equations to partial derivatives that Cartan ensures the existence of the functions sought after, but by the prior algebraic study of structural coefficients of Pfaffian forms attached to the proposed equations. We encounter here considerations that will be further developed in Chapter 4: to a certain extent, the existence of continuous functions depends on the structural properties of a discontinuous system of algebraic magnitudes. The calculus of non-commutative magnitudes in analysis thus carries out a reconciliation of the continuous and the discontinuous in which the distinction between the two mathematics is erased.

CHAPTER 4

The Continuous and the Discontinuous: Analysis and the Theory of Numbers

We've already seen in the Introduction to this paper how the distinction between Weyl's two mathematics could be interpreted in terms of an opposition between the mathematics of the finite and that of the infinite, or more exactly the countable and the continuous. These are the two essentially distinct mathematics, since operations can be defined in the domain of the continuous that have no meaning in the discontinuous, like analytic continuity in the theory of functions. To establish a theory of relations of the continuous or the discontinuous, of arithmetic and analysis, is then a fundamental problem for mathematical philosophy.

Two different mathematical disciplines can be interrogated in this regard: mathematical logic and number theory, and both cases are found to be faced with a similarly complicated situation.

One of the most important facts in the reconstruction of the foundations of mathematics undertaken by the school of set theorists, Dedekind, Cantor, Frege and Russell, was certainly to support the definition of whole and finite numbers of ordinary arithmetic over the consideration of infinite sets or an infinity of sets. It is thus that the simple addition of whole numbers in Russell is based on the application of the transfinite axiom of choice. While paying tribute to the work of his predecessors, Hilbert judged the route that they were committed to as impractical and showed how the axiomatization of analysis presupposed the axiomatization of arithmetic, without the latter supporting the former. It cannot yet be asserted that Hilbert's position has eliminated all possibility of envisaging the primacy of the infinite over the finite, and this for reasons arising first of all from mathematical logic itself. Cavaillès has moreover extensively studied the meaning of Gödel's results which demonstrated that it was impossible to prove the consistency of arithmetic using only means borrowed from arithmetic. This logical discovery takes on a singular importance when it is approached from the historical development of effective arithmetic. A very large number of arithmetic results could only have been achieved by appealing to very powerful analytic means. An analytic theory of numbers has thus developed since Dirichlet, and whatever hope some arithmeticians have of one day eliminating transcendent proofs from number theory, the acquired fact remains of the close connection between arithmetic and analysis.

The problem of the relations between the continuous and discontinuous is therefore presented under a new aspect. It is no longer a matter of whether or not the existence of an analytic number theory is compatible with the logical priority of the axioms of the discontinuous with respect to the axioms of the continuous, but rather to study, within analytic number theory itself, the mechanism of the connections that are asserted between the continuous and the discontinuous.

Certain modern mathematicians distinguish in this respect two kinds of problems: there are those, such as Fueter, in which the methods of the theory of functions are of use in the solution of purely arithmetic problems, and in which the theory of ideals (that is, higher arithmetic and algebra) is essential for the construction of certain analytic functions (Fueter 1932, 83). There is would be in short a distinction between problems in which analysis is of use to arithmetic and in which arithmetic is of use to analysis. That the two kinds of problems exist is certain, but they are, as we shall see, reciprocal to one another to the extent that it is impossible to study them separately. This seems to be the opinion of other authors, such as Hecke, who, a student of certain functions of considerable importance in arithmetic, considers that advances in analytic number theory now require the deliberate departure from arithmetic to construct these functions (Hecke 1927, 213). It is impossible to proceed here with a study of the whole of analytic number theory, which is among the most difficult theories of all mathematics. We would simply like to show, with a few precise examples, how it can be argued that if certain analytic functions are of use to arithmetic, then the definition of these functions is already based on the arithmetic structure of these fields whose study is contributed to by these functions. The two theories distinguished by Fueter, analytic number theory and what might be called the theory of the arithmetic origins of certain analytic functions, are probably only one because the latter absorbs the former. In this domain, as with most of those discussed above, the unity of algebra and analysis is effected by the role of the structural and finitist characteristics of the algebra in the genesis of the continuous.

Let k be the field of rational numbers. Riemann defined on this field a function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

in which n is any rational number > 0 of k, and in which s is a complex variable whose real part is always > 1. This function also admits a representation in the form of a product:

$$\zeta(s) = \prod \frac{1}{1 - \frac{1}{p^s}}$$

in which p is any prime of k. We can guess by this the close link that unites the function $\zeta(s)$ to the distribution of primes p in the field k. Thus Riemann was able, by means of this function, to construct a function $F(x)$ which gives the number of primes less than an arbitrary positive number.

Dirichlet and Dedekind generalized the function $\zeta(s)$ by defining, for an any field K, a function $\zeta_K(s)$ also able to be represented as a sum and as a product of an infinity of terms. These terms are the ideal norms of the field K. Here's what should be understood by this: what's called the ideal of a field of algebraic numbers is any set of numbers of the field such that:

a) If α is part of the set, then so is $\lambda\alpha$, whatever the integer λ.
b) If β is another element of this set, $\alpha + \beta$ is also a part.
c) There is an integer $\mu \neq 0$ such that for all α of this set, $\mu\alpha$ is an integer.

Let $\mathcal{A}$ be an ideal of an algebraic field K. The norm $N(\mathcal{A})$ of this ideal is a fixed number, attached to $\mathcal{A}$ in the field K in which it is considered.[1]

That being said, for the function $\zeta_K(s)$ of any field we have:

$$\zeta_K(s) = \sum \frac{1}{[N(\mathcal{A})]^s} = \prod \frac{1}{1 - \frac{1}{[N(\mathcal{P})^s]}}$$

the sum being extended to all ideals numbers of the field K different from 0, the product has all the prime ideals of K. It is generally difficult to use these formulas to express the function $\zeta_K(s)$ of any field. An expression like

$$\prod \frac{1}{1 - \frac{1}{N(\mathcal{P})^s}}$$

can in effect be formed from the function

$$\zeta_K(s) = \prod \frac{1}{1 - \frac{1}{\mathcal{P}^s}}$$

relative to the field of rationals only if the laws of decomposition in K of rational primes are known. On the other hand, if we can know the function $\zeta_K(s)$ of a field K, or at least some of its properties, other than by the knowledge of the way that rational primes decompose into prime ideals of K, this function can then be used to study the laws relative to the prime ideals of K. For example, we can prove by means of the function $\zeta_K(s)$ and certain others that are connected to it, that there exist an infinity of prime ideals in any class of ideals[2] of the field K.[3]

This result is characteristic of analytic number theory. An analytic function is seen in effect to be of use in the determination of results relative to arithmetic notions of the discontinuous: numbers and prime ideals. This priority of analysis with respect to arithmetic is, however, only apparent, and we'll see how the arithmetic utilization on the field K of the function $\zeta_K(s)$ is possible only because the determination of this function already implies the knowledge of certain structural and discontinuous properties of the base field. For example, the theorem mentioned above and relative to the distribution of prime ideals according to the classes of ideals of K, is based, as has been said, on a prior knowledge of certain properties of $\zeta_K(s)$, which are those that result from the formula:

$$\lim_{s=1}(s-1)\zeta_K(s) = h\chi$$

in which h is the number of classes of the field K and χ is another invariant also attached to the field K. What in our view is essential, in the previous equality, is that the determination of the function $\zeta_K(s)$ is thus based on the prior arithmetic decomposition of the ideals of K in h ideal classes.

This is no longer, strictly speaking, analytic number theory, but rather a fact of the arithmetic theory of origins of certain analytic functions. In sum the following schema is obtained: the decomposition into classes of field K permits the conclusion that the function $\zeta_K(s)$ exists, the knowledge of which can, by a rebound effect on the base field, be of use in a more profound study of this field.

The previous example again shows only a global relation between the number of classes of a field and the analytic functions $\zeta_K(s)$. It is possible, in certain cases, to associate the genesis of certain analytic functions more closely to the discontinuous domain that results from the decomposition of a set into classes of equivalent elements. It may then be that these functions are later likely to be of use in exploring the more hidden arithmetic properties of this field on which they depend, but their very existence as analytic functions connected to a domain of discontinuity is of considerable mathematical and philosophical interest. Considering that Weyl described the primacy of a domain with respect to algebraic entities defined on this domain as a characteristic fact of the new mathematics, we see a dependency with respect to a basic domain appear by analogy with the *Abhängigkeit vom Grundkörper* (Dependency on the base-field) of algebra, even within the theory of functions, in which the priority of the idea of geometric domain with respect to that of number or function is equally asserted in analysis.

Let us further study in this regard the case of automorphic functions. Consider in the plane of the complex variable z the substitutions defined by relations:

$$z' = \frac{\alpha z + \beta}{\gamma z + \delta}$$

$\alpha, \beta, \gamma, \delta$ being whole numbers such that $\alpha\delta - \beta\gamma = 1$

These substitutions are demonstrated to form a discontinuous group G, what's called the nodular group, and any function $f(z)$ such that:

$$f(z) = f\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right), \alpha\delta - \beta\gamma = 1$$

is called an automorphic function with respect to this group.

To better see the connection between the discontinuity of the group and the existence of a continuous automorphic function, we will define what is called the domain of discontinuity of the group.

The group operates a division into classes of points in the complex plane, and two points belong to the same class, when a substitution S of the group exists for which we have $z' = Sz$. The transferable set of points by a substitution of the group obviously forms a discontinuous set. It is difficult to demonstrate such a set, but the discontinuity of the group can nevertheless be materialized as follows: to do this let us envisage a domain such that every point in the complex plane is equivalent to one and one only point of this domain. A similar domain which contains only points that are non-equivalent attests to the discontinuity of the modular group since any transformation of the group concerned with a point of the domain of discontinuity transforms this point into a point that does not belong to this domain.

The figure below thus represents a domain of discontinuity or fundamental domain of the modular group. It is located outside of the unit circle and is limited by the straight lines (Figure 1):

$$z = \pm\frac{1}{2} + i\gamma$$

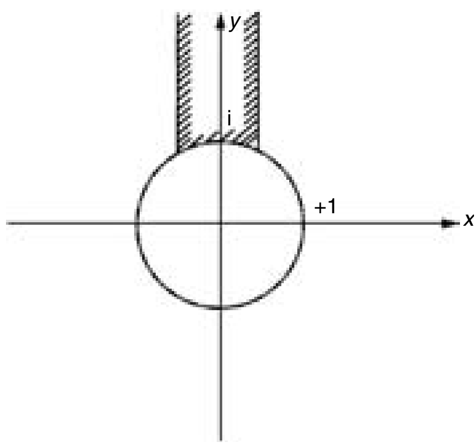

Figure 1

This domain of discontinuity permits the immediate definition of an automorphic function connected to this modular group. There exists in effect a function $w = J(z)$ (the modular function) that assures the conformal representation of the domain of discontinuity of the modular group on the totality of the complex plane w with two cuts $-\infty \to 0$ and $1 \to +\infty$, and this modular function is an automorphic function of the group G. We have in effect $J(z) = J(Sz)$, S being a substitution of G.

The modular function $J(z)$ whose determination is thus based on the discontinuity of a basic domain, is, by what we've called a kind of rebound effect, amenable to very rich applications in number theory. In effect, let K be an imaginary quadratic field (obtained by adjoining a number $\sqrt{m}, m < 0$, to the field of rationals). Each class of ideals of this field is defined by a number of the form

$$\theta = \frac{c + \sqrt{m}}{a} \quad [a, c, m \text{ are integers and rational, in addition } m < 0 \text{ and } c^2 - m \equiv 0].$$

For two classes of ideals to be equal, it is necessary and sufficient that the numbers θ and θ' which correspond to them are equivalent to each other with respect to the modular group G. There are therefore as many numbers $\theta_1 \ldots \theta_h$ of the field k' in the domain of discontinuity of G as there are classes of ideals in the field. Let h be the number of classes. The function $J(z)$, for the h values $\theta_1 \ldots \theta_h$, takes h different 'singular' values $J(\theta_1) \ldots J(\theta_h)$, and the algebraic interest of these values is considerable. They belong in effect to an extension field k' of major importance: the field of classes (*Cf.* Lautman 1938b, ch. 3).

We therefore obtain a schema of relations of the continuous and the discontinuous analogous to the one described above. The continuous function $J(z)$ is, without doubt, of use in determining a discontinuous set of numbers of a new algebraic field,[4] but the very existence of this function is already based on a fundamental discontinuity: that of the domain of discontinuity of the modular group. In this pure example of analytic number theory, we can therefore observe how the essential moment of the theory resides in the genesis of a continuous function from the discontinuous structure that gives rise to it.

Conclusion

The introduction to this essay showed how the distinction between Weyl's two mathematics tended to oppose the synthetic methods of modern algebra to the analysis of the infinite. Then, over the course of four chapters, it was shown that this distinction should not be conceived in the sense of an essential opposition between irreducible disciplines, since it is possible, in modern theories of analysis, to retrieve the points of view that characterize algebra. The theory of equations with infinitely many variables, the arithmetic theory of algebraic functions, the theory of non-Euclidean metrics that are invariant by a discontinuous group, the calculus of Pfaffian forms and continuous group theory and analytic number theory, these are many of the intermediate theories between algebra and analysis whose methods are algebraic and whose results apply to analysis. It is therefore legitimate to oppose the mathematics of the twentieth century to that of the nineteenth century, but they are opposed like the physics of the continuous and the discontinuous. Until about 1905, the coexistence of two distinct areas of facts could have been believed in. That which presented itself as continuous, light waves; and that, on the contrary, in which a discontinuous structure appeared, material atoms. Since Einstein introduced discontinuity into the study of light, and de Broglie the continuity of waves into the study of matter, it is impossible to maintain the old idea of domains of physical facts that are separate from one another. The physics of the continuous represents a mode of treatment by differential equations of physical facts. The physics of discontinuity represents a mode of treatment of the same facts by other methods: group theory, calculation of matrices, quantum statistics, etc. There thus exists a certain analogy between contemporary physics and contemporary mathematics, in that they offer each other the spectacle of facts amenable to being studied at the same time by the calculus of the continuous and by the calculus of the discontinuous. But considering that this duality of methods relative to the same facts is the source of the main problems of contemporary physics, it is contrary to the testimony of the profound unity of the mathematical sciences.

The continuous and discontinuous in quantum physics, are, in effect, complementary points of view in Bohr's sense, that is, our knowledge of continuous aspects of matter increases to the extent that those aspects of the discrete decrease, and conversely. They are, writes de Broglie,

> like the faces of an object that cannot be contemplated at the same time, and that nonetheless requires each in turn to be envisaged in order to describe the object completely. These two aspects, which Bohr called 'complementary aspects,' understanding by this that these aspects, on the one hand, contradict each other and, on the other hand, complement each other. (De Broglie 1937b, 242)

The relations of the continuous and the discontinuous, and of the finite and the infinite, are very different in mathematics. We do not merely consider special problems, like those of analytic number theory, which was discussed in Chapter 4, but fully intend to envisage here the general problem of the relation between analysis and algebra. There exist, in this respect, two classical positions. For one, the continuous emanates from the discontinuous like the infinite from the finite, by a kind of progressive enrichment of the finite and the discontinuous. What then happens is that, in the conditions where the passage to the limit is legitimate, new properties, which are connected to the continuous and the infinite, and which have no equivalent in the finite, are suddenly discovered. Consider for example a polynomial in z (z being a complex variable) that contains a finite number of terms. A convergent series with an infinity of terms, in certain cases, represents an analytic function, and, as noted by Montel (1927), the transition from a finite to an infinite number of terms introduces the new fact of the existence of a singular point without which the analytic function is reduced to a constant. Analysis thus presented goes beyond algebra and its results have long seemed to analysts incapable of being retrieved by simple methods.

The priority of the continuous and the infinite can also be asserted, and the finite and the discontinuous seen either as a limit of infinity, or as an approximation of infinity. This attitude is perhaps more philosophical than mathematical: the Cartesian infinite is first with respect to the finite, like continuity in Bergson is first with respect to the discontinuous. It has in addition been shown how the infinite is retrieved especially in those disciplines of mathematics that are most in contact with philosophical thinking: Cantor's set theory and the mathematical logic of Frege and Russell. It is no less true that an image of such relations can be retrieved in the authentically mathematical theorems of approximation. We were reminded, for example, in Chapter 1 of our principal thesis (Laumtan 1938b), how a continuous function can be approached by polynomials; the continuous function in this theory is therefore correctly conceived as given anterior to the infinite discontinuous polynomials that it approaches.

We think it possible to observe in the movement of the mathematics of the twentieth century a third way of conceiving the relations between analysis and algebra, the continuous and the discontinuous, the infinite and the finite. By seeking to give rise to the infinite by the dilation of the finite, or the finite by the constriction of the infinite, the finite is always considered to be in extension as a part of the infinite, and the impossibility of exhausting the infinite can only incessantly be come up against. A very different attitude is held by many contemporary mathematicians,[1] who see in the finite and the infinite not the two extreme terms of a passage to be carried out, but two distinct kinds of entities, each endowed with a specific structure, and disposed to support relations of imitation or expression between them. We understand by relations of imitation the cases where the internal structure of the infinite mimics that of the finite, and by relations of expression, the cases where the structure of a discontinuous finite domain envelops the existence of another continuous or infinite domain, which thus finds itself expressing the existence of this finite domain to which it has adapted. Some of the examples have shed light on relations of imitation; others on relations of expression. When Hilbert transports the dimensional methods of decomposition of algebraic origin into analysis, he imposes a structure on the functional space which mimics that of a space that has a finite number of dimensions. When Poincaré envisages a discontinuous group of transformations and the continuous automorphic function attached to this group, he brings together two kinds of entities that are entirely foreign in nature, but the existence of the continuous function expresses no less the properties of the domain of discontinuity used to define it. By thus focusing each time, not on the quantity of the elements, but on the existence or framework of the entities being compared, structural analogies and reciprocal adaptations are thus discovered between the finite and the infinite, with the result that the unity of mathematics is essentially that of the logical schemas that govern the organization of their edifices. We thus rediscover considerations that agree with those that are developed our principal thesis (1938b). Our *Essay on the notions of structure and existence in mathematics* (1938b) tends in effect to show that it is possible to retrieve in mathematical theories, logical Ideas incarnated in the very movement of these theories. The structural analogies and adaptations of existence between analysis and algebra, that we have tried to describe here, have no other aim than to contribute to shedding light on the existence of logical schemas at the heart of mathematics, which are only knownable from within mathematics itself, and to secure both its intellectual unity and spiritual interest.

BOOK II

Essay on the Notions of Structure and Existence in Mathematics

Introduction: On the Nature of the Real in Mathematics

This book arises from the sentiment that in the development of mathematics, a reality is asserted that mathematical philosophy has as a function to recognize and describe. The spectacle of most modern theories of mathematical philosophy, in this regard, is extremely discouraging. In most cases, mathematical analysis reveals only very little and that very poorly, like the search for identity or the tautological character of propositions.[1] It is true that in Meyerson the application of rational identity to a variety of mathematics presupposes a reality that resists identification. There seems therefore to be an indication here that the nature of this real is different from the too simplistic schema that is used to try to describe it. On the contrary, the development of the notion of tautology in Russell's school completely eliminated the idea of a reality specific to mathematics. For Wittgenstein and Carnap, mathematics is no more than a language that is indifferent to the content that it expresses. Only empirical propositions refer to an objective reality, and mathematics is only a system of formal transformations allowing the data of physics to connect to each other. If one tries to understand the reasons for this progressive disappearance of mathematical reality, one may be led to conclude that it results from the use of the deductive method. By trying to construct all mathematical notions from a small number of notions and primitives logical propositions, we lose sight of the qualitative and integral character of the constituted theories. Now, what mathematics leaves for the philosopher to hope for is a truth which would appear in the harmony of its edifices, and in this domain as in all others, the search for the primitive notions must yield place to a synthetic study of the whole. It seems to us in this regard very strange that after having lead the most complete investigations on theories related to number and space, Poincaré had claimed to see in mathematics only a game of symbols devoid of meaning (*Cf.* Poirier 1932). He seems to have approached them with the intention of asking of them an enrichment of the indications that suggest the external perception or inner sense of the real. The real is foremost to him that of immediate experience, and abstract theories give us no hold over it. Poirier almost reproaches these theories for their excessive perfection. The ease with which they correspond to one another gives the aspect of each of them an arbitrary character, possible among many others. Nowhere is there impressed upon the mind the sentiment of a necessity resulting from the nature of things, and one never finds only formal procedures, which do not respond to a 'natural and intuitive classification' of their objects.

We believe it is possible to arrive at less negatives conclusions, and contemporary mathematical philosophy has moreover committed, on two different routes, to a positive study of mathematical reality. This reality can in effect be characterized by the way in which it can be grasped and organized, which it can equally be intrinsically, in terms of its own structure. We will first try to briefly summarize the basic ideas of the two methods.

There is no philosopher today more than Brunschvicg who has developed the idea that the objectivity of mathematics is the work of intelligence, in its effort to overcome the resistance that is opposed to it by the matter on which it works. This matter is neither simple nor uniform, it has its folds, its edges, its irregularities, and our conceptions are never more than a provisional arrangement that allows the mind to go further forward. Mathematics is constituted like physics: the facts to be explained were throughout history the paradoxes that the progress of reflection rendered intelligible by a constant renewal of the meaning of essential notions. Irrational numbers, the infinitely small, continuous functions without derivatives, the transcendence of e and of π, the transfinite had all been accepted by an incomprehensible necessity of fact before there was a deductive theory of them. They had the fate of these physical constants like c or h which were essential in an incomprehensible way in the most different domains, up until the genius of Maxwell, Planck and Einstein knew to see in the constancy of their value the connection of electricity and light, of light and energy. We thus understand the defiance of Brunschvicg vis-a-vis any attempts that would deduce the whole of mathematics from a small number of initial principles. Brunschvicg, in *Les étapes de la philosophie mathematique,* rose up against the reduction of mathematics to logic, just as much as against the idea that there might be general mathematical principles like Poncelet's principle of continuity or Hankel's principle of the permanence of formal laws. Any effort of *a priori* deduction tends for him to reverse the natural order of the mind in mathematical discovery. Brunschvicg's mathematical philosophy should not, however, be interpreted as a pure psychology of creative invention:

> Between the vicissitudes of invention that are solely of interest to an individual consciousness, and the forms of discourse that concern above all the pedagogical tradition, mathematical philosophy delimits the terrain where the collective acquisition of knowledge is produced, it will recognize the pathway traced by creative intelligence. (Brunschvicg 1912, 459)

Between the psychology of the mathematician and logical deduction, there must be room for an intrinsic characterization of the real. It must partake both of the movement of the intelligence and of logical rigor, without being mistaken for either one, and this will be our task, to attempt this synthesis.

The structural point of view to which it is thus also our duty to refer is that of Hilbert's metamathematics. We know the difference that separates the Hilbertian conception of mathematics from that of Russell and Whitehead's *Principia Mathematica* (1910). Hilbert has replaced the method of genetic definitions with that of axiomatic definitions, and far from claiming to reconstruct the whole of mathematics from logic, introduced on the contrary, by passing from logic to arithmetic and from arithmetic to analysis, new variables and new axioms which extend each time the domain of consequences. Here is, for example, according to Bernays (see Bernays 1935, 196–216), who in the complete works of Hilbert published a study of all his work on the foundations of mathematics, all that is necessary to be given to formalize arithmetic: the propositional calculus, the axioms of equality, the arithmetic axioms of the 'successor' function $(a+1)$, the recurrence equations for addition and multiplication, and finally some form of the axiom of choice. To formalize analysis, it is necessary to be able to apply the axiom of choice, not only to numeric variables, but to a higher category of variables, those in which the variables are functions of numbers. Mathematics thus presents itself as successive syntheses in which each step is irreducible to the previous step. Moreover, and this is crucial, a theory thus formalized is itself incapable of providing the proof of its internal coherence. It must be overlaid with a metamathematics that takes the formalized mathematics as an object and studies it from the dual point of view of consistency and completion (see Chapter 4). The duality of planes that Hilbert thus established between the formalized mathematics and the metamathematical study of this formalism has as a consequence that the notions of consistency and completion govern a formalism from the interior of which they are not figured as notions defined in this formalism. To express this dominant role of metamathematical notions with respect to formalized mathematics, Hilbert writes:

> The axioms and provable theorems (i.e. the formulas that arise in this alternating game [namely formal deduction and the adjunction of new axioms]) are images of the thoughts that make up the usual procedure of traditional mathematics; but they are not themselves the truths in the absolute sense. Rather, the absolute truths are the insights (*Einsichten*) that my proof theory furnishes into the provability and the consistency of these formal systems. (Hilbert 1923; 1936, 180 [1996, 1138])

The mathematical theory thus receives its value from the metamathematical properties that its structure incarnates.

The structural conception and the dynamic conception of mathematics seem at first to be opposed: one tends in effect to consider a mathematical theory as a completed whole, independent of time; the other, on the contrary, does not separate it from the temporal stages of its elaboration. For the former, the theories are like entities qualitatively distinct from one another, whereas the latter sees in each an infinite power of expansion beyond its limits and connection with the others, by which the unity of the intellect is asserted. In the pages that follow, we would however like to try to develop a conception of mathematical reality which combines the fixity of logical notions and the movement with which the theories live. In Hilbert's metamathematics, we propose to examine mathematical theories from the point of view of the logical notions of consistency and completion, but this is only an ideal toward which the research is oriented, and it is known at what point this ideal currently appears difficult to attain (*Cf.* Cavaillès 1938). Metamathematics can thus envisage the idea of certain perfect structures, possibly realizable by effective mathematical theories, and this independently of the fact of knowing whether theories making use of the properties in question exist, but then only the statement of a logical problem is possessed without any mathematical means to resolve it.

This distinction between the position of a logical problem and its mathematical solution has sometimes seemed not very fertile, because what matters is not knowing that a theory could be non-contradictory, but rather being able to effectively decide whether or not it is. It seems to us nevertheless possible to envisage other logical notions, equally likely to be potentially linked to one another within a mathematical theory, and which are such that, contrary to previous cases, the mathematical solutions to the problems they pose can entail an infinity of degrees. Partial results, comparisons stopped midway, attempts that still resemble groupings, are organized under the unity of the same theme, and in their movement allow a connection to be seen which takes shape between certain abstract ideas, that we propose to call dialectical. Mathematics, and above all modern mathematics, algebra, group theory and topology,[2] have thus appeared to us to tell, in addition to the constructions in which the mathematician is interested, another more hidden story made for the philosopher. A dialectical action is always at play in the background and it is towards its clarification that the following six chapters tend.

The first three deal more specifically with the notions of mathematical structure. In Chapter 1 ('The Local and the Global'), we study the almost organic solidarity that pushes the parts to organize themselves into a whole and the whole to be reflected in them. Then, in Chapter 2 ('Intrinsic Properties and Induced Properties'), we examine if it is possible to reduce the relations that a mathematical entity maintains with the ambient milieu to properties of inherence characteristic of that entity. In Chapter 3 ('The Ascent towards the Absolute), we show how the structure of an imperfect entity can sometimes preform the existence of a perfect entity in which all imperfection has disappeared. Then come three chapters on the notion of existence. In Chapter 4 ('Essence and Existence'), we try to develop a new theory of the relations of essence and existence which shows the structure of an entity interpreted in terms of the existence of entities other than the entity whose structure is being studied. Chapter 5 ('Mixes') describes certain mixed intermediaries between different kinds of entities, whose consideration is often necessary to effect the passage from one kind of entity to another kind of entity. The last chapter ('On the Exceptional Character of Existence') describes finally the processes by which an entity can be distinguished from an infinity of others.

We claim to show that the ideas which are inscribed at the head of each of the chapters and which seem to dominate the movement of certain mathematical theories, though conceivable independently of mathematics, are nevertheless not amenable to direct study. They exist only with respect to a matter that they penetrate with intelligence, but it can be said, on the other hand, that it is they who confer on mathematics its eminent philosophical value. The method that we follow is essentially a method of descriptive analysis, mathematical theories constitute for us a given within which we try to identify the ideal reality with which this matter is involved.

SECTION 1

The Schemas of Structure

CHAPTER 1

The Local and the Global

One of the characteristic traits of the development of mathematics since the mid-nineteenth century is that the most diverse mathematical research has been able to be pursued from a dual point of view, the local point of view and the global point of view. The local study is directed towards the element, most often infinitesimal, of reality, which it seeks to determine in its specificity. Then, following its course step by step, gradually establishing strong enough connections between its different parts thereby recognized, so that an idea of the whole emerges from their juxtaposition. The global study seeks instead to characterize a totality independently of the elements that compose it. It immediately tackles the structure of the whole, thus assigning elements their place before even knowing their nature. It tends mainly to define mathematical entities by their functional properties, arguing that the role they play confers on them a much more assured unity than that resulting from the assemblage of parts.

The duality of the local point of view and the global point of view was first presented to mathematicians as an opposition between two modes of study, irreducible to one another. It seemed necessary to choose between these two incompatible conceptions, and in fact, the division that was thus set up in mathematics still remains today in many domains. We would like to briefly show what is in the theory of analytic functions, in geometry and in the theory of differential equations.

The conception of the analytic function according to the ideas of Cauchy and Riemann is a global, or at least regional, conception. It is based in effect on the consideration of a whole domain of the plane of the complex variable $z = x + iy$. A complex expression $\zeta = u + iv$, for Cauchy, represents an analytic function over the whole of this domain, if, at each point of the domain, the existence of a unique derivative of ζ in relation to the complex variable z can be defined. We know that for such a derivative to exist, it is necessary that the functions u and v are continuous functions of x and y possessing continuous first order partial derivatives and satisfying the differential equations (of Riemann):

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}.$$

The analytic function thus defined by the unicity of the derivative at each point is not yet a defined notion of the global point of view, but it leads to the theory of the integral, which is a global notion of the highest degree. The value of an analytic function at an interior point z of a domain limited by a closed curve C is determined by the value of the function on the boundary curve:

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0} dz,$$

if the domain in question is simply connected, that is, only has as its boundary the sole closed curve C. The structural properties of simple connection relative to the topology of the basic domain are thus seen to play a part in this definition. We will discuss in a later chapter the importance that these topological considerations have taken with Riemann and how they have allowed linking the existence of analytic functions to the existence of basic domains, defined in their totality by their topological properties. The conditions relative to the existence of the derivative at each point no longer plays the primary role, and the function is no longer so much defined by its properties at each point of the domain as because it is appropriate to the entire domain.

The global conception of Riemann is opposed to the local conception of Weierstrass. An analytic function is essentially defined for Weierstrass in the neighborhood of a complex point z_0, by a power series with numerical coefficients, which converges in a 'circle of convergence' around the point z_0. The method of 'analytic continuity' in turn helps construct step by step a whole domain in which the function is called 'analytic' and this is done in the following manner: take as a new center a point inside the first circle, both a new series and a new circle of convergence is thus obtained that can extend beyond the first. The new series extends the first if their values coincide in the common part of the two circles. The series can thus be extended in all directions up to the points in the immediate neighborhood in which the series obtained diverge. Thus we see that in this method the domain is not circumscribed in advance, but rather results from the infinite succession of local operations.

Weierstrass's theory is developed in deliberate opposition to the integral conception of Cauchy and Riemann. If some authors today, like Bieberbach, give a presentation of the whole of the theory of analytic functions in which the two points of view are intimately intertwined, others on the contrary, like Goursat or Courant (*Cf.* Courant 1925), argue that it is necessary to maintain a separation between the conceptions of Cauchy-Riemann and those of Weierstrass. We shall have an opportunity later to see the importance that this separation of points of view also has for Hermann Weyl.[1]

As concerns geometry and group theory, we can do no better than to be inspired by the famous articles published by Cartan on the question.[2] At the forefront of his general exposés on geometry, he always puts the profound differences that, up until the theory of relativity, separated the global conception of space as defined by Felix Klein in his famous 'Erlanger program' of 1872 (Klein 1893) and Riemann's infinitesimal conception developed in the essay of 1854, *On the hypotheses which lie at the bases of geometry* (Riemann 1868). A geometry by Klein's definition is the study of the properties of figures that are conserved when space taken as a whole is subjected to certain transformations, forming what is called a transformation group. Euclidean geometry is thus the study of properties of figures that are conserved when all the points of the space are subject to the same displacement. It is found that these displacements conserve all the properties of the figures and in particular their metric properties. Affine geometry studies the properties that are conserved by a linear transformation: the parallelism of two straight lines, for example. Projective geometry studies the invariant properties in relation to a homographic transformation like the anharmonic ratio of four points on a straight line or the degree of an algebraic curve. Whatever the properties with which the invariance in relation to a group of transformations is researched, the essential characteristic of Klein spaces is their homogeneity. The group operates in the same way on all points of space. Riemann spaces are on the contrary devoid of any kind of homogeneity. Each one is characterized by the form of the expression which defines the square of the distance between two infinitely near points. This expression is called a quadratic differential form which generalizes the Euclidean formula of the distance between two points: $ds^2 = du_1{}^2 + du_2{}^2$. The Riemannian ds^2 to two dimensions is of the following form $ds^2 = g_{11}du_1{}^2 + g_{12}du_1du_2 + g_{21}du_2du_1 + g_{22}du_2{}^2$. In an n-dimensional manifold we have the general formula:

$$ds^2 = \sum_{i,j}^{n} g_{ij} du_i du_j$$

The g_{ij} are the absolutely arbitrary coefficients, which vary from point to point. The result, as Cartan said, is that 'two neighboring observers can locate the points in a Riemann space that are in their immediate neighborhood, but they cannot, without new convention, be located with respect to one another' (Cartan 1924, 297). Each neighborhood is therefore like a small bit of Euclidean space, but the connection from one neighborhood to the next neighborhood is not defined and can be done in an infinity of ways. The most general Riemann space is thus presented as an amorphous collection of juxtaposed pieces that aren't attached to one another. The distinction that thus exists between a Klein geometry and a Riemann geometry is found between the special theory of relativity and the general theory of relativity. Special relativity is of the Kleinian type, it studies, in the Minkowski four-dimensional universe, the invariants of the Lorentz group. General relativity is a Riemannian geometry in which the g_{ij} depend at each point of the distribution on the matter at that point. The space of the general theory of relativity however does not present this complete absence of organization that characterizes the most general Riemann spaces. A physics in which the laws of the universe would vary from point to point is in effect inconceivable. Einstein's Riemannian space has what Cartan calls a Euclidean connection, that is, it is possible to locate step by step the different positions of an observer from each other. We discuss in the next chapter the philosophical problems that are related to this Euclidean connection of Riemann spaces. If the purely local point of view is exceeded, no knowledge of the universe as a whole is even obtained. The gap between the local point of view and the global point of view still remains, and, for Cartan, it is from this disparity that the principal difficulties of the unified field theory arise, as presented by Einstein in 1929. The metric of the universe gives rise to a system of partial differential equations for which Einstein sought solutions without singularity existing in all of space. This would require the knowledge of the topological properties of space–time taken in its totality, like knowing for example whether it is open or closed. 'This shows,' Cartan said, 'that the search for local laws of physics cannot be dissociated from the cosmogonical problem. It cannot in addition be said that the one precedes the other. They are inextricably mingled with one another' (Cartan 1931, 18). Global integration is not an extension of local integration. The solution of the local problem requires prior knowledge of the structure of the universe.

Departing from an opposition of points of view which seemed proper to geometry, the same conflict is found in problems of considerable philosophical importance, because the interpretation of the determinism of physics depends on their solution. They are problems that are related to the conditions of the existence of solutions of differential equations or partial differential equations.

The analysts of the nineteenth century were able in most cases to establish the theorems of existence that allowed the assertion of the existence and possibly the uniqueness of the solution to a differential equation or a partial differential equation, defined in the whole domain in which a certain inequality holds, and this by relying solely on the knowledge of local data, in a point of origin for example.

It is thus that a second order differential equation[3] of the form:

$$\frac{d^2y}{dx^2} = f\left(x, y\frac{dy}{dx}\right)$$

admits in general one and only one solution corresponding to the given initial conditions, namely that for $x = a$, y takes a given numerical value b and

$$\frac{dy}{dx}$$

takes the value b'. The solution to such a problem is thus determined by the local conditions, according to Cauchy. Kovalevsky (1875) established a theorem for second order partial differential equations analogous to Cauchy's theorem for differential equations: If the equation of partial derivatives

$$F\left(x, x_1, \ldots x_n, u, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_k}\right) = 0 \quad \begin{matrix} i,k = 0,1,2\ldots n \\ x_0 = x \end{matrix}$$

can be resolved with respect to $\frac{\partial^2 u}{\partial x^2}$ so that we have

$$\frac{\partial^2 u}{\partial x^2} = f\left(x, x_1, \ldots x_n, u, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_k}\right), i, k = 1 \ldots n$$

function f being holomorphic[4] with respect to x, x_1, . . . x_n, u, and to all the other derivatives, this equation admits one and only one solution, holomorphic in x, x_1, . . . x_n, and satisfying, for $x = 0$, the conditions

$$u = g(x_1 \ldots x_n), \frac{\partial u}{\partial x} = h(x_1 \ldots x_n)$$

functions g and h being holomorphic in x_1 . . . x_n. If the set of points $x = 0$ are then envisaged as determining a plane or a surface of n-dimensional space (n being the number of independent variables in the equation), the Kovalevsky theorem can be interpreted[5] in terms of classical determinism. Knowing the value of a function and one of its derivatives at all points of a surface S allows the assertion of the existence and the regularity of this function in a certain neighborhood of the surface S.

While the analysis thus establishes the local theorems of existence, the direct study of certain physical phenomena led to the consideration of very different problems These problems are all the same type as the famous Dirichlet problem, in which one is led to prove the existence of a function at the interior of a volume V satisfying a certain partial differential equation,

$$\text{(the Laplace equation: } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0\text{)}$$

and, at the boundary of the domain, taking the values given in advance. Similar functions are encountered in the study of electric or calorific equilibrium, when it is a matter of, for example, determining the temperature which will eventually be established within a domain, on the boundary of which were distributed a continuous succession of temperatures invariable over the course of time. The physical fact that an equilibrium temperature ends up effectively being established gave mathematicians the certainty of the existence of the sought after function even before they had a rigorous proof of it.

If one reflects upon the nature of the conditions at the limit of the Dirichlet problem, one realizes, remarks Hadamard, that, between them and the initial data of the Cauchy-Kovalevsky theorems, there is a profound contradiction. Since knowledge of the value of the function at each point of the surface S completely determines this function in the case of the

Dirichlet problem, 'it appears obvious that one has no right to give the length of *S* the value of *u* and that of one of its derivatives' as one does in the general statement of the theorem. The apparent contradiction from the mathematical point of view is avoidable, on the one hand, because the initial data of the general theorem of existence is subjected to the rigorous conditions of analyticity, while the data 'at the limits' is of a much more general nature, and, on the other hand, because the solution of which Kovalevsky's theorem asserts the existence is only defined for a more or less immediate neighborhood of the surface *S*, while in the case of a problem 'à la Dirichlet' the desired solution must be defined and regular in the whole volume V whose surface *S* is the boundary. It is no less true that the theory of partial differential equations governs the clearly different physical processes. Some, like the propagation of light, are in free evolution, and the deterministic schema can be perfectly applied to them. Others, on the contrary, are circumscribed by the data at the limits. Not only must the initial data be known in advance, but also the extreme limits between which the phenomena studied can oscillate. If the function sought after is not a constant, it is in effect shown to be on the boundary of the field, and that, for the values given in advance, in the case of a Dirichlet problem, it attains its maximum and minimum. It seems then that for a physical phenomenon whose evolution is 'directed' and that precise limits enclose all parts, it is necessary to call upon philosophical interpretations, unexpected in mathematics, in which the physical system is comparable to an organic unity. We will find in addition considerations of this kind in Chapter 4, and see again how the search for maximum and minimum could sometimes suggest the idea of a finality inherent in certain mathematical theories and in certain physical phenomena.

The study of these questions, which in addition emerge more from the philosophy of physics, will not be discussed here, we will confine ourselves to the purely mathematical aspect of the duality of the local study and the global study.

The observation of this duality naturally suggests to mathematicians the search for a synthesis. Given that the elements composed by some process of progressive development cannot give rise to any entity amenable to global characteristics, it is necessary, to be sure to reach a result, that the topological structure of the whole is reflected in the properties of its parts. This can give rise to two kinds of problems: either we start from the set whose structure is known and we search for the conditions that must be satisfied for the elements to be elements of this set; or we provide elements having certain properties and try to read in these local properties the structure of the set in which these elements are able to be arranged.[6] In both cases we try to establish a connection between the structure of the whole and the properties of the parts by which the organizing influence of the whole to which they belong is manifested in the parts.

Considerations are thus met in mathematics that may at first sight seem foreign to mathematics and brought there as reflections of certain specific conceptions in biology or sociology. It is evident that the mathematical entity as we understand it is not without analogy to a living being. We however believe that the idea of the organizing action of a structure on the elements of a set is entirely intelligible in mathematics, even if transported to other domains it loses its rational clarity. The obstructions that the philosopher sometimes meets with in regards to arrangements that are too harmonious comes not so much from the subordination of the parts to the idea of a whole which organizes them, than from the manner in which this organization of the whole is carried out sometimes as a naive anthropomorphism and sometimes as a mysterious obscurity. Biology like sociology often lack in effect the logical tools necessary to constitute a theory of the solidarity of the whole and its parts: we shall see on the contrary that mathematics can render to philosophy the eminent service of offering the example of interior harmonies whose mechanism satisfies the most rigorous logical requirements.

Three theories will briefly be reviewed that will provide us as many models in which this implication of the whole in the part is realized: differential geometry in its relation to topology; group theory; and the theory of the approximate representation of functions. These three examples seem to us to be particularly suggestive because they allow the same conclusion to be arrived at regarding the conditions that must be fulfilled by the structure of a mathematical entity so that within this entity reigns like an organic solidarity.

1. DIFFERENTIAL GEOMETRY AND TOPOLOGY

The study of the relations between topology, an eminently synthetic study of geometrical objects,[7] and differential geometry has given rise to a large amount of methodically pursued research under the leadership of W. Blaschke in Hamburg. What is proposed here is a brief analysis of a paper by Hopf (1932) which contains the essential ideas that dominate the question. Posing the general problem of knowing what the connections are that can exist between the topological properties and the differential properties of a surface, Hopf is led to distinguish two reciprocal problems whose interest has been pointed out above: a problem of metrization and a problem of extension. Their statement is of such logical interest that we cannot but reproduce the very text of the author:

> The problem of metrization is as follows: given a topological surface *F*, it is a matter of determining on this surface a differential metric (a ds^2). . . . What are the conditions to be satisfied by the metric of the surface? What metric properties are prescribed in advance by the topology of *F*? By what limitations is the arbitrariness with which I can fix the g_{ik} to the place where I begin to fix the metric of the surface restrained?

The inverse problem of extension is as follows:

> Given a small piece of a surface *F*, I can examine this piece with all the precision possible, but, on the other hand, it is not possible to study the surface as a whole. What conclusions can I draw from the knowledge that I have of the small piece of the surface, as regards the total surface and in particular its topological structure? (Hopf 1932, 209)

These problems can only be addressed if the meaning of the expression 'the total surface' is specified. A surface is only total if it cannot be 'extended' in turn into another surface, and for this to be so Hopf and Rinow (1931) state four equivalent conditions each of which is sufficient to make the surface an independent 'whole'.

Only one of these conditions will be focused on because the necessity of an analogous condition will be found in all the examples in this chapter. It is necessary that the surface is complete as defined by the metric. Here's what should be understood by complete surface as defined by the metric. A fundamental sequence on a surface is called an infinite sequence of points $a_1, a_2, \dots a_n \dots$, so that, from a certain rank, the distance between two points is infinitely small. The sequence is said to be convergent to a limit *A* if from a certain point the distance of points of the sequence to this point *A* also becomes infinitely small. If they happen to be real numbers, every fundamental sequence would be convergent, by virtue of Cauchy's theorem, that is, it tends towards a limit which would also be part of the set of real numbers. When it comes to points on a surface, it is no longer always the case and the surface is said to be rightly complete as defined by the metric, when any fundamental sequence converges towards a limit also located on the surface. When a set of points, a surface for example, is not complete, it can be completed by adding to it the points which it lacks, namely the limits of its fundamental sequences. Now what is essential in the results of Hopf and Rinow is that the topological properties of a surface are only reflected in the properties of the parts if the surface is not likely to be completed. It is only on this condition of completion that the results that we will now present are valid.

Let it be the case that a surface is simply connected, that is, such that any closed curve by continuous deformation can be reduced on this surface to a point. This is a topological property of the surface. A surface is said to have a constant curvature if a certain quantity is defined at each point using the coefficients g_{ik} of ds^2 of the surface and that the so-called curvature is the same for all values of g_{ik}. The curvature being a purely local notion, the constancy of the curvature is also a local property, defined for each element of the surface. That being so, we have the following theorem: For any curve K there exists, to a near isometry, a sole simply connected surface of constant curvature K, namely the surface of a sphere, the Euclidean plane or hyperbolic plane according to whether we have $K > 0$, $K = 0$ *or* $K < 0$. If the requirement of simple connection is abandoned, the surfaces can be arranged in three topological classes, the classes C_+, C_0, C_-. The class C_+ contains two types: the sphere and the projective plane. The class C_0 contains five types: the plane, the cylinder, the torus, the non-orientable cylinder, and the non-orientable closed surface of genus zero. The class C_- contains all surfaces with the exception of the four closed surfaces contained in C_+ and C_0 (sphere, projective plane, torus and non-orientable closed surfaces just defined). We then have the following theorem: only surfaces of class C_+ can have a constant positive curvature, the surfaces of class C_0 have a constant zero curvature, and those in class C_- have a constant negative curvature.[8] The metric properties are therefore closely related to the topological class of the total surface. It is thus that, for example, a surface of constant positive curvature is necessarily closed (problem of extension) or that on a closed surface of constant curvature, the sign of the curvature is the same as that of a global topological invariant of this surface, the Euler characteristic (problem of metrization) (Hopf 1932, 213). The importance of such results can be seen for the cosmogonical problems which were discussed earlier. The ds^2 of the space is not determined by the form of the space, but the choice of ds^2 is subject to very restrictive conditions that fix the topology of the global space.

2. THE THEORY OF CLOSED GROUPS

The results due to Weyl and Cartan will now be presented where the metric, in certain cases, can be entirely determined by a global property of the group associated with the space in question. The importance of these results comes from them sometimes allowing the reconciliation of Kleinian type spaces and Riemannian type spaces, despite all the differences in structure that have been recognized in them above.

The Weyl theorem used by Cartan is as follows: a closed linear group with real coefficients leaves invariant a quadratic form defined as positive.[9] We immediately see the immense interest of this theorem: the group is defined by global characteristics since it is closed; the quadratic form can be used on the other hand as a local metric, of ds^2, in a space associated in a convenient way with the group in relation to which this quadratic form is invariant. In the problem studied by Hopf, the global properties of the surface could only be registered in the nature of the infinitesimal elements of the surface if this surface was complete. We end up with an condition of completion analogous to the theorem of Weyl. A continuous group can in effect be conceived as giving rise to a topological manifold if a point of the manifold is associated with each transformation of the group. It is not necessary for this manifold to be defined for it to possess a metric, it is sufficient that it is constituted by a set of 'neighborhoods' satisfying certain conditions set forth by Cartan (Cartan 1930, 3). For the manifold of a group, a condition of closing or closure can then be defined that plays, in the case of spaces defined by a system of neighborhoods, the role of completion, which, in the case of metric spaces, is played by the property of being complete: a point *A* being the accumulation point for an infinite set of distinct points of a manifold, if every neighborhood of *A* contains at least one point of the distinct set of *A*, the manifold is said to be closed if every infinite set of points on this manifold admits an accumulation point. This property of completion as defined by topology is distinct from completion as defined by the metric, but is at least sufficient to confer on the structure of a group a character of closure whose presence is indispensible for this structure to reflect itself locally, in the metric of the space with which the group is associated. Weyl's theorem cannot be applied directly to Klein groups that operate on a given homogeneous space *E*, since these groups are in general not linear. Also Cartan proceeds as follows: he adjoins to group *G* the transformations that operate on the space *E*, the group Γ of automorphies internal to group *G*, this adjoint group is found to be linear, as are all its subgroups. Cartan then considers a certain subgroup γ of Γ,[10] and proves the following theorem: If linear group γ is closed, there exists in the homogeneous space *E* a Riemannian metric invariant under *G* (Cartan 1930, 30, 43). There is here an essential connection between the topology of the Klein group and the local metric of a Riemann space from which Cartan has identified enough consequences to be able to write in the terms of an article that we have been cited above:

> Departing from the antagonism between Kleins geometries and Riemann geometry, we arrive after a long detour at this finding that it is in the Riemannian form that Kleins geometries best show their fundamental properties. (Cartan 1927, 222)

3. APPROXIMATE REPRESENTATION OF FUNCTIONS

The domains in which we are now going to study a solidarity between the structure of the set and the individual nature of the elements are sets of functions. We will first consider the set $K(\varepsilon)$ of continuous functions, the value of which is a complex number and which is defined on a closed set of points ε. It is possible to envisage this set of closed functions as forming a space $\Gamma(\varepsilon)$, satisfying the axioms of vector spaces, and which is called the space of continuous functions on ε. The global properties of this space are essentially the completion properties comparable to those already defined above. The individual properties of functions, which are elements of this space, concern a mode of decomposition that is applied to each of them, and we see the close link that unites the properties of the set to the properties of the elements.

Completion as defined by the metric is defined for a function space in an analogous way to the completion of a space of points. It is sufficient to define for any two functions *f* and *g* a number $\| f - g \|$, which is said to be the distance of these two functions. The fundamental sequences and the converging sequences (as defined by uniform convergence) can then be defined on the space of functions.[11] The space is complete if every fundamental sequence is convergent, and the space $\Gamma(\varepsilon)$ of continuous functions is precisely such a complete space.

To study the individual properties of decomposition of continuous functions we begin with the simple case in which the space of points ε is a closed interval $[a, b]$ of Euclidean space. Weierstrass proved that if $f(x)$ is continuous in $[a, b]$, then this function can be approximated as closely as desired by a polynomial in x

$$P(x) = \sum_0^n c_k x^k$$

We thus have for every $\varepsilon > 0$

$$\| f(x) - P(x) \| < \varepsilon$$

This approximation of any continuous function by a polynomial is immediately interpretable in terms of decomposition for the function in question. It proves in effect that one has the right to deduce from the inequality[12]

$$\| f - P \| < \varepsilon \text{ the equality } f(x) = \sum_{k=0}^{k=\infty} c_k x^k$$

The function is thus decomposed into a uniformly convergent series of infinite terms. If we now try to clarify the meaning of this result we will see that the global point of view of completion and the point of view of individual decomposition are united in it. The equality

$$f(x) = \sum_0^\infty c_k x^k$$

concerns the particular mode of decomposition of the function f(x), but if the series

$$\sum_0^\infty c_k x^k$$

is considered, an infinity of polynomials $P_0 \ldots P_n \ldots$ of a finite number of terms

$$\sum_{k=0}^{k=n} c_k x^k$$

and of increasing degree can be distinguished. The uniform convergence of this sequence of polynomials towards a limit $f(x)$ results this time, no longer in the individual properties of this limit, but in the global property of closure of the space of continuous functions. In other words, the fact that a polynomial $P(x)$ is found in the infinitesimal neighborhood of $f(x)$ rightly concerns the function $f(x)$ considered in isolation, but this fact is immediately related to the totality of analogous cases. In the space of continuous functions, the set of polynomials in x is everywhere dense, that is, in the neighborhood of every continuous function a polynomial is found, and the proof of this theorem appeals to the closure of the space.

There exist, in analysis, other examples in which such a connection between the global structure of a set of functions and the mode of individual decomposition of these functions is shown. In this way, for example, any continuous and differentiable function in the interval $-\pi < x < \pi$ is representable as a convergent series of trigonometric polynomials (or Fourier series):

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

the a_n and the b_n being connected to the expression of $f(x)$ in the interval considered by the formulas

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx;\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

This theorem can be presented as a consequence of Weierstrass's theorem: the set of continuous and differentiable functions in the interval $-\pi < x < \pi$ also forms a complete space, and this global property of completion is translated by a new mode of individual decomposition of functions that belong to this set.

The preceding studies show that it is impossible to consider a mathematical 'whole' as resulting from the juxtaposition of elements defined independently of any overall consideration relative to the structure of the whole in which these elements are integrated. There thus exists a descent from the whole towards the part, as a ascent from the part to the whole, and this dual movement, illuminated by the idea of completion, allows the observation of the first aspect of the internal organization of mathematical entities. If one claims to admit that the study of such structural connections is an essential task for mathematical philosophy, one cannot fail to notice the differences that separate mathematical philosophy thus conceived from the entire current of logicist thought that developed after Russell had discovered the paradoxes of set theory. The logicians have since always claimed to prohibit non-predicative definitions, that is, those in which the properties of an element are supportive of the set to which that element belongs. Mathematicians have never been willing to admit the legitimacy of this interdiction, rightly showing the necessity, to define certain elements of a set, to sometimes call upon the global properties of this set. The whole chapter that has just been read tends towards showing the fecundity of that point of view. We thus hope to make evident this idea that the true logic is not *a priori* in relation to mathematics but that for logic to exist a mathematics is necessary.

CHAPTER 2

Intrinsic Properties and Induced Properties

When the properties of a geometric object are studied, one is led to distinguish between properties which result in the consideration of this entity's intrinsic nature, and those that confer on it its relations with the environment that surrounds it. The intrinsic properties of an entity are independent of the position of this entity in space; they are even independent of the existence of other entities. They belong literally to the entity under consideration. The properties of relation on the contrary can only be attributed to a mathematical entity if one is referred to something other than it. It is sometimes a reference system common to several entities, sometimes an ambient space whose properties can be defined independently of any content, sometimes again a certain number of other entities that support, with the former, relations of neighborhood, of impact, of orientation, etc. The properties of relation in sum translate the solidarity between one entity and the universe within which it is embedded.

It is known that one of the essential differences between the mathematical philosophy of Leibniz and that of Kant resides in the opposition of their conceptions relative to the extrinsic properties of geometric entities. We would like to briefly summarize here a debate that, we will see later, retrieves the relevance of this opposition today in differential geometry and in topology.

The Leibnizian conception of the monad, just as much as concerns geometry as concerns the existence of created things, is based on the reduction of relations that the monad sustains with all other monads to internal properties, enveloped in the essence of the individual monad. There are two important moments in this reduction: that in which Leibniz conceives the universal sympathy of all substances; and that in which he inscribes this sympathy in the law of the internal becoming of each monad considered as isolated from all others. In the letter to Arnauld of September 1687, he explains that one thing can express another as 'a projection in perspective expresses a geomtric figure', and further on he writes: 'every substance sympathizes with all the others and receives a proportional change corresponding to the slightest change which occurs in the whole world' (Leibniz 1969, 339). The notion of monad only takes shape, however, in *A New System of the Nature and Communication of Substances, and of the Union of the Soul and Body* published in 1695:

> [E]very substance represents the whole universe exactly and in its own way, from a certain point of view, and makes the perceptions or expressions of external things occur in the soul at a given time, in virtue of its own laws, as if in a world apart, and as if there existed only God and itself . . . There will be a perfect agreement among all these substances. (Leibniz 1989, 202)

All Kant commentators have shown the importance, for the formation of the Kantian theory of space, of two texts of the pre-critical period, one from 1768 *Concerning the ultimate ground of the differentiation of directions in space*, the other from 1770, the dissertation *Concerning the form and principles of the sensible and intelligible world*, which sketch the ideas of the transcendental aesthetic. The respective positions that bodies occupy in space with respect to one another are not described uniquely in terms of mutual relations between these bodies. They are necessarily referred to a system of privileged and universal references that establish in space the fundamental distinctions of left and right, top and bottom, front and rear of the human body. The distinction of the left hand and right hand is the most important. It imposes on space like lines of cleavage to which all bodies in space are subjected. If we look in effect at the way in which the hair 'grows in a spiral on the head', the spirals of snail shells uncoil, or 'hops wind around their poles' (Kant 1768, 368), we observe everywhere in nature a privileged movement of left to right, which no artifice can erase. One of the more famous consequences of this opposition of the left and the right, is the incongruity of symmetric figures. The left hand and right hand possess, when examined in themselves, exactly the same parts, they are arranged in each hand in the same way, both hands are therefore identical and similar (*gleich und ähnlich*) but if they are by all means symmetric, they are not superimposable. This incongruity of symmetrical figures that relates to the structure of our own body is found in pure geometry, and, in support of his thesis, Kant invokes the case of spherical triangles that can be perfectly identical and similar, without however overlapping. There are here sensible facts, of which no rational analysis of internal properties of bodies can account for and which result from the difference of place that these bodies occupy in sensible space. The dependence of bodies with respect to ambient space is therefore narrowly connected for Kant to the fact that reason can only characterize the intrinsic properties of geometric bodies in an abstract way, and that those arising from their position in space can only be grasped by sensible intuition which refers to the orientation of space as a whole: 'Which things in a given space lie in one direction and which things incline in the opposite direction cannot be described discursively nor reduced to characteristic marks of the understanding by any astuteness of the mind . . . ' and later: 'It is, therefore, clear that in these cases the difference, namely, the incongruity, can only be apprehended by a certain pure intuition' (Kant 2003, 396). We shall return later to these possible intrinsic limitations of geometry.

It is hardly necessary to recall how the constitution by Gauss and Riemann of a differential geometry that studies the intrinsic properties of a manifold, independently of any space in which this manifold is embedded, eliminates any reference to a universal container or to a center of privileged coordinates. Gauss's *General Investigations of Curved Surfaces* (1827) envisages a real surface of two dimensions and defines a metric on that surface from the point of view of an observer bound to the surface and who consequently could not consider it from a position of space exterior to it. Riemann's point of view, which was presented in the previous chapter, generalizes, in the case of a ds^2 of n variables, Gauss's 'superficial' point of view. The notions of distance, curvature and geodesic have an intrinsic meaning, since they are defined step by step, without exiting the manifold. The distinction between space and the manifold disappears; it subsists only as the space of this manifold. It is known how the theory of relativity reinforces this identification of the container and the contents: matter is no longer considered to be located in space, the properties of space at each point being determined by the density of matter at this point. Geometry and physics are constituted interdependently, so that it is impossible to separate space, the Riemannian manifold and matter.

Such an intrinsic characterization of geometric objects is not brought about without causing some difficulties of interpretation among philosophers.

The idea that this Riemann manifold, which is the Einstein space, would be closed often evokes the image of a closed surface that the intuition could not help but locate in an infinite 3-dimensional space, and yet outside of this surface, by an incomprehensible paradox, there could be no matter, nor even space. The paradox disappears when one realizes that a manifold on which a ds^2 of more than two dimensions is defined is in no way amenable to an intuitive comparison with a surface. The notions of intrinsic differential geometry are purely intellectual, they characterize a mode of mathematical exploration of a manifold by following a path on this manifold, in opposition to the extrinsic method which considers this manifold as embedded in a Euclidean space to a sufficient number of dimensions. It is in effect always possible to realize a ds^2 in a Euclidean space, but it is a Euclidean space of:

$$\frac{n(n+1)}{2}$$

dimensions whose geometry is as abstract as that of the manifold it contains. What is of interest here is the existence of two points of view that are as clearly distinct from one another as the intrinsic point of view and the point of view of insertion. This new duality leads in effect to the first of the problems that we intend to study in this chapter: Is it possible to reduce the induced properties on a Riemann surface by the ambient Euclidean space to the purely intrinsic properties of this manifold?

1. PARALLELISM ON A RIEMANN MANIFOLD[1]

The question posed is authentically mathematical and the technical aspects of it can be seen in Cartan's expositions that have been cited above. It is of no less considerable philosophical interest that Cartan himself stresses in sober terms, which confer on his remark all the precision of a scientist's assertion. The point of view of induced properties, he tells us, is 'philosophically inferior' to the intrinsic point of view. The lesser result, in the sense of the reduction of the extrinsic to the intrinsic, tends in effect to inscribe in the structure of an entity the relations it maintains with the ambient space and thus to restore the vision of the Leibnizian monad. Here again, it seems that mathematics offers a privileged domain to the movement of a thought which attempts to reconcile two opposed logical notions.

We saw the beginning of the first chapter how, in a Riemann space defined by a certain ds^2, two neighboring observers can, by means of a trirectangular trihedron, 'locate the points that are in their immediate neighborhood, but cannot, without new convention, locate with respect to one another their trihedrons of reference' (Cartan 1924, 297). We also indicated the necessity, where the theory of relativity is found, to endow the Riemann spaces that it considers with a certain homogeneity so that the laws of physics can be independent of any attachment to particular points in space. This Euclidean connection of Riemann spaces is defined by Levi-Civita (1917) with his conception of parallelism on any manifold. The connection of neighborhoods of different points is no longer indeterminate, the pieces of space are oriented closer and closer to one another, so that it is always possible to define the parallelism of two vectors issuing from two infinitely near points. Cartan explains how, in Levi-Civita's theory, this parallelism is 'induced' on the manifold by the Euclidean space to

$$\frac{n(n+1)}{2}$$

dimensions in which it is embedded. To represent intuitively the way that properties can be induced on a manifold of ambient space, Cartan envisages first of all a curve in Euclidean space. This curve differs from a straight line only for an observer exterior to it. The kinematics of a mobile point on this curve is identical to the kinematics of a mobile point on a straight line. This correspondence between axes of the curve and segments of the straight line comes from the possibility of unrolling the curve on a straight line which would initially be a tangent at a point on the curve. But the operations of rolling or unrolling a curve on a straight line are only possible by a series of successive projections of the curve on the straight right, effected in the space that contains the two. Let us now define the parallelism of two vectors tangent to a Riemann surface V_n and issuing from two infinitely near points A and A'. Cartan presents the point of view of Levi-Civita by considering the tangent to the surface at A (Cartan 1924, 297). This plan contains the tangent vector V_n issuing from A. We can project orthogonally on this plane the vector issuing from the plane infinitely near A', and the two vectors from A and A' are called parallel if the projection of the latter on the tangent plane at A is said to be parallel, in the ordinary meaning of the word, to the vector issuing from A. Knowledge of the ds^2 of the surface is sufficient to determine the coordinates of A' with respect to A, but if local systems of references are attached to A and A', the latter axes are subject to a rotation with respect to the former and it is possible to determine the angles of this rotation so that the conditions of parallelism of two infinitely near vectors is always satisfied. As in the case of the curve embedded in Euclidean space, the exterior tangent planes, the projections and the rotations implied by the parallelism of Levi-Civita only make sense with respect to the space in which the manifold is embedded. Cartan shows later in his article the differences that separate this view from that of Weyl, who could give the parallelism a purely intrinsic definition. In the case of two dimensions, it presents itself as follows: two directions issuing from two adjacent points *A* and *A´* are parallel if they form the same angle with the geodesic (line of minimum length) passing through *A* and *A´* (Cartan 1924, 298). This definition does not call upon any operation in the exterior space and it works out to be the same result as that of Levi-Civita.

This result is all the more remarkable because, when both the intrinsic point of view and the extrinsic point of view are possible, it is not at all necessary that they work out to confer a same connection to a same surface. Cartan in effect generalizes the point of view that is derived in the work of Levi-Civita by seeking to attribute to a manifold embedded in a space other than the Euclidean space (the affine space, projective or conformal) a connection which gives the simplest account of the relations of this manifold with ambient space (Cartan 1924, 317). Let, for example, a surface be embedded in conformal space (in which the notion of plane is replaced by that of sphere). It is possible to consider a 'conformal connection' induced on this surface by the ambient conformal space as follows: instead of tangent planes in two neighboring points, envisage the spheres of curvature passing through two points and connect the two spheres by a kind of orthogonal projection of one on the other. This extrinsic conformal connection differs essentially from the purely intrinsic conformal connection since the latter identifies every plane and every surface with a sphere. The reduction of the extrinsic to the intrinsic thus conflicts with the facts that show the limits faced by the elimination of any reference to a universal container, and confer an even greater interest on surprising cases in which this elimination succeeds.

2. STRUCTURAL PROPERTIES AND SITUATIONAL PROPERTIES IN ALGEBRAIC TOPOLOGY

The duality of the extrinsic point of view and the intrinsic point of view that has been observed with respect to certain problems of differential geometry is rediscovered in algebraic topology and occupies such an important place there that the authors of the most recent treatises on topology, Seifert and Threlfall (1934), like Alexandroff and Hopf (1935), put it at the center of their general considerations relative to topology.

The geometric properties studied by topology are those which are conserved by biunivocal and bicontinuous transformations. Two figures are said to be homeomorphic in this regard if such a correspondence can be established between the points of one and those of the other. Two types of homeomorphism must then be distinguished: those that are realizable by a deformation of the two figures that makes them coincide in space; and those that exist between the points of two figures that no deformation in space can get them to coincide. Consider any two closed curves; they are homeomorphic in the sense that it is possible to establish a biunivocal and bicontinuous correspondence between the points of one and the points of the other. If they are situated in the same plane, this correspondence can always be realized by bringing one of the two curves, in a series of intermediate positions, to coincide with the other. If they are not situated in the same plane, it may be impossible to get them to coincide without tearing. Consider, for example, the circle and the node in the form of a clover (Figure 1).

These are two homeomorphic curves since a punctual correspondence can be established from one to the other and yet it is impossible to get them to coincide by a continuous deformation in space. There thus exist for the figures: internal properties (*Eigenschaften Innere* in Seifert and Threlfall 1934) or structural properties (*Eigenschaften Gestaltliche* in Alexandroff and Hopf 1935) like being a closed curve, and which are independent of any

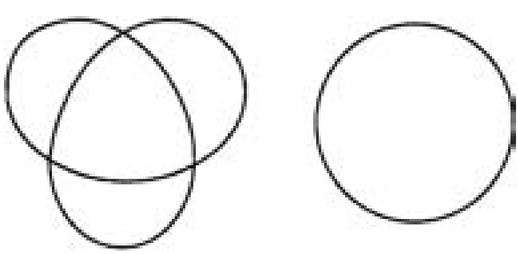

Figure 1. (Seifert and Threlfall 1934, Figure 2)

reference to the ambient space; and insertion or situational properties (*Einbettungseigenschaften*, *Lageeigenschaften*) that are conferred on a figure by the relations it maintains with all of the other figures in the space. The results that we will present all relate to this eminently Leibnizian problem: Is it possible to determine the properties of situation by the knowledge of the structural properties? The very term *Analysis situs*, due to Leibniz, expresses well this hope of determining what concerns the 'situation' by an analysis of the internal properties of the figure.

There is, at least in 3-dimensional space, a classic case of situational property completely reducible to a property of intrinsic structure. Consider what's called the Mobius ring. It is the figure obtained by welding the two ends of a band twisted once on itself. If a line is drawn on this surface by stopping only when it rejoins the starting point, and then if the ring is split to unfold it in its full length, a band is obtained whose two sides are traced on by the line drawn. If the ring were to be colored step by step, it would be seen, by unfolding the ring again in space, that both sides of the obtained band have been colored (Figure 2).

The Mobius ring therefore only has a single side, and that is an essentially extrinsic property since, to be realized, it is necessary to split the ring and untwist it, which implies a rotation around an axis exterior to the surface of the ring. It is nevertheless possible to characterize this 'unilaterality' by a purely intrinsic property. Consider in effect an arrow perpendicular to the line traced on the ring and move this arrow along the line, so that it is

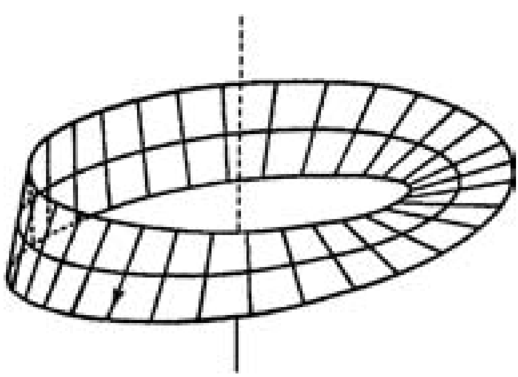

Figure 2. (Seifert and Threlfall 1934, Figure 12)

always situated on the surface. By presuming the surface transparent enough we realize that the arrow arrives at a point to cover its departure position with an inverse orientation. The surface is said to be non-orientable and this property could be observed by an observer bound to the surface, which would neither split the ring, nor untwist it. In an n-dimensional orientable space, the 3-dimensional Euclidean space for example, it can be proved that for an $(n - 1)$-dimensional manifold, a two dimensional ring for example, there is equivalence between the fact of being two-sided and the fact of being orientable, or between the fact of being unilateral and that of being non-orientable. This junction of the situational point of view and the intrinsic point of view is, in this case again, all the more interesting because it does not necessarily take place in all cases. Orientability or non-orientability being intrinsic properties cannot be removed at a surface by a modification of its relations with the ambient space, but it is not as before due to being bilateral or unilateral, properties that depend on this ambient space. It is thus that an orientable surface is bilateral in Euclidean space but could cease to be so in another 3-dimensional space.

This example hints at the philosophical interest of algebraic topology (or even combinatorial topology): the geometric properties of relation to a very large extent let themselves be expressed in intrinsic algebraic properties and, to the extent that the intellectualization of the relations of a figure and of ambient figures is successful, the Kantian distinction between an aesthetic and an analytic is seen to vanish. The most triumphant success of topology in this regard is the theory of the duality that we will now present.

3. DUALITY THEOREMS

If a polyhedron is envisaged in Euclidean space R^n, the situational properties that we are led to attribute to it concern in the first place the repercussions that involve, on the global structure of the space, the presence in it of the geometric entity considered. The specific structure of the polyhedron is an intrinsic property of the polyhedron, independent of all the possible situations in which it can be found, but its introduction[2] within a space changes the internal structure of this space, by submitting the elements to new connections and thereby establishing the relations between the space and itself that characterize its mode of insertion in the space. Duality theorems in topology allow the determination of this action of the polyhedron on the space solely from the structural knowledge of the polyhedron. The polyhedron thus enjoys certain properties of the Leibnizian monad, and we will try to show this by drawing upon only the indispensable definitions.

The object of study of topology is not the polyhedron but the simplicial complex. Here's what is meant by that: a point is a 0-dimensional simplex; a line segment *AB* is a 1-dimensional simplex determined by the 2 points *A* and *B*; a triangle is a 2-dimensional simplex; more generally $n + 1$ vertices determine an n-dimensional simplex. This simplex possesses faces of 0, 1, 2 . . . $n - 1$ dimensions obtained by considering successively its vertices, its edges, its sides. Now consider a figure formed by a set of simplices satisfying the following conditions: a) any point of the figure belongs to at least one simplex; b) any point belongs only to a finite number of simplices; c) given two simplices, either they are without common points, or they have a common face; d) the neighborhoods of a point in the different simplices to which it belongs are 'reunited' in a single neighborhood. This figure is what is called a simplicial complex, whose dimension is that of the highest simplex that appears in the 'simplicial decomposition' of the complex.[3] Naturally, there is no distinction between a complex and the figures that are topologically equivalent to it.[4] Thus, for example, the surface of a sphere is a 2-dimensional complex whose simplicial decomposition is obtain by considering the tetrahedron inscribed *ABCD*. The tetrahedron is a 3-dimensional simplex: the 4 triangles that limit it form a 3-dimensional complex topologically equivalent to the surface of the sphere where 4 2-dimensional simplices occur: the 4 triangles *ABC*, *ABD*, *BCD*, *ACD*, six 1-dimensional simplices (6 edges) and four 0-dimensional simplices (vertices). It can even be demonstrated that most of the important figures of topology are complexes. Any structural study of a complex is based on knowledge of certain numbers, called Betti numbers, attached to this complex, that are invariant under topological transformation. Despite the abstract character of this theory it is absolutely necessary to clarify the nature of the geometrical objects that these numbers measure on a complex. On an n-dimensional complex, certain combinations of 0-dimensional, 1-dimensional, to n-dimensional simplices are defined, forming, relative to each dimension, what are called cycles.[5] Then, what is understood as the independence of several cycles of the same dimension is defined and, for each dimension from 0 to n, the Betti numbers measure the maximum number of independent cycles of this dimension. It is possible to give an intuitive representation of the meaning of Betti numbers of dimension 0 and dimension 1. The Betti number of dimension 0 measures what's called the components of the complex, that is, the number of isolated parts that constitute it.

Thus, for example, the Euclidean space from which the points situated on a circular annulus are removed has its Betti number of dimension 0 equal to 2, because a point on the shaded exterior part and a point of the interior parts cannot be connected by a continuous path. They therefore belong to two distinct components (Figure 3).

The Betti number of dimension 1 measures the maximum number of independent closed curves that are not reducible to a point by continuous deformation. Consider for example the Euclidean plane pierced by the hole *z*. The edges of the holes are curves irreducible to a point, the Betti number of dimension 1 is equal to 2.

We can now turn to the presentation of duality theorems. The first duality theorem, due to Poincaré, refers exclusively to the intrinsic structure of a complex. Alexander's duality theorem is a direct consequence of Poincaré's theorem, and immediately attains the end to which we intended to arrive. It reduces the structural study of the space that receives the complex to a structural study of a complex (Figure 4).

It is Alexander who operates the reduction of situational properties to intrinsic properties, and it is necessary for us to show how this is latent in Poincaré's 'internal' theorem.

Poincaré's theorem proves that for an n-dimensional closed multiplicity (complex satisfying certain conditions) the Betti numbers of dimension k

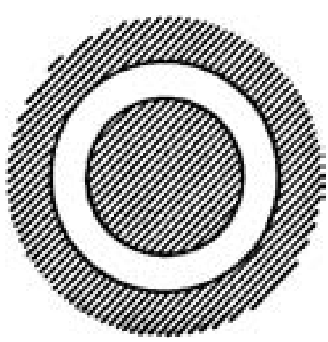

Figure 3.

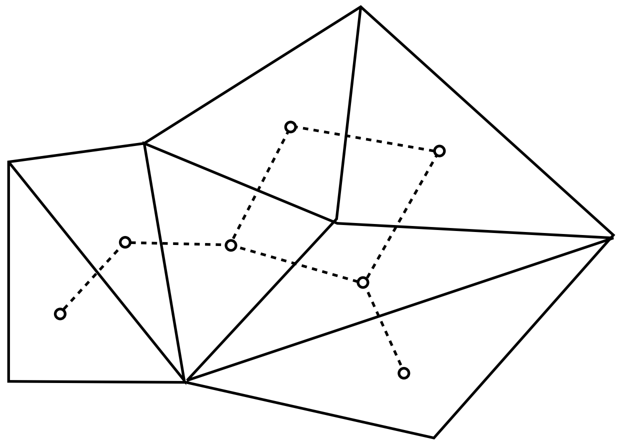

Figure 4. (Lefschetz 1930, Figure 13)

are equal to the Betti numbers of dimension $n - k$. This internal symmetry between the Betti numbers of a same complex Q comes from the Betti numbers of dimension k being Betti numbers of dimension $n - k$ for a different complex to the first, its dual complex. The dual complex Q^* of a complex Q results in another cell decomposition of the same points as those figured in the first cell decomposition of Q (Figure 4). Each $(n - k)$-dimensional cell of Q^* is in intersection with a k-dimensional cell of complex Q. This notion of the dual complex is essential to assure the passage from the internal case to the external case. Considering that we were party to a conception in which the Betti numbers of a multiplicity were the very characteristics of this multiplicity, here's the place where the studied object divides in two and that the Betti numbers of the new complex can be determined from those of the former. This duality within the same entity therefore already has the sense of a relation between two discernable entities albeit still indissolubly connected to one another. A simple change of perspective will dissociate and transform an internal symmetry into a true correspondence of two distinct entities (Figure 5).

Consider in effect a k-dimensional complex Q embedded in an n-dimensional Euclidean space R^n. Let $R^n - Q$ be the complementary space of the complex Q, that is, the space whose points belonging to Q are removed. Considering that there was earlier a duality between the Betti numbers of

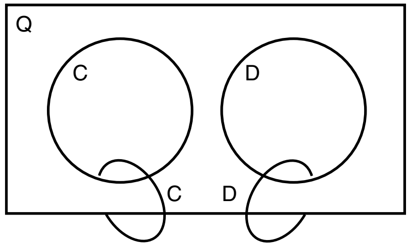

Figure 5. (Alexandroff and Hopf 1935, Figure 35)

the same complex, Alexander's theorem proves a new duality between the Betti numbers of Q and those of its complementary space $R^n - Q$.[6] Let $p^r(Q)$ be the Betti number of the dimension r of the complex Q, we have $p^r(Q) = p^{n-2-1}(R^n - Q)$. The Betti number of dimension r of Q is equal to the Betti number of dimension $n - r - 1$ of $R^n - Q$, except in the case in which $r = n - 1$, for which we have $p^{n-1}Q = p^0(R^n - Q) - 1$. It is necessary to insist on the meaning of this result. In n-dimensional Euclidean space, all the Betti numbers are zero except that of dimension 0 which is always equal to 1 in the spaces composed of a single part. It is only because a complex Q is introduced in R^n that a more complicated structure than the primitive space results for the space $R^n - Q$. Thus in the case of Figure 5, it is only because the space Q, pierced with two holes C and D, is removed from the space R^n that, for $R^n - Q$, there results the existence of cycles C' and D', of dimension 1, *enlaced* with C and D, and not reducible to a point by continuous deformation. The Betti numbers of $R^n - Q$, which express the structure of this space, therefore express in the same way the nature of the action that the complex Q exercises over R^n. Alexander's theorem can therefore predict in advance the result of this action of Q on R^n by the knowledge of the specific structure of Q. How Alexandroff and Hopf can write the following can now be understood:

> Alexander's duality theorem belongs without doubt to the most important discoveries of topology in recent years. Everything we know about the situational properties of polyhedra and closed sets in multidimensional space is derived from him. The situational properties of a set F in a space R are primarily the structural properties of the complementary

> space $R - F$, and the duality theorem teaches us to determine these properties, in the case of a polyhedron embedded in Euclidean space, provided that they are allowed to be expressed by Betti numbers and torsion groups[7]. (Alexandroff and Hopf 1935, 449)

Here is the simplest case in which these considerations are applied: Jordan's theorem teaches that a closed curve in the plane divides the plane into two separate regions of which it is the common boundary. The theorem manifestly characterizes the action of the curve on the space. Brouwer and Lebesgue have shown, long before Alexander had proclaimed his general theorem, how it was possible to link this action of an n-dimensional closed multiplicity on an $(n + 1)$-dimensional ambient space to the intrinsic properties of the multiplicity. With Alexander's formula, Jordan's result is immediate: a closed curve C is a 1-dimensional complex, its Betti number of dimension 1, p^1, is equal to 1. In the plane in which $n = 2$, by calling $p^0(R^2 - C)$ the number of isolated components of the space whose points situated on the curve C are removed, we therefore have:

$$p^0(R^2 - C) = p^1(i) + 1 = 2$$

The action of the curve on the space is therefore determinable from the knowledge of the structural invariants of the curve considered intrinsically. What this result had from the moment of its discovery that is extraordinary is highlighted by Pontrjagin in the introduction to a very important article on duality theorems. Pontrjagin writes in effect:

> In his celebrated memoir of 1895 on *Analysis situs* published in the *Journal de l'Ecole Polytechnique*, Poincaré proved the duality theorem that now bears his name and which established the identity of the r^{th} and the $n - r^{\text{th}}$ Betti number of an n-dimensional orientable multiplicity. At about the same time, Jordan stated for the first time the theorem related to closed curves. No one suspected then that two totally different theorems belonged to the same circle of ideas. (Pontrjagin l931, 165)

Nobody could indeed have suspected, before the development of algebraic topology, that the properties of internal structure discovered by Poincaré would someday explain the extrinsic situational properties expressed by Jordan's theorem.

4. THE LIMITATIONS OF REDUCTION

Duality theorems allow the determination of the structural properties of the complementary space of a polyhedron embedded in Euclidean space by the knowledge of the structure of the polyhedron in question 'provided that they are allowed to be expressed by Betti numbers and torsion groups' (Alexandroff and Hopf 1935, 449). Now the facts have come to show that this restriction was essential and that the reduction of situational properties to structural properties could never be completed. It seems as though topology could only have developed by siding with Leibniz, but it always encounters facts that side with Kant and require the search for new methods. The discovery in question is that of Louis Antoine (Antoine 1921, 221). Studying the case of two Jordan curves F and f each situated in spaces F_1 and f_1, he examines to what extent the homeomorphism that always exists between two Jordan curves can be extended into their neighborhood. Three cases are possible: 1) that in which the homeomorphism of the curves extends to any space; 2) that in which the homeomorphism extends to a neighborhood that exceeds the curves without covering any space; 3) that in which the homeomorphism of the curves can be extended for any region exterior to the curves. In the case of plane curves, they are always of the first case and the structure of the curve completely determines the structure of the space that contains it. On the other hand, for curves in 3-dimensional Euclidean space, three cases can be presented and Antoine effectively constructs a Jordan arc on a torus whose correspondence with a line segment does not extend to any neighborhood. The two curves F and f being homeomorphic, their structural invariants, by virtue of Alexander's theorem, determine structural invariants identical to their respective complementary spaces $F_1 - F$ and $f_1 - f$, but the identity of these invariants is not sufficient for the spaces $F_1 - F$ and $f_1 - f$ to be homeomorphic. They are not determined univocally by their Betti numbers, therefore by the curves that can be inserted into them, and their structural differences are irreducible. If it can be observed that the two 3-dimensional Euclidean spaces F_1 and f_1 are identical and that it is the introduction of homeomorphic curves F and f in each of them that makes them profoundly dissimilar, this accounts for all that the invariant structure of curves lets escape from the relations between the curve and the space. The situational properties, reducible to structural properties in the case of two dimensions, cease to be so in the case of three dimensions. At this level of reality, the distinction of an aesthetic and an analytic subsists.

CHAPTER 3
The Ascent towards the Absolute

In Cartesian metaphysics there is an essential dialectical reasoning: the passage from the idea of imperfection to the idea of perfection and to God who is the cause of the presence in us of the idea of the perfect. We will focus mainly on two stages in this passage, as found in the fourth part of the *Discourse on Method*. The first is where Descartes asserts the logical anteriority of the idea of the perfect with respect to the idea of the imperfect. The imperfect being can only be understood by reference to the perfect being whose existence is thus enveloped in its very own:

> Next, reflecting upon the fact that I was doubting and that consequently my being was not wholly perfect (for I saw clearly that it is a greater perfection to know than to doubt), I decided to inquire into the source of my ability to think of something more perfect than I was; and I recognized very clearly that this had to come from some nature that was in fact more perfect. (Descartes 1985, 127)

Gilson compares this text to a passage from the interview with Burman where the statement of this rule is found: 'And every defect and negation presupposes that of which it falls short and which it negates'.[1] Not only does imperfection presuppose perfection but, and this is the other point that we insist on, imperfection being only a privation, it is possible, solely by consideration of the imperfect, to determine the attributes of the perfect being. This is what the text of the *Discourse* shows:

> For, according to the arguments I have just advanced, in order to know the nature of God, as far as my own nature was capable of knowing it,

> I had only to consider, for each thing of which I found in myself some idea, whether or not it was a perfection to possess it; and I was sure that none of those which indicated any imperfection was in God, but that all the others were. (Descartes 1985, 128)

The distance that separates perfection from imperfection is thus inscribed in the very nature of the imperfect being. The mind rises to the absolute in a movement whose steps are controlled by the end that is perceived from the starting point. The structure of the imperfect being then takes on its true meaning: its complication or its obscurity are only deviations with respect to the transparent simplicity of the final vision, the ascent towards perfection seems to go through in reverse the stages of an anterior degradation.

It is possible to retrieve in some theories of modern algebra similar relations between perfection and imperfection, we will study in this chapter the necessity of this reference to an absolute which lets itself be seen in the imperfect nature of certain mathematical entities, and this ascent toward it in a series of steps each of which effaces some impurity, up to the last where every defect is rectified. This is a very different mode of thought to those arising in ordinary arithmetic. Arithmetic is in effect the domain of recurrence to infinity, whereas what is characteristic of the movement of the theories that will be considered is the existence of an end conceived in advance as a term of the ascent. In class field theory, that goal is the absolute class field, in the theory of the uniformization of algebraic or analytic functions on a Riemann surface, it is the universal covering surface. The logical framework of each of these two theories is derived from Galois' theory of algebraic equations. Although several authors[2] have also already presented the philosophical importance of this, we think it should be shown anew how it contains the mathematical tools necessary for the passage to the absolute.

1. GALOIS'S THEORY[3]

Let there be a number field k, that is, a set of arbitrary numbers satisfying the axioms of addition, multiplication and division, and let $f(x)$ be a polynomial in x of degree n and with coefficients in k. This polynomial is irreducible in k if k contains none of its roots $\alpha_1 \ldots \alpha_n$. Thus, for example, the polynomial $x^2 - 2$ is irreducible in the field of rational numbers since its two roots $+\sqrt{2}$ and $-\sqrt{2}$ are not contained in it.

Let K be the field obtained by adding to k a root α_1 of the polynomial. This extension K of k is written $k(\alpha_1)$. If the n fields conjugated $k(\alpha_1) \ldots k(\alpha_n)$ coincide, the unique field K thus defined contains all the roots of the polynomial $f(x)$ and is said to be Galoisian over k. The degree of the extension K over k is equal to the degree n of the irreducible polynomial $f(x)$. The Galois group G of polynomial $f(x)$ is formed by the internal transformations (or automorphies) of the field K which leaves the elements of k (all contained in K) fixed and permutes between them the roots α_i of the proposed polynomial. As n conjugated roots $\alpha_1 \ldots \alpha_n$ can correspond to a root α_i, the group contains n substitutions. The order of this group is thus equal to the degree of the extension field K.

These definitions will allow us to understand what could be called the 'imperfection'[4] of the base field with respect to a given polynomial. This imperfection resides in that it requires an extension of degree n to pass from field k to field K which contains all the roots of the polynomial in question and is measured by order of the Galois group attached to the equation. We are going to see how, by 'ascending' from k to K, the intermediate field k' can be considered such that $k \subset k' \subset K$ and the imperfection of which decreases as K is approached. In effect, let k' be such an extension field, such that K is no more than degree m with respect to k' $(m < n)$. Galois's theorem univocally associates a subgroup g' of G to this intermediate field, defined as follows: g' leaves all the elements of K which are in k' invariant and permutes only those that remain to be integrated. In addition, the order of this subgroup is equal to the degree m of the extension that has yet to pass from k' to K, and thus measures what imperfection remains in k'. The 'ascent' from k to k' is therefore accompanied by a 'descent' in the order of the groups attached to these fields. And, to the final field K, in which all imperfection has disappeared, since it contains all the roots of $f(x)$, corresponds the smallest subgroup of G, the unit 1 group which only contains the identical transformation. The interest of the logical schema of Galois's theory is considerable. A certain number of other theorems are encountered in algebra that, for a basic domain imperfect from a certain point of view, assert the existence of an extension in which this imperfection has disappeared. Thus, for example, there exists for all number fields, an extension Ω which is 'algebraically closed', that is, such that any polynomial in x with coefficients in Ω is completely decomposable into first-degree factors with coefficients in Ω, but the completion theorem of Galois's theory is infinitely richer. It associates with each stage of the ascent of k to K a number measuring the remaining difference between the stage in question and the final stage. The forecast of the end is even more precise when the equation $f(x) = 0$ is solvable by radicals, that is, when the field K that contains all the roots of the proposed equation can be constructed by successive adjunctions of magnitudes $\sqrt[n]{a}$, a belonging each time to the field already obtained. In this case a increasing series of fields can be ordered from k to K having special properties $k \subset k' \subset k'' \ldots \subset K$ to which correspond term for term a decreasing series of groups $G \supset g' \ldots \supset I$ such that this series cannot be extended by the adjunction of any inserted element. The initial data implies then not only the existence of the end and the difference that separates it from the base field, but also the exact number of stages to be executed to arrive at it. The two essential moments of the passage to the absolute are indeed found in the Cartesian Meditations: first, the vision of the perfect being whose existence is implicated by that of the imperfect being; and secondly, the consciousness that the reasoning to be effectuated to attain the absolute is, to some extent, given with the imperfect being proposed, whose structure is thus called a complete model in which its defects are effaced.

It is by placing ourselves at this Cartesian or Galoisian point of view that, class field theory will now be examined.

2. CLASS FIELD THEORY[5]

Class field theory issuing entirely from the genius of Hilbert is an extremely abstract theory of algebra that calls upon a large number of notions that are difficult to grasp, but it stands out as one of the clearest examples of the new mathematics in which successive edifices tend toward an end that their movement anticipates. It presents once again the base of an ascent from fields to extension fields up to a maximum entity: the absolute class field that has, with respect to an initial base field, the greatest simplicity of which an extension is possible. Its philosophical richness is moreover inexhaustible in this ascent towards the absolute, because this structural solidarity between the elements of a whole and the whole to which they belong, which was described in Chapter 1, can be found in it. The two problems are also so closely connected that it is impossible to present the contribution of one or the other separately.

The 'elements' of a 'whole' are here the 'ideals'[6] of an algebraic number field. They remain 'ideals' in the successive extensions of this field, but their internal structure varies with the extension that each time is considered and expressed as a global characteristic of the field in which they are embedded. If any of these fields can then be conceived as the ultimate term of an ascent, it is because in it finally all the ideals of the base field find the most uniform and simplest internal structure. More precisely, given a prime ideal of a base field k, this ideal does not remain prime in an extension K of k. Knowing the degree of extension of K with respect to k, can the mode of decomposition of prime ideals of k in K be predicted? Conversely, knowing the laws of decomposition of ideals in an extension field, can the nature of the envisaged extension be characterized? These are two reciprocal problems of solidarity between a whole and its parts, problems comparable to problems of metrization and extension envisaged above (Chapter 1), but there is a new element added, namely the ascent towards a maximal field which establishes a connection between the problem of Galois's theory.

Hilbert stated in 1898, without proof, the principal theorems of class field theory in the case of particular extension fields (non-ramified abelian extension fields). These results have been proven and extended to much more general categories of extension fields by Furtwängler in 1907 and Takagi in 1920. We confine ourselves here to simple cases envisaged by Hilbert and would like to briefly highlight the connection established between the set of extension fields, which contain the field k, and the set of groups of ideals that are contained in this field.

The definition of groups of ideals of a field k can be made as follows: let A be the group of all ideals of the field k and let S be the group of principal ideals of the field, which is evidently a subgroup of the group A. All subgroups H of A, which contains the group S of principal ideals, can be envisaged as groups of ideals. These sub-groups are thus arranged between a maximal group A and a minimal group S, and they each determine a division into classes of ideals of A, two ideals being in the same class with respect to H when their quotient is contained in the considered subgroup H.

The notion of 'groups of ideals' being thus clarified, Hilbert called 'class fields', for the group of ideals H of a field k, a certain algebraic extension field K such that only the prime ideals of k belonging to H decompose in K following a certain simple law that he indicates.[7] The other ideals are decomposed according to a more complicated law. Furtwängler (1907) and Takagi (1920) then prove the two reciprocal theorems that allow for a biunivocal correspondence to be established between groups of ideals in k and the extensions over k: for any group of ideals H situated in k, there exists an extension K of k that is a class field for H. And conversely: any field K which is relatively abelian over k is a class field for a certain group of ideals H situated in k. The degree of the extension K over k is equal to the order of the group A/H of classes of ideals determined by the division into classes that the subgroup H operates in the group A of all the ideals of the field k. In addition, this biunivocal correspondence of fields and groups, like the Galois theory, establishes a connection between the ascent in the extension field and descent in the groups. It shows in effect that the relation $K' \supset K''$ entails the inverse relation $H' \subset H''$ for the groups of ideals H' and H'' corresponding to two distinct class fields K' and K''.

These results therefore establish first of all a close solidarity between the laws of decomposition in K of prime ideals of k and the overall features of the extension K, but they also give us a richer result. They let us foresee the existence of an absolute class field that contains all the class fields for all groups of ideals H of k, and that is not contained in any. Since there is in effect a minimal group in the series of subgroups, the group S of principal ideals, and that to a descending hierarchy between groups corresponds an ascending hierarchy between corresponding fields, there is a maximal class field, one corresponding to the group of principal ideals. It turns out then that the maximal class field is such that all the ideals of k will be subject to the same modification of structure: they all become principal ideals. To the fact of being the last, the maximal class field combines the fact of being the simplest: the ascent comes to an end at a stage where certain of the most important differences between the ideals of the base field have disappeared, and the existence of this stage was implicated from the moment the function was established between the groups of ideals and class fields. From the theory in its entirety emerges the same Cartesian movement as Galois's theory.

3. THE UNIVERSAL COVERING SURFACE

The theory that will now be presented has a far greater philosophical importance than the previous theories, because the ascent towards the absolute has as a consequence, not only to confer on a mathematical entity the greatest simplicity of internal structure possible, but to make it able to give rise to entities other than itself. The entire second part of this essay will be dedicated in effect to the study of the procession of mathematical entities one with respect to the other, and, as we shall see, this movement is only possible if the structure of the entity from which the other entities proceed was brought to a certain prior state of perfection. In so far as we will have shown how the existence of a universal covering manifold is implied by the structure of any manifold whatsoever, the ascent towards this maximal surface is still a problem concerning the completion of an internal structure, and the elimination of entanglements that it could primitively present. But, insofar as the universal covering manifold immediately gives rise to certain functions whose existence was impossible on all surfaces covered by this universal manifold, the passage to the absolute confers on the surface a power of production that it did not possess at earlier stages; the structure of the perfect entity is radiant in surprising richness.

To study the covering, we can taken as an object of study either: the n-dimensional complex of combinatorial topology, as defined in Chapter 2 in the section devoted to duality theorems; or, the n-dimensional manifold of 'set theoretical' topology, defined by the axioms of neighborhood that Threlfall presents as follows: 1) to any point P of a set is associated a neighborhood of points of the set such that any subset containing this neighborhood is equally a neighborhood of P; 2) to each neighborhood V is associated a biunivocal correspondence between the points of V and the points of a hypersphere in n-dimensional Euclidean space; 3) the manifold is in one piece (2 points can always be reunited by a continuous path). We will confine ourselves in the following to manifolds of two dimensions, that is, to surfaces.[8] What we will consider as an inherent imperfection in the structure of a given surface is the fact that this surface cannot be simply connected. A surface is said to be simply connected if any closed curve on this surface is reduced to a point by continuous deformation. The Euclidean plane is simply connected, but the surface of the torus is not. By arranging in effect all closed curves reducible to one another by a continuous deformation in the same class, at least two classes of closed curves on the surface of the torus that are not reducible to one point are discovered: those surrounding the central 'hole' and those that go around the surface like the circle AOB (Figure 1). The set of classes of closed curves issuing from any point O of a surface forms a group: the fundamental group of the surface, defined by Poincaré, and whose structure thus measures as it were the multiplicity of the internal connection of the surface.

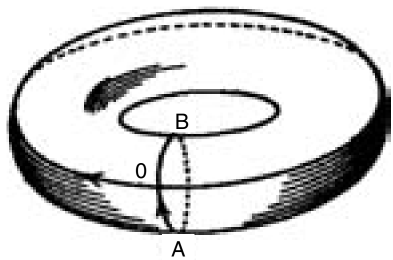

Figure 1. (Seifert and Threlfall 1934, Figure 4)

We'll now see how it is possible to define on a multiply connected surface *F* a series of manifolds, each of which 'covers' the surface *F*. These manifolds lie between *F* and a universal covering manifold that covers all the preceding without being covered by any, and on which the multiplicity of connection to the basic surface is completely erased.

A surface $\overline{F}$ is said to cover a given surface *F* if:[9] 1) at any point of *F* there corresponds at least one point of $\overline{F}$ which covers it; 2) for each point $\overline{P}_1, \overline{P}_2, \ldots \overline{P}_n$ that covers in $\overline{F}$ a point *P* of *F* there exist neighborhoods $V(\overline{P}_1), V(\overline{P}_2), V(\overline{P}_n)$ that can be represented topologically on the neighborhood $V(P)$ of *P* in *F*; 3) if *Q* is a point in the neighborhood $V(P)$ of a point *P* on *F*, and $\overline{Q}$ a point covering *Q* in $\overline{F}$, then $\overline{Q}$ belongs to one of the neighborhoods $V(\overline{P}_1), V(\overline{P}_2), V(\overline{P}_n)$ that cover $V(P)$ in *F*.

There exist in general several covering surfaces of the same surface. It is possible for them to become covered by each other and for each to possess like the basic surface a fundamental group that expresses its degree of connection. The essential theorem of the theory then establishes an isomorphism between the fundamental groups of different covering surfaces of a given surface *F* and the subgroups of the fundamental group of *F*. The set of these covering surfaces is thus in biunivocal correspondence with the set of subgroups of the fundamental group. By considering then the smallest subgroup of the fundamental group, that is, the unit group, which corresponds to the class of paths reducible to one point by continuous deformation, the dual result expected is obtained. There exists a maximal or universal covering surface that covers all the other covering surfaces, and this surface possesses a character of absolute simplicity since any closed path on it, its fundamental group being the unit group, is reducible to a point. This universal surface, the only one of all the surfaces that cover the initial surface, is therefore simply connected. It can be characterize as follows: to a given point P on the initial surface F, there correspond as many conjugate points P_1, P_2 . . . on the covering surface $\overline{F}$ as there exist on F of closed contour issuing from P, not reducible to one another. In sum, each point of the multiply connected surface is only unique in appearance, it possesses a hidden complexity, which is reflected by the degree of connection of the surface and by dissociating all the confounded points to scatter them separately on the covering surface, the ideal simplicity of perfect surfaces is restored to the surface. We will see in the next paragraph how by restoring to the surface its simplicity, its fertility has simultaneously been restored.

4. THE UNIFORMIZATION OF ALGEBRAIC FUNCTIONS ON A RIEMANN SURFACE

The problem of the uniformization of analytic functions (we'll only be occupied with analytic algebraic functions)[10] is in itself a problem whose logical signification is comparable to that of the preceding problems: it is still a matter of eliminating the imperfections of certain mathematical entities by the passage from what they are primitively to an ideal of absolute simplicity, whose existence is implicated in the entanglements of their structure. An algebraic function $\zeta = f(x)$ is said to be multiform around an algebraic point of ramification of order m if, when the variable turns around this point, the function can take different m values successively before returning to the first of these values. It is therefore said that m branches of the function end up in this point. A uniform function on the contrary can only take one value at one point. The problem of uniformization consists then in finding a complex parameter t such that ζ and z are left to be expressed as uniform functions $\zeta = \varphi(t)$ and $z = \psi(t)$ of the new variable t. The passage from the old to the new variable therefore has the consequence of making the points of ramification of the algebraic function disappear, and, by presenting the theory of Riemann surfaces of algebraic functions, it will be shown how the possibility of solving the problem of uniformization is connected to the fact that these Riemann surfaces contain within them the elements of the construction of a universal covering surface.

If the complex plane z is itself taken as a domain of existence of the function $\zeta = f(x)$, we have said that the function could then be ramified into several branches around certain points: thus the elliptic function

$$\zeta = \sqrt{A_0 z^4 + A_1 z^3 + A_2 z^2 + A_3 z + A_4}$$

is divided around each of the 4 supposed distinct roots of the polynomial under the radical, in 2 distinct branches. Instead of admitting that the distinct branches of ζ end up in a same point of the complex plane, Riemann imagines a surface composed of different sheets of the complex plane, welded in crosses along the cuts joining two by two the points of ramification. In the case of the elliptic function just considered, the Riemann surface is composed of two sheets welded in a cross along the 2 cuts uniting two by two the 4 points of the ramification. This gives, in a general way, the Riemann surface of the algebraic function in question. This surface, from the point of view of its structure, is infinitely more complicated than the primitive complex plane. The cuts which it is subject to, the cross weld of the edges of each cut, make it not simply connected. It is possible in effect to trace closed curves, called retrosections, on this surface, none of which divides the surface into two regions, so that to pass from one to the other it is necessary to meet the curve, and which are irreducible to a point by continuous deformation. These topological complications of Riemann surfaces are compensated for by the fact that the comportment of the algebraic function on this surface is much nearer to uniformization than on the complex plane. If a global uniformizing function still doesn't exist, there is already at each point on this surface a 'local uniformization'.

Let there be for example the function $\zeta = \sqrt[m]{z}$. The Riemann surface of this function possesses, at the complex point $z = 0$, m superposed sheets connected so as to establish a circular permutation of m values of the function at the origin. If we let $t^{m} = z$, we have $\zeta = t$, there is punctual biunivocal correspondence between the m sheets of the surface welded together at the point $z = 0$ and m separated portions of the simple complex plane of the variable t, in the neighborhood of the point $t = 0$. The variable t thus uniformizes the function ζ, near the origin. The Riemann surface can even be defined as it was by Weyl (1913), without proceeding to this complicated superposition of sheets and by only appealing to the existence, in the neighborhood of each point p_0, of a local uniformization $t(p)$ such that if $f(p)$ is a regular or branched analytic function in the neighborhood of p_0, $f(p)$ can be represented in the form of a power series in $t(p)$:

$$f(p) = a_0 + a_1 t(p) + a_2[t(p)]^2 + \dots$$

The Riemann surface thus constituted (Weyl 1913, 36 [1964, 36]) by a juxtaposition of neighborhoods in which local uniformizations are defined, seems to Weyl comparable to those *n*-dimensional multiplicities of Riemannian differential geometry, defined by the value of their ds^2 in the infinitesimal neighborhood of each point. This comparison is only possible thanks to the new definition of the Riemann surface proposed by Weyl, which itself recognizes in addition (Weyl 1913, 36 [1964, 36]) that nothing in the writings of Riemann suggests that Riemann had established the connection between the spaces that he introduced in geometry and the surfaces that he introduced in analysis. An essential difference can even justifiably be seen between Riemann spaces and Riemann surfaces. The spaces defined by their ds^2 are explored in a purely local way, and we have indicated which richness of connections their elements remained amenable to. The principal characteristic of the Riemann surface of an algebraic function, on the contrary, is to possess that global topological structure which confers on it the cuts and the retrosections mentioned above, and whose consideration is essential to the problem of uniformization. It is possible in addition to intuitively represent this structure of the Riemann surface of an algebraic function. It can be shown in effect that a similar surface, by continuous deformation, can always take the form of the surface of both sides of a disk with holes (Figure 2).

If the disc has *p* holes, there are 2*p* possible independent retrosections, that is, 2*p* closed curves, irreducible to one another by continuous deformation, and none of which divide the surface into two distinct regions.

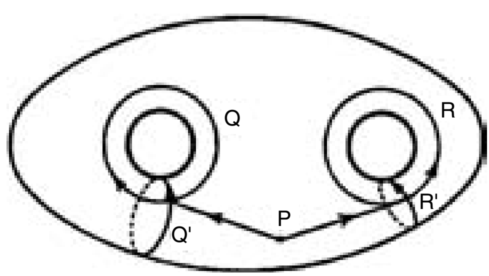

Figure 2.

Consider, for example, the case in which $p = 2$. At each of the two holes there corresponds a pair of retrosections Q and Q' for the first, R and R' for the second: it is easy to see that the surface (when the retrosections have not been cut) is not simply connected and possesses at each point a fundamental group in the sense defined by Poincaré. Let us join in effect any point P to the point common to two conjugated retrosections. Already four classes of closed paths are obtained issuing from a same point and irreducible to one another. These classes generate the fundamental group of the surface. All that remains for us is to show that all the elements necessary to prove the existence on this Riemann surface of a global uniformization are thus possessed.

For this we will turn to the considerations announced at the beginning of this section. There is a close connection between the fact of a surface being simply connected and the fact of giving rise to a function that ensures the conformal[11] representation of this surface either on the totality of the complex sphere, on this sphere from which a point is removed, or on the unit circle of the complex plane. This is a theorem of considerable mathematical importance, as much by the difficulty of the means that Riemann, Poincaré, Hilbert, Koebe and many others made use of to prove it, as by the immensity of the new horizons that it allowed to be discovered. Let p_0 be any point on the simply connected surface, t_0 the corresponding point of the complex plane, the function that ensures the conformal representation of the surface on the plane has the value t_0 for the value p_0 of its variable, and there exists an inverse function which, taking the value p_0 on the surface for the value t_0 of its variable thus ensures the reciprocal representation of the plane on the simply connected surface. The problem of uniformization is therefore solved for a simply connected surface. The theorem of conformal representation in effect provides this function $t(p)$ whose inverse $p(t)$ is a uniform function on the complex plane t. In addition, any function $f(p)$ on the simply connected surface is also a uniform function of t. The new variable is therefore a global uniformization in the sense defined above.

When the Riemann surface of the algebraic function in question is not simply connected, the brilliant idea of Poincaré and Koebe was to consider the universal covering surface of this surface: since this surface is itself simply connected, the theorem of conformal representation is itself applicable, and it gives rise to a global uniformization of the function $f(p)$ attached to the primitive Riemannn surface, which solves the problem completely.

We therefore reach the conclusion of this study on the uniformization of algebraic functions at a dual result. The universal covering surface is presented first with a character of perfection with respect to the primitive Riemann surface, because the ascent towards it has made the ramifications of the algebraic function under question disappear. But this is still only an internal perfection that results from a simpler structure letting itself to be seen in the design of the primitive structure. Another perfection is manifested that is no longer only of interior completion, but of creative power. It is that privilege that certain domains have to give rise to a new world of functions and integrals, which will occupy the second part of this essay.

The studies that we have pursued during these three chapters thus show the variety of logical connections that are manifested within mathematics. The solidarity of the whole and its parts, the reduction of relational properties to intrinsic properties, the passage from imperfection to the absolute, here are so many attempts at structural organization that confer on mathematical entities a movement towards completion by which it can be said that they exist. But this existence is not only manifested in what the structure of these entities imitate, the ideal structures to which they allow themselves to be compared. It happens to be that the completion of an entity is at the same time the genesis of other entities, and it is in the logical relations between essence and existence that the schema of new creations is inscribed.

SECTION 2

The Schemas of Genesis

CHAPTER 4
Essence and Existence

1. THE PROBLEMS OF MATHEMATICAL LOGIC[1]

The problems of the passage from essence to existence, which will occupy us henceforth until the end of this essay, belong to questions that have been raised for a long time by the development of mathematical logic. It doesn't seem to us however that logic has the benefit in this regard of a special privilege. It is in effect only one mathematical discipline among others, and the geneses that are manifested there are comparable to those observed elsewhere. The presentation that we will make of what could be called the metaphysics of logic therefore has above all the value of an introduction to a general theory of connections that unite the structural considerations to assertions of existence.

Two periods can be distinguished in mathematical logic: one, the naive period, ranging from the early work of Russell until 1929, the date of the metamathematical work of Herbrand and Gödel which marks the beginning of what could be called the critical period. The first period is that where formalism and intuitionism are opposed in discussions that extend those raised by Cantor's theory of sets. Proponents of the actual infinite claim the right to identify, for a same mathematical entity, the essence of this entity, as a result of its implicit definition by a system of non-contradictory axioms, with the existence of this entity. We know, on the other hand, the attitude adopted towards an entity whose construction would require an infinite number of steps, or a theorem that is impossible to verify, by those that Poincaré called the pragmatists: in his *Last Essays*, the famous mathematician says that 'they see in it only unintelligible verbiage' (Poincaré 1913, 66). Poincaré relies mostly on the assertion of existence contained in the Zermelo theorem: 'there is a way of well ordering the continuous,' to show how this statement could be meaningful only if the manner in which it is necessary to proceed for well ordering the continuous was really known. Asserting the possibility of an unrealizable operation is to assert something which is either meaningless, or false, or at least unproven.

In his essay on the Infinite, Hilbert (1926) justifies the introduction of transfinite elements in mathematics in putting them side by side with the ideal elements introduced by Kummer in algebra, which were mentioned in Chapter 3. Ideal numbers are no more parts of the numbers of a field than transfinite elements of axiomatized mathematics are determinable in a finite number of steps. But, just as the consideration of ideal numbers is essential to generalize the theorem of decomposition into prime factors, the consideration of transfinite elements is necessary to generalize the application of the excluded middle. Requiring their elimination would imply abandoning the rule of contradiction in logic. Here's how, in an essay from 1923, the connection is presented between the transfinite axiom of choice and the excluded middle for infinite sets. The axiom is introduced as follows:

$$A(\tau A) \rightarrow A(a)$$

The object τA is an object such that if the property A agrees with it, then it certainly agrees with all the objects a. For this, it is necessary to conceive the object τA as that in which there is the least chance that property A is applied. Thus, for example, if A is the property of being bribable, τA is the least bribable of men (Hilbert 1923; 1936, 183 [1996, 1141]). It is then evident that if this ideal man (in every sense of the word) is also bribable, then all men will be. Hilbert and Bernays proved that it is possible to deduce the axiom of the excluded middle from the transfinite axiom

$$(\overline{a})A(a) \rightarrow (\mathrm{E}a)\overline{A}(a)$$

(If all a do not possess the property A, there exists at least one a that does not possess it.)

The object τA, distinguished in the set of entities that have property A, is obviously not likely to be constructed in a finite number of steps, since it is defined by comparison with the infinity of entities of the set. It is no less true that the admission of the axiom, where it is formally defined, is equivalent to the use of the excluded middle. If therefore it is possible to prove the consistency of a system of axioms containing the transfinite axiom, it is, in the same way, as legitimate and advantageous to talk of the existence of the object τA as the existence of the point at infinity in geometry, of imaginary numbers, or of ideal elements of a number field. It can therefore be said that in 1926 the problems of mathematical logic arose again in the same terms as the discussions at the beginning of the century, relating to the existence of the transfinite. True to its Leibnizian origins, formalism always considered that the passage from essence to existence should consist uniquely in the proof of the 'compossibility' of essences, of the consistency of the axioms that define them.

It is precisely in the research of the critical period with respect to the consistency of arithmetic that we seem to see asserted a theory of the relations between essence and existence as different from the logicism of the formalists as from the constructivism of the intuitionist. We will first recall the principal features of this evolution internal to logic and then try to extract a philosophy of mathematical geneses, whose scope goes far beyond the domain of logic. The considerable work that Bernays wrote on Hilbert's logic is in this regard a source of inexhaustible richness.[2]

We have already indicated at the beginning of this essay how Hilbert conceived the necessity of a metamathematics whose object would be to prove the consistency and completeness of systems of axioms of formalized mathematics. Metamathematical research was undertaken from two different points of view, and the connection between the two methods is in our view an essential fact: one of these methods employs the extensive processes of set theory, the other more in line with the directives given by Hilbert himself, constitutes what Hilbert called 'proof theory'.[3]

From the point of view of proof theory, a mathematical theory contains, as given with the initial system of axioms, the set of propositions that can be obtained from these axioms by applying the rules of substitution and of passage, just like the 'conclusive schema':

$$S \xrightarrow[T]{S} T$$

In these conditions, to prove the consistency of a system of axioms amounts to proving the compatibility of the set to its consequences, that is to obtain the certainty that it is not possible to encounter in this set the contradictory proposition: $p.\overline{p}$ (p and not p). Similarly, the system is said to be complete if any proposition of the theory is either provable, or refutable (by proof of its negation), that is, if one always knows which proposition p or $\overline{p}$ is provable. The properties of 'consistency' and 'completion' for a system of axioms, and of 'provable', 'refutable', 'irrefutable' for the propositions of a theory, are called structural properties (*Cf.* Carnap 1934) because their attribution to a system or to a proposition requires an internal study of the set of consequences of the considered system.

We will now define the second method of metamathematical study of a formalized theory, the 'extensive' method, which adjoins to the system of axioms studied the consideration of individual domains that may be used as values for the arguments of logical functions of a formula of the theory. Certain metamathematical notions exist that can only be defined in the extensive method. The principals are those of 'universal validity' (*Allgemeingüligkeit*) and 'possibility of realization' (*Erfüllbarkeit*) (Hilbert and Bernays 1939, 8, 128). A formula is said to be universally valid if, whatever the manner in which predicates are substituted for the variables representing logical functions and whatever the individual domains substituted for the argument variables, a true proposition is always obtained. A formula is said to be realizable if there is at least one individual domain and one mode of similar substitutions capable of making the formula a true proposition. Thus, for example, the formula:

$$(x)P(x,x) \rightarrow (x)(\mathrm{E}y)P(x,y)$$

is universally valid. Whatever the relation P and the considered domain, for all x in this domain such that the relation P takes place from this object to that object, there is always at least one object y such that P takes place from object x to object y, the latter could be none other than x itself. On the other hand, consider the system of the following three axioms (Hilbert and Bernays 1939, 14):

$$\overline{(x)R(x,x)}$$

$$(x)(y)(z)R(x,y) \ \& \ R(y,z) \rightarrow R(x,z)$$

$$(x)(\mathrm{E}y)R(x,y)$$

Suppose that R means 'to be smaller than'. The first axiom requires that the relation 'smaller' does not take place from one individual to that individual itself, the second, that it is transitive, the third, that there always is, for all x, at least one y such that $x < y$. It is impossible to verify this system in a finite domain, since the third axiom cannot be verified, the last number of a finite collection not being smaller than any number. The system is thus achieved only in an infinite domain. The properties of universal validity and of possibility of realization are in duality: a proposition is universally valid only if its negation is not realizable and, by supposing the excluded middle to be admitted, a formula is realizable only if its negation is not universally valid.

The structural conception and the extensive conception being thus defined, we will now present the importance of the passage that modern mathematical logic operates from one to the other.

Consider first of all the links that exist between the structural conception of consistency and the extensive notion of 'realization in a given domain'. When the given domain only contains a finite number of individuals, by trying all possible combinations, the question (*entscheiden*) of knowing whether there is a choice of values of the variables allowing the proposition obtained by the conjunction of all the axioms of the proposed system to be verified can be settled. It can be said that this choice realized the system of axioms, and it can easily be proved by relying on the principle of the excluded middle in the finite, that if a conjunction of axioms is thus realizable, its negation is not provable. This latter property is none other than the consistency of the system. There is therefore equivalence in the finite between the non-contradictory structure of a system of axioms and the existence of a domain containing a number of determined individuals and such that the system is realizable in this domain (Hilbert and Bernays 1939, 17). This result, so simple that it seems almost obvious, contains the seed of a new theory of the relations of essence and existence. These two notions cease to be in effect relative to the same mathematical entities. It is no longer a matter of knowing whether the definition entails existence, but of inquiring whether the structure of a system of axioms can give rise to a domain of individuals who support among themselves the relations defined by the axioms. The envisaged essence is rightly that of the system of axioms, but the existence that the internal study of the system allows to be asserted is that of the 'interpretations' of the system, the domains in which they are realized.

When the system of axioms can only be realized in an infinite domain of individuals, there is no longer necessarily equivalence between the consistency of the system and the existence of an interpretation of this system. The principle of the excluded middle is not necessarily applicable in the narrow form that it takes in the case of finite domains, and the conjunction of the axioms of a system can be irrefutable, hence the non-contradictory system, without there existing in fact an interpretation of the axioms of the system. In sum, the possibility of realization is a stronger requirement than consistency. A dissociation is established between the extensive point of view and the structural point of view. The problem of realization must leave room, according to Bernays, for a purely internal search for irrefutability. This is what Bernays calls the negative conception of consistency. Bernays has in this sense proven (Hilbert and Bernays 1939, 156) that for the predicate calculus there was a strict equivalence between consistency and irrefutability of the conjunction of axioms, without implying in the least the existence of an interpretation of the system.

The facts show that it is nevertheless difficult to develop a purely structural theory of consistency. In this regard, the most authentic attempt at a structural proof of the consistency of arithmetic is that by Gentzen (1936). We do not propose to discuss it here in detail, referring for this to Cavaillès's thesis (1938b). We would like only to recall a key element of it. Far from seeking to realize the axioms of arithmetic, as in the extensive method, Gentzen remains faithful to the conception of Hilbert's proof theory and intends to follow each of the operations involved in a proof, to prove that a contradictory proposition could not slip in at any moment whatsoever. For this, Gentzen undertook, as Herbrand had moreover already tried, to reduce each of the operations of a proof to increasingly simple operations until the final formula of the proof had been put in the form of an evidently true expression. For this, Gentzen coordinates an ordinal number to each proof and proves an essential theorem that amounts roughly to this: the reductions succeed because the set of these ordinal numbers are 'well ordered' as defined by set theory. The proof is thus not based on the existence of an interpretation of the system, it resorts no less to the consideration of a set of numbers whose existence is equivalent to the completion of the proof of consistency. Whatever the intentions of the author, the extensive point of view reappears associated with the structural point of view. The structure of a theory is expressed anew by the existence of entities other than the elements of the theory.

These considerations apply in an even more manifest way when one moves from the study of consistency to that of the completion of a system of axioms. In the purely structural sense of Hilbert, the completion of a system is equivalent to the fact that any formula of the theory possesses, as we have seen, either the property of provability, or the property of refutability. In regard to the predicate calculus, Gödel proved a theorem of completion that immediately had extensive repercussions (1930). He in effect established that any formula of predicate calculus is either refutable, or realizable in the domain of whole numbers. So, for the envisaged system of axioms, there is equivalence between the structural properties of irrefutability and the extensive property of realization. In fact, says Bernays, a finitist theorem of completion can be extracted from Gödel's proof, according to which there always exists, for an irrefutable formula, a formal realization within the framework of arithmetic (including the excluded middle). Conversely, Herbrand's theorem allows the deduction of the irrefutability of a proposition from the existence of a domain of individuals who realized this proposition. The domain envisaged by Herbrand to tell the truth does not realize a logical formula like the numerical values realize an algebraic value. Nor do they contain the possible values of arguments and of logical functions in the general sense defined above. We shall see however in the following chapter how it is possible nevertheless to consider them from a certain point of view as interpretations of the system of axioms. In any case, they are constructed in a rigorously intuitionistic way, they possess a nature proper to domains, fundamentally heterogeneous to the nature of the propositions of the system of axioms, and their interest results from their existence being linked closely to the internal structure of the system to which they are associated. Herbrand's theorem in effect can be summarized as follows: the existence 'in extension' of an infinite domain in which $\overline{P}$ is realizable is equivalent to the structural fact that P is not part of the set of provable propositions of the theory.[4] As in the study of consistency, we encounter, at the threshold of the theory of completion, theorems that identify the structural properties of a system of axioms with the existence of a domain in which the propositions of the system are realized. The connection in that sense is complicated by the fact that the structural properties of P are expressed in extensive properties of $\overline{P}$ and conversely. It does not appear to us any less like the first schema of all the mathematical expansions that we will describe later. It is offered to us in effect like a pure case of solidarity between a set of formal operations defined by a system of axioms and the existence of a domain in which these operations are realizable. It seems that a certain restriction still adheres to this logical schema; the genesis only takes place in effect in one sense, the operations of the domain. Now, if, between the domain and the operations definable on it, a rigorous appropriation can be established, then one can seek just as well to determine the operations from the domain as the domain from the operations. We have just seen how in logic the operations rely spontaneously on a domain that they seem to summon by the very organization of their structure. We will see that conversely, in a very large number of cases, the domain appears already prepared to give rise to certain abstract operations.

Our intention being to show that completion internal to an entity is asserted in its creative power, this conception should perhaps logically imply two reciprocal aspects: the essence of a self realizing form within a matter that it would create; the essence of a matter giving rise to the forms that its structure designs.

This symmetric presentation of things sometimes seems justified when it is a matter of certain simple axiomatic theories like arithmetic or domain theory: there is absolute reciprocity between the domain of whole numbers and the Peano axioms, between the domain of rational numbers and the axioms that define the four arithmetic operations. One can look for the mathematical entities that satisfy the conditions of axioms just as well as for the axioms that implicitly define the domain in question. The expression of a closed domain with respect to certain operations shows well in this case the tight equivalence between the 'material' point of view of the domain and the 'formal' point of view of the operations.

In fact, the schema of geneses that we are going to describe within more complicated theories abandons the too simplistic idea of concrete domains and abstract operations that would possess in themselves for example a nature of matter or a nature of form. This notion would tend, in effect, to stabilize the mathematical entities in certain immutable roles and ignore the fact that the abstract entities that arise from the structure of a more concrete domain can, in their turn, serve as a basic domain for the genesis of other entities. It is therefore only within a determined problem that distinct functions can be assigned to different kinds of entities. The essential fact is the genesis of kinds of entity from one another, and it is in a purely relative sense, to account for these mutual relations, that we always use the term 'domain' for the given structure and that of 'creation' for the final object of the described genesis. Thus, for example, in regards to mathematical logic, despite all the 'material' meaning attached to the expression of 'realization', it seems to us more congruous with the logical schema that we describe to say that these are the axioms that form the domain and that the interpretations come from the domain, as the entities that it creates.

2. EXISTENCE THEOREMS IN THE THEORY OF ALGEBRAIC FUNCTIONS

We will now turn to the study of purely mathematical examples, in which one sees a structure preform the existence of abstract entities on the domain that this structure defines. Our first example is borrowed from Riemann's theory of algebraic functions. We saw in Chapter 3 how he had been led to associate, to an algebraic function, a surface composed of a certain number of sheets welded in crosses along certain cuts, and at each point of which the function is locally uniformizable. By continuous deformation, this Riemann surface can be given the form of a two sided disk, pierced with holes. The number p of these holes is a topological invariant of the surface that determines the $2p$ retrosections of the surface, that is, the maximum number of independent closed curves that can be traced on this surface without dividing it into two separate regions. This number p defines the genus of surface. That being posed, Riemann's train of thought is not so much to construct the Riemann surface of an algebraic function that would be defined by its algebraic equation, but to choose an arbitrary surface to later find out if an algebraic function exists of which this surface is the Riemann surface. The Riemann method does not proceed directly, it is true, to the determination of the algebraic functions of a surface, but to the search for functions that are integrals of algebraic functions. These integrals are called abelian integrals, and there are three kinds of them: the integrals of the first kind that never become infinite; those of the second kind that have poles; and those of the third kind that have logarithmic singularities.[5]

Riemann's essential theorem is the following: the number of linearly independent integrals of the first kind is equal to the topological genus of surface. There is a connection of extraordinary importance here between the topological structure of the surface and the existence of abelian integrals that are everywhere finite on this surface. Also, mathematicians cannot help but use the most admiring terms to describe this dual role of the genus. Here is what Weyl writes, for example:

> One cannot refuse to recognize the importance of the number p in the theory of functions, and yet this number is, as for its nature and its origins, a magnitude of *Analysis situs*. The main lines of the theory of functions . . . are all inspired by the divine lawgiver (*göttliche Gesetzgeberin*) who hovers unseen above the functional reality: *Analysis situs*. (Weyl 1913, 134)[6]

We would like to go into the detail of this genesis of the integral from the domain and show the moment in which, imperceptibly, the passage from the structure of the surface to the existence of integrals that are everywhere finite is carried out. It is proven that if any two points *a* and *b* are taken on the surface, and if the surface is cut from *a* to *b*, there exists a potential function[7] $u_{a,b}$ that admits the two points *a* and *b* as logarithmic singularities, and that, by crossing the cut that goes from *a* to *b*, is subject to a determined discontinuity and is continuous, finite and uniform everywhere else on the surface. It is easy by departing from here to define a potential function that has no logarithmic singularity (nor any pole): all that is required is to apply the preceding theorem of existence to the case of a succession of points z', z'', z^{n-1}, z', such that the last coincides with the first, and which are situated on one of the 2*p* closed curves that does not divide the surface into separate regions (a retrosection of the surface). The following functions are thus obtained

$$u_{z'z''}, \ldots\ldots u_{z^{n-1}z'}$$

which, by virtue of a theorem of addition, give rise to the potential function

$$u(z) = u_{z'z''} + \ldots\ldots + u_{z^{n-1}z'}$$

This potential function admits as a line of discontinuity the entire closed cut $z'z''$. It therefore no longer possesses the logarithmic singularities since the singularities of the functions of which it is the sum cancel each other out in pairs, and it is not uniformly zero since, when the variable *z* jumps over this cut, the value of the function undergoes a determined increase. In addition, the cut does not divide the surface into two regions, the function is finite, continuous and uniform over the whole surface. As there are 2*p* retrosections, the potential functions that are everywhere finite can thus be defined, and by associating them in pairs, *p* linearly independent abelian integrals of the first kind are thus obtained. It can then be demonstrated that others can be obtained, which completes the proof of the dual role of the number *p*.[8]

The importance of the 'canonical cutting' of the surface is seen in this theory: the brute surface leaves nothing to appear on it. On the other hand, the structure it receives from its retrosections renders it apt to a creation. The precise moment of the genesis resides in the act by which it confers on the structure a dual interpretation: insofar as the retrosections render the surface simply connected, they have a topological value, insofar as they require the variables of certain functional expressions to jump, they already distribute in advance these integrals on the surface. The passage from essence to existence thus becomes a connection between the structural decomposition of an entity and the existence of other entities that this decomposition gives rise to. We will show that this result is reflected in the theories in which the problems of decomposition play an essential role: algebraic theories.

3. EXISTENCE THEOREMS IN CLASS FIELD THEORY

An aspect of class field theory has already been discussed in Chapter 3. We shall now see how the theorems of existence of this theory are closely related to a decomposition into classes of ideals contained in the base field k. The set of ideals of k forms a group with respect to multiplication. In this group a division into classes is established as follows: envisage the subgroup of principal ideals (those which are generated by the multiples of a number of the field) and two ideals are said to belong to the same class when their quotient is a principal ideal. In the group of ideals of a field, h distinct classes of equivalent groups are thus obtained. This number h therefore gives the most important structural notion relative to the ideals of the field k. In class field theory it plays an analogous role to the genus p of the theory of Riemann surfaces, and like it, can be immediately interpreted in terms of being created.

In dealing with the universal covering surface, we have seen how certain extension fields K of k, called class fields over k, could be put in relation with the subgroups H of the group of all the ideals of k. Hilbert's quadratic class field theory studies the class fields that can be generated from the base field when the square root of an element μ contained in k is adjoined to this base field. In sum, k contains μ, but does not contain $\sqrt{\mu}$, and the conditions under which the extension $k(\sqrt{\mu})$ can be a class field for a group of ideals contained in k are sought. An essential result of the theory, that we can only indicate without proof, establishes a very precise connection between the number h of the classes of ideals in k and the existence in k of a number μ such that $k(\sqrt{\mu})$ is a class field: it is necessary, for such a number to exist, that the number of classes h is even.[9] Just as in the theory of Riemann surfaces, in which the existence of abelian integrals that are everywhere finite was linked to the number p, in class field theory the properties of the number h, which measures the classes of an internal decomposition of the field, have repercussions by the assertion of existence relative to a number μ which generates a quadratic class field $k(\sqrt{\mu})$.

We compared a theory in which existence arises from the cutting of a surface, to a theory in which existence arises from the division into classes of ideals of a field, because the analogies of these theories seemed to shed light on a same conception of the relations of essence and existence, in which the existence of an entity emerges from the structural decomposition of a basic domain. We showed how this notion could be related to the metamathematical distinction between structural properties and extensive properties. Now these notions have only taken shape during the last ten years, and since 1900, at the Paris Congress, Hilbert established the comparison of the two mathematical theories which we have just spoken about! Mathematicians have always admired the prophetic power of a genius who, in 1900, could state 23 problems to be solved, most of which were later solved by him. We will quote a fragment of this Conference that concerns the problem that we are currently dealing with, and where it seems that Hilbert outlines the philosophy of mathematical genesis that should suggest to us the logic that he went on to develop twenty years later:

> The analogy between the deficiency of a Riemann surface and that of the class number of a field of numbers is also evident. Consider a Riemann surface of deficiency $p = 1$ (to touch on the simplest case only) and on the other hand a number field of class $h = 2$. To the proof of the existence of an integral everywhere finite on the Riemann surface, corresponds the proof of the existence of an integer α in the number field such that the number $\sqrt{\alpha}$ represents a quadratic field, relatively unbranched with respect to the base field [that is, a class field]. (Hilbert 1900; 1935, 312 [2000, 423])

This text does not perhaps contain the terms of the philosophical commentary that we have given, it certainly contains the spirit. From the genus of the surface or of the number of class fields to the existence of the integral or to that of the number α, there is in effect a passage from one kind of entity to another kind of entity, and the possibility of this passage results from the discovery of a way to structure a domain that renders it apt to a creation.

4. THE THEORY OF THE REPRESENTATION OF GROUPS

The theory of the representation of groups will show us how the role of domain and that of the created entities, do not depend on the intrinsic nature of the mathematical entities in question, but are conferred on them by their respective functions in a process of genesis. Historically, the notion of 'representation' has a more 'concrete', more 'material' sense than the notion of group, but as representations arise from the structure of groups, it will be more consistent to our conception to assign to the group the nature of domain, and to see in the representations, the abstract entities created on this domain that is called the space of the group. We will see, in addition, that this change of perspective corresponds well with the current evolution of the notion of representation.

The notion of group has its origin in the groups of transformations. A group of transformations presupposes a space of points E on which are defined the operations of the group. A set of transformations forms a group if, given a transformation S that passes from point p to point p' and a transformation T which passes from point p' to point p'', a transformation exits, considered as the product ST of the first two, that passes from point p to point p''. It is necessary in addition that the group contains the identical transformation, which makes that point itself correspond to each point, and finally, that for any transformation, an inverse transformation exists. This being posed, in the transformations, the sense of operations defined over a space of points can be neglected, and the elements of an abstract set within which a certain law of composition is defined are no longer seen. What is then obtained is an abstract group whose elements no longer have any intrinsic meaning and can be regarded as the points of an abstract space: the space of the group. We can then inversely pass from this abstract group to a group of transformations by restoring a nature of transformations to elements of the group. It is in this sense that a group of transformations *realises* an abstract group. In particular, when the realization of the group makes a linear and homogeneous transformation of the space of primitive points (see Chapter 1) correspond to each element of the group, we have what is called a *representation* of the group. A representation is therefore a correspondence between the elements s of an abstract group and the transformations U operating on the points of the space E, and is defined by equations of the following type:

$$\begin{aligned} x'_1 &= u_{11}x_1 + \ldots\ldots u_{1n}x_n \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x'_n &= u_{n1}x_1 + \ldots\ldots u_{nn}x_n \end{aligned}$$

Weyl's train of thought is without any doubt to consider (1913, ch. 3) that, from transformations to the abstract group, there is impoverishment, and that, in the opposite direction, the descent of the abstract group to representations that operate on a space of points, nourish with matter the slightly empty form of the abstract group. The representations of the group thus play the role of interpretations of a system of axioms in logic with respect to the abstract group. We will see that despite this concrete character that comes from their geometric origins, the representations can be considered as abstract entities defined on the group space. We will then see how their existence is linked to the structure of this space.

The representations that will be envisaged are those that are unitary and irreducible. A representation is unitary if the transformations that constitute it leave invariant a Hermitian form $x_1\overline{x_1} + x_n\overline{x_n}$.[10] It is in addition irreducible if it does not leave invariant any subspace of the space on which it operates.

Let there then be s, a variable element of the group, and U the corresponding transformation for this element s in a unitary and irreducible representation of the group. This transformation is characterized by an array of coefficients

$$\begin{array}{c} u_{11} \ldots\ldots u_{1n} \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ u_{n1} \ldots\ldots u_{nn} \end{array}$$

and we designate any element of this table by $u_{ik}(s)$ to show that the coefficients of a transformation depend on an element s of the group. They are thus presented as functions of one variable that is none other than the variable element of the group. These are therefore like functions defined on the space of the group, and an essential theorem of the theory establishes that the coefficients of all unitary representations, irreducible and non-equivalent[11] among themselves, form on the space of the group a complete system of basic functions. This means that all functions definable on the space of the group can be represented as a linear combination of coefficients of these representations. If a geometric sense is still attached to the notion of representation, this sense is totally absent from the notion of function defined on a group space, and under this new aspect of basic functions, the envisaged coefficients of representations are seen to have in effect the sense of being defined on the abstract group space.

All that now remains for us is to indicate how, in the case of finite groups, these abstract entities are not only defined on the group space but emanate directly from its structure. The structure of a group is essentially the number of its classes. Let a be a fixed element and s any element of a group, the element asa^{-1} is said to be conjugated by s with respect to a. Two conjugate elements are said to belong to the same class, and the group is thus decomposed into classes of conjugate elements. This division into classes can be interpreted in terms of genesis, in regard to the representations of the group. For finite groups, we have in effect the following theorem: the number of representations that are irreducible and non-equivalent among themselves[12] is equal to the number of classes of the group (*Cf.* Weyl 1913, 130 [1964, 147]). The number of these classes characterise an internal decomposition of the group. It also has a creative sense since it determines the number of representations whose coefficients form a basic system for the functions of the space of the group. Its role is therefore in all respects comparable to that of the number p in the theory of Riemann surfaces.

We have just studied the clearest case in which the structure of a domain is immediately interpretable in terms of existence for certain functions defined on that domain: these are cases in which this structure is characterized by a number attached to the decomposition of the basic domain. The mathematical geneses are based upon the dual meaning of this number, structural and creative. We therefore have here a means to distinguish, in the relations that support these two different kinds of mathematical entities, what plays the role of concrete domain and what can be conceived as abstract entities created on this domain. It is perfectly possible that, in a schema of genesis, a same kind of entity plays the role of abstract with respect to a concrete base and is, in another genesis on the contrary, the concrete of a new abstract. Thus, for example, the group space can be conceived both as being abstract with respect to a space of fundamental points, and as a concrete domain with respect to the representations of the group. The essential element in the passage from essence to existence is not so much the nature of the role assumed by each kind of entity present, than the very existence of the passage between two kinds of the entity.

A presentation of things like this implies such a reversal as regards to the habits of classical thought, that, in closing, we must again insist upon the new meaning that the expressions concrete and abstract receive here. The 'abstract group' in the mathematical sense of the term, just as the systems of axioms of formalized mathematics are often conceived as purely formal structures, 'abstract' as defined by the concept of classical philosophy, and independently of their arithmetic or geometric realizations. It seemed to us, as we have seen, that these structures, abstract in the classical sense, were so profoundly engaged in the genesis of their realizations that it was better to abandon any reference to an ontology in the use of the expressions concrete and abstract in order to designate respectively by these terms only the basic structures and the entities whose existence is determined by these structures.

This conception of the relations of the concrete and the abstract seems to be particularly adapted to express not only the engagement of the concrete in the genesis of the abstract, but also, the relations of imitation that can exist between the structure of this abstract and that of the concrete base. We shall give only one example: for certain systems of numbers (called hypercomplex systems), as well as their representations, a mode of complete decomposition into irreducible elements can be defined. An essential theorem of the theory then establishes a close relation between the complete decomposition of certain of these hypercomplex systems and that of all their representations (*Cf.* van der Waerden 1931, 180). We thus see how, in certain cases, the genesis of the abstract from a concrete base is asserted up to the realization of an imitation of structure between these kinds of entities that arise from one another.

CHAPTER 5
'Mixes'

What characterizes the geneses described in the previous chapter is that created entities originate directly from the basic domain, and that from one kind of entity to another there is thus immediate passage. The structure of the domain plays a dual role in these creations, since it is interpretable both as a decomposition of the domain and as a distribution of new entities on this domain. It is not however a third kind interposed between the domain and the created entities because it is entirely inherent to the domain that receives it.

Certain mathematical geneses are however able to be described by schemas of this type. They obey more complicated schemas in which the passage from one kind to another requires the consideration of mixed intermediaries between the domain and the entity sought after. The mediating role of these mixes is going to result from their structure imitating that of the domain on which they are superimposed, so that their elements are already the kind of entities that arise on this domain. Requiring the adaptation of these radically heterogeneous realities to one another, mathematics recognises in its own development the logical necessity of a mediation, comparable to that of the schematism of the Transcendental Analytic, intermediate between the categories and intuition. The text in which Kant defines the schematism is in this respect of an importance that goes far beyond the special problem of the philosophy of the understanding. It contains as a general theory the mixes that we'll see be applied almost perfectly to the needs of mathematical philosophy. Here is the text:

> Now it is clear that there must be a third thing, which must stand in homogeneity with the category on the one hand and the appearance on

> the other, and makes possible the application of the former to the latter. This mediating representation must be pure (without anything empirical) and yet intellectual on the one hand and sensible on the other. Such a representation is the transcendental schema . . . Now a transcendental time-determination is homogenous with the category (which constitutes its unity) insofar as it is universal and rests on a rule *a priori*. But it is on the other hand homogenous with the appearance insofar as time is contained in every empirical representation of the manifold. Hence an application of the category to appearances becomes possible by means of the transcendental time-determination which, as the schema of the concept of the understanding, mediates the subsumption of the latter under the former. (Kant 1998, 272)

The essential moment of this definition is that in which the schema is conceived from two different points of view as homogenous to the nature of two essentially distinct realities and between which it serves as a necessary intermediary for any passage from one to the other. The mixes of the mathematical theories assure the passage from one basic domain to the existence of entities created on this domain by the effect of a similar internal duality.

It is possible to find a preliminary outline of these mixes in the theories we have mentioned in the previous chapter. Consider first of all domain theory in mathematical logic. The extensive property for a formula to be realizable in a domain of k individuals implies, as we have seen at the beginning of Chapter 4, that the domain of variation of the formulas variables contains at least k individuals, and that it is possible to find a choice of values of the variables that make the formula, a true formula in the ordinary sense. This research relies on the fact that in the case of finite domains (in the ordinary sense) any logical formula can be put in the form of a disjunction of terms, none of which contain any variables. If any of these terms is true, and it is determinable by a finite number of steps, the disjunction is an identity and the original formula is then realizaible. For example, in a domain containing only the two variables 1 and 2, the expression $(\exists x)\ P(x, x)$ is equivalent to $P(1, 1) \vee P(2, 2)$. The simple inspection of two arithmetic expressions $P(1, 1)$ and $P(2, 2)$ allows one to see if either of these expressions is true, and, in the same way, resolves the problem of knowing if the formula is realizable. We have already indicated that when the variables can take an infinity of values, it becomes impossible to form the disjunction that results from all the possible substitutions.

The substitution that realizes the given proposition cannot be discovered by a finite process, and the passage from the extensive point of view of the realization to the structural point of view of irrefutability is due to this impossibility, as we have shown. The extensive point of view however reappears in Herbrand's research, but under the form of metamathematical domains, intermediate between the signs of formulas and mathematical domains of their effective values.

Herbrand's intention is to reduce any arithmetical proposition, and this irrespective of the extension of the domain of definition of the variables, to a finite disjunction of terms, no longer containing variables and whose examination thus permits, as in the case of finite mathematical domains, the decision as to whether this disjunction is a logical identity. It was therefore necessary for him to find a way to define, for a variable that can take an infinity of mathematical values, a finite number of metamathematical values that thus symbolize the existence of this infinity so difficult to handle. Consider for example the variable y playing a part in the 'particular' form: $(\exists y)$, (there is a y such that . . .). This y can be equal to any of the terms of an infinite set. Herbrand (1930, 99 [1971, 150]) however only envisaged as a domain of values of this variable a domain C_1 composed of a sole value a_1. Now let the variable z play a part in the 'general' form: (z) (for any value of z). Herbrand associates a certain index function f_z to this general variable that has the restricted variables preceding the general variable in question for arguments. The values of this index function are situated in a domain C_2 which only contains the number of elements necessary so that all elementary functions (descriptive functions given in the formula and index functions) make the different values $a_1 \dots a_n$ of C_1 correspond to the different values $a_{n+1}, a_{n+2}, a_{n+3} \dots$ in the domain C_2. If we have, for example, the following succession of signs of variables: $(\exists y)$ (z) (there is a y such that for any value of z), and if a_1 is the value of $(\exists y)$, to z corresponds the function $f_z(y)$ to which, as a value, is given by convention the letter a_2 of a domain C_2, a_2 represents the 'value' of z for the value a_1 of y. A finite series of domains of ascending order is thus constructed by induction, each of which contains the values of descriptive functions and index functions to arguments taken in the domains of lower order whose value has not yet been given. The elements of these domains are therefore in close correspondence with the signs of the variables of the formulas. They constitute a system of new signs that are substituted for the first, rather than a set of true values for the variables designated by these signs. On the one hand, they do not possess any less a nature of domains independent of the formula they realize, and thus present a first aspect of mixes insofar as they are intermediate between the formal signs and their effective mathematical value. From this first character, a second can be drawn, infinitely precious for the general conception of mixes in mathematics. Intermediate between the signs and their values, these domains are, on the one hand, homogeneous to the finite discontinuity of the signs, since to a sign of variable $(\exists y)$ there only corresponds a value a_1, and, on the other hand, they symbolize an infinity of mathematical values, since the letter a_1 represents any current mathematical value of the variable y when it plays a part in the particular form $(\exists y)$. A mediation therefore takes place via these domains from the finite to the infinite, which, in the cases treated by Herbrand, allows the domination of the infinite, and this is the role that we recognize in the mixes that will now be considered. We will see in effect with respect to the Hilbert space how, between the continuity of a basic domain and the discontinuity of solutions to certain expressions defined on this domain, a mix is interposed that takes the continuous by the origin and the topology of its elements, the discontinuous by its structural properties, and allows the connection of the one to the other.

1. HILBERT SPACE

We saw in Chapter 1 how certain differential equations or partial differential equations could only be solved if the boundary conditions, initial and final, are also given, which the functions that are solutions of the proposed equation must satisfy. Consider for example a vibrating membrane whose edges are firmly embedded in a fixed wall. The equation that translates the oscillation of a point of the membrane is the classic equation $\Delta u + \lambda u = 0$. It is evident that any solution u must be zero at the boundary of the membrane. Consider also in quantum mechanics the Schrödinger equation which expresses the amplitude a of a corpuscle of mass m oscillating in an exterior domain of intensity V and possessing an energy E. This equation[1] is written

$$\Delta a \frac{8\pi^2 m}{h^2}[E - V(x,y,z)]a = 0$$

Given that the amplitude a is the amplitude of a wave associated with a corpuscle, it is natural to assume that it occupies only a limited region of the x, y, z space, and that it is zero at the boundaries of this region. The basic domain, in these cases, is evidently the domain D of variation of the variables. The entities whose existence is trying to be determined are the functions that are solutions of the problem, but the structure of the domain is not directly adapted to the identification of these solutions. The schemas of genesis of the previous chapter are thus powerless to describe the passage to existence for equations of this type.

These equations are of considerable importance because their solution presents as a remarkable synthesis of continuous notions and discontinuous notions. They are in effect amenable to solutions $u_1 \ldots u_n \ldots$, continuous over the whole domain of variables and zero at the limits for certain discrete values of the parameter, $\lambda_1, \lambda_2 \ldots \lambda_n$, which forms the spectrum of eigenvalues of the equation. The solutions $u_1 \ldots u_n \ldots$ corresponding to these eigenvalues have an essential property for the whole theory that we will present. They form a complete system of continuous functions in the basic domain D. This means that any continuous function in this domain, twice differentiable and zero at the limits, can be represented as a linear combination of these basic functions. It can again be said that any arbitrary function $f(x, y, z)$ on the domain can be expanded into a convergent series in the system of basic functions in the following form

$$f(x, y, z) = c_1 u_1 + c_2 u_2 \ldots c_n u_n \ldots$$

An analogous situation was met in the study of integral functions.[2] Consider first of all the linear and homogeneous integral equation (that is, without a second member)

$$\text{(I)} \qquad \varphi(s) - \lambda \int_0^1 K(s,t)\varphi(t)\,dt = 0$$

The variable s varying from 0 to 1, the unknown function sought after is the function $\varphi(s)$. The function $K(s, t)$ is a given continuous function of two variables. It is the kernel of the integral equation, which we will assume in the symmetrical series in s and in t: $K(s, t) = K(t, s)$. As in the previous problems, it is shown that for $0 < s < 1$, the equation (I) only has solutions for certain discontinuous values of the parameter, the eigenvalues $\lambda_1, \lambda_2 \ldots$. The solutions $\varphi_1, \varphi_2 \ldots$ that correspond to them are again called solutions or eigenfunctions of the homogeneous equation. We shall see later that the functions and the eigenvalues depend essentially on the kernel $K(s, t)$. What is of interests to us here is that in the case in which a countable infinity of eigenfunctions exists, they form a complete basic system for the set of functions $f(s)$ that satisfy an equation of the type:

$$f(s) = \int_0^1 K(s,t)x(t)\,dt$$

$x(s)$ being an arbitrary continuous function.

We therefore have

$$f(s) = \sum_{n=1}^{n=\infty} c_n \varphi_n$$

and this theorem of expansion in uniformly convergent series of eigenfunctions of the kernel $K(s, t)$ applies particularly to functions that are solutions of the non-homogeneous integral equation (with second member)

$$\varphi(s) - \lambda \int_0^1 K(s,t)\varphi(t)\,dt = f(s)$$

Any solution of the non-homogeneous equation is a limit of the finite linear combinations of eigenfunctions of the corresponding homogeneous equation. The central idea that guided Hilbert in his general theory of linear integral equations can then be seen to appear: on the domain $0 < s < 1$, construct a space of functions that have global characteristics such that, by an appropriate decomposition of this space, a system of basic functions can be distinguished that are at the same time eigenfunctions of the proposed equation. Insofar as they form a basic system for the functions of the functional space, they are connected to the structure of this space. Insofar as they are eigenfunctions of the proposed equation, they are detached from the set of other functions of the space in order to appear with their new sense of solutions. There may be no clearer case in which the structural decomposition of a space is equivalent to the assertion of the existence of the functions sought after, and we will see that the functional space of this theory lends itself to similar geneses, because, on the one hand, it is endowed with a topology as a set of points, and, on the other hand, its elements are already homogeneous to the solutions sought after, which are thus found included in this space before being recognized there.

We will first examine the schemas of genesis in the Hilbert space H, which is a vector space, and then see the application of the results thus found to the functional spaces of the theory of integral equations.

Here is the axiomatic definition of the Hilbert space H according to Von Neumann:[3]

A. H is a vector space. It is a set of elements $x, y \dots$, in which the addition $x + y$ of two vectors and multiplication $\alpha . x$ of a vector by a complex number α are defined;

B. H is Hermitian. This means that given any two vectors x and y of H, a 'scalar product' (x, y) of two vectors exists having the following properties:

   1) (x, y) is linear in
   $x(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 (x_1, y) + \alpha_2 (x_2, y)$
   2) (x, y) possesses the hermitienne symmetry[4]
   $(x,y) = \overline{(y,x)}$
   3) the scalar product (x, y) generates the existence of a positive Hermitian form, defined
   $(x,x) = |x|^2 > 0$ unless $x = 0$.

This form $|x|^2$ defines a metric in the space H.

C. H is complete as defined by the metric.

D. H is to a countable infinity of dimensions.

In a similar space, an infinity of complete systems of base vectors $\varphi_1, \varphi_2 \dots\dots$ exist, such that for any vector x of H we have:

$$x = \sum c_i \varphi_i .$$

We shall now consider in H, an n-dimensional closed linear manifold, that is, a subspace of the Hilbert space. 'Hermitian forms' can be attached to this manifold, that is, the algebraic expressions formed from the vectors of the manifold and amenable to being decomposed in a particularly remarkable way when a choice is made to express the coordinates of the vectors from a 'proper' system of base vectors. It is the connection of this choice of the basic system to the decomposition of Hermitian forms that is the essential stage of schemas of genesis that we claim to describe in this chapter.

Consider a linear operator A acting on the elements x of the manifold, that is, such that Ax is also an element of the manifold. The Hermitian forms that will be envisaged are the scalar products (x, Ax). What results from the axioms of Hilbert space is that

$$(x,Ax) = A(x,x) = \overline{A(x,x)}.$$

If any basic system in the manifold is chosen, the vectors x on the axes of the chosen coordinates have components x_i, and we have:

$$A(x,x) = \sum_{i,k=i}^{i,k=n} a_{ik} x_i \overline{x_k}; \quad a_{ik} = \overline{a_{ik}}.$$

In the adopted basic system, the Hermitian form $A(x, x)$ is therefore characterized by the matrix of coefficients

$$A^* = \begin{vmatrix} a_{11} \ldots\ldots\ldots a_{1n} \\ \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots \\ a_{n1} \ldots\ldots\ldots a_{nn} \end{vmatrix}$$

This matrix is comprised of rectangular terms, for which both indices are different, and square terms ($i = k$), the diagonal terms. It is possible then, and this is the central point of the theory, to find a system of privileged base vectors $\varphi_1, \varphi_2 \ldots \varphi_n$, such that the components of a vector x having become in this system $x'_1, x'_2 \ldots x'_n$, the Hermitian form is transformed into a sum

$$\sum_{i=1}^{i=n} \alpha_i x'_i \overline{x'_i}$$

containing only the square terms. The change in the basic system therefore has the effect of transforming the matrix A^* into a matrix containing only diagonal terms:

$$A^{*\prime} = \begin{vmatrix} \alpha_1 & & 0 \\ 0 & \ddots & \\ 0 & & \alpha_n \end{vmatrix}$$

The α_i are the 'eigenvalues' of the operator A, to which correspond the eigenvectors φ_i.

These results which are due to Hermite (2009) in the case of an n-dimensional manifold, were extended by Hilbert to the infinitely dimensional space H. We have not entered here into the study of the extremely delicate conditions of convergence that allow the passage to infinity, we only import the final aspect of the theory. In sum, the space H is decomposed into subspaces orthogonal to one another H_1, H_2 . . . Each of these subspaces is underpinned by the eigenvectors corresponding to an eigenvalue α_i. The Hermitian form $A(x, x)$ then becomes equal to the sum of its projections

$$A(x,x) = \sum \alpha_i x_i' \overline{x_i'}$$

in each of these different subspaces. This shows the dual role of eigenvectors: they have a structural role in the space whose orthogonal parts they underpin; they also have a sense of existence for the operator A, since they constitute the system of privileged vectors that allow the reduction of the Hermitian form into a sum of squares.

These considerations apply immediately the functional space of the theory of integral equations. Let there be a fundamental space E. This will be the basic domain of our schemas of genesis, the functional space being a mix interposed between E and functions sought after. Let μ be a Lebesgue measure defined on E. Let us consider the set Ω of functions $f(p)$ of complex value such that

$$\int_E f(p)\, d\mu \text{ exists and that } \int_E f_n(p) |^2 d\mu$$

has a finite value. This gives the space of summable square functions, which is, according to the work of Riesz (1907) and Fischer (1907), isomorphic to the Hilbert space. This space therefore possesses a topology identical to that of space H. Let,

$$\int_E | f_m(p) - f_n(p) |^2 d\mu$$

be the expression of the distance between two functions of the space. The space is complete as defined by the metric, and a countable infinity of basic systems exist such that any function of Ω can be expanded into a convergent series (as defined by the metric defined in Ω),

$$f(s) = \sum_{i=1}^{i=\infty} c_i \varphi_i .$$

All that remains is to see the role of this mix, which is the space Ω of the summable square functions, in the genesis of unique solutions of an integral equation with symmetric kernel. We only envisage the study of eigenfunctions of the kernel $K(s, t)$ which are situated in space Ω, their determination will be made by a privileged decomposition of this space, which, in regards to the kernel $K(s, t)$, plays the role that, in the Hilbert space, is played by the decomposition into eigenspaces corresponding to a Hermitian form $K(x, x)$. To do this, and this is the central point, Hilbert associates to the space of summable square functions a space H such that to kernel $K(s, t)$ there corresponds a Hermitian form

$$K(x,x) = \sum k_{pq} x_p x_q .$$

The space H is decomposed, as we have just seen, into subspaces each defined by the eigenvectors corresponding to an eigenvalue of the operator K. We next go immediately from eigenvectors and eigenvalues of the operator K in Hilbert space to the eigenfunctions and eigenvalues of the kernel $K(s, t)$ in space Ω, and thus obtain the solutions of the integral equation.

The genesis of these solutions therefore happens as follows: the basic domain E is the domain of variation of the variable. On this basic domain, a functional space is superimposed which, by its topology, is, to some extent, comparable to a space of points whose elements are functions homogeneous to the functions sought after. A decomposition of this space into eigenspaces then brings to the fore the existence of eigenfunctions of the equation. The essential moment of this genesis is therefore contained in the decomposition of the space, or what is equivalent, in the reduction of a quadratic form to a sum of squares. And here again, a text of Hilbert's in 1909 characterizes in the clearest way the connection of a structural decomposition and the existence of solutions:

> A quadratic form $Q(x)$, when it is continuous, can always be transformed by an orthogonal transformation into a sum of squares of new variables x' such that we have: $Q(x) = k_1 x'^2_1 + k_2 x'^2_2 \dots$ From this theorem the ensuing important theorem follows: a system of linear equations to an infinity of unknowns – thus an integral equation – as concerns the number and existence of its solutions, possesses all the properties of a system to a finite number of equations possessing the same unknowns. (Hilbert 1909, 59)

The importance of the theory of Hilbert space developed by Hilbert and his students, in particular Weyl and von Neumann, is considerable in the new physics. It happens to be that this theory, constituted between 1904 and 1906, was the tool the most adapted to the interpretation of quantum mechanics. Let us take, for example, the Schrödinger equation:

$$\Delta a \frac{8\pi^2 m}{h^2}[E - V(x,y,z)]a = 0$$

The variable parameter being energy E, this equation only has a solution for discontinuous levels of energy which are the eigenvalues of the equation. The physics of the continuous has often been opposed to the physics of the discontinuous. We see that the problem of the genesis of the discontinuous on the continuous is not a problem of physics, the drama is played out much higher, at the level of mathematics. The continuous is the domain of variation of the variable, the discontinuous are the solutions, and the mix that is the Hilbert space is homogeneous to the continuous by the nature and topology of its elements, and to the discontinuous by its structural decompositions. The theory of mathematical relations between the continuous and the discontinuous therefore seems to receive its entire meaning from the fact that it is incarnated in the more abstract schema of genesis, in which the passage from the continuous to the discontinuous happens through the intermediary of mixes whose fecundity results from the properties of their dual nature.

2. NORMAL FAMILIES OF ANALYTIC FUNCTIONS

We will now examine another model of mixes, also intermediate between a basic domain and a sought after function. These are the 'normal' families of analytic functions whose consideration led Montel to a theory of extreme richness.[5] We'll give an example of this mediating function of normal families by relying on the presentation that Montel made of a theorem of Caratheodory (1932). It is a matter of proving that any simply connected domain D of the plane of the complex variable Z can be conformally represented (see Chapter 3) on a circle d of the plane of the complex variable z when the boundary of the domain does not reduce to a point. The basic domain, as defined by our schemas of genesis, is obviously the simply connected domain in question, the sought after function is the function $z = G(Z)$, which ensures the conformal representation of the domain on the circle, and to which corresponds the inverse function $Z = f(z)$, which ensures the conformal representation of the circle on the domain.

What is interesting for us in this problem is the moment in which the necessity of a mix intervenes, that is, the moment in which the modes of passage to existence offered by the simplest theories do not agree with the envisaged conditions. We therefore admit the classical theorems that establish the existence of a conformal representation of limited domains by rectilinear polygons on a circle, and depart from the fact that these theorems cannot be applied to the case of a domain D, bounded, simply connected and whose boundary does not reduce to a point. Caratheodory's method consists then in interposing between the domain D and the function $z = G(Z)$ a family of functions such that the internal structure of this family immediately entails the existence within this family of the function sought after. The condition of structure that is employed is that of compactness. It is a topological notion that, when it is relative to a family of functions, allows comparing this family to a space of points and ensures a certain homogeneity between the nature of the family of functions and the nature of the basic domain. On the other hand, as the family of functions is composed of functions and not of points, its elements are, in another sense, homogeneous to the function sought after, so that it presents the aspect of a mixed intermediate between two heterogeneous realities. Here's the definition of compactness: given an infinite sequence of points, this sequence is said to be compact, that is, it is always possible to extract from this sequence a subsequence which converges uniformly to a limit point. As concerns the sequences of analytic functions, the property of compactness is no longer general. In order for it to be applied, it is necessary that the function satisfies certain restrictive conditions, that of being bounded as a whole for example. It is said that the functions $f(z)$ holomorphic in a domain D are bounded as a whole in this domain if we have $f(z) < M$, M being a fixed number regardless of the function $f(z)$ and regardless of z interior to d. The following theorem due to Montel then establishes the existence of compact, or normal, families of analytic functions: let a family of functions be holomorphic and bounded as a whole in a domain d, from any infinite sequence of this family a subsequence can be extracted which converges uniformly in each domain interior to d towards a limit function (*Cf.* Montel 1927, 21).

In the problem of conformal representation that we are occupied with here (*Cf.* Montel 1927, 98–102) the solution is going to be obtained by building a compact family on the domain D of a kind such that a limit function, whose existence is implied by the compactness of the family, is the function sought after. The structure of domain D is therefore not directly adapted to bringing to the fore the function sought after $z = G(Z)$, but we're going to see how this structure is adapted to the genesis of the mixed intermediary: we cover the plane Z with a grid formed by squares with sides equal to 1. All the edges are considered to be completely interior to the domain. They form a simply connected domain D_0 whose boundary is a rectilinear polygon. The grid with 1/2 sides obtained by conserving the previous gridlines is then envisaged, which allows a domain D_1 to be defined in the same way, containing all points of D_0. With the grid with $1/2_n$ sides, a domain of D_n is likewise obtained. The infinite sequence of domains D_0, D_1, . . . D_n . . . is such that: 1) each of these domains is completely interior to D; 2) D_n contains all points of $D_{n\text{-}1}$; 3) any point interior to D is interior to the domain D_n starting from a certain rank.

The domain D is thus structured by an increasing infinity of separate limited domains of rectilinear polygons: D_0 . . . D_n . . . By virtue of the theorems admitted at the beginning of the previous paragraph, there therefore exist functions $Z_0 = f_0(z)$, $Z_1 = f_1(z)$ . . . that ensure the conformal representation of the circle d on each of the domains D_n thus defined. The functions $f_n(z)$ are holomorphic in d, they are bounded since the values of Z_n are interior to D for any n, they therefore form a normal family in d.

The equations defining the correspondence can be solved with respect to z, we obtain:

$$z = G_0(Z_0),\ z = G_1(Z_1)\ .\ .\ .\ z = G_n(Z_n).$$

The functions $G_n(Z)$ are holomorphic and bounded in D_0 since the corresponding values are the affixes of the points interior to d. They thus form a normal family in D_0, which is truly the essential mix of the theory.

Montel then proves that a subsequence can be extracted from this family which converges uniformly in every domain D_n. Let $G(Z)$ be the limit function of this sequence and $f(z)$ the inverse limit function of the corresponding sequence in the family of functions $f_n(z)$. The function $G(Z)$ ensures the conformal representation of the domain D on the circle d, and its existence is thus determined within the normal family defined on the basic domain.

We thus see how the structural decomposition of *D* can be interpreted in terms of existence, not of the entity sought after, but of a mix, and how it is the structure of this mix that results in the existence of this entity sought after. One remark is required. In the case of Hilbert space, the structure of the mix gave rise to the values and to the eigenfunctions sought after by an internal decomposition analogous to the cuts and the divisions of the basic domains examined in Chapter 4. We are here dealing with a slightly different schema of genesis. The normal family proceeds from a sort of decomposition of the basic domain, but the limit function is no longer related to a mode of decomposition of the mix. It results from a selection that allows, within the normal family, the property of infinite subsequences to be convergent. This point of view leads to those that we will place in the next chapter to examine as a whole the schemas of genesis in which the existence of an entity results from this entity having the benefit of exceptional properties that allow it to be distinguished from among others. It is no less true that in the case of normal families, the selection of the limit function is determined by the compact structure of these normal families with the same logical rigor as the existence of eigenfunctions in the Hilbert space were connected to the possibilities of decomposition of this space.

The two problems that have just been briefly explained show that the discovery of the existence of an entity often presupposes the existence of a set that contains the entity being sought after even before one knows to see it there. The construction of this set is in effect easier because its nature is adapted directly to the domain on which it is superimposed. The problem of mixes in mathematics is therefore divided into a problem of imitation, the mixes imitating a structure of the domain, and a problem of selection. The mixes are thus situated well between two realities in the nature of each of which they participate.

CHAPTER 6
On the Exceptional Character of Existence

In most of the problems examined in Chapter 4 and Chapter 5, the passage from the structure of the domain to the existence of the entities defined on this domain results from a structural decomposition of the domain interpretable in terms of existence with respect to other entities, those that are created on this domain. We will now consider a new mode of connection between structure and existence, that in which the sought after entity results from the selection within a domain of a element distinguished among all others by its unique properties. Maurice Janet published an extremely valuable article in *Research philosophiques* (Janet 1933) on problems of this sort, the most important from the philosophical point of view: the exceptional properties that characterize the existence of the entity sought after are in effect extremal properties of maximum or minimum. This determination of existence by the research of maximum or minimum suggested to Janet essential connections with the thought of Euler and Leibniz. Janet cites the text of Euler:

> As the construction of the world is perfect and due to an infinitely wise creator, nothing happens in the world that doesn't have some property of maximum or minimum. Also there is no doubt that it is possible to determine all effects of the universe by their final cause, with the aid of the method of maxima and minima, with as much success as by their efficient causes. (Janet 1933, 1)

This text expresses the same ideas as Leibniz's famous treatise *On the Ultimate Origination of Things*, 1697:

> From this it is obvious that of the infinite combinations of possibilities . . . the one that exists is the one through which the most essence or possibility is brought into existence. In practical affairs one always follows the decision rule in accordance with which one ought to seek the maximum or the minimum: namely, one prefers the maximum effect at the minimum cost, so to speak. (Leibniz 1989, 150)

There is in this text as a dual program the application of the calculus of variations to mathematics and physics, and all the success of variational considerations in physics is known: Janet briefly redoes the history of Fermat's principle for light, Maupertius's principle for mechanics, and shows that the analogy of these two principles of the minimum dominate the flow of ideas ranging from Hamilton to Louis de Broglie.

We intend in this chapter to confine ourselves exclusively to mathematics and would like to show that, in the determination of the existence of a mathematical entity by considerations of extremum the logical schema of a novel solution to the problem of the passage from essence to existence is realized, in which, as in the schemas of previous chapters, essence and existence are concerned with distinct mathematical entities. When an entity is determined by the properties of maximum or minimum, it is necessary in effect to consider it as embedded in a whole and then to show that the structure of the whole is such that it allows the entity sought after to be distinguished. Insofar as the properties that render the selection possible are the properties of maximum or minimum, they confer on the entity obtained an advantage of simplicity and an appearance of finality, but this appearance disappears when an account is given of that which ensures the passage to existence. It is not the fact that the properties in question are extremal properties, it is that the selection they determine is implied by the structure of the given group. We shall see moreover how it is of other exceptional properties, very different from extremal properties and that also allow an entity within a whole to be distinguished. In all these cases, we will focus on showing how the structure of the whole was prepared to bring to the fore the distinguished element. The logical problem which dominates these theories is therefore still here that of the connections to be established between the structure of a domain and the existence of entities defined on this domain, and it is essential for us to be able to observe in mathematics the different solutions to this same problem. One of the main theses of this essay asserts in effect the necessity to separate the supra-mathematical conception of the problem of the connections that support certain notions and the mathematical discoveries of these effective connections within a theory. The schemas of genesis by selection are distinct from the schemas of genesis by decomposition. Both nevertheless carry out the passage from the essence of the domain to the existence of functions defined over this domain.

First, the two problems described by Janet will be envisaged: that of the Dirichlet problem, and that of the eigenvalues of an operator in Hilbert space. The Dirichlet problem consists in proving, given a domain D and a continuous function on the boundary of D, that a continuous function V in D exists just as its partial derivatives to two prime orders taking on the boundary of D the given values and satisfying the Laplace equation:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Riemann had used the fact that such a function gives the minimum expression:

$$I(U) = \int \int_D \left[\left(\frac{\partial U}{\partial x}\right)^2\right] + \left[\left(\frac{\partial U}{\partial y}\right)^2\right] dxdy$$

to assert the existence of this function, supposing without proof that the integral $I(U)$ effectively attains its minimum. Weierstrass has shown that it is not at all obvious that a given expression effectively attains its minimum so that if the sought after function exists it rightly has the benefit of the minimum property in question, but the fact that it has it is not sufficient to ensure its existence. It is necessary to have recourse to other processes. The principle of the method discovered by Hilbert, as in the last example presented in Chapter 5, consistsis in interposing between the domain D and the solution V a compact family of functions, which here again plays the role of an intermediate mix between the structure of the domain and the existence of the function. Recall that a family of holomorphic functions bounded as a whole in a domain D is compact if, for any infinite sequence of this family, a subsequence can be extracted converging uniformly in each domain interior to D towards a limit function. The theorem that asserts the compactness of a sequence of functions is therefore both an existence theorem for the limit function of a convergent sequence extracted from the family and the fact that the limit function can have in addition the extremal properties playing no part at all in its determination. This is the existence theorem whose statement is borrowed from Lebesgue: suppose as given the continuous and monotonic functions $u_1, u_2 \ldots u_n \ldots$ (which means they have both an upper and a lower limit in a closed domain and on its boundary, and this irrespective of the domain considered) in a domain *D*, all equal to each other on the boundary of *D*, having at any point interior to the domain finite and continuous first order partial derivatives, such that finally, for any *i*, the expression:

$$I(u_i) = \int \int_D \left[\left(\frac{\partial u_i}{\partial x}\right)^2\right] + \left[\left(\frac{\partial u_i}{\partial y}\right)^2\right] dxdy$$

has a meaning and is lower than a finite number *H*, independent of *i*. Then, among $u_1 \ldots u_n \ldots$, a sequence can be chosen that converges uniformly towards a continuous limit (Lebesgue 1907, 386). The existence of the function sought after is therefore connected to the compact structure of the family of functions $u_1 \ldots u_n \ldots$, and the exceptional property that distinguishes it is no longer a property of extremum but the property of being the limit of a convergent sequence.

Analogous considerations emerge from the study of eigenvalues of an operator. Recall that given an expression:

$$H(X,X) = \sum h_{ik} \overline{x_i} x_k,$$

the determination of eigenvalues of *H* amounts to the transformation of the expression

$$\sum h_{ik} \overline{x_i} x_k$$

into a sum of squares

$$\sum \lambda_i \overline{y_i} y_i$$

by a transformation of the base vectors of the Hilbert space. One of the methods for determining the eigenvalues λ_i is a maximum method. It relies on the following reasoning: Every real and continuous function with *k* real variables in a bounded and closed domain is uniformly continuous. Therefore it is bounded, therefore it has a maximum attained for a point at least of the domain of variables or its boundary. The expression *H*(*X*, *X*) is then considered as a function of X, in which X is any vector satisfying $|X^2| = 1$.

The domain of variation of the coordinates x_i of X is bounded and closed, therefore $H(X, X)$ is a real and continuous function of x_i in this domain and presents a maximum λ_i attained for $X = e_1$. Then all vectors X, of length equal to 1, orthogonal to e_1 are considered. The X variables still describe a bounded and closed domain in which H attains a maximum $\lambda_2 \leq \lambda_1$ for $X = e_2$ and so on.[1] The essential point of the reasoning is one that ties the fact that the domain of variation of x_i is bounded and closed to the existence of a maximum of the function

$$H(X,X) = \sum h_{ik} \overline{x_i} x_k$$

on this domain. The extremal value only appears each time because the structure of the domain is adapted to bringing it to the fore. The schema of genesis therefore still carries out the passage from the essence of the domain to the existence of the maximum.

1. THE METHODS OF POINCARÉ

We will now envisage the exceptional properties that are going to allow us to operate on a domain of very different selections. These are the ones that date back to the methods pioneered by Poincaré in his paper *Memoire sur les courbes definies par une equation differentielle* (1881). This paper has a capital importance for the logical questions that we examine because it is here that Poincaré characterized the structural aspect that in his eyes the modern problem of the integration of differential equations was bound to deal with. In 1881, Poincaré only envisaged the substitution of the global point of view for the local point of view, but we will show how these new methods later led to determine the existence of certain solutions by their exceptional properties. Here are the texts to which we refer:

> Fuchs, Briot, Bouquet and Kovalevsky . . . instead of studying the manner of being of the integrals of differential equations or of partial differential equations for all the values of the variable, that is, in the whole plane, they were at first occupied with determining the properties of the integrals in the neighborhood of a given point . . . (Poincaré 1915, 38; 1928, iii)

To this quantitative method, Poincaré opposes a qualitative method:

> To study an algebraic equation, begin with the aid of Sturm's theorem by trying to find the number of real roots, this is the qualitative part. Then calculate the numerical value of these roots, which constitutes the quantitative study of the equation. Likewise, to study an algebraic curve, begin by constructing this curve, as they say in the special Mathematics courses, that is, try to find the branches of the closed curve, the infinite branches, etc. After this qualitative study . . . a certain number of points can then be exactly determined. (Poincaré 1881, 376; 1928, 4)

This qualitative study is therefore essentially a topological study of curves in which one tries to determine their saddle points, their nodes, their focus, their centers. The most important curves are those that are presented in the form of closed cycles around a center. Their mathematical importance comes from what their knowledge determines at the same time, the knowledge of the curves that are their neighbors. In the case of first order differential equations with real variables, they are for example spirals that approach this limit asymptotically. The physical significance of closed cycles in celestial mechanics is obvious: they correspond to periodic trajectories, that is, to celestial bodies that are stable in their orbit. Generally one tries to determine these periodic solutions by analytical methods, but the genius of Poincaré was able to discover infinitely simpler methods of selection, borrowed from topology. We will briefly summarize this research according to the presentation by Bieberbach (1923, 202).

Poincaré's new methods that are of interest here are concerned with the solutions of differential equations in relation with a problem of the calculus of variations. All these solutions are therefore extremal with respect to this problem, and it is a matter for Poincaré to determine the closed extremals. It is possible, by fairly simple transformations to represent these extremals as curves of a 3-dimensional space, some of which are closed, those that correspond to periodic solutions, and the others which are wound infinitely many times around these closed curves. The whole forms like a sheaf of curves whose intersection is envisaged with a 2-dimensional surface. The closed curves only meet this surface at a finite number of points. That the closed curves only meet the surface at a point can even be arrived at by an appropriate choice of coordinates, but all the other curves wind infinitely many times around the closed cycles meeting the surface of intersection at an infinite number of points. That being so, if a curve interior to the sheaf is followed from one of its points of contact with the surface until it meets a second point on the surface, and if this is carried out in the same way for all the curves of the sheaf, a transformation is carried out of all the points interior to the surface of intersection, which to each point *a* makes the nearest point of intersection a' of the curve passing through *a* with the surface correspond. Only the corresponding points to the closed curves remain fixed in this transformation. A characteristic property of this transformation is to leave invariant a certain integral expression. The problem of determining the periodic solution is therefore reduced to the determination of points of a domain that remains fixed when a transformation is carried out on all interior points of this domain. We thus see appear a new exceptional property, that of the fixed points whose existence is immediately interpretable in terms of existence of periodic solutions of differential equations. It must now be shown how the existence of these fixed points is related to the structure of the domain on which the internal transformation takes place.

Poincaré had stated a theorem shortly before his death, the proof of which was given by Birkhoff a few months later. Here's this 'last theorem' of Poincaré: if a biunivocal continuous transformation conserving the areas transforms a ring in itself, by realising two rotations in opposite directions for the two bounded circumferences of the ring, at the interior of the ring two points always exist that are not affected by the transformation (see Poincaré 1912; Husson 1932, 50; Bieberbach 1923, 205).

The conditions of existence of fixed points in this statement do not depend solely on the structure of the basic domain, they also depend on the nature of the internal transformation carried out since this transformation must conserve an invariant surface integral. The recent development of topology has permitted the determination, for most topological domains, by the sole consideration of the topological structure of these domains, whether or not an internal transformation of their points admits fixed points. It is known that among the structural characteristics of a complex figure, above all the Betti numbers of this complex (see Chapter 3), that is, the maximum number of linearly independent cycles of 0 dimension, 1 dimension . . . *n* dimension, *n* being the dimension of the complex. It is then possible, through the formulas obtained by Hopf and Lefschetz, to determine the existence or absence of fixed points of an internal transformation solely by the consideration of Betti numbers. Here are some results that are significant in this regard: let *P* be a contiguous polyhedron all the Betti numbers of which are zero except that of dimension 0 ($p_0 = 1$).

Any internal transformation of the polyhedron admits at least one fixed point. Let N be the Euler characteristic of a complex, that is, a structural invariant of this complex, which is equal to the algebraic sum of the Betti numbers of the complex

$$\left[-N = \sum_{k=0}^{k=n} (-1)^k p^k\right]$$

So that an internal transformation of a complex, reducible to identity by continuous deformation, admits fixed points,[2] it is sufficient that the Euler characteristic of the complex is different to 0, etc. All these theorems show how the existence of fixed points of an internal transformation is implied by the structural invariants of the complex. There is a close analogy between the assertion of existence of the maximum of a function on a domain from the fact that the domain is bounded and closed, and the assertion of the existence of fixed points from the fact that the Euler constant of the complex is different to 0. If, from these theorems, the logical schema that they lay out is abstracted, a schema of genesis is obtained, from the essence of the domain to the existence of the entity distinguished on this domain, which is as characteristic as it could be of the schemas of genesis that establish a connection between decomposition and existence.

2. THE SINGULARITIES OF ANALYTIC FUNCTIONS

Poincaré considered, we have just seen, that the qualitative study of the shape of integral curves is opposed to the processes of his predecessors who were preoccupied primarily with studying the nature of the solution in the neighborhood of a given point. One can however show that the knowledge of the value of a solution around certain points also allows the global characterization of the function that is the integral of a proposed equation, because the points chosen are not arbitrary; they correspond in effect to the singularities of the solutions. The singular points are, insofar as they are points, points in the basic domain as any other point in this domain, but, in modern theories they increasingly have a dominant and exceptional role. We will see briefly that in them a large number of logical points of view that we have characterized separately in the course of this work come to coincide.

Let the differential equation be $w'' + p_1(z)w' + p_2(z)w = 0$, whose coefficients $p_1(z)$ and $p_2(z)$ have an isolated singularity at point a. It can be proven that any solution of the equation can be obtained at any regular point of the domain of the variable z by linear combination of two particularly independent solutions $w_1(z)$ and $w_2(z)$, forming what is called a fundamental system. It is therefore a matter above all of determining two fundamental solutions of the equation, and this is going to be done first in the neighborhood of the singular point. To determine these two fundamental solutions in the simplest cases, let us suppose that the coefficients $p_1(z)$ and $p_2(z)$ had in a only the poles that are respectively of the first and second order, that is, that we have:

$$p_1(z) = \frac{P_1(z-a)}{z-a} \quad \text{and} \quad p_2(z) = \frac{P_2(z-a)}{(z-a)^2}$$

In these conditions, the equation:

$$w'' + w'\frac{P_1(z-a)}{z-a} + w\frac{P_2(z-a)}{(z-a)^2} = 0$$

always admits, in the neighborhood of the point $z = a$, a fundamental system of the form:

$$w_1 = (z-a)^{l_1} P(z-a)$$

$$w_2 = (z-a)^{l_2} P*(z-a) + \text{possibly a logarithmic term.}$$

The numbers l_1 and l_2 are roots of an equation called the fundamental equation, formed with the prime coefficients of the series expansion of $P_1(z-a)$ and $P_2(z-a)$. If an equation admits several singular points, the same method applies: it is necessary to determine in the neighborhood of each of them a system of two fundamental solutions. The passage of the solutions w_1 and w_2 defined in the neighborhood of singular points to the solutions defined at each regular point is done according to the ideas of Weierstrass by analytic continuation. We are going to see how these local solutions are nevertheless amenable to characterizing a global function. The cases that we are going to envisage are those of the functions admitting 3 singular points: the points 0, 1 and ∞. It can still in effect be obtained by a linear transformation that the singularities are produced at these points here.

Consider the plane of the complex variable z cut on the real axis from 0 to $-\infty$ and 1 to $+\infty$. The extension from a local solution can never surround more than one singular point at a time. Let w_1 and w_2 be a fundamental system corresponding to the singular point $z = 0$. Let w be the quotient w_1/w_2. It can be proven that the function $w(z)$ ensures the conformal representation of the upper half-plane of the complex variable z on a curvilinear triangle of the plane of the variable w. If we then try to extend the function $w(z)$ on the lower half-plane of the variable z, a curvilinear triangle of the plane of w is obtained which has a common side with the first, and by indefinitely repeating the extension of the function $w(z)$ from one half-plane to the other of the variable z, an infinite curvilinear triangle is obtained which in certain cases tends to completely cover all the points situated at the interior of a fundamental circle of the plane of w. It is thus that, for example, on the plane w the figure below said to be a modular figure can be obtained.

The function $w(z)$ therefore carries out the representation of the plane z on this modular figure,[3] and its inverse function, $z(w)$, called the modular function, carries out conversely the representation of the modular figure on the complex plane z.

If we recall that the global point of view in the theory of analytic functions is that which regards a function $\zeta = f(z)$ as carrying out the representation of the domain of the variable z on the domain of the variable ζ, we see that the functions $w(z)$ and $z(w)$ defined from the local solutions of a differential equation take no less an immediate place in Riemann's theory.

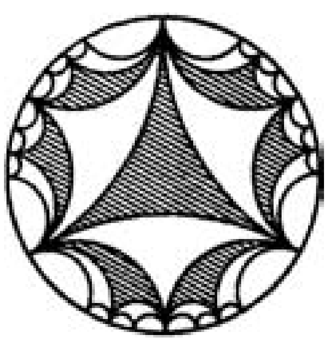

Figure 1. (Hurwitz and Courant 1925, figure 104)

Indeed one of their properties accentuates the global character of the result obtained: in Chapter 3 we showed that the universal covering surface corresponding to a given algebraic function $\zeta = f(z)$ was complete, in the sense that a function t always existed on this surface such that the functions $\zeta = \varphi(t)$ and $z = \psi(t)$ are uniform. Now these functions are automorphic functions, that is, they conserve the same value for certain transformations of the variable t:

$$\varphi\left(\frac{\alpha t + \beta}{\gamma t + \delta}\right) = \varphi(t);\ \psi\left(\frac{\alpha t + \beta}{\gamma t + \delta}\right) = \psi(t).$$

It then happens to be that the modular function, obtained from a system of purely local solutions, is the type of automorphic function that globally uniformizes the algebraic functions.[4]

We will therefore see the multiple aspects under which singular points are presented in this theory. (1) They allow the determination of a fundamental system of solutions, analytically extendable on any path encountering no singularities: it is in this sense that their singularity is expressed in terms of the existence of solutions. (2) They allow a structural cutting of the complex plane such that a function can exist that represents the similarly cut plane on the modular figure. In this sense their role is to decompose a domain in a way that the function which ensures the representation is definable on this domain. (3) They allow the passage from the local integration of differential equations to the global characterization of analytic functions which are solutions of these equations. We announced above that a large number of logical notions that we've studied separately are found to be mixed in the problems related to singular points: we can in effect see the role they play in terms of the synthesis of the local and the global (Chapter 1), in terms of the connection between the structural decomposition of a domain and the existence of functions on this domain (Chapter 4), and finally how the nature of singular points on a domain determines, at each point of the domain of the variable z, the existence of solutions of the proposed equation, in the same way that the maxima or minima studied at the beginning of this chapter entail, in other cases, the existence of specific solutions.

This encounter of distinct logical points of view, within the same problem, has been noted often in the previous chapters: it shows, as paradoxical as the reconciliation of the terms might be, both the intimate union and the complete independence of the dialectical logic, as we conceive it, and mathematics. Mathematical theories are developed by their own force, in close reciprocal solidarity and without any reference to the Ideas that their movement reconciles. The logical schemas that the philosopher then discovers in this movement cannot have the sharpness of contours that rules given anterior to the experience would have. They only have existence united to these theories which are made at the same time as them, and they all impinge upon one another, to be better able to support the magnificent and universal network of mathematical relations.

Conclusion

The preceding studies allow us to propose an answer to the problem that was considered in the introduction. Trying to define the nature of real mathematics, we are going to first examine the possible solutions to the problem. The nature of mathematical reality can be defined from four different points of view: the real, it is sometimes the mathematical facts, sometimes the mathematical entities, sometimes the theories, and sometimes the Ideas that govern these theories. Far from being opposed these four conceptions fit naturally together: the facts consist in the discovery of new entities, these entities are organized in theories, and the movement of these theories incarnate the schema of connections of certain Ideas.

The first two points of view are adopted in the book by Boutroux: *L'Idéal scientifique des mathématiciens* (1920). In his chapter on the mathematical analysis of the nineteenth century, Boutroux situates mathematical reality in the resistance that mathematical matter opposes to the logic with which one tries to describe it, and in the chance discovery of a new path that allows overcoming encountered obstacles. Thus, for example, examining the problem of integration, Boutroux shows that the discovery of elliptic functions, abelian functions, automorphic functions is due to the fact that certain definite integrals or certain differential equations are not able to be integrated with the methods that succeeded in the simpler cases. Winter (1911) develops analogous ideas: the expression

$$\int \frac{dx}{\sqrt{R(x)}}$$

can be integrated by means of elementary functions if $R(x)$ is a polynomial of the first or second degree. If $R(x)$ is of the third or fourth degree, a new transcendent has to be dealt with:

> It is sufficient that the degree of the polynomial contained under the radical of the denominator increases by one unit, so that suddenly we find ourselves in the presence of a new transcendent that requires considerable special study . . . Here is a fact that, it seems, formal logic cannot account for. (Winter 1911, 85)

This definition of mathematical facts is thus intimately bound up with the discovery of new entities, and Boutroux, in his chapter on the objectivity of mathematical facts, adopts a much more conceptualist point of view. Here is the very sentence in which the two points of view are recognizable, without Boutroux perhaps having claimed to establish the differences:

> In order to account for this resistance opposed by mathematical matter to the will of the scientist, we are obliged to assume the existence of independent *mathematical facts* of scientific construction, we are forced to attribute a true objectivity to mathematical *notions*. (Boutroux 1920, 203)[1]

Facts are thus organized under the unity of the notion that generalizes them. The real ceases to be the pure discovery of the new and unforeseeable fact, in order to depend on the global intuition of a supra-sensible entity. Boutroux takes as an example the reality of the ellipse. The ellipse is for him neither the locus of points such that the sum of their distances to the foci is constant, nor a curve defined by its algebraic equation, nor a curve to the projective properties of conics. It is all that and much more. It is, he says,

> a whole that does not include parts, . . . a sort of Leibnizian monad. This monad is pregnant with the properties of the ellipse; I mean that these properties, even though they have not been explicitly formulated (and they cannot be since they are infinite in number) are contained in the notion of ellipse. (Boutroux 1920, 208)

It must be admitted that the thought of Boutroux in the rest of the chapter is rather difficult to specify. The example of the ellipse tends to show that the reality of mathematical entities exist for him in the intuition, independently of the way the properties of these entities are presented logically, and yet he refuses to admit that the difficulties of the logical exposition are exterior to the specific reality of mathematics. These are the two attitudes that seem, once again, to be opposed and yet here is a passage in which both can be found:

> That the idea of pure intuition, separated from logical reasoning, poses difficulties is undeniable and it would be highly desirable to be able to remove these difficulties by eradicating their root. But the distinction between opposing tendencies in the work of mathematics, appears to us to require being maintained in one form or another; and we cannot believe that it has been devised solely for the purposes of the discussion engaged in by the logicists. (Boutroux 1920, 228)

We cannot understand why it would be desirable to separate the intuition from reasoning if this duality is inherent in the very nature of mathematics.

The source of these difficulties seems to reside in two of Boutroux's opinions that we would like to discuss. The first is relative to the power that the intuition would have to reveal to us the existence and objectivity of mathematical entities. If the example of the ellipse in his thought has a general bearing (and all indications in the text are that this is so), Boutroux seems to assert, as a matter of course, a solidarity between the local properties and the global properties of an entity, while bringing this solidarity to the fore is only possible, in most cases, as we have shown in Chapter 1, when the entities in question have certain properties of closure or completion. Take, for example, the case of a closed surface with constant curvature, the sign of this curvature is determinable by the global topological properties of the surface (its Euler characteristic). There is no theorem of this kind for open surfaces. If therefore the reality of mathematical entities resided in this inner harmony, a distinction is made between surfaces, sets, groups. Some would exist 'more really' than others, and distinctions would be seen to reappear in topology analogous to those which prevailed for a long time in the history of mathematics, between good and bad functions, good and bad roots.

One point of view however exists from which the multiplicities that are either completed, complete, closed, enclosed or compact, have the benefit of a privilege over those that are not. It is that the examination of the latter is most often reduced to the study of the former. We are going to give two examples of this: we have shown in Chapter 2 how the structural properties of a closed complex Q were in duality with those of its complementary space $R^n - Q$. But if Q is closed, $R^n - Q$ is open, in such a way that the duality theorem allows the study of certain open complexes to be partially reduced to that of closed complexes. Here's another example: Cartan, constituting the symmetrical, irreducible and non-Euclidean theory of Riemann spaces, initially examined enclosed spaces, and then associated to each open space an enclosed space, so the study of open spaces is completely determined by the enclosed space (Cartan 1937, 51). These processes can be compared to those employed in the resolution of systems of linear equations. Initially, the homogeneous system is resolved and the study of the non-homogeneous system is reduced to that of the corresponding homogeneous system. The mathematical reality therefore does not reside in the differences that separate the completed entities from the incomplete entities, perfect entities from imperfect entities. It resides rather in the possibility of determining one from the other, that is, in the mathematical theory that asserts these connections. Thus we see that the reality in question is not that of static entities, objects of pure contemplation. If qualitative distinctions exist in mathematics, they characterize the theories and not the entities.

We end up at the same conclusion by discussing another of Boutroux's ideas, that of the independence of mathematical entities with respect to the theories in which they are defined. Boutroux writes: 'The mathematical fact is independent of the logical or algebraic clothes in which we seek to represent it,' and later, 'mathematical facts are totally indifferent to the order in which they are obtained' (Boutroux 1920, 203–205). Boutroux has above all in view the analysis or geometry of the nineteenth century. It is evident that in Euclidean geometry, an ellipse is or is not, and that it possesses all its properties as soon as it is defined in any way. On the other hand, in modern algebraic theories, entities can be envisaged that are amenable to belong to distinct basic domains, and whose properties vary with the domain in which they are considered. It is impossible then to consider such 'mathematical facts', as defined by Boutroux, independently of the axioms that define the basic domains in question, and these axioms nevertheless constitute what Boutroux only claims to recognize as the supplementary character of 'logical or algebraic clothes'. We find here instead an essential dependence between the properties of a mathematical entity and the axiomatic of the domain to which it belongs. We won't describe it at length because we have examined, in our secondary thesis (Lautman 1938a), the principal aspects of this *Abhängigkeit vom Grundkörper* [Dependence on the base field] of the German authors, and we will be content to show this in an example. Let us examine the properties of divisibility of the number 21. If the field K of rational numbers is considered as a base field, 21 is only decomposable in one way into a product of prime factors: 21 = 3 x 7. If the field $K\sqrt{-5}$ obtained by the adjunction of $\sqrt{-5}$ to the field K of rational numbers is considered as a base field, two different decompositions of 21 into a product of prime factors are obtained:

$$21 = 3 \times 7 \text{ and } 21 = (1 + 2\sqrt{-5})(1 - 2\sqrt{-5})$$

The properties of the number 21 are therefore not all given with the simple construction of this number. They can only be studied within the field in which the number is embedded, and this involves the whole axiomatic of the theory of algebraic fields and of their successive extensions.

Whether therefore by studying the relations that unite certain entities to other entities or certain entities to certain axioms, we see that the problem of mathematical reality is posed neither at the level of facts, nor at that of entities, but at that of theories. At this level, the nature of the real divides into two. We have shown during the preceding chapters in effect how mathematical theories are amenable to a dual characterization, one that focuses on the unique movement of these theories, the other on the connections of ideas that are incarnated in this movement. These are the two distinct elements whose reunion constitutes in our opinion the reality inherent to mathematics and we are going to show how this reunion seems possible.

The reading of *Étapes de la philosophie mathématique* (Brunschvicg 1912) [*The Stages in the Philosophy of Mathematics*] teaches the philosopher to associate in an indissoluble way the elaboration, or even only the comprehension of mathematical theories, and the experience that the intelligence has of its own power. Any logical attempt that would profess to dominate *a priori* the development of mathematics therefore disregards the essential nature of mathematical truth, because it is connected to the creative activity of the mind, and participates in its temporal character. It is on the other hand indisputable that, since the development of non-Euclidean geometries, mathematics is not presented as an indefinitely progressive and unifying extension. The theories are rather figures of an organic unity, and are suitable for these global metamathematical considerations announced in the work of Hilbert. The point of view of a new mathematics gradually asserts itself, which substitutes for the infinitist process of the analysis of the nineteenth century, the structural schemas of algebra or of topology. Elsewhere we have described this evolution of modern mathematics and the gradual penetration of the methods of the finite in the infinite. We shall only mention here the idea of a 'qualitative' mathematics as defined by Poincaré or an 'integral' mathematics as defined by Severi (see Severi 1931), being developed around structural logical schemas.

Our task therefore is to reconcile the irreducibility of mathematics to an *a priori* logic and its organization around similar logical schemas. To do this, we will therefore attempt to show that it is possible to conceive what we will call the exigency of a logical problem, without the consciousness of this exigency implying in any way an attempt at solution. One can even say that a dialectic which would be engaged in the determination of solutions that these logical problems can bring about, would be involved in constituting an entire set of subtle distinctions and artifices of reasoning that imitate mathematics to this point, that it would be conflated with the mathematics itself. Such is the fate of mathematical logic in its most recent development. It is possible to conceive the problem of consistency in arithmetic without redoing all of arithmetic, but as soon as you try to establish an effective proof of consistency of arithmetic, you are obliged to employ in this proof mathematical means that exceed in richness those of the theory whose validity you are trying to guarantee. These results, due to Gödel, show definitively that the consistency of arithmetic is not able to be reduced to the consistency of a simpler theory, and, in the present state of science, any metamathematical proof of the consistency of arithmetic necessarily uses transfinite means. It seemed therefore that this problem had lost all logical interest until Gentzen managed to envisage it under another aspect:

> It remains quite conceivable that the consistency of arithmetic[2] can in fact be verified by means of techniques which, in part, no longer belong to arithmetic, but which can nevertheless be considered to be *more reliable* than the doubtful components of arithmetic itself. (Gentzen 1936, 500 [1969, 139])[3]

We see in this way how the problem of consistency makes sense, even though we are unaware of the mathematical means necessary to resolve it.

This seems to us to be the case for all the logical problems that we have successively considered. The logical schemas that we have described are not anterior to their realization within a theory. They lack in effect what we call above the extra-mathematical intuition of the exigency of a logical problem, a matter to dominate so that the idea of possible relations gives rise to the schema of true relations. The fate of the problem of the relations between the whole and the part, of the reduction of extrinsic properties to intrinsic properties, of the ascent towards completion, the constitution of new schemas of genesis, depends on the progress of mathematics itself. The philosopher has neither to extract the laws, nor to envisage a future evolution, his role only consists in becoming aware of the logical drama which is played out within the theories. The only *a priori* element that we conceive is given in the experience of the exigency of the problems, anterior to the discovery of their solutions.

It is necessary to now clarify the nature of the *a priori* that we introduce into the philosophy of mathematics. We do not at all pretend to support that the ideas of logical problems that mathematics resolves can be deduced systematically according to the requirements of a rationalist idealism. We understand this *a priori* in a purely relative sense, and with respect to mathematics. It is uniquely the possibility of experiencing the concern of a mode of connection between two ideas and describing this concern phenomenologically independent of the fact that the connection sought after may, or may not be carried out. Some of these logical 'concerns' are found in the history of philosophy: as, for example, the concern of the connections between the same and the other, the whole and the part, the continuous and the discontinuous, essence and existence. But mathematical theories can conversely give rise to the idea of new problems that have not previously been formulated abstractly. Mathematical philosophy, as we conceive it, therefore consists not so much in retrieving a logical problem of classical metaphysics within a mathematical theory, than in grasping the structure of this theory globally in order to identify the logical problem that happens to be both defined and resolved by the very existence of this theory. A spiritual experience is thus attached anew to the effort of the intelligence to create or understand. But this experience has a content different to that of mathematics which is made at the same time as it. Nor is it the consciousness of the infinite power of thought. Beyond the temporal conditions of mathematical activity, but within the very bosom of this activity, appear the contours of an ideal reality that is governing with respect to a mathematical matter which it animates, and which however, without that matter, could not reveal all the richness of its formative power.

Before concluding, we would like to show how this conception of an ideal reality, superior to mathematics and yet so willing to be incarnated in its movement, comes to be integrated into the most authoritative interpretations of Platonism. Certain historical explications are indispensible on this subject, given the sense that the expression of Platonism in mathematics generally receives. In the open debate between formalist and intuitionist, since the discovery of the transfinite, mathematicians have become accustomed to summarily designate under the name Platonism any philosophy for which the existence of a mathematical entity is taken as assured, even though this entity could not be built in a finite number of steps. It goes without saying that this is a superficial knowledge of Platonism, and that we do not believe ourselves to be referring to it. All modern Plato commentators on the contrary insists on the fact that Ideas are not immobile and irreducible essences of an intelligible world, but that they are related to each other according to the schemas of a superior dialectic that presides over their arrival. The work of Robin, Stenzel and Becker has in this regard brought considerable clarity to the governing role of Ideas–numbers which concerns as much the becoming of numbers as that of Ideas. The One and the Dyad generate Ideas–numbers by a successively repeated process of division of the Unit into two new units. The Ideas–numbers are thus presented as geometric schemas of the combinations of units,[4] amenable to constituting arithmetic numbers as well as Ideas in the ordinary sense.

The Ideas–numbers therefore constitute, as Stenzel says, the principles which serve both to 'dialectically order the arithmetic units to the place that suits them in the system, and to explicate the different degrees of progressive division of Ideas' (Stenzel 1923, 117). The schemas of division of Ideas in the *Sophist* are organized according to the same planes as the schemas of generation of numbers, both can be traced to a 'metamathematics'[5] that is superior to both the Ideas and the numbers.

The existence of Ideas – numbers, governing in regards to arithmetic numbers, therefore has the consequence of ordering a generation of numbers as of Ideas, which, though not being in the time of the created world, are produced no less according to an order of the anterior and the posterior. Robin shows how the constitution of bodies in the *Timaeus* assumes a matter which, before the existence of the world, has already been the receptacle of a geometric qualification. 'There is therefore a generation and becoming anterior to the generation and the becoming of the world' (Robin 1935, 235).

The introduction of becoming within Ideas, in Stenzel's work, takes all its value from the genius of the text of the *Nicomachean Ethics* (*EN* 1.4, cited by Stenzel 1923, 118), where Aristotle says that Platonists did not admit the ideas of numbers because they did not admit as ideas things in which there exists the before and the after. Stenzel gives an explanation of this text the importance of which cannot be exaggerated: ideal-numbers being the principle of determination of essences according to the order of the before and the after, it is not possible that there are 'numbers of numbers', that is, 'a principle of the division of essences that is superior to this numerical division itself' (Stenzel 1923, 119).[6] Metamathematics which is incarnated in the generation of ideas and numbers does not give rise in turn to a meta-metamathematics. The regression stops as soon as the mind has identified the schemas according to which the dialectic is constituted. We can thus see how well our reference to Platonism is justified, as regards the relations that exist between mathematical theories and the Ideas that govern them.

These relations also appear comparable to those that could be established between mathematics and physics. A simple empiricism tends sometimes, currently, to be installed in the philosophy of physics, according to which a profound dissociation should be established between the experimental findings of fact and the mathematical theory that links them to each other. Any criticism of contemporary science shows the philosophical weakness of such an attitude[7] and the impossibility of considering an experimental result outside of the mathematical framework in which it makes sense. On the other hand, critical thinking sometimes leads, conversely, to an idealistic dogmatism, from the fact that mathematics increasingly penetrates the domain of physics, reality has become so abstract that the scientist has the impression of no longer ever happening to be in front of their own mind. Such is the idea that seems to emerge from Eddington's celebrated sentence:

> We have found a strange foot-print on the shores of the unknown. We have devised profound theories, one after another, to account for its origin. At last, we have succeeded in reconstructing the creature that made the foot-print. And Lo! it is our own. (Eddington 1920, 201)

This idealism of mathematical physics does not moreover exclude, for Eddington, the notion of reality, but it is no longer physics that will have the mission to make it known, it is a direct contact with the supernatural.

In any case, the neo-positivism of the Vienna Circle, like the idealism of the English metaphysical physicists, separate mathematics and reality quite distinctly, while the philosophy of physics essentially has as its task the problem of their union. We do not pretend to treat this problem here, which is quite different from those we have envisaged throughout the course of the preceding pages. We are simply going to show how, to some extent, and to take up the expressions that Robin made use of in regards to Plato, the process of connecting theory and experience symbolizes the connection of Ideas and mathematical theories.

Just as we had recognized that mathematical reality was not at the level of mathematical entities, but at that of the theories, the problem of physical reality does not intervene at the level of an isolated experience, but at the level of what might be called a physical system. The notion of system in physics implies that the set of phenomena that occur within a certain process are considered globally, and that a series of measurements are carried out relating to magnitudes playing a part in this process. In addition, it is not necessary that the measurements are concerned with different magnitudes; they can be relative to only a same magnitude. For there to be a system, it is sufficient that the same magnitude is measured several times. It is perhaps possible to suppose that the finding of a single measurement is anterior to any mathematical elaboration, but whatever the sentence or the relation by which two measurements are coordinated, the expression obtained is already situated within mathematics. To speak of simultaneous phenomena is to adopt the language of special relativity, to speak of successively measurable magnitudes is to speak the language of permutable operators in quantum mechanics. To measure the intervals between two levels is to set out the results obtained according to a matrix array. To speak in classical mechanics of the constancy of a certain magnitude in time is to take a derivative with respect to time. To speak of the invariance of a certain magnitude with respect to certain variations that other magnitudes undergo is to use the language of group theory. To note the periodicity of a phenomenon is to make use of trigonometric functions to represent this phenomenon. To suppose that measurements converge towards a limit is to adopt the point of view of the calculus of probabilities. Just as the movement of a mathematical theory was at the same time a logical schema of relations between ideas, the description of the state of a system at any given moment or of the evolution of this system over time, amounts to noting that the magnitudes of the system are orderable according to a structural law of mathematics. Physical reality is therefore not indifferent to this mathematics which describes it. Experimental findings call for a mathematics whose outline they already imitate (*Cf.* Juvet 1933, *passim*), sometimes even before an adequate mathematics has been developed for them.

What results from the preceding is that the structure of experience is not detachable from the experience itself, and that in understanding by experience the global experience relative to a system, this mathematical structure coincides with the system of effectuated or possible experimental measurements. Here again reality resides in the discovery of a structure which is organizing with respect to a matter that it animates with its relations. The philosophy of physics would therefore also amount to the Platonic conclusion to which we have led mathematical philosophy, as we conceive it. The nature of the real, its structure and the conditions of its genesis are only knowable by ascending again to the Ideas whose connections are incarnated by Science.

BOOK III

New Research on the Dialectical Structure of Mathematics*

The essays of this series will be devoted to dogmatic philosophy without restriction of any kind, neither for the objects considered, nor for the processes employed. The unity, if it appears, – it will in any case never be sighted consciously – could only be original – influences undergone –, or also aspirational, a certain concern by which we have a sense of community. To be more specific about what was only a method would be altogether in vain: any requirement for clarification that does not satisfy the current instruments of scientific techniques, or their normal development, is philosophical. It is possible that these change and render our problems devoid of meaning: as was the case for rational evidence after the crisis of mathematical infinity. In the contemporary system of notions and processes of authentic thought, that is, confirmed by the least experience, the philosopher has their own activity that recognizes its contents, by discovering the true consequences and the relations. It is not a matter of subordination to science or renunciation of the

* This essay, published in 1939, was the first of a series called *Essais philosophiques*, created by Jean Cavaillès and Raymond Aron at Hermann publishers. It is known from a letter by Cavailles to Lautman that this essay was closely associated with the drafting of this Introduction that is worthy of a manifesto. In the series there were only four volumes in total that appeared. In 1939, in addition to this one, *The Emotions: Outline of a Theory* by Jean-Paul Sartre (1939), and in 1946 two that were posthumous: *Transfini et Continu* by Jean Cavaillès (1947); *Symmetry and dissymmetry in mathematics and physics* by Albert Lautman (1946, included in this volume).

> *fundamental questions of metaphysics or morality, but an effort to address these issues in an effective way. At the level of specific difficulties, arising from effective research, subject to the sanctions of failure, reflection for us begins. Whether they are of an original nature, we become conscious of them by exercising them: neither pure intuition nor abstract dialectic, it is initially critical ordeals, doubt and care for the other. What we are trying to say has the ambition to be true and, as such, must participate in the destiny of all knowledge, informed by a conceptual architecture whose structural rigor as well as power of change lend themselves to communication:* philosophia perennis sed in actione hominis manifesta.[1]
>
> Jean Cavaillès and Raymond Aron

Foreword

This essay consists of two distinct parts: in the first, developing the ideas of our principal thesis (Lautman 1938b), relative to the participation of Mathematics in a Dialectic that governs it, we try to show in an abstract way how the understanding of the Ideas of this Dialectic is necessarily extended in the genesis of effective mathematical theories. We rely for this on certain essential distinctions in the philosophy of Heidegger who seems to agree in a remarkable way with the metaphysical problem envisaged. In the second part, instead of descending from the abstract to the concrete, we work backwards: we examine a particular mathematical theory, the analytic theory of numbers, in which it is possible to grasp in a concrete way the necessity, in order to understand the reason of certain results, of relating them to the structural Ideas of a higher dialectic.

It may seem strange to those who are used to separating the 'moral' sciences from the 'exact' sciences, to see, reunited in the same work, reflections on Plato and Heidegger, and remarks on the law of quadratic reciprocity or the distribution of prime numbers. We would like to have shown that this rapprochement of metaphysics and mathematics is not contingent but necessary.

CHAPTER 1
The Genesis of the Entity from the Idea

We tried in a previous work (Lautman 1938b) to show in a few examples how in mathematics the ideal relations of a dialectic abstract and superior to mathematics is realized in concrete ways. It is in this sense that the intrinsic reality of mathematics appeared to us to reside in its participation in the Ideas of this dialectic which governs them. We do not understand by Ideas the models whose mathematical entities would merely be copies, but in the true Platonic sense of the term, the structural schemas according to which the effective theories are organized. This distinction between dialectic and mathematics leads us to a more precise analysis of the nature of the 'governing' (*domination*) relation that exists between dialectical Ideas on the one hand, and mathematical theories on the other.

The most habitual sense of a governing relation between abstract Ideas and their concrete realization is the cosmological sense, and a cosmological interpretation of such a relation is based almost entirely on a theory of creation. The existence of a matter that is the receptacle of the Ideas is not implied by the knowledge of the Ideas. It is a sensible fact, known by some bastard reasoning, as Plato said, or by a kind of natural revelation, as Malebranche thought. The Ideas are then like the laws according to which this matter is organized to constitute a World, but it isn't necessary that this World exist to realize in a concrete way the perfection of the Ideas.

Such an epistemology can make sense in regard to physical reality; it certainly does not in regards to mathematical reality. The cut between the dialectic and mathematics cannot in effect be envisaged. It is necessary on the contrary to clarify a mode of emanation from one to the other, a kind of procession that connects them closely and does not presuppose the contingent interposition of a Matter heterogeneous to the Ideas. This engagement of the abstract in the genesis of the concrete is a 'transcendental'[1] interpretation of the governing relation that can be better accounted for. We intend to show by this that an effort of understanding adequate to the dialectical Ideas, by the very fact that it applies to knowing the internal connections of this dialectic, is creative of systems of more concrete notions in which these connections are asserted. The genesis is then no longer conceived as the material creation of the concrete from the Idea, but as the advent of notions relative to the concrete within an analysis of the Idea. This transcendental conception of the relation of governing that is narrowly applied to the case of the relation between the dialectic and mathematics is not however limited to this case. It happens to be in effect that independently of any reference to mathematical philosophy, Heidegger has presented analogous views to explain how the production of notions relative to concrete existence arise from an effort to understand more abstract concepts.[2]

Heidegger's analysis is based on the distinction between *being* and *entity*. The truth of being is, in the vocabulary of phenomenology, an *ontological* truth, relative to the essence. The truth of what exists is *ontic*, and relative to the effective situations of concrete existence. The distinguishing feature of the entity is to manifest itself, to be revealed, but this revelation is only possible 'guided and clarified by an understanding of the being (the constitution of being: what something is and how it is) of entities' (Heidegger 1969, 21, 23 [1938, 56]). In this way therefore, if it belongs to the ontic truth to be the 'manifestness of the entity' (21), the ontological truth is quite different, it is 'disclosure' (117) understood 'as the truth about being' (23). We are going to see how, in the analysis of the disclosure of being, a general theory of these acts is constituted which, for us, are geneses, and that Heidegger calls acts of transcendence or of surpassing.

Here are the principal moments of the disclosure of being: it comes primarily from the act of *asking a question* about something, this does not necessarily mean that the thing to which a question is thus posed is conceived in its essence. This prior delimitation to that about which knowledge is possible, but which is not yet knowledge of the concept of being, Heidegger calls the *pre-ontological understanding*. It precedes the formation of the concept of being, an act by which a structure is disclosed to the intelligence that thus becomes capable of outlining the set of concrete problems relating to the being in question. What then happens, and this for us is the fundamental point, is that this disclosure of the ontological truth of being cannot be done without the concrete aspects of ontic existence taking shape at the same time:

> One characteristic stage is the project of the constitution of the Being of the entity whereby a determinate field of being (perhaps nature or history) is, at the same time, marked off as an area that can be objectivized through scientific knowledge. (Heidegger 1969, 23 [1938, 57])

According to this text, a same activity is therefore seen to divide in two, or rather act on two different planes: the constitution of the being of the entity, on the ontological plane, is inseparable from the determination, on the ontic plane, of the factual existence of a domain in which the objects of a scientific knowledge receive life and matter. The concern to know the meaning of the essence of certain concepts is perhaps not primarily oriented towards the realizations of these concepts, but it turns out that the conceptual analysis necessarily succeeds in projecting, as an anticipation of the concept, the concrete notions in which it is realized or historicized.

The distinction between essence and existence, and in particular the extension of an analysis of the essence in genesis of notions relating to the entity, are sometimes masked in the philosophy of Heidegger by the importance of existential considerations, relating to being-in-the-World, as they appear in *Being and Time*. But in the second part of *The Essence of Reason* (1969 [1929]), Heidegger relies precisely on the distinction between the ontological point of view and ontic point of view to explain the link that exists between human reality and existence-in-the-World. The concept of World for Heidegger does not signify the simple totality of entities in general. It is an *ontic* notion, exclusively related to human reality, to effective situations of man in the world. Heidegger does not support the thesis according to which being-the-World would properly belong to the essence of human reality, 'for it is not necessary that the sort of being we call human . . . exists factically. It can also *not* be' (43, 45 [67]). On the other hand, it is necessary to recognize that being-in-the-world cannot be attributed to human reality. Whether it is a matter of the finite world in which the religious destiny of man is hanging in the balance, the world organized into subjective knowledge of phenomena by the human mind, or the world in which it is given to man to be man, the World always reveals itself, on the *ontic* plane of existence, as 'that for the sake of which [human reality] exists' (85 [88]). Conversely, by considering the idea or essence of human reality this time on the *ontological* plane, it is necessary to recognize that its 'basic constitutive feature' is 'being-in-the-world' (81, n. 55). If the concept of the World is revealed as that 'for the sake of', it is of the essence of human reality to be the 'intention' directed towards this purpose. In 'the essence of its being', human reality 'is such that it forms the world' (89 [90]). The purpose returns to the intention, just as the ontological analysis of intention outlines the notions that constitute the purpose in which this intention is realized.

Heidegger thus describes, as we have just seen, a genesis of the ontical concept of the World from the idea of human reality. His primordial interest is therefore concerned with the problem of the Self, but this primacy of anthropological preoccupations in his philosophy should not prevent his conception of the genesis of notions relating to the Entity, within the analysis of Ideas relating to Being, from having a very general bearing: he himself applies them moreover to physical concepts:

> The preliminary definition of the being (understood here as what something is and how it is) of nature is established in the 'basic concepts' of natural science. . . . space, locus, time, movement, mass, force, and velocity are defined in these concepts . . . (23, 25 [57])

These scientific concepts concern the entity, and not being. They do 'not include authentic ontological concepts of the being of the entity' (25 [57–58]) that is put into question by this science.

Whether it is therefore about the Self, or about Nature, Heidegger identifies the *grounding*, rational activity of founding (*Begründung*), which is knowledge by Ideas, and the creative activity of *foundation* (*Gründung*) which constitutes, in the complexity of its internal relations, the world of the entity. Grounding intervenes as soon as the Leibnizian question of knowing *why* something exists is posed. 'A preconceptual, prior understanding of what something is, of how it is, and even of being (nothing) lies implicit in the Why, no matter how it is expressed' (115 [103]). The *Why* is therefore not a pure interrogation. The notion of being that it implies is already an answer and grounding, disclosure of ontological truth. The passage from this disclosure of the essence to the different possible forms of existence appears as soon as it is realized that the enquiry into the *why* is inseparable from the consideration of the possibles implied in *rather than* (*why* something exists *rather than* another thing, or *rather than* nothing). Grounding is in effect naturally extended in outline of potential entities: 'By its very essence, ontological founding opens marginal realms of the possible' (125 [108]). But there is more, and the principle of sufficient reason is only a logical principle, destined to legitimize the real by removing what the real is not. For Heidegger it is a transcendental principle that the determination of the entity necessarily relies on a creative freedom rooted in the ontological constitution of the being that determines. It is thus that grounding gives rise to the formation of a project of the World, in which the creative freedom of the founding power is asserted (in the dual sense of founding and foundation). If to this is added, about human reality at least, that there is reciprocity between the constitution of the World by the project that descends towards it and the fact that existence in the world only makes sense by ascending again to human reality whose world constitutes the purpose, one understands 'the threefold transcendental dispersion of grounding in the project of world, preoccupation with the entity, and the ontological founding of the entity' (121 [106]).

It is possible, in the light of these conceptions of Heidegger, to see the utility of mathematical philosophy for metaphysics in general. Whereas for all the questions that do not come out of the anthropology, Heidegger's indications remain, despite everything, very brief, one can, in regards to the relations between the Dialectic and Mathematics, follow the mechanism of this operation closely in which the analysis of Ideas is extended in effective creation, in which the virtual is transformed into the real. Mathematics thus plays with respect to the other domains of incarnation, physical reality, social reality, human reality, the role of model in which the way that things come into existence is observed.

1. THE GENESIS OF MATHEMATICS FROM THE DIALECTIC

The notion of genesis implying a certain order of the before and the after, it is first necessary to specify the type of anteriority of the Dialectic with respect to Mathematics. It cannot be, we have already seen, a matter of the chronological order of creation. This is not the order of knowledge, because the method of mathematical philosophy is analytical and regressive. It arises from the global apprehension of a mathematical theory to the dialectical relations that this theory incarnates, and it is not a question of determining an *a priori*, prior knowledge of which would be necessary to understanding mathematics. The order implied by the notion of genesis is not about the order of the logical reconstruction of mathematics, in the sense in which from the initial axioms of a theory follow all the propositions of the theory, because the dialectic is not part of mathematics, and its notions are unrelated to the primitive notions of a theory. We have already defined, in our thesis (Lautman 1938b), the priority of the dialectic as that of 'concern' or the 'question' with respect to the response. It is a matter of an 'ontological' anteriority, to use the words of Heidegger, exactly comparable to that of the 'intention' in regards to that 'for the sake of'. Just as the notion of 'for the sake of' necessarily refers to an intention oriented towards this purpose, it is of the nature of the response to be an answer to a question already posed, and this, even if the idea of the question comes to mind only after having seen the response. The existence of mathematical relations therefore necessarily refers back to the positive Idea of the search of similar relations in general.

Having established this point allows us to specify that which constitutes the essence of the Ideas of the Dialectic. First of all let us point out that we distinguish *notions* and dialectical *Ideas*. The *Ideas* envisage possible relations between dialectical *notions*. Thus we examined in our thesis (Latman 1938b) the Ideas of the possible relations between pairs of notions such as whole and part, situational properties and intrinsic properties, basic domains and the entities defined on these domains, formal systems and their realization, etc. The essential difference between the nature of mathematics and the nature of the Dialectic can be inferred from these definitions.

While the mathematical relations describe the connections that in fact exist between distinct mathematical entities, the Ideas of dialectical relations are not assertive of any connection whatsoever that in fact exists between notions. Insofar as 'posed questions', they only constitute a problematic relative to the possible situations of entities. It then happens to be once again exactly as in Heidegger's analysis, that the Ideas that constitute this problematic are characterized by an essential insufficiency, and it is yet once again in this effort to complete the understanding of the Idea, that more concrete notions are seen to appear relative to the entity, that is, true mathematical theories. First of all, insofar as Ideas of possible relations, these Ideas are free from all constraints that always bring, in an effective realization, the matter upon which one works. They do not participate as either more or less, and ignore all determinations of sense, sign, degree, without which nothing would exist. Moreover, being only outlines of possible positions, they do not necessarily bring about the existence of particular entities capable of supporting the relations they sketch. To think them fully, it is necessary then to rely on some example, perhaps foreign to their very nature, but that thus gives shape, at least for thought, to the necessary matter. So that the example supports the idea, it is then necessary to provide a whole series of precisions, limitations, exceptions, in which mathematical theories are asserted and constructed. Heidegger seems moreover to have meant something like this in writing about scientific concepts:

> Original ontological concepts must instead be obtained *prior* to any scientific definition of 'basic concepts', so that only by proceeding from them will we be in a position to evaluate the manner in which the basic concepts of the sciences apply to being as graspable in purely ontological concepts. (Heidegger 1969, 25 [1938, 58])

The restrictions and delimitations in question in this text should not be conceived as an impoverishment, but on the contrary as an enrichment of knowledge, due to the increase in precision and the certainty provided. The more the individual character and the very structure of particular mathematical theories is asserted, the more fertile the Ideas thereby appear which, as defined by a non-lived history, the philosopher recognizes at the origin of theories.

All that remains now is the elucidation of one point, that of the transcendence of Ideas. Note first the special sense that Heidegger gives to this term in his philosophy. When it is of the essential nature of a thing to go beyond itself in order to go towards an entity exterior to it, without which this thing would no longer be conceived as existing, this going beyond of the subject towards the entity, this is transcendence. It follows that transcendence properly belongs to human reality, which could not be conceived otherwise than as oriented towards the world. In thus describing transcendence as an act of bringing together, and not as a state of separation, Heidegger does not mitigate the ontological distinction that separates the disclosure of being and the manifestation of the entity, but he insisted on the fact that the genesis and the development of the entity was the necessary extension of an effort of disclosure of being. In regards to the relation of the Ideas of the Dialectic to mathematics an analogous situation can be described. Insofar as posed problems, relating to connections that are likely to support certain dialectical notions, the Ideas of this Dialectic are certainly transcendent (in the usual sense) with respect to mathematics. On the other hand, as any effort to provide a response to the problem of these connections is, by the very nature of things, constitution of effective mathematical theories, it is justified to interpret the overall structure of these theories in terms of immanence for the logical schema of the solution sought after. An intimate link thus exists between the transcendence of Ideas and the immanence of the logical structure of the solution to a dialectical problem within mathematics. This link is the notion of genesis which we give it, at least as we have tried to grasp it, by describing the genesis of mathematics from the Dialectic.

CHAPTER 2
The Analytic Theory of Numbers

It follows from the preceding pages that, while it is necessary that mathematics exists, as examples in which the ideal structure of the dialectic can be realized, it is not necessary that the examples which correspond to a particular dialectical structure are of a particular kind. What most often happens on the contrary is that the organizing power of a same structure is asserted in different theories; they then present the affinities of specific mathematical structures that reflect this common dialectical structure in which they participate. It is in this regard that we propose to envisage certain results of analytic number theory.

It is a matter of the results relative to whole numbers, therefore to an essentially discontinuous set, and whose proof calls for the continuous functions of analysis. Some of these results, like those on quadratic reciprocity, are also capable of being proved in a purely algebraic way and without any analytic means. Others on the contrary, for example those that are related to the distribution of prime numbers, have never been proved other than using the famous Riemann function $\zeta(s)$. Mathematicians are sometimes concerned, for 'aesthetic' reasons, to eliminate analysis from arithmetic. On the other hand, it is for reasons that depend on the very nature of their conception of the problem of the foundation of mathematics that most theorists of contemporary mathematical logic judge this 'purification' of arithmetic of any analytic elements as extremely desirable. Whether it is an undertaking to reconstruct all of mathematics from the single notion of the whole number, or the requirement to at least reduce the consistency of analysis to that of arithmetic, it seems that it will always rely on the idea that arithmetic is metaphysically anterior to analysis, and that calling upon analysis to prove arithmetic results is consequently contrary to the natural order of things. In any case, whatever the effort dispensed on this route, it does not currently appear that it will ever be possible, for example in the theory of prime numbers, to eliminate analysis from arithmetic. If this negative result is then compared with the proof by Gödel of the impossibility of formalizing a supposedly consistent proof of the consistency of arithmetic without appeal to means that exceed arithmetic, it can lead to thinking that it is incorrect to consider arithmetic as fundamentally simpler than analysis.

The relations that support these two mathematical disciplines will perhaps appear more clearly if, instead of trying to eliminate analysis from arithmetic, we wonder why it is possible to prove arithmetic results by analysis. At least in regards to the facts that will be considered later, it is possible to provide an extremely precise response to this problem: the demonstration of certain results related to whole numbers relies on the properties of certain analytic functions, because the structure of the analytic means employed is already accorded to the structure of the arithmetic results sought after. More precisely, from a close structural imitation between analysis and arithmetic, the idea of certain dialectical structures can be identified, anterior to the diversification of distinct theories in mathematics, and such that arithmetic and analysis, far from being respectively simpler and more complex than one another, are on the same plane and of equal status, the realizations of this dialectic which governs them.

1. THE LAW OF RECIPROCITY

Here is what the quadratic reciprocity consists of for ordinary whole numbers (that is, in the field of rational numbers): m is said to be a quadratic residue modulo p (m and p are relatively prime) if an integer x exists such that $m \equiv x^2 \pmod{p}$, that is, such that $m - x^2$ is a multiple of p. Legendre introduced into arithmetic a symbol of quadratic residue (m/p) which is +1 if m is quadratic residue mod. p and –1 in the contrary case. Consider now two positive odd integers a and b. They satisfy the fundamental law of reciprocity

(I) $$\left(\frac{a}{b}\right) = \left(\frac{b}{a}\right)(-1)^{\frac{a-1}{2}\cdot\frac{b-1}{2}}$$

Assuming for example $a \equiv b \equiv 1 (\mathrm{mod}.4)$, we have, by virtue of I:

(II) $$\left(\frac{a}{b}\right) = \left(\frac{b}{a}\right) \text{ or } \left(\frac{a}{b}\right) = \left(\frac{b}{a}\right) = 1$$

In this case, if a is quadratic residue modulo b, b is quadratic residue mod. a. The relations (I) and (II) therefore allow those cases to be recognized in which two whole numbers are likely to exchange their respective roles in a same algebraic relation. By considering the symbols

$$\left(\frac{a}{b}\right) \text{ and } \left(\frac{b}{a}\right)$$

as the inverse of each other, a mathematical situation can therefore be seen in the relations established by the law of quadratic reciprocity responding to the following structural problem: given two terms that are the inverse of one another, in what way can an exchanging between their respective roles be conceived?

It has been possible to generalize the relations of reciprocity in two different ways: first by considering the algebraic integers of any field and not only the ordinary integers of the field of rational numbers. And then by defining symbols for the integers of this field, analogous to those of Legendre, but that are attached to more general properties: it is in this way that Hilbert envisaged not only the numbers congruent to a square, but the numbers congruent to a power of degree $l > 2$ or the numbers congruent to the norm of a number of the field. The particular case of quadratic reciprocity is thus related to the general law of reciprocity of which there currently exist, through the work of Hilbert, Takagi, Artin, Hasse, Herbrand and Chevalley, purely algebraic proofs. There also exists, as regards quadratic reciprocity in the case of any field, transcendental methods, due to Hecke, that have considerable philosophical interest for us, in regards to a problem that is just as important, to show that the use of continuous functions allows arithmetic results to be obtained, because these functions satisfy the structural relations which preform those that are sought after to be obtained between whole numbers.

Here is the main part of Hecke's method, according to the summary that he himself gives:[1] let ω be a number of the considered field, whole or fractional, but distinct from 0; to ω a certain sum of terms are coordinated, this sum defines a new number called a Gauss sum $C(\omega)$. The Gauss sums

$C(\omega)$ attached to ω are then considered, and $C(-1/4\omega)$ attached, not directly to the inverse $1/\omega$, but the number $-1/4\omega$ formed from this inverse. Between $C(\omega)$ and $C(-1/4\omega)$ there exists a reciprocal relation that unites the value of the first to that of the second, and it is from this relation of reciprocity between Gauss sums that the law of quadratic reciprocity between whole numbers is proved.

The crucial point of the method therefore lies in the establishment of the reciprocal relation between $C(\omega)$ and $C(-1/4\omega)$. For this (by taking the field of rational numbers as the base field) the function of a complex variable $\theta(\tau)$ is envisaged defined as follows:

$$\theta(\tau) = \sum_{m=-\infty}^{m=+\infty} e^{-\pi\tau m^2} \tag{III}$$

This theta series converges for all values of τ of positive real parts, and the function $\theta(\tau)$ admits as a singular line the imaginary axis. It then happens to be that in the neighborhood of a singular point $\tau = 2ir$ (r being a rational number) the function becomes infinite, but the expression

$$\lim_{\tau=0} \sqrt{\tau}\,\theta(\tau + 2ir) \tag{IV}$$

takes a finite value. This limit is precisely, to a number of factors of no great importance, the Gauss sum of $C(-r)$ corresponding to the number $-r$. It will be further demonstrated that the function $\theta(\tau)$ obeys the functional equation:

$$\theta\left(\frac{1}{\tau}\right) = \sqrt{\tau}\,\theta(\tau) \tag{V}$$

so that a relation can in particular be obtained between the comportment of $\theta(\tau)$ at the point $\tau = 2ir$ and the comportment of $\theta(\tau')$ at the point

$$\tau' = \frac{1}{2ir} = -\frac{2i}{4r}$$

However, if for $\tau = 2ir$ the expression (IV) tends towards a limit which is of the order of $C(-r)$, for $\tau' = 1/\tau = -2i/4r$, it tends towards a limit which is of the order of $C(1/4r)$. A reciprocal relation between $C(r)$ and $C(-1/4r)$ can therefore be deduced from the functional equation (V) from which the law of ordinary quadratic reciprocity follows.

The transcendent proof of the law of quadratic reciprocity is essentially built on the functional equation (V). This equation represents a 'transformation formula' of theta functions, absolutely independent of arithmetic theorems of reciprocity. However a possible exchange of roles between notions is also observed, which to a certain extent can be considered as the 'inverse' of one another: the value of a function at a point τ and the value of this function at point $1/\tau$. It is in this sense that we could say above that the analytical tool, that is, the functions, serves to prove an arithmetic result, because the structure of the tool and that of the result both participate in a same dialectical structure, one that poses the problem of the reciprocity of roles between elements inverse to one another.

This dialectical idea of reciprocity between inverse elements can be so clearly distinguished from its realizations in arithmetic or in analysis that it is possible to find a certain number of other mathematical theories in which it is similarly realized. There is at least one, which not only thus presents the close affinities of structure with the previous theories, but is directly involved in the proof of the functional equation corresponding to (V), in the more general case of an arbitrary field *k*. This is the theory of the 'difference' of algebraic fields. In this theory two series of numbers belonging respectively to two distinct ideals of the field *k* are brought together, such that: (1) the ideals in question can be conceived as the inverse of one another (in a broad sense, distinct from the usual narrow sense); (2) The numbers belonging respectively to these inverse ideals, in the broad sense, can exchange the roles of coefficients and unknowns in a set of linear equations.

Here's how this theory plays a part in the proof of the general formula of transformation of theta functions. Let *n* be the relative degree of the field *k*, all the conjugates $k^{(p)}$ of which are assumed to be real. Consider in place of the function defined in III, the function

(VI) $$\theta(t,\mathbf{a}) = \sum e^{-\pi(t_1\mu^{(1)^2}+\dots t_n\mu^{(n)^2})}$$

in which $t_1 \dots t_n$ are complex variables with real positive parts, and in which $\mu^{(1)}\dots\mu^{(n)}$ represent the n conjugated values of a number μ that goes through all the numbers of an ideal **a** of the field once and once only. The theta function thus defined is therefore attached to this ideal **a**. The transformation equation that it satisfies is the following:

(VII) $$\theta(t,\mathbf{a}) = \frac{1}{N(\mathbf{a})|\sqrt{d}\,|\sqrt{t_1\dots t_n}}\theta\Big(\frac{1}{t},\mathbf{b}\Big)$$

$N(\mathbf{a})$ is the norm of the ideal **a**, d the discriminant of the field and **b** an ideal that may be characterized as both the inverse and as the reciprocal of **a**. It is defined in effect as follows: let α be any number of **a**. The ideal **b** is formed of all the numbers β of the field that satisfies the equation:

$$\text{(VIII)} \qquad \beta^{(1)}\alpha^{(1)} + \ldots \beta^{(n)}\alpha^{(n)} = \text{whole number}$$

and that for all α of **a**. It is proven that the product of two ideals **ab** forms an ideal independent of **a** and only depends on the field k. This ideal being the inverse (in the usual sense) of an ideal integer of the field, the different **d** is represented by l/**d**. We therefore have

$$\mathbf{ab} = \frac{\mathbf{1}}{\mathbf{d}}; \mathbf{a} = \frac{\mathbf{1}}{\mathbf{bd}}; \mathbf{b} = \frac{\mathbf{1}}{\mathbf{ad}}$$

It is in this broad sense that **a** and **b** can be considered as the inverse of one another. They are in addition reciprocal since in equation (VIII) the α can also be taken as given and the β as unknown which reverses their respective roles. This reciprocity of the two ideals is therefore defined independently of its application to the proof of quadratic reciprocity, and nevertheless plays a part in this proof as shown in formula (VII). Just as it passes from a function of t to a function of l/t, it passes from a function attached to the ideal **a**, to a function attached to the inverse and reciprocal ideal **b** = **l/ad**.

Hecke's transcendent proof therefore shows the convergence of three orders of mathematical facts: the facts of analysis – the transformation formulas of θ functions; the facts of algebra – the definition of the different of a body; of facts of arithmetic – quadratic reciprocity; and this convergence is explained only by the common dialectical structure in which these three theories participate.

It is impossible to conceive this problem of the reciprocity of the roles of inverse elements without also seeing its link with the duality theorem in topology. We have already, in our thesis (Lautman 1938b), insisted on the logical framework of this theory. It defines invariants called Betti numbers for an n- dimensional 'manifold' of dimension 0, of dimension 1 . . . of dimension n, and Poincaré's theorem asserts, in the case of an orientable manifold, the identity of the Betti numbers of dimension $n - m$ and those of dimension $m (0 \leq m \leq n)$.

The numbers $n - m$ and m, whose sum is constant, can still be conceived, in a new sense it is true, as the inverse of one another, and the identity of Betti numbers corresponding to the dimension $n - m$ and m can be conceived as an exchange of roles between these dimensions. The structural affinity of this theory with the theories of reciprocity which have been discussed above is manifest, and until very recently, however, it seemed that there was no link between them. A recent indication by Andre Weil (1938, 86) nevertheless shows that in certain cases, an approximation can be carried out between the laws of reciprocity and the duality theorems. Here again, the convergence of different mathematical theories results from the affinity of their dialectical structure.

2. THE DISTRIBUTION OF PRIMES AND THE MEASUREMENT OF THE INCREASE TO INFINITY

The existence of primes in the series of whole numbers has always seemed to present the type of mathematical facts as objective, as independent of any prior intellectual construction, as the most manifest physical facts. The passage from 15 to 16 and that from 16 to 17, for example, are done by the same act: the addition of unity to the preceding number, and yet the second operation gives a very different result from the first, since 17 is prime and 16 is not. What thus confers on the primes their objective character is the unpredictability of their coming. Arithmeticians since Euclid, but especially since Euler, have sought to progressively reduce this unpredictability in the occurrence of primes, and have tackled the problem from several sides. What was the *n*th prime, was sought to be determined *a priori* for any *n*; what was the interval separating two consecutive primes; how are primes distributed within the different geometric progression of reason k; what was the number of primes below a given number, etc. Some of this research provided extremely rich results, some was less advanced. It is thus, for example, that 'there are strong indications . . . that there exists an infinity of pairs of primes differing only by 2 (such as 17, 19 or 10 006 427, 10 006 429)'[2] but there is still no proof of the hypothesis in question.

We intend to examine briefly here the results related to the best approximate determination of the number of primes below a given number x, and this for two reasons. First, in accordance with the purpose of this essay, because the theory relative to this number $\pi(x)$ is essentially an analytic theory, it even constitutes the most classic part of analytic number theory. And second, because we can verify, at least in the particular case of this theory, one of the claims of our principal thesis (Lautman 1938b): mathematical reality does not lie in the greater or lesser degree of curiosity that can present isolated mathematical facts, but only in the dependence of a mathematical theory with respect to a dialectical structure that it incarnates. Research relating to the number $\pi(x)$ proceeds from a first concern of curiosity about primes, but its success is due to the fact that it was possible to free oneself from this easy and sensible aspect of things, and to express the problem of determining $\pi(x)$ in abstract terms that puts a more hidden structure into play, and of which the theory of analytic functions already offered a mathematical realization. It is in this connection of the arithmetic problem to a dialectical problem already resolved in an analytic theory that the reality inherent to the arithmetic theory lies, and by this the problem of knowing why the results of this theory are proven analytically is also resolved.

It is easily noticed that $\pi(x)$, which represents the number of primes that do not exceed x, increases to infinity with x, and the problem arises of expressing, for x tending towards infinity, $\pi(x)$ as a function of x. It is this problem that has been resolved, each time in a more precise way, by the work of Legendre, Gauss, Tchebicheff, Riemann, Hadamard, La Vallée Poussin, Landau, etc. A first result gives,

$$\text{for } x \to \infty,\ \pi(x) \to \frac{x}{\log x}.$$

This arithmetic theorem is closely connected to the properties of Riemann's continuous function $\zeta(s)$

$$\left[\zeta(s) = \sum \frac{1}{n^8}\right]$$

n goes through the whole numbers from 1 to infinity, and $s = \sigma + it$ with $\sigma > 1$. The previous formula can in effect be proven by relying on the fact that $\zeta(s)$ has no zero on the straight line $\sigma = 1$ of the complex plane, and conversely, from the proposition

$$\pi_{x \to \infty}(x) \to \frac{x}{\log x}$$

assumed to be true, the proposition related to the absence of zeros of $\zeta(s)$, on the straight line $\sigma = 1$, can be proven. These two propositions are therefore equivalent, but scarcely any intuitive logical reason for this equivalence can be seen. We are going to see on the contrary a reason of this kind appear in a more advanced study of $\pi(x)$.

If the nature of the link sought after between $\pi(x)$ and x is examined, it is realized that it is a matter of comparing the respective rapidity of two quantities increasing to infinity, constantly increasing, and one of which, x, is always the upper possible limit of the values of the other. Instead of envisaging two increasing quantities, one of which is the upper limit of the other, two quantities can still be considered, one of which is an independent variable increasing to infinity, and the other is a constantly increasing function of the first. Thus, for example, the radius r and the area of the circle for r tending towards infinity. The comparison of their mode of increase is a problem logically analogous to the one that arises in connection with $\pi(x)$ and x. Just as it is always $\pi(x) \leq x$, the determination of the circumference of any circle of radius r, *ipso facto* sets the maximum area of any circle of radius ρ, such that $0 \leq \rho \leq r$. Instead of envisaging the area of the circle, any function $f(r)$ can still be considered, provided that it is constantly increasing as r tends towards infinity, and it is thus precisely the point of view of the measurement of their rapidity of increasing with respect to the radius of their definitional circle, that certain analytic functions connected to the function $\zeta(s)$ are introduced in the theory of primes.

It is a matter of analytic integral functions, that is, which only have singularities at infinity. Let $f(z)$ be such a function and M the maximum of its modulus in the circle $|z| \leq r$. It is proved that the function reaches the maximum of its modulus $M = M(r)$ for $|z| = r$ and that consequently this function $M(r)$ is a constantly increasing function of r when r tends towards infinity. Consider then the expression:

$$\text{(IX)} \qquad \frac{\log M(r)}{r^{\beta}} \text{ for } r \to \infty \quad .$$

If a real value of β exists, such that the expression (IX) always remains lower than a constant K, there is a unique number $\omega \geq 0$, such that it also has place for $\beta > \omega$ for any β. The number is called the order of the function $f(z)$. The research of the order of an integral function $f(z)$ therefore comes back to comparing the rapidity of increase of an increasing function, dependant on the integral function [of the kind $\log M(r)$], with the rapidity of the increase of successive powers of the radius. We find here again the structural idea, identified above, in regards to any magnitude $f(r)$, increasing function of an independent variable r.

The integral function, attached to $\zeta(s)$, which plays a part in the theory of primes is the following:[3]

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s)$$

By calling $M(r)$ the maximum modulus of $\zeta(s)$ for $|s| = r$ we have:

(X) $$\log M(r) \sim \frac{1}{2} r \log r$$

or again $$\frac{\log M(r)}{\frac{1}{2} r \log r} \to 1 \quad \text{for } r \to \infty$$

This fundamental relation allows the comparison of the rapidity of increase of $M(r)$ and of r. From this relation another relation is deduced concerning the increase of

$$\frac{\zeta'(s)}{\zeta(s)}$$

in a certain region of the complex plane $s = \sigma + it$, when t tends towards $\pm\infty$. This region is such that it always has $\sigma \geq 1 - \alpha\eta(|t|)$, with $0 \leq \alpha \leq 1$ and in which $\eta(t)$ is a decreasing function of t satisfying certain conditions, which it is unnecessary to present here. It is sufficient to have indicated that for a function $\eta(t)$ satisfying these very specific conditions, we have

(XI) $$\frac{\zeta'(s)}{\zeta(s)} = 0(\log^2|t|) \quad \text{with} \quad \underset{t \to \pm\infty}{\sigma \geq 1 - \alpha\eta(|t|)}$$

which means that

$$\frac{\frac{\zeta'(s)}{\zeta(s)}}{\log^2|t|}$$

always remains below a constant quantity.

$$\left(\frac{\zeta'(s)}{\zeta(s)} \text{ is of the order of } \log^2|t|\right).$$

All that we now have to show is how the transposition of these measures of increase in arithmetic is done. Just as we compared M(r) to r, we claim to compare $\pi(x)$ to x. Instead of taking $\pi(x)$ on directly, arithmeticians first envisage another arithmetic function $\psi(x)$, whose increase with respect to x is closely connected to that of $\pi(x)$. This function is in addition, like $\pi(x)$, a discontinuous arithmetic function. It is defined as

$$\psi(x) = \sum_{p^m \leq x} \log p \quad ;$$

$\psi(x)$ is therefore the sum of logarithms of primes p, such that any power p^m, for positive integer m, is less than x. The essential point of the theory consists therefore in passing from the consideration of the increase of the continuous function

$$\frac{\zeta'(s)}{\zeta(s)}$$

with respect to the increase of the variable t, to the consideration of the increase in the discontinuous function $\psi(x)$ with respect to that of the variable x. The passage from the continuous to the discontinuous is carried out through the formulas that allow $\psi(x)$ to be expressed according to

$$\frac{\zeta'(s)}{\zeta(s)}$$

For this the continuous function

$$\psi_1(x) = \int_0^x \psi(u)\,du$$

can be considered and the essential relation:

$$\psi_1(x) = \frac{1}{2\pi i}\int_{0-\infty i}^{c+\infty i} \frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)ds \qquad \begin{matrix} x>0 \\ c>1 \end{matrix}$$

is proved. The link between the continuous and the discontinuous being established, here are the results:

We deduce from (XI), for x tending to infinity

$$\psi(x) = x + 0(xe^{-a\sqrt{\log x}})$$

$$\pi(x) = \int_2^x \frac{du}{\log_u} + 0(xe^{-a\sqrt{\log x}})$$

in which *a* is a positive constant absolute.[4]

We thus see in what sense it has been possible to speak of imitation or structural affinity between analysis and arithmetic, the continuous and the discontinuous. The idea of comparing the rapidity of increase to infinity of two constantly increasing quantities, and connect one to the other, is rationally anterior to the distinction between the continuous and the discontinuous. It can therefore be realized in theories as distinct as the theory of the increase of integral functions and the theory of the distribution of primes, just as we have seen above, the search for the possible exchange of roles between inverse notions realized in arithmetic, in algebra, in analysis, in topology. Each time it is a common participation in the same dialectical Idea, which explains the use of the analytical tool in arithmetic.

The convergence of theories which manifest the same structure thus confirms, in a concrete way and as *a posteriori*, the results of the deduction that we attempted in the first part: thought necessarily becomes involved in the elaboration of a mathematical theory as soon as it claims to resolve in a precise way a problem that could be raised in a purely dialectical way, but it is not necessary that the examples are taken from a particular domain, and in this sense, the various theories in which the same Idea is incarnated, similarly find in it the reason for their structure and the cause of their existence, principle and origin. As in the philosophy of Heidegger, in the philosophy of mathematics, as we conceive it, the rational activity of foundation can be seen transformed into the genesis of notions relating to the reel.

3. CONCLUSION

If the geneses that have been described in the preceding pages are compared with those that have been studied in the second part of our principal thesis (Lautman 1938b), one cannot fail to notice the differences between these two kinds of genesis. Those described here concern the coming of mathematics from the Dialectic, whereas the schemas of genesis of our thesis concern the different ways, within mathematics proper, in which the structure of certain basic domains are adapted to bringing to the fore certain new entities defined on these basic domains. To understand the reasons which led us to thus distinguish in mathematical philosophy at least two kinds of genesis, it is necessary to recall the two ways in which the reunion, in a same synthetic theory, of notions related to the continuous and notions related to the discontinuous can be interpreted.

Following a study on the penetration of algebraic methods in analysis (See Lautman 1938a) we distinguish two types of possible relations between the theories of the continuous and those of the discontinuous: relations of imitation and the relations of expression. There is imitation when the structure of an arithmetic or algebraic theory and that of an analytic theory present affinities comparable with those observed above in analytic number theory. We have seen in this case how the analysis of certain dialectical Ideas are extended in the genesis of a reality distinct from the Ideas: the theories in which these Ideas are realized. There is on the contrary a relation of expression between the discontinuous and the continuous when the algebraic or topological structure envelops the existence of analytic entities definable on this structure. It is thus that the topological properties of Riemann surfaces envelop the existence of abelian integrals attached to the surface, or more generally that a correspondence is established between algebraic entities and transcendental entities, for example between discontinuous groups and continuous functions. The structure of the former envelops the existence of the latter and inversely the existence of the former expresses or represents the structure of the latter. It is then in accordance with the general theory of geneses to also define the passage from the structure of the domain to the existence of representations as a genesis, since this passage from essence to existence takes place from the structure of an entity to the assertion of the existence of other entities than the one whose structure was originally in play.

The result of this comparison is that no such difference separates the genesis of mathematical theories as a result of the Dialectic, from the geneses that are carried out, within mathematics, from structures to existence. One could say in Platonic terms that the participation of Ideas among themselves obey the same laws as the participation of Ideas in the supreme genus and ideal numbers; in both cases, mathematical philosophy essentially offers its services as the object to witness the eternally recommencing act of the genesis of a Universe.

Letter to Mathematician Maurice Fréchet

1st February 1939

Léon Brunschvicg, president of the Société française de philosophie, had made the rare choice on Saturday, 4 February 1939, to have two presentations that dealt with mathematical thought presented respectively by Jean Cavaillès and Albert Lautman. The account of the meeting was published in the Bulletin of the Société *in 1946 (Cavaillès and Lautman 1946). It has been included in the volume* Oeuvres complètes of Jean Cavaillès *by Hermann (1994). Among the interventions are those notably of Elie Cartan (who had been a member of the thesis jury for each of them the previous year), Maurice Fréchet, Claude Chevalley, Charles Ehresmann, and Jean Hyppolite.*

Below is an exchange, previously unpublished, between the great mathematician Maurice Fréchet and Albert Lautman that took place on the eve of this meeting. The comparison between the response of Albert Lautman and his presentation three days later allows this letter to be read as a first version of the presentation.

Maurice Fréchet to Albert Lautman, 30 January 1939:

> My dear colleague,
> Brunschvicg invited me to participate next Saturday in the discussion of your thesis. The summary of your communication concerns in addition perhaps also your other publications.
>
> Not being accustomed to the philosophical language, I have some difficulty grasping the precise meaning of your summary. Thus in the first paragraph I understand the mathematical examples that you give, but I don't quite grasp exactly the general idea they are charged with

illustrating and that is indicated at the beginning of the last sentence of this paragraph.

In the second paragraph, I understand that you have gone from certain material realizations to a very formal system, but I cannot conceive the reverse (except of course after the fact).

I understand the genesis of the Idea from the real but not vice versa. I do not assimilate the second and third sentence from the end of the summary. If you had the time to explain in one or two pages these different points in ordinary language, I would draw more profit from it.

I'll send you the only copy I have left (while asking you to return it to me on Saturday) of my report to the conference in Zurich (December 1938) on the foundations of math.

Please accept my dear colleague, the expression of my highest consideration.

Albert Lautman responded on the 1st of February, by announcing the forthcoming publication of the essay *New Research on the Dialectical Structure of Mathematics* (Lautman 1939). He continues:

Your first question concerns the way in which particular mathematical theories, for example those I cite in my summary, seem to receive all of their meaning from the fact that they provide examples of solution to the problems that are not strictly mathematical but dialectical (as defined by Plato).

I call notions the Whole, the part, the container, structure in the topological or algebraic sense, existence etc. I call Ideas the problem of the elaboration of relations between notions thus defined. So I conceive the Idea of a dialectical problem of the relations between the Whole and part as knowing if global properties can be inscribed in local properties. I even conceive of the Idea or the problem of knowing if the situational properties can be expressed as a function of structural properties, and it is to the extent that a mathematical theory provides a response to a dialectical problem definable but not solvable independently of the mathematics that the theory seems to me to participate, in Plato's sense, in the Idea, in comparison to which it is in the same situation as the Response with respect to the Question, Existence with respect to essence. *Even if, historically or psychologically, it is the existence of the response which* *suggests the Idea of the question (the existence of mathematical theories allow the identification of the dialectical problem to which they respond), it is in the nature of a question to be rationally and logically anterior to the response*. Thus the problem of knowing whether it is possible to determine the existence of an Entity by the indication of the exceptional properties that it would have the use of if it existed seems to me to dominate mathematical theories as different as the calculus of variations or the topological methods of Poincaré-Berkhoff, determining the existence of periodic solutions by the determination of fixed points in an internal transformation of the points of a surface. To the exceptional properties of extremum or fixed points, assertions of existence are, in certain cases, attached. The two theories are different responses to the same problem. Your second question concerns the passage from a formal system to material realizations. Let us call the properties of a system of axioms, definable independently of any realization whatsoever in a domain of objects, formal or structural; for example, consistency, saturation. The concern of knowing whether a system of axioms is consistent has a purely formal meaning and does not require the existence of a realization. But experience shows that the proofs of consistency nonetheless rely most often on the consideration in extension of domains of objects in which the hypotheses of the theory are realizable (*cf*. Herbrand's theorem).

The consideration of the possibility of realisation (*Erfüllbarkeit*) can therefore be conceived as led necessarily by a primarily oriented concern towards the study of the structural properties of a formal system. Domain theory is the necessary detour to arrive at conclusions related to formal consistency. It is in this sense that I see the existence of a realization as the manifestation of an internal consistency. Formal, it is made explicit by the material existence of realization. Similarly in the representation of abstract groups, if one were to consider as the formal or structural property of an abstract group the number of its classes, this number determines at the same time the number of irreducible and non-equivalent representations of the group. One can therefore envisage the number of classes of the group as carrying out the passage from the structure of the abstract group to the existence of its representations. The linear representations of the group constitute a material realization. There is here in group theory a passage from the formal to existence that I compare in my thesis to the passage that the notion of genus of a Riemann surface allows to be carried out between the topological structure of this surface and the existence of abelian integrals of the first kind on the said surface. These analogies have led me to substitute for the usual terminology Form and Matter another terminology in which the systems of axioms, as well as the abstract group, constitute a basic domain (by analogy with the Riemann surface) while the representations or realizations constitute the entities defined on a basic domain. This does not alter the fact that there exists a passage in the usual sense from the formal to the material.

Your third question is concerned with the genesis of the real from the Idea. Just as you admitted the passage from the material to the formal but not the inverse, you admit the passage from the reel to the idea, by abstraction evidently, and not the inverse. In understanding by Idea, the idea of a possible dialectical problem, *one can envisage abstractly the Idea of knowing whether relations between abstract notions exist, for example the container and the content, but it happens that any effort whatsoever to outline a response to this problem is ipso facto the fashioning of mathematical theories. The question of knowing whether forms of solidarity between space and matter exist is in itself a philosophical problem, which is at the center of Cartesian metaphysics. But any effort to resolve this problem leads the mind necessarily to construct an analytic mechanics* in which a connection between the geometric and dynamic can in fact be asserted. Here again logical anteriority, the philosophers would even say ontological, of the Idea with respect to real mathematics. The interest for me of General Relativity taken as a pure mathematical theory and not physical comes from what appears to me to be a response to a problem that is able to be formulated independently of mathematics: to what extent do the properties of space determine those of matter? Einstein's theory is not the only possible response: it is a model of a possible solution among others, but what is necessary is the constitution of a mathematical theory as soon as the dialectical question is raised. That's why I wrote that mathematics is an example of incarnation, in the sense in which mathematical concepts constitute for example a matter on which relations envisaged as possible by the dialectic are effectively drawn. The comprehension of a mathematical theory or its elaboration when it is under development has a dual meaning: mathematical from the point of view of the results it provides; philosophical from the point of view of the constitution, in the process of a schema of response to a dialectical problem being carried out. *It is the spectacle of the constitution of these structural schemas that seemed to me to found the*

> *philosophical interest in mathematical thought. In sum, while Cavaillès searches in mathematics itself for the philosophical sense of mathematical thought, this sense appears to me rather in the connection of mathematics to a metaphysics (or Dialectic) of which it is the necessary extension. It constitutes the matter closest to the Ideas. It seems to me that this is not a diminution for mathematics. It confers on it, on the contrary, an exemplary role.*

On 4 February 1939, Cavaillès finished his presentation: 'Clear, rigorous, mathematical knowledge prevents us from posing objects as existing independently of the system performed on these objects and even independently of a necessary sequence from the beginning of human activity' (Cavaillès and Lautman 1946, 12).

Albert Lautman: 'Cavaillès seems to me, in what he calls mathematical experience, to attribute a considerable role to the activity of the mind, determining in time the object of its experience. General characteristics constitutive of mathematical reality would therefore not exist. The latter would be asserted at every moment like a simultaneously necessary and singular event. . . . I admit the impossibility of an immutable Universe of ideal mathematical entities. The properties of a mathematical entity depend upon the axioms of the theory, that which deprives them of the immutability of an intelligible universe.

'I consider no less numbers and figures as possessing an objectivity as certain as that which the mind comes up against in the observation of physical nature, but this objectivity of mathematical entities, that is manifested in a sensible way in the complexity of their nature, only reveals its true meaning in a theory of the participation of Mathematics in a higher and hidden reality which constitutes a true world of Ideas' (13, abridged). 'That this experience is the *sine qua non* of mathematical thought is certain, but I think that there is in experience more than experience . . . to grasp beyond the temporal circumstances of the discovery, the ideal reality, independent of the activity of the mind.' (p. 39, abridged)

Cavaillès: 'Personally I am reluctant to posit something else that would govern the actual thinking of the mathematician, I see the exigency in the problems themselves. Perhaps this is what he calls the Dialectic that governs; if not I think that, by this Dialectic, one would only arrive at very general relations. . . . The future will show which of us is right.' (p. 36)

This is surely not a purely verbal comment since a few weeks before, he had written to Lautman: 'Basically you may be right. I am for myself so engrossed in (basically the same) problem of mathematical experience that I cannot see the connection with any other way of positing it. But maybe we will concur in the end . . . I'd very much like to' (Benis-Sinaceur 1987, 123–4).

BOOK IV

Symmetry and Dissymmetry in Mathematics and Physics*

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* The original edition (Lautman 1946) reprinted as an overture the biographical notice written by Suzanne Lautman, also a qualified philosophy teacher, for the *Annuaire des anciens eleves de l'École normale supérieure* (Obituaries edition published in 1945). This moving text, written for the Normalien community even before the end of the war, is not reprinted here, the author already having withheld consent to reproduce it in the 1977 edition (Lautman 1977).

CHAPTER 1
Physical Space

The important role played by the so-called paradox of symmetric objects in Kant's philosophy is well known (Kant 1768). The difference in orientation of symmetric figures with respect to a plane in ordinary space appeared to Kant as a sensible intuition, irreducible to any conceptual determination, and this necessary intervention of sensibility in the knowledge of left and right is at the origin of the Kantian distinction between sensibility and understanding. The specificity of the sensible is therefore marked by the incongruence of symmetric figures, and it is truly a remarkable achievement that Kant had characterized the sensible as early as 1770 by a property that contemporary science has found at the centre of all investigations of the structure of phenomena.

In the language of crystallographers, two isomeric crystals, symmetric to one another with respect to a plane, and not superimposable, are said to be enantiomorphs. For two symmetrical crystals to be enantiomorphs, it is necessary that each presents in isolation a certain internal dissymmetry, as for example the absence of a centre of symmetry. The importance of dissymmetry by enantiomorphy appeared with the pioneering work of Pasteur, when he recognised the connection that exists between the difference of geometric orientation of two enantiomorphic (or hemihedral) crystals, and the inversion of their effects on the plane of polarization of light. Following these discoveries, Pasteur conceived the theory of molecular dissymmetry, manifesting itself essentially in the hemihedry of two isomers, and characteristic of living phenomena:

> Artificial products therefore have no molecular dissymmetry and I cannot indicate the existence of a more profound separation between the

> products formed under the influence of life and the others. (Pasteur 1861 [1922, 333])

Even if the conceptions of Pasteur on the connection of life and dissymmetry by enantiomorphy can no longer currently be defended, they nevertheless gave rise to the theory of asymmetric carbon, which is at the origin of all the structural theories of modern stereochemistry.

The idea of enantiomorphy presents itself to analysis, we have seen, as a close union of symmetry and dissymmetry. This same requirement for the mixture of symmetry and dissymmetry is found in the work of Pierre Curie on symmetry in physical phenomena. Curie no longer understands this to characterize only biological phenomena as opposed to physical phenomena. The mixture of symmetry and dissymmetry becomes for him a necessary condition of physical phenomena in general. The determination of the elements of symmetry of a physical phenomenon is carried out, as in crystallography, by the searching for the center, the axes and the planes of internal symmetry that present the phenomenon. To any physical phenomenon is tied the idea of a saturation of the symmetry, of a maximal symmetry compatible with the existence of this phenomenon and which characterizes it. A phenomenon can only exist in an environment possessing its characteristic symmetry or a lesser symmetry. Therefore, if the absence of an element of symmetry is called an element of dissymmetry, it is conceivable how Pierre Curie could write:

> Certain elements of symmetry can coexist in certain phenomena, but they are not necessary. What is necessary is that certain elements of symmetry do not exist. It is the dissymmetry that creates the phenomenon. (Curie 1894; 1908, 126 [1982, 21])

Thus the presence of an electric field is incompatible with the existence of a centre of symmetry and of a plane of symmetry normal to the axis of the field, and the presence of a magnetic field excludes the existence of planes of symmetry passing through the axis of this field. The dissymmetry constitutive of the physical phenomena is therefore defined by Curie in the idea of a limited symmetry, of a presence of elements of symmetry to which the absence of other elements is necessarily conjoined, and the enantiomorphy of Pasteur is only one of these dissymmetries within the symmetry which generate the sensible world.

In seeing the sensible thus defined by a mixture of symmetry and dissymmetry, of identity and difference, it is impossible not to recall Plato's *Timaeus* (1997). The existence of bodies is based there on the existence of this receptacle that Plato calls the place and whose function consists, as Rivaud has shown in the preface to his edition of the *Timaeus* (Plato 1932), in making possible the multiplicity of bodies and their alternation in a single place in the sensible world, just as the role of the Idea of the Other in the intelligible world is to ensure, by its mixture with the Same, both the connection and the separation of types. This reference to Plato enables the understanding that the materials of which the universe is formed are not so much the atoms and molecules of the physical theory as these great pairs of ideal opposites such as the Same and the Other, the Symmetrical and Dissymmetrical, related to one another according to the laws of a harmonious mixture. Plato also suggests more. The properties of place and matter, according to him, are not purely sensible, they are, as Rivaud goes on to say, the geometric and physical transposition of a dialectical theory. It is also possible that the distinction between left and right, as observed in the sensible world, is only the transposition on the plane of experience of a dissymmetrical symmetry which is equally constitutive of the abstract reality of mathematics. A common participation in the same dialectical structure would thus bring to the fore an analogy between the structure of the sensible world and that of mathematics, and would allow a better understanding of how these two realities accord with one another.

The development of modern mathematical physics offers in this regard an extremely suggestive lesson. If we consider the theories elaborated to account for sensible facts, in which the distinction between the left and right plays a crucial role, that is, essentially phenomena of rotation, of the electromagnetic field, of the polarization of light, we realize that they bring into play abstract mathematical theories developed independently of any concern for physical application, and which nevertheless present the following aspect: we find there either a duality of opposing elements capable of being permuted with one another as the left and right are permuted in a symmetry; or, which is even more characteristic of the mixture of symmetry and dissymmetry, a division of mathematical entities into two classes, a rigorously symmetric class as an ambidextrous being would be and a class that mathematicians call antisymmetric, that is, that changes orientation by symmetry, as the right hand or the left hand, or the sign by the permutation of two variables, as a straight line AB in space changes orientation when it is traversed from B to A.

Our first example is taken from the theory of spinors, in its relation with the *spin* of the electron. The physical requirement that is at the origin of the theory of electron *spin* is the necessity of experimental origin to endow the electron with a moment of rotation, or *spin*, which could at any time take two opposing values with two different probabilities. The elaboration of this conception, in Dirac's theory, resulted in the attachment to an electron of a wave of four components, or rather two groups of two components, constituting what is called a spinor. These two groups of two components, these two semi-spinors, play a role with respect to one another that can be considered as the left and the right in the space-time of special relativity. To any turning of space in space-time, which conserves the sense of time and changes the sign of the directions of space, an algebraic operation can be made to correspond which permutes the two semi-spinors, and thus represents in the abstract space of the spinor components, the sensible operation of a change in orientation in physical space. These abstract entities, the spinors that are internally divided in two, whose role in quantum mechanics is so important, had not however been invented by physicists, since Elie Cartan had discovered them in 1913 in his research on the linear representations of the rotation group of a space of an arbitrary number of dimensions. This is a typical case where the physics of the dissymmetric symmetry refers to an algebra in which there is an exchange of roles between opposite terms.

The example of the unitary theories of the electromagnetic and gravitational fields offers the analogous case of a physical theory based on the dissymmetry of certain mathematical entities discovered anew by Cartan, well before their use by Einstein. In his theory of 1928, Einstein associates the phenomena of electro-magnetism and the phenomena of gravitation by considering spaces endowed with *torsion*, which were introduced into science by Cartan in 1922. Here is a simple example, due to Cartan, which may suggest the notion of torsion. Consider two systems of curves on a surface, such as the meridians and the parallels on the sphere. Suppose that a ship moves a distance d along a meridian going from point A to point A´, then a quantity δ along a parallel from A´ to A˝. Now suppose that the ship moves in reverse order and traverses first δ on the parallel passing through A, which leads it to A''', and d on the meridian passing through $\overline{A}''$, which leads it to a point A''' distinct from A˝. If the succession of operations in each case is therefore designated respectively by $d\delta$ and δd, then we see that $d\delta \neq \delta d$. The existence on the sphere of a vector of torsion $A''\overline{A}''$ (or of the opposite vector $\overline{A}''A''$) thus realizes the non-commutativity of the operations d and δ. In a more general way, the distinction between left and right in this case is only the sensible expression of an algebra with non-commutative multiplication, in which the product AB differs from the product BA.

It is necessary to emphasize the fashion in which the distinction between left and right in the sensible world can symbolize the non-commutativity of certain operations of abstract algebra. The fundamental property of symmetry with respect to a plane, applied once, gives a figure distinct in its orientation from the original figure, and repeated a second time, gives the original figure again. It is for this reason that symmetry is said to be an *involutive* operation. Let us now consider an algebraic operation concerned with two quantities X and Y, and which can be written (XY), the parentheses denote an ordinary product, or any other operation defined on the two variables. It is a non-commutative operation if $(XY) \neq (YX)$ and the most fruitful non-commutativity in mathematics is that in which $(XY) = -(YX)$. The operation (XY) is dissymmetrical, in X and Y, but it is easily verified that it defines an involutive operation, as does ordinary symmetry. The expressions (XY) and (YX) are said to be antisymmetric, and this word reflects well the mixture of symmetry and dissymmetry which is thus installed deeply in the heart of modern algebra. The whole theory of continuous Lie groups is based on the non-commutativity of the product of two infinitesimal operations of the group. This theory, which is closely associated with the theory of Pfaffian forms, expressions with antisymmetric multiplication, allowed Cartan to discover a profound analogy between the generalized Riemann spaces which play a part in the physic–geometrical theories of relativity and the space of Lie groups.

The examples of mathematical physics that have been cited so far show how the sensible fact of dissymmetrical symmetry could be conceived as the equivalent, in the world of phenomena, of the antisymmetry inherent to certain mathematical entities. Recent developments in the wave mechanics of systems of particles have shown that the distinction between symmetry and antisymmetry served as the foundation for sensible qualities of matter, perhaps even more important from the philosophical point of view than the properties of orientation. It is a question of the constitution and stability of molecular structures, of the very notion of system, of the 'whole', in the sense that a whole possesses the global properties which characterize it qualitatively, and make something other, and more, than the sum of its parts.

The wave mechanics of systems of particles considers a system composed of particles of the same nature, and attaches to this system a wave function

$\psi(1,2,\dots n)$, which is a function of n particles of the system. When the trajectories of these particles of the same nature impinge upon one another, the particles become indiscernible, and the system must also be described equally well by any function obtained by permuting in any way the n variables of the function ψ as by the function ψ itself. The result of this theorem is that for the systems containing only two particles of the same physical nature, the functions ψ which describe the evolution must necessarily be either symmetric or antisymmetric with respect to these two particles, that is, we have either $\psi(1,2) = \psi(2,1)$ in the symmetric case, or $\psi(1,2) = -\psi(2,1)$ in the antisymmetric case.

As far as systems with an arbitrary number of particles are concerned, an analogous result follows from experimental data. According to de Broglie, it is certain that for each type of particle the wave functions are either symmetric or antisymmetric, and the antisymmetry seems to play a much more fundamental role in Nature than symmetry. In fact, if the elementary particles are distinguished from the composite particles, the systems of elementary particles such as the electron, the proton, the neutron, the neutrino if it exists, are observed in an antisymmetric state. On the other hand, composite particules formed by the union of several elementary particles, like photons or α particles, are in a symmetric or antisymmetric state depending on whether the number of constituents is even or odd (See de Broglie 1939). These results have allowed the notion of chemical valence, and consequently the constitution of molecules, to be linked to the antisymmetry of the spin of two electrons belonging to two distinct atoms. Antisymmetry thus plays a crucial role in explaining the molecular bond.

If the mathematical basis of this distinction of wave functions into symmetric functions and antisymmetric functions is examined, it is found in an internal dissymmetry in the group of permutations of n objects. This group is decomposed into two families: the subgroup of even permutations and the family of odd permutations, which does not form a group since it does not contain the identity permutation. Consider for example the group of permutations of three variables 1, 2, 3. The subgroup of three even permutations is obtained by circular permutation: 1, 2, 3; 2, 3, 1; 3, 1, 2; the three odd permutations are 1, 3, 2; 3, 2, 1; 2, 1, 3. Here we have the second type of dissymmetry discussed above, that of an entity which is divided into a symmetrical part and a dissymmetrical part. The example of the wave mechanics of systems of particles, in which we see the saturation of electronic levels in an atom and the formation of chemical molecules related to a mathematical dissymmetry as essential in its simplicity as that of the group of permutations of *n* objects, shows in the clearest way in what sense it is possible to speak of the common participation of the sensible world and of certain mathematical theories in a same dialectical structure, composed of the mixture of symmetry and dissymmetry.

We would like to go further and show the importance of this structure, not only for those mathematical theories that are applied to the sensible universe, but, in a general way, for the most abstract domains of mathematics. In order to do this, reconsider for a moment the analysis of the distinction between left and right. We find there two ideas: 1) the division of a complete entity into two distinct parts, at least by the inversion of their orientation, and 2) the existence of an involutive relation between the two parts, such that A is to B as B is to A, the symmetrical of the symmetrical restoring the original element. Transposed into a more abstract language, this situation is equivalent to the possibility of distinguishing within the same entity two distinct entities X and X´, which are said to be in duality, first, if an orientation or an order inverse to that of the other can be defined for each of them, and second, if an involutive relation exists between them, that is, if X is to X´ as X´ is to X; that is if $(X')' = X$.

A certain number of mathematical theories that are based on this kind of structure of duality have been known for a long time. The most famous is the calculation of propositions established by Boole in 1847, which is at the basis of modern mathematical logic. Let Σ be the set of all possible propositions of the theory. This set is subdivided into two parts with no common elements S and S´ which are complementary to one another, that is, their logical product (in symbols: $S \cap S'$, the set of elements common to S and S´) is empty, and their logical sum (in symbols: $S \cup S'$, the set of elements belonging to either S or S´) is equal to the total set Σ. These two subsets can be considered, if we wish, as the set of true propositions and the set of false propositions, but this particular interpretation is not at all necessary. It is sufficient to consider them as two complementary sets. The involutive character of this complementation is made evident by the fact that the complement of the complement of S is equal to S. The global duality between S and S´ allows a duality to be established between a formula P of S and a formula P´ of S´, which will be called the negation, the contrary, or the complement of P, and is such that the logical sum $P \cup P'$ is always true and the logical product $P \cap P'$ always false. The essential property of this duality is to interchange the symbols of logical addition and logical multiplication. Given a formula constructed with elementary propositions, *p*, *q*, . . . etc., and the logical symbols of the sum, the product, and the negation,

$\cup$, $\cap$, $'$, the negation can be obtained, which is also called, as we have seen, the contrary or the complement of this formula, by replacing all the elementary propositions of the formula by their negation and by permuting the signs $\cup$ and $\cap$. Thus we have

$$(p \cup q)' = p' \cap q' \text{ and } (p \cap q)' = p' \cup q'.$$

If, by using the notion of implication, an order is introduced between two propositions, it can easily be proved that the duality changes the meaning of the implication

$$(p \supset q)' = q' \supset p'$$

which helps identify the duality of geometric symmetry and the duality of logical negation.

There is another mathematical theory in which the notion of duality has played a fundamental role since its discovery by Poncelet in 1822. It is projective geometry. It is well known that in projective plane geometry, in any true proposition constructed with the notions of point and line and the relation of inclusion, a true proposition can be obtained anew by interchanging the words: point and line, and by changing the meaning of the relation of inclusion. To the points situated on a line there correspond the lines passing through a point. Thus a curve can be defined by means of the points which compose it, or by the lines tangent to it at each point.

Here is the algebraic expression of this duality. Consider the equation, $u_1x_1 + u_2x_2 + u_3x_3 = 0$. This equation can be interpreted in two ways. If the three quantities u_1, u_2, u_3 are considered as defining a line in the space of the projective plane, the coordinates of all the points of this line are defined by the variables x_1, x_2, x_3 satisfying the equation. Conversely, if we take the three coordinates x_1, x_2, x_3 of a fixed point, the equation is that of all the lines u_1, u_2, u_3 passing through this point. The quantities u_1, u_2, u_3 and x_1, x_2, x_3 can therefore interchange their roles as coefficients and variables in the proposed equation, and it follows from this fact that the projective plane can be considered either as a set of points, or as a set of lines, and these two sets are said to be in *correlation* with one another. More generally, in an n-dimensional projective space S_n, the points (elements of 0 dimension) and the hyperplanes (elements of $n - 1$ dimension) of this space are in correlation, and this correlation constitutes a true duality, as defined above. Modern axiomatic research has, in effect, allowed the sum and the intersection, or product, of two projective spaces S_p and S_q of respective dimensions p and q to be defined rigorously, and to associate in this way, to every subspace S_m of a space S_n, a dual space S_{n-m-1}, such that

$$S_m \cup S_{n-m-1} = S_n \text{ and } S_m \cap S_{n-m-1} = 0.$$

The sum of a subspace and its dual gives the whole space, and their intersection is empty. It can easily be demonstrated that this duality is involutive and it reverses the inclusion relation:

$$\text{if } S_p \subset S_q, S_{n-p-1} \supset S_{n-p-1}.$$

There is therefore a projective duality as well as a logical negation or a geometric symmetry.

It seems at first that the aim of the calculus of propositions and that of projective geometry are different, and yet the logical structure of these two disciplines, as we have shown, present many analogies. The reason for this analogy only appeared recently in light of the latest research in the domain called abstract algebra. Inspired by Dedekind, a large number of contemporary mathematicians, including Birkhoff, von Neumann, Glivenko (see Glivenko 1938), Ore and others, have constructed a general theory of structures (English authors call lattices)[1] that includes the theory of sets, number theory, projective geometry, combinatorial topology, probability theory, mathematical logic, the theory of functional spaces, etc. Here are the basic notions of the theory. Each time a set S is considered, the parts of this set that are selected are not the individual elements of this set, but the subsets of the set. Between any two parts there is defined either an order (in the case of a set composed of a finite number of parts), or at least a partial order, through relations such as magnitude, dimension, inclusion, implication, boundary, and the two operations of sum and product. The set S* is then considered, obtained by inverting the order between the parts and the permutation of the symbols for sum and product. The new set thus obtained is called the dual of the first, and, in the most interesting cases, the dual set is none other than the original set in which all the order relations are reversed. Duality thus establishes an *anti-isomorphism* (or inverse isomorphism) between S and S itself. It is in addition an involutive operation, since the dual of a dual set restores the original set. The general theory of lattices is therefore based on the possibility of structuring the same set in two mutually inverse ways. To see this internal duality of two antisymmetric entities, distinguishable within the same entity, is a result of major philosophical importance, forming the principle generator of an immense harvest of mathematical reality. The theories that have been cited above, according to Glivenko, allow us to consider as lattices, the totality of rings, the set of convex fields, the set of subgroups of any group, the set of all positive whole numbers, the set of all the elements of projective geometry, the set of all simplexes subordinate to a topological simplex, the set of events of probability theory, the set of propositions of propositional calculus, etc.

This development of the theory of lattices has naturally led to the establishment of distinctions between all the structures satisfying the law of duality. Thus, for example, in the case of mathematical logic, the complementary formula of a given formula is determined in a unique way, whereas, given $m + 1$ points that define a subspace S_m of a projective space S_n, there exist an infinity of ways to choose $n - m$ other points that determine a complementary subspace S_{n-m-1} of S_m. The determination of the complementary element is therefore possible in both cases (this not the case for all lattices), but the uniqueness of this element, established in the propositional calculus (or Boolean algebra), does not hold for projective geometry.

These represent two different realizations of the same dialectical structure of both complementary and antisymmetric duality, and it is interesting to emphasize for a moment the foundation of this difference.

Boolean algebra satisfies the following laws of distribution for sum and product

$$x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$$

$$x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$$

while projective geometry satisfies only a weaker distributive law, called the modular identity, discovered by Dedekind:

$$\text{If } x \leq z, \qquad x \cup (y \cap z) = (x \cup y) \cap z$$

By replacing the symbols ∪ and ∩ with the usual symbols of the product and sum, we see that in the distributive lattice $x(y + z) = xy + xz$, while in the modular lattice we have, for $x \leq z$, only $(x + y)z = x + yz$.

In a famous article, 'The logic of quantum mechanics', Garret Birkhoff and von Neumann (1936) relied upon this difference to establish that the calculus of propositions relative to the observations of classical mechanics has the structure of a Boolean algebra, while the calculus of propositions relative to the observable facts of quantum mechanics would satisfy modular identity, but not the distributive law, and would thus have the structure of a projective geometry. Berkhoff and von Neumann provide the following example in support of this thesis: If proposition a corresponds to the observation of a train of waves a' on one side of the plane, proposition ψ to the observation of ψ on the other side of the plane, and b to the observation of ψ in a symmetric state with respect to the plane; neither $b \cap a$ nor $b \cap a'$ can be observed simultaneously, therefore we have $b \cap a = b \cap a' = 0$ (0 represents the identically false proposition). Thus $(b \cap a) \cup (b \cap a') = 0$. On the other hand, since $a \cup a'$ is an identically true expression, the conjunction $b \cap (a \cup a')$ is equivalent to b. Since clearly $b > 0$, we conclude that:

$$b \cap (a \cup a') > (b \cap a) \cup (b \cap a').$$

The distributive law is therefore not satisfied (Birkhoff and von Neumann 1936, 823).

The logical difference between two lattices both satisfying the law of duality would therefore be translated in the sensible world into the difference between classical mechanics and quantum mechanics. These applications of the theory of lattices to physics cannot substitute at present for mathematical physics properly speaking. They nevertheless appear to us to justify the hypothesis of a similar importance of dissymmetrical symmetry in the sensible universe and antisymmetric duality in the mathematical world. Moreover, it is remarkable that even in the mathematical theories that do not at present seem to be related to lattice theory, there are laws of reciprocity comparable to the duality that we have just studied. In certain cases, this reciprocity presents itself as a possible exchange of roles between arguments of a same relation or of a same function, accompanied by a change in sign of the relation. This reciprocity is therefore comparable in every way to the antisymmetry of non-commutative products. Just as we had (XY) = –(YX), we have $f(x, y) = -f(y, x)$. In other cases, the relation of reciprocity appears as a total symmetry that is not accompanied by any change of sign, that is, by any dissymmetry. Nevertheless, it seems that the symmetry that we will call symmetric is only a limiting case of an antisymmetric symmetry which remains the general case. Just as when (XY) = –(YX), we nevertheless have (XY) = (YX) when (XY) has the particular value 0, an antisymmetric relation between two elements related to the numbers p and $n - p$ whose sum n is constant, can become symmetric in the particular case in which $n - p = p$, that is to a say if $n = 2p$.

Something analogous is found in the extremely important theorems of quadratic reciprocity in arithmetic. Legendre introduced into arithmetic the symbol (m/p) which equals +1 if m is a quadratic residue modulo p, that is, if there exists an integer x such that $m - x^2$ is a multiple of p, and –1 in the opposite case. Consider now two positive odd integers a and b. They satisfy the fundamental law of reciprocity:

$$\left(\frac{a}{b}\right) = \left(\frac{b}{a}\right)(-1)^{\frac{a-1}{2}\cdot\frac{b-1}{2}}$$

This law therefore includes at once all the cases of reciprocity in the strict sense of the word, that is, the cases in which a is to b as b is at a, and the cases in which reciprocity does not hold. The general law contains an element of dissymmetry (the factor –1), which disappears in the particular cases in which reciprocity actually holds.

Seeking to determine the nature of mathematical reality, we have shown in a previous work (Lautman 1938b) that mathematical theories can be interpreted as a matter of choice destined to give substance to a dialectic ideal. This dialectic seems to be principally constituted by pairs of opposites and the Ideas of this dialectic present themselves in each case as the problem of connections to be established between opposing notions. The determination of these connections can only be made within the domains in which the dialectic is incarnated, and it is thus that we have been able to follow in a great number of mathematical theories the concrete outline of the edifices whose effective existence is constituted as a response to problems posed by the Ideas of this dialectic. It seems certain in this regard that the idea of the mixture of symmetry and dissymmetry plays a dominant role, not only with respect to physics, but as we have tried to show, with respect to mathematics. The two realities are thus presented in accord with one another as distinct realizations of a same dialectic that gives rise to them in comparable acts of genesis.

CHAPTER 2
The Problem of Time

In the previous chapter, an essential property of physical space, the difference of orientation of symmetrical figures, is interpreted as the sensible manifestation of a dialectical structure that is as much the generator of abstract mathematical realities as of the conditions of existence for the world of phenomena. Such an analysis thus situates at the level of Ideas what seemed to be one of the characteristics of spatiality, and this is perhaps the most current sense that the notion of intelligible extension can take today. The success of the spatial problem leads us to pose an analogous problem for time: Is it possible to describe within mathematics a structure that is like a first outline of the temporal form of sensible phenomena? At first glance, this problem appears much more difficult than the previous problem, since time, more so than space, seems to be linked to the sensible existence of the Universe. That it is defined in effect by the movement of the earth, the order of causality, biological aging, thermodynamic irreversibility, the duration of consciousness, one always makes use of notions that are meaningful only for a sensible observer in a sensible world. Our task therefore does not consist in introducing change or becoming into the unchanging world of mathematical truths, but in distinguishing from sensible time an abstract form of time whose necessity is essential to the intelligible Universe of pure mathematics as to the concrete time of Mechanics and Physics. The importance of this attempt is easily seen for the problem of reasons for the application of mathematics to the physical universe.

Such a study is comprised naturally of several moments. First, we will describe the sensible properties of time that are inscribed in the equations of mathematical Physics. We will then show that the mathematical structure of these equations comes to them not from the physical domain to which they are applied, but from the mathematical domain from which they proceed. We will then search for the dialectical bearing of these results. As this regressive analysis proceeds, time is assuredly stripped of its unstable and lived aspect, but on the other hand we will reach this uncreated germ that contains within it both the elements of a logical deduction and an ontological genesis of becoming sensible.

1. SENSIBLE TIME AND MATHEMATICAL PHYSICS[1]

Before studying the relations between sensible time and mathematical physics, it is first necessary to pose as a preliminary point: all theorists of modern science are in agreement in recognizing that the notions of space and time, such as they result from sensible experience only make sense in classical mechanics. On the other hand, relativistic mechanics and more so quantum mechanics and wave mechanics require in their domain of validity the development of radically new notions relating to time and space. Is it still possible in these conditions to try to characterize in a unique and general way the manner in which sensible properties of time are expressed in the symbols of a mathematical physics that cannot be considered as one? This objection however is not of great weight, since if there is a rupture between the physical sense of classical space and time, space-time relativity and the uncertainty of quantum relations, there is continuity in the mathematical form of these various mechanics. The same authors who insist on the fact that modern conceptions are unrepresentable in the classical framework are still trying to find in the apparent harmony of the classical theories, the origin of all the mathematical complications of more recent theories. It is thus that the distinction between opposing points of view in modern science very often proceed from the strict equivalence of two modes of presentation of certain classical results. We can therefore infer the common structural conditions that are essential to the succession of physical theories, and that, given both the sensible aspects of time and the exigency of a dialectic ideal, carry out the connection between the sensible and the intelligible.

The sensible properties of time, which are constituted as experimental facts that any physical theory must account for, can be expressed in the following propositions:[2]

1) Time always flows in the same direction. This property establishes a dissymmetry between time oriented from the past towards the future, and space which knows neither direction nor privileged meaning.
2) Material objects persist over time. This property relates the existence of material objects to the flow and the direction of time. A material object can in effect be independent of other material objects which exist simultaneously elsewhere, but its existence at a given time is inextricably linked to its own past and its own future. The continuity of time is thus an essential element of the permanence of objects.
3) Magnitudes, other than the time which characterizes physical systems, vary as a function of time.

The first two properties are distinct in the sense that one concerns the irreversibility of what could be called pure time, and the other, the spreading out in time, according to this irreversible order of the before and the after, of the physical objects of the universe, but they are nevertheless closely linked. Joined together, they make of time an oriented direction, necessarily associated with the directions of space for the location of physical phenomena. This direction of time possesses no less than space a special dissymmetry which comes precisely from its orientation from the past towards the future. The third property, on the contrary, does not concern the direction of time. It only indirectly concerns time, since it is relative to the changes undergone by the other physical magnitudes of the universe. It happens in effect to be with respect to time that the physicist studies the changes of position, speed, temperature, density, energy, etc. . . . within any physical system, but these changes do not at all obey in themselves any requirement of irreversibility. This was the opinion of Boltzmann, and still is the opinion of Schrödinger when he wrote:

> Even the laws of nature, that are called irreversible, by themselves imply no temporal direction when they are interpreted statistically. The predictions they allow to be stated dependent in effect on the boundary conditions in two temporal sections t_0 and t_1 and are absolutely symmetrical with respect to these two sections, without the order of time playing any role. (Schrödinger 1931, 152)

Time, which plays a part in the statement of the laws of nature, in kinematics, in dynamics, in thermodynamics, etc. . . . is therefore not the irreversible time of the duration of things. The role it plays is simply that of a factor of evolution and it would be perfectly possible to study the evolution of physical phenomena in terms of another magnitude taken as an independent variable. The result of these considerations is that two kinds of properties of sensible time can be clearly distinguished, those connected to the notions of dimension and orientation are the geometric properties of time, and those connected to the notion of evolution determine in particular the dynamic properties of bodies. We will now study the mathematical aspects that these sensible properties of time take within the various physical theories.

First of all, let us consider the distinction between time and space that results from the existence of an orientation of time. It is known that the development of the theory of special relativity has led to the constitution of a spatio–temporal synthesis in which the time coordinate, in certain transformations, plays a symmetric enough role to that of the space coordinates. It is nonetheless true that a fundamental difference subsists between time and space. While in Euclidean space the square ds^2 of the distance between two points is given by a sum of three squares preceded by the + sign, $ds^2 = dx^2 + dy^2 + dz^2$, in space–time the square of the distance between two points is given by a sum of three positive squares and one negative square $ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2$, or conversely by calling ds^2, for reasons of convenience of calculation, the quantity $-ds^2$, $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$. Here is a geometric interpretation of this formula, the speed of light being taken as unity. To each point P of coordinates x, y, z, t of space–time a cone can be attached that has two nappes of vertex P which has the time axis as the axis of revolution, is open towards the future, and whose generators are define by the equation $ds^2 = dt^2 - dx^2 - dy^2 - dz^2 = 0$. These lines of zero length define the luminous trajectories in space–time. All other lines of the universe are interior to the two nappes of the cone, oriented from the nappe of the past to that of the future and satisfying the condition $ds^2 > 0$. This direction and sense imposed on the lines of the universe constitute what de Broglie calls the fibrous structure of space–time. The dissymmetry of time and space is therefore expressed in two equivalent ways. One, algebraic, connected to the idea of default or difference, is the possibility of attaching to each point in space-time a quadratic form composed of a positive square and three negative squares. The other, geometric, connected to the idea of orientation, is the possibility of attaching to each point a set of oriented directions constituting the generators of the cone of the future relative to this point. With the connected notions of difference of sign and oriented direction attached to the points of the universe, we thus possess the mathematical expression of the dimensional properties of time in classical mechanics and relativistic mechanics. We shall see what it is in quantum mechanics later.

Let us now shown how the role of time conceived as a parameter of evolution is presented in classical mechanics. Let us now place ourselves in the simple case of a single material point moving in ordinary Euclidean space whose three coordinates x, y, z are three functions $x(t), y(t), z(t)$ of the parameter t which represents time. Let T be the kinetic energy of this point, V its potential energy which is a function of the three coordinates x, y, z and of time. Consider the Lagrange function $L(x, y, z, t) = T - V$. Suppose that at the instant t_0 the material point is at A and at the instant t_1 at B. Let C be the curve representing the trajectory of the point. Hamilton's principle asserts that the real movement that leads the point from A to B along the curve C is such that the expression $\int_{t_0}^{t_1} L dt$ taken along C is extremal with respect to infinitely near curves connecting the same point A to the same point B, that is, we have in symbols $\delta \int_{t_0}^{t_1} L dt = 0$. All the laws of classical mechanics can be deduced from this extremum principle.[3] Similarly, all the laws of classical electro-magnetism and the whole of the theory of special relativity can be deduced from an analogous variational principle. It is likely that this is also the case for the theory of general relativity, and as concerns the theory of quanta, the importance of Hamilton's principle has been fairly well demonstrated by the work of de Broglie. Suffice it to say that when there is conservation of energy, from Hamilton's principle $\delta \int L dt = 0$, Maupertuis' principle of least action $\delta \int \overrightarrow{mv} ds = 0$ can be deduced, in which $\overrightarrow{mv}$ represents the momentum vector, and ds the element of the arc of the curve traversed, and it is through action that quanta are introduced in physics. It is known in addition that the analogy of the Maupertuis principle for the dynamics of material points, and Fermat's principle for optics, is the basis of wave mechanics.

The Hamilton and Maupertuis principles are therefore currently among the most general principles of all theoretical physics, and it is important for us to note that if time figures as a parameter of evolution in the statement of the first, $\delta \int L dt = 0$, this role is played by the element of the arc of a curve in the statement of the second, $\delta \int \overrightarrow{mv} ds = 0$. We thus grasp in a particularly simple example in what sense the fact for time to be the parameter of evolution, as a function of which the other physical magnitudes vary, is independent of the geometric properties of time, since this role can in certain cases suit another variable. In the domain of classical mechanics, the difference between the geometric aspect and the dynamic aspect of time should not however be exaggerated since there is strict equivalence between them. Hamilton's dynamic principle can in effect be expressed in a geometric form in which time figures as a coordinate associated with the coordinates of space but, due to its characteristic dissymmetry, marked by a different sign. For this,[4] consider a configuration space of $n + 1$ dimensions defined by the n spatial coordinates $q_1 \ldots q_n$ and time t. Each state of the system defined by a value of these $n + 1$ coordinates corresponds to a point in this space. Let L be the function which generalizes the Lagrange function defined above, L = T – V, and E = T + V, the energy of the system. Associate to each coordinate q_i the canonically conjugated quantity $p_i = \frac{\delta L}{\delta \dot{q}_i}$. We have seen that with this notation the system energy can be written $E = \sum p_i \dot{q}_i - L$.

From this is deduced $\int_{t_0}^{t_1} L dt = \int_{P}^{Q} p_1 dq_1 + \ldots p_n dq_n - E dt$, by calling P the point of the configuration space of $n + 1$ dimensions, which corresponds to the instant t_0 and to the state of the system at that instant, and Q the point corresponding to time t_1 and to the state of the system at that instant. The expression $p_1 dq_1 + \ldots p_n dq_n - E dt$, whose integral from P to Q thus presents a minimum with respect to the infinitely near trajectories of the configuration space–time, can be regarded as the work in this space of $n + 1$ dimensions $q, \ldots q_n, t$ of a vector which would have as spatial components the n ordinary components $p_1 \ldots p_n$ of the momentum, and as components following time, the energy with sign changed. A spatio–temporal synthesis is thus found again analogous to that of special relativity in which time imposes a difference of sign with respect to space.

The equivalence thus shown in classical mechanics between the dynamic aspect and the geometric aspect of time no longer easily subsists in wave mechanics in which we will see instead their opposition asserted. We can then state a fundamental principle of our whole study: when it is a question of two distinct notions, their equivalence or their opposition appears on the same plane as subsequent to the fact of their simple duality,[5] conceived as indifferent again to the assertion of any relation between them. To reuse terminology which we have made use of elsewhere, we will call *Idea* the problem of determining a connection made between distinct *notions* of a dialectical ideal. It is in these conditions that we conceive the existence of a theory of Ideas, of a dialectic common to wave mechanics and classical mechanics, even though the former throws into opposition the notions whose agreement is evident in classical mechanics.

It is necessary to now envisage the relations of geometric time and dynamic time in wave mechanics. In this presentation of these questions, we will follow the analysis of de Broglie cited above and in particular that of the last chapter of his book: *L'Électron magnétique* (1937c). De Broglie addresses the problem of the relations of space and time in wave mechanics by starting with the fact that Dirac's wave mechanics equations are invariant under a Lorentz transformation operating in the space-time of special relativity, and noting nevertheless that in wave mechanics time plays a very different role to that of the coordinates of space. Here is a summary of the main considerations that he develops in this respect: to any physical magnitude, coordinates of space, momentum, energy, torque, etc. . . . there corresponds, in wave mechanics, no longer an accurate measurement, as in ordinary mechanics, but a mathematical entity called a 'Hermitian operator'. The measurement of any physical magnitude A can only give one of the 'eigenvalues' $\alpha_1 \ldots \alpha_n$ of the corresponding operator $A_{(op)}$, and the probability that an observation attributes the precise value α_i to the magnitude A is equal to the square $|c_i|^2$ of the module of the coefficient c_i of the eigenfunction φ_i corresponding to α_i in the development $\psi = \sum c_i \varphi_i$ of the wave function ψ according to the eigenfunctions $\varphi_1 \ldots \varphi_n$ of the operator $A_{(op)}$. In general these probabilities are functions of time and that is why the system evolves, but time thus remains a regular number and no operator corresponds to it. It therefore appears as a parameter of evolution in the study of probabilities attached to no matter which physical magnitude, and not as a coordinate likely to be seen as a proper probability distribution in space-time. We thus find a distinction analogous to that already encountered above. De Broglie later envisages the notion of 'average value' of a magnitude in wave mechanics. It is a very important notion since if it is impossible to speak of the precise value of a magnitude A at a determined moment, we can calculate at each instant an average value $\overline{A}$ as the sum of the mathematical expectations of each of the possible values at the considered time. We thus have $\overline{A} = \sum \alpha_i |c_i|^2$ in which again, by a simple proof $\overline{A} = \int \psi * A_{op} \psi dxdydz$. The integration takes place in an area of space Δ for which $\overline{A}$ is consequently a constant. This definition makes sense because it is possible to consider a particle as practically isolated from the rest of the universe. The influence of fields of forces exterior to the atom on the waveform ψ is entirely negligible because these waves tend towards zero when moving away from the atomic domain (De Broglie 1937c, p. 306). Whereas, the integration in a space-time domain *dy dx dz dt* would suppose a static physics from which all evolution would be banished, which is evidently absurd. De Broglie finally considered uncertainty relations. It is known that Heisenberg's relations establish the impossibility of measuring accurately a coordinate of space q_i and the corresponding component of momentum p_i: $\Delta q_i \Delta p_i \geq h$. There exists a fourth relation concerning energy and time $\Delta E \Delta t \geq h$, but it has a very different meaning to the previous ones. It defines, not an error committed in the measurement of E, but the minimum duration of the experiment that would allow a value to be attached to the energy of the particle marked by a minimum uncertainty equal to ΔE.

This difference in nature between time and space in the equations of wave mechanics is related for de Broglie to the fibrous structure of space–time that has already been described above. To each particle there corresponds a line of the universe oriented in the direction of time, and this cleavage of the universe allows the operation of the spatial sections of all the lines of the universe in order to obtain a coexistence of independent systems in space. It does not appear however that this reason is sufficient to also explain the difference between the evolution parameter of time and the oriented dimension of time. De Broglie also seems to be led to the idea of two distinct times:[6]

> It is evident that it would be desirable to introduce into quantum theory the idea that the coordinate *t* is also linked to a probability distribution, but it should be done by keeping in the theory a variable of evolution, and we have said, this does not seem very easy. (De Broglie 1941, 200)

It is interesting to emphasize the mathematical source of this duality of the roles of time in wave mechanics. The work of de Broglie starts with the relativist conception in which energy E and the three components of the momentum of a particle, p_x, p_y, p_z, constitute the four elements of a four-vector of space–time. If, according to the general principles of the theory of quanta and wave mechanics, $E = h\nu$ and $\lambda = h/m$ are posed, in which ν is the frequency of the wave and λ the wavelength attached to a particle, m represents its mass and v its speed, then the plane monochromatic wave ψ associated with an isolated particle, in the case of a constant external field, is given by the formula

$$\psi(x,y,z,t) = A^{e})\left\{\frac{2\pi}{h}[Et - p_x x - p_y y - p_z z]\right\}$$

The function $Et - p_x x - p_y y - p_z z$ represents the phase of the wave and associates time to space with this difference of sign which always characterises the dimensional conception of time. The following relations can be deduced from the preceding equation, obtained by derivation, and valid in the conditions indicated:

$$\frac{-h}{2\pi i}\frac{\partial\psi}{\partial x} = p_x\psi, \quad \frac{-h}{2\pi i}\frac{\partial\psi}{\partial y} = p_y\psi, \quad \frac{-h}{2\pi i}\frac{\partial\psi}{\partial z} = p_z\psi, \quad \frac{h}{2\pi i}\frac{\partial\psi}{\partial t} = E\psi.$$

It is these relations which led to replacing the quantities p_x, p_y, p_z, by the operators

$$\frac{-h}{2\pi i}\frac{\partial\psi}{\partial x}, \frac{-h}{2\pi i}\frac{\partial\psi}{\partial y}, \frac{-h}{2\pi i}\frac{\partial\psi}{\partial z}.$$

The relativist analogy between space and time also suggests replacing the energy E by the operator

$$\frac{h}{2\pi i}\frac{\partial\psi}{\partial t}$$

but this is only permissible for quantified states of energy. The operator which corresponds in all cases to energy is the Hamiltonian operator $\mathcal{H}$ obtained as follows:

$$\text{let } E = H(q_i, p_i, t) = \frac{1}{2m}[p_x^2 + p_y^2 + p_z^2] + V(q_i, t)$$

be the expression of classical Hamiltonian energy. In H, replace p_x, p_y, p_z, with the corresponding operators

$$\frac{-h}{2\pi i}\frac{\partial}{\partial x}\cdots$$

we obtain the Hamiltonian operator

$$\mathcal{H}\left(x, y, z, \frac{-h}{2\pi i}\frac{\partial}{\partial x}, \frac{-h}{2\pi i}\frac{\partial}{\partial y}, \frac{-h}{2\pi i}\frac{\partial}{\partial z}, t\right)$$

corresponding to energy, in all cases. It is now essential to note that the operators thus defined are conceived only as operating on functions satisfying certain conditions which are made functions of what is called a

Hilbert space. Thus, in wave mechanics, the operators corresponding to physical magnitudes are always applied to wave functions ψ that belong to a Hilbert space. By then applying the operator $\mathcal{H}$ to a function ψ, and by drawing on the particular relation

$$\frac{h}{2\pi i}\frac{\partial\psi}{\partial t} = \mathrm{E}\psi$$

the fundamental equation of propagation of wave mechanics is obtained by posing:

$$\mathcal{H}(\psi)\frac{h}{2\pi i}\frac{\partial\psi}{\partial t}$$

This equation is an equation of wave propagation, since at each instant the partial derivations of the function ψ with respect to time are envisaged. The evolution of this function can therefore be calculated from its acquaintance with an initial instant t_0. It is nevertheless necessary to point out the very special nature of this equation of evolution. The derivation with respect to time plays a part there in effect as the formal result of the application of a privileged operator $\mathcal{H}$, the energy operator, to a function ψ defined in a Hilbert space. This process therefore allows the evolution of the function ψ, which is not a physical magnitude, to be studied, but does not apply to the study of the evolution of an arbitrary physical magnitude attached to the particle. In fact, the study of the evolution of a physical magnitude in wave mechanics always leads to the consideration of certain relations of the operator corresponding to this magnitude with the energy operator. It is thus that we have the following very important theorem: The necessary and sufficient condition so that a physical magnitude A is constant during the motion defined by the Hamiltonian operator $\mathcal{H}$ is that the operators A and $\mathcal{H}$ permute. In classical mechanics, Hamilton's canonical equations already relate the evolution in time of mechanical magnitudes p_i and q_i to the consideration of a privileged Hamiltonian function. We had in effect

$$\frac{dq_i}{dt} = \frac{\partial \mathrm{H}}{\partial p_i}, \frac{dp_i}{dt} = \frac{\partial \mathrm{H}}{\partial q_i}$$

without this connection of evolution in time of a mechanical magnitude and of energy being the result of two different roles of time. Whereas, in wave mechanics it seems that it is necessary to distinguish between the operator

$$\frac{h}{2\pi i}\frac{\partial}{\partial t}\cdots$$

applied uniquely to the wave function ψ, and the derivation with respect to time of all magnitudes, mechanical and physical, attached to the particle. The first operator is associated with energy as the operators

$$\frac{-h}{2\pi i}\frac{\partial}{\partial x}\cdots$$

are associated with components p_x of momentum. Whereas, the time that plays a part in ordinary derivatives is unrelated to a dimension of space-time and serves only in the study of the evolution of the magnitudes of the system.

2. THE THEORY OF PARTIAL DIFFERENTIAL EQUATIONS

We have seen the equations of mechanics bring to the fore two aspects of time, sometimes equivalent, sometimes distinct, but both adapted by their intrinsic characteristics to the sensible dissymmetries of experience. The fundamental result of the *a priori* deduction that we will present is then the following: this duality of aspects of time, each endowed with its own dissymmetries, does not appear only in the application of mathematics to the physical universe, that is, in mechanics, but it already exists at the level of pure mathematics, independently of any concern for its application to the universe. Whatever the dialectical origin of this duality of time that is found inherent in the theory of differential equations and partial differential equations, mechanics, insofar as the problem of time is concerned, does not provide any arrangement whose schema lets itself be seen in the pure abstractions of which it is the application. Physical time in all its forms, is only the sensible realization of a structure which is already manifested in the intelligible domain of mathematics.

The method that we are going to follow in this proof is a method of regressive analysis. We will proceed from the concrete to the abstract, from the composite to the simple, in order to always refer the crux of the problem to a higher level in the hierarchy of Ideas. And it is this incorporation of empirical data in an ideal structure that constitutes for us the *a priori* deduction of the sensible dissymmetries related to time. This method presents, in our eyes, a considerable advantage over the deductive syntheses of time that the idealist philosophies of the nineteenth and twentieth centuries had attempted. We do not carry out an arbitrary *a priori* deduction of time, we observe, in the order of the universe, the stages constituted by that deduction.

Such a requirement of analysis explains the reasons why we will consider the theory of partial differential equations before the theory of differential equations. The equations of mathematical physics are generally in effect partial differential equations so that the general theory of these constitutes for example the first abstract domain in which the pure play of relations that support the different aspects of physical time is found.

Consider[7] the first order partial differential equations, to 2 independent variables, $F(x_1, x_2, u, p_1, p_2) = 0$, in which u is an unknown function of two variables x_1 and x_2 and in which

$$p_1 = \frac{\partial u}{\partial x_1}, p_2 = \frac{\partial u}{\partial x_2}. \text{ Suppose that } \left(\frac{\partial F}{\partial p_1}\right)^2 + \left(\frac{\partial F}{\partial p_2}\right)^2 \neq 0.$$

At each point of the 3-dimensional space, x_1, x_2, u, this equation defines a family of possible tangent planes to an integral surface $u(x_1, x_2)$ passing through this point. This family of planes envelops a cone, at each point of the envisaged space the Monge cone attached to this point is defined, so that the equation $F = 0$ can be considered to associate, to all points of the space, a cone or even a sheaf of characteristic directions, the generators of the Monge cone at that point. To integrate the proposed equation consists in finding a surface tangent at each point to a characteristic direction passing through this point, and this integration is done by considering the characteristic curves of the equation. Here is what is meant by a this: a characteristic curve of the equation $F = 0$ is tangent at each of its points to a characteristic direction passing through this point. It shows that an integral surface $u(x_1, x_2)$ of the equation $F = 0$ is generated by a family of characteristic curves, so that the proposed problem of the integration of partial differential equations is reduced to the problem of the integration of differential equations that define the characteristic curves of this equation. It is very important for us to consider the differential equations of these characteristics. For this in general, the coordinates x_1, x_2, u, of the points of a curve in the space envisaged are considered as functions $x_1(t)$, $x_2(t)$, $u(t)$

of a parameter t. The characteristic curves of the equation $F(x_1, x_2, u, p_1, p_2) = 0$ then satisfy a set of three differential equations of which we retain here only the following:[8]

$$\text{(I)} \qquad \frac{dx_1}{dt} = \frac{\partial F}{\partial p_1}, \frac{dx_2}{dt} = \frac{\partial F}{\partial p_2}.$$

A surface integral is generated, we said, by characteristic curves, so that given a characteristic curve defined by differential equations (I), this curve can be subject to the supplementary condition of being situated on a surface integral of the equation $F = 0$. This condition gives rise to two new equations that we write under the simplified form they take when the function F does not contain the variable u.

$$\text{(II)} \qquad \frac{dp_1}{dt} = -\frac{\partial F}{\partial x_1}, \frac{dp_2}{dt} = -\frac{\partial F}{\partial x_2}.$$

The determination of surface integrals is therefore reduced to the integration of a system of differential equations in which we will retain only equations (I) and (II) written in the form

$$\text{(III)} \qquad \frac{dx_i}{dt} = -\frac{\partial F}{\partial p_i}; \frac{dp_i}{dt} = -\frac{\partial F}{\partial x_i}.$$

The equations (III) have exactly the form of the canonical equations of the dynamics of the material point in which F represents the energy of the particle, and t the time. Here, in the *a priori* deduction of the laws of the sensible universe, is a result of considerable importance, that of seeing the equations of the trajectories of the dynamics result *a priori*, without special physical hypothesis, from the problem of the integration of partial differential equations. Geometric space x_1, x_2, u thus contains, like a state of possibles, with its directions and its characteristic curves of the equation $F = 0$, the form of trajectories and the dynamic law of motion that material particles will take when a physical interpretation, transmuting the function F into energy and the parameter t into temporal variable, thus projects, fully armed, into sensible existence, a mathematical universe already equipped with all the necessary richness of organization.

So far we have only shown the possibility of an *a priori* deduction of the laws of mechanics in which time plays a part as a parameter of evolution. We now come to the central point of the proof announced above by showing within the same theory of partial differential equations the genesis of a dimensional conception of physical time, which is in principle distinct from the parametric conception but serves to resolve the same problems. Let us now place ourselves in the general case of a space of $n + 1$ dimensions, and envisage the following partial differential equations, in which the function F explicitly does not contain the function $u(x_1 \ldots x_n, x_{n+1})$ sought after:[9] F $(x_1 \ldots x_{n+1}, p_1 \ldots p_{n+1}) = 0$. Following the presentation of Hilbert and Courant (1937), we are going to *distinguish* a variable, for example x_{n+1}, and resolve the proposed equation with respect to the corresponding derivative p_{n+1}.

We obtain the equation:

$$p_{n+1} + \mathrm{H}\ (x_1 \ldots x_n, x_{n+1}, p_1 \ldots p_n) = 0$$
$$\left(p_{n+1} = \frac{\partial u}{\partial x_{n+1}}; p_i = \frac{\partial u}{\partial x_i}; i = 1 \ldots n\right)$$

The expression $p_{n+1} + \mathrm{H}$ therefore replaces the expression F of the given equation. The first group of equations of characteristic curves, $\frac{dx_i}{dt} = \frac{\partial \mathrm{F}}{\partial p_i}$, with $i = 1 \ldots n + 1$ give for $i = n + 1$, $\frac{dx_{n+1}}{dt} = 1$.

The variable x_{n+1} is therefore always equal, to within an additive constant, to the parameter t, and we can replace the parameter t of the preceding theory by the independent variable distinguished in the new theory. The characteristic equations are then written

$$\frac{dx_i}{dx_{n+1}} = \frac{\partial \mathrm{H}}{\partial p_i}; \frac{\partial p_i}{dx_{n+1}} = -\frac{\partial \mathrm{H}}{\partial x_i}; i = 1 \ldots n,$$

and the abstract form of Hamilton's canonical equations is retrieved. In the space of $n + 1$ dimensions defined by the variables $x_1 \ldots x_{n+1}$, the fact of solving the proposed equation with respect to a partial derivative concerning one of these variables makes this variable play the role of the temporal variable. There is in this case absolute equivalence between the role of the parameter and the role of the dimensional coordinate of the same distinguished variable, but it remains no less the fact that the purely mathematical theory of partial differential equations thus allows the emergence of two different conceptions of the same variable that are at the origin of the duality of the sensible properties of time. In addition, if the second conception presented, which is that of Hamilton and Jacobi, is at the base of the classical theory of Hamilton's canonical equations when the distinguished and variable parameter is being identified, it is also the starting point of wave mechanics in which the derivation with respect to time, symmetric with a change of sign of the deviation with respect to the coordinates of space, has an operational meaning quite distinct from the parametric sense of time. Let us in effect go back to the equation:

$$p_{n+1} + \mathrm{H}\ (x_1 \ldots x_n, x_{n+1}, p_1 \ldots p_n) = 0$$

This equation can be written, with

$$x_{n+1} = t \text{ and } p_{n+1} = \frac{\partial u}{\partial x_{n+1}} = \frac{\partial u}{\partial t}.$$

(IV)
$$\mathrm{H}(x_1 \ldots x_n, t, p_1 \ldots p_n) = -\frac{\partial u}{\partial t}$$

If it is posed that,

$$\frac{\partial u}{\partial t} = \mathrm{E} \text{ and } p_i = \frac{\partial u}{\partial x_i},$$

the function H of equation (IV) gives the classical Hamiltonian expression of energy:

$$\mathrm{H}(x_i, t, p_i) = \frac{1}{2m}[p_x^2 + p_y^2 + p_z^2] + \mathrm{V}(x, y, z, t) = \mathrm{E}.$$

Whereas, as we have also seen above, by replacing p_i in the function H of equation (IV) by the operators

$$-\frac{h}{2\pi i}\frac{\partial}{\partial x_i},$$

the Hamiltonian operator $\mathcal{H}$ of the wave mechanics is obtained. If this operator is then applied to the wave function ψ, playing the role of the function u of equation (IV), the Schrödinger equation is obtained:

$$\mathcal{H}(\varphi) = \frac{h}{2\pi i}\frac{\partial \psi}{\partial t}.$$

The Jacobi equation thus gives the Schrödinger equation directly when the operator

$$\frac{h}{2\pi i}\frac{\partial}{\partial t},$$

plays a part, operating on the wave function ψ, and these are the characteristic equations associated (as defined by the theory of partial differential equations) to the Jacobi equation which gives the derivatives of magnitudes x_i and p_i in terms of a parameter t, conceived this time independently of any relation with the coordinate x_{n+1}. The derivation with respect to the time coordinate therefore appears in the partial differential equation itself, and it is only in the characteristic equations of this equation that the derivation with respect to the time parameter appears. The first of these derivations is concerned with the wave function sought after, the others with the mechanical magnitudes attached to a mobile point that describes the trajectories defined by the characteristic equations. The distinction of the two times is therefore attached to the distinction of partial differential equations and characteristic equations. This is a fundamental and ineffable distinction of which we will find the full meaning by studying the theory of differential equations.

There remains one point to consider. We have just rediscovered *a priori* in the theory of partial differential equations the distinction between the time parameter and the time coordinate. We must now show *a priori* how the time coordinate plays a part in a spatio–temporal synthesis affected by this special dissymmetry with respect to space which is manifested by a difference of sign. The same problem will allow us to rediscover both the distinction between two conceptions of time and the special dissymmetry of dimensional time. For this, we will envisage the theory of second order partial differential equations, and we will restrict ourselves to the cases in which these equations are linear, that is, of the form

(V) $$\sum a_{ik} u_{ik} + \sum b_i u_i + cu + d = 0$$

with

$$u_{ik} = \frac{\partial^2 u}{\partial x_i \partial x_k}; u_i = \frac{\partial u}{\partial x_i},$$

u being the unknown function sought after.

It can be shown that a characteristic form can be associated to any equation of this type,

$$\sum \alpha_i X_i^2, \ (i = 1 \ldots n) \text{ with } \alpha_i = \pm 1.$$

The characteristic form associated with the equations that describe propagation is said to be of the hyperbolic type, and is composed of $n - 1$ positive squares and one negative square:

$$X_1^2 + \ldots + X_{n-1}^2 - X_n^2.$$

We are occupied here only with the hyperbolic case. As in the case of first order partial differential equations, a surface integral of equation (V) is generated by the characteristic manifold $\varphi = 0$ which here satisfies the equation

$$\sum \alpha_i \varphi_i \varphi_k = 0$$

and these *characteristic manifolds* are in turn generated by the *characteristic radius* that defines in n-dimensional space the differential equations

$$\frac{dx_i}{dt} = \sum_{k=1}^{k=n} a_{ik} \varphi_k$$

A variable x_n can be distinguish anew and solve the equation $\varphi = 0$ with respect to x_n to obtain $x_n = \psi(x_1 \ldots x_{n-1})$. Under these conditions the equation of the radius gives:

$$\frac{dx_n}{dt} = 1$$

and the dimensional variable can be identified anew with the parameter of evolution. A concrete example nevertheless shows the character of each of these two aspects of time. Consider the classical wave equation $u_{44} - u_{11} - u_{22} - u_{33} = 0$. In 4-dimensional space-time a distance between two points can be defined

$$d\sigma^2 = dx_4^2 - dx_1^2 - dx_2^2 - dx_3^2.$$

The lines of zero length ($d\sigma = 0$) issuing from an arbitrary point define the characteristic cone attached to this point and any direction situated at the interior of the cone satisfies the inequality $d\sigma^2 > 0$. In particular, for the time axis x_4 defined by the relations

$$dx_1 = dx_2 = dx_3 = 0 \text{ we have } dx_4^2 > 0,$$

and this shows the 'oriented' character of the variable x_4 in its role as dimensional time. Let us now place ourselves in 3-dimensional space. The distance between two points is given by the formula

$$d\rho^2 = dx_1^2 + dx_2^2 + dx_3^2.$$

If t designates in this case the length counted on a radius from the origin, the speed of propagation being taken as unity, the surfaces t = constant defines the wave surfaces which represent the wave front at each instant. There is equivalence in the results obtained between the time, as a dimension of 4-dimensional space, and the time as a length in 3-dimensional space, but each of these two conceptions of time brings with it a specific element: the conception of the oriented time dimension is linked to the existence of a difference of sign in terms of a sum of squares; the parametric conception is linked to the kinematic notions of speed and displacement.

3. THE THEORY OF DIFFERENTIAL EQUATIONS AND TOPOLOGY[10]

The relations that support the different notions embraced by the theory of partial differential equations become much more intuitive in the theory of differential equations. The geometric meaning of the partial differential equations that we have envisaged is in effect the following: the given equation defines, at each point of a space, a cone characteristic of this point, and the integration of the equation consists in finding the curves and the surfaces tangent to each point of the cone characteristic of this point. The distinction of the time dimension and the time parameter is related to a certain extent to the following duality: the direction of opening of the characteristic cone assigns at each point of the space a privileged role to one direction which can be considered as a temporal direction, and on the other hand, the curves and surfaces tangent to this field of cones are a function of a parameter that can also be considered, but in another sense, as a time variable. In the theory of differential equations, the situation is even simpler. An equation

$$\frac{dy}{dx} = f(x, y)$$

defines in the x, y plane a field of directions, that is, one directly attached to each point, and solutions of this equation are the curves tangent to each point in the direction that passes through this point.

The geometric interpretation of the theory of differential equations therefore brings to the fore two absolutely distinct realities. There is the field of directions and the topological accidents that can occur on it, as for example the existence in plane of singular points to which no direction is attached, and there are the integral curves with the form they take in the neighbourhood of the singularities of the field of directions. Consider for example the equation

$$\text{(I)} \qquad \frac{dy}{dx} = \frac{\mathrm{Q}(x,y)}{\mathrm{P}(x,y)}$$

in which the functions P and Q are supposed uniform, continuous and bounded in absolute value. Under these conditions the equation (I) defines a field of directions. If the variables *x* and *y* are related to a parameter *t*, we have instead of (I) the following system

$$\text{(II)} \qquad \frac{dx}{dt} = \mathrm{P}(x,y) \quad \frac{dy}{dt} = \mathrm{Q}(x,y).$$

This system no longer defines only a direction at each point but a direction and a meaning, that is, a vector attached to each point of the plane *x*, *y* for which P and Q do not vanish simultaneously. The points of indetermination in which P = Q = 0, constitute the singularities of the vector field. In his famous *'Mémoire sur les courbes définies par une équation différentielle'* (1881), Poincaré established a classification of these singularities according to the bearing of the integral curves in the neighbourhood of these points. He distinguishes: saddle points, through which two and only two curves defined by the equation pass; nodes, in which an infinity of curves come to be crossed; foci, around which the curves turn by drawing constantly closer to a logarithmic spiral; centers, around which the curves present themselves in the form of closed loops enveloping each other and surrounding the center. The existence and distribution of singularities are notions relative to the vector field defined by the differential equations. The form of the integral curves is relative to the solutions of this equation. The two problems are most certainly complementary because the nature of singularities of the field is defined by the form of the curves in their neighbourhood.

It is no less true that the vector field on the one hand, and the integral curves on the other are two essentially distinct mathematical realities. It might seem that we are here in purely abstract domains in which any reference to physical problems of time has vanished. In fact the parameter t as a function of which coordinates x and y are defined can be conceived anew as a parameter of temporal evolution. As for the notion of vector field, it is linked in an essential way to the dimensional aspect of time. We will show this by presenting the results due principally to Ehresmann (1943). We will see the distinction between time and space within a 4-dimensional manifold, interdependent with the existence in this manifold of a field of directions without singularity and determined by the global topology of this manifold. In opposition with the parametric properties of time which can only relate to a limited evolution within a well defined interval of time without reference to any overall structure, the dimensional properties of time have a cosmogonical meaning and reflect the general form the Universe.

Here are the principal stages of Ehresmann's reasoning. Consider a differentiable manifold, that is, a topological manifold V_n in which a set of systems of local coordinates are defined such that every point of V_n is found in at least one of them, and such that two systems of coordinates which are defined in the same domain of V_n correspond to one another by a transformation of continuously differentiable coordinates. On this manifold, a precise, positive quadratic differential form can always be defined, that is, a sum of n squares formed by the differentials of coordinates:

$$dx_1^2 + dx_2^2 + \ldots dx_n^2$$

that is reducible at each point to a sum of

$$n \text{ squares } w_1^2 + w_2^2 + \ldots w_n^2 \text{ in which } w_1, w_2, \ldots w_n$$

refer to n independent linear differential forms. The problem posed is to know under what conditions there exists at each point of this manifold a quadratic form reducible to a sum of p positive squares and $n - p$ negative squares, as in the case of the invariant of the theory of special relativity

$$ds^2 = dx_4^2 - dx_1^2 - dx_2^2 - dx_3^2, \text{ in which } p = 1 \text{ and } n = 4.$$

Ehresmann has proven that the existence at each point of V_n of a quadratic form composed of p positive squares and $n - p$ negative squares is equivalent to the existence of a p-dimensional field of contact elements without singular points.[11] In the case of a 4-dimensional space, the existence of a relativistic invariant of the universe is therefore equivalent to the existence in this Universe of a field of directions presenting no singularity at any point. Now, a fundamental theorem due to Hopf established the fundamental topological conditions under which a continuous vector field without singularities on a closed differentiable manifold can be defined. It is necessary and sufficient that the Euler characteristic[12] of this manifold is zero, which is the case for all closed manifolds with an odd number of dimensions, but which on the other hand eliminates a large number of manifolds with an even number of dimensions. For example, the sphere S^4 and projective plane P^4 of 4-dimensional space, having a Euler characteristic different from 0, cannot be provided with a vector field without singularity (*Cf.* Ehresmann 1943). We can therefore not define the Lorentz invariant at each point of these manifolds and they would not constitute a possible Universe for the theory of relativity. On the other hand, if a 4-dimensional universe is compact and such that a distinction between past and future at each point can be defined by continuity, it can be concluded that the Euler characteristic of this universe is equal to 0. The determination of the meaning of time in the universe is therefore supportive of the global structure of the universe.

Let us try to identify the scope of the results thus obtained. We started with the distinction of the sensible properties of the time reference point, which is a dimension associated dissymmetrically to space in a 4-dimensional synthesis, and with the time factor of evolution which is a parameter. The mathematical study of these two aspects of time led us to envisage in the theory of differential equations the distinction between two kinds of mathematical entities: vector fields and solution curves. Neither one nor the other of these two realities is by nature a temporal reality, but the problems posed by the study of each are directly interpretable in terms of time.

The theory of vector fields shows a necessary connection between the existence of a privileged direction at each point of a geometric universe and the default of a quadratic form, the difference in sign between terms of ‘space’ and terms of ‘time’. The same theory shows in addition that the distinction at each point of a privileged direction is possible only if the entire Universe has satisfied certain global conditions. The application of these results to physical theories is, we have seen, obvious: the orientation of time, the duration of things have mathematical meaning only if there are no holes, of interruption, in this continuity of openness towards the future, and this exigency is cosmogonical. The aspect of time that we called geometric, throughout this chapter, is thus in conjunction with the general form of the whole Universe.

Now let us envisage the other aspect of time, that to which the study led, on a field of vectors, of integral curves defined as a function of a parameter. The principal problems of this theory are immediately interpretable in terms of time. The problem of determinism can be cited, that of finality, that of the return of things. In certain cases, the evolution of a magnitude as a function of time can be described by a differential equation, or partial differential equation, such that the knowledge of initial data, at a time t_0, determines the subsequent evolution of the magnitude in question. This is a local determinism, operating step by step, as the propagation of light for example. In other cases, in which partial differential equations of the elliptic type are encountered, as in the phenomena of thermal equilibrium, the knowledge of initial data is insufficient to determine the whole evolution of the phenomena. It is necessary to be given both the initial and final conditions. The problems of this kind allow a theory of finality to be established in mathematics, as Maurice Janet has shown in his article on finality in mathematics and physics (1933, 1). As for the problem of the return of things, it presents itself in all the cases of closed trajectory, in celestial mechanics for example, where to a same value of position coordinates correspond several distinct values of the variable *t*. This demonstrates how the study of integral curves defined as a function of a parameter led to consider the laws of evolution of physical systems, which clearly brings to the fore the dynamic meaning of this parameter.

The mathematical distinction between cosmogonical time and dynamical time is therefore the expression of a duality inherent in theories as abstract as the theory of differential equations, and it is consequently highly probable that it corresponds to an intimate structure of things that has its source in the structure of Ideas. This conclusion leaves us with the feeling of the limits that the reduction of dynamics to cosmogony must necessary encounter. In a forthcoming book, Tonnelat (1955) shows the history of the unitary theories of the electromagnetic field and gravitation since the geometrization of gravitation by Einstein in 1916. The impossibility of geometrizing the electromagnetic field in a physically acceptable way led Tonnelat to the contrary attitude, that is, to give a physical sense, a dynamic sense back to gravitation. There can be no question of going back to a pre-relativist conception of a geometric container definable independently of its physical contents, but the identification of the two notions seems equally impossible. Here again the notion of complementarity seems to play a significant role.

Notes

Introduction

1. Testimony of Gilbert Spire, a student at the *Ecole Normale Supérieure* in 1936, prisoner with Lautman at Oflag IV D, where he also attended Lautman's lectures at the camp university. Spire provided valuable help in the escape of the group of which Lautman was a part; Spire would later be Inspector General of Philosophy.

2. Blay 1987, dedicated to Jean Cavaillès and Albert Lautman, with articles by Catherine Chevalley, Gerhard Heinzmann, Jean Petitot and some unpublished letters of Jean Cavaillès to Albert Lautman, presented by Hourya Sinaceur.

3. Gonseth 1997. A work that brings together various texts and in which Lautman is cited several times.

4. WWII.

5. Jacques Herbrand, 1908–1931, presented a thesis in 1930 for a doctorate of Science entitled *Investigations in Proof Theory* (1930), which was considered of premier importance by Hadamard, professor at the Collège de France, but his colleagues from the Sorbonne found it too philosophical. It took the insistence of Vessiot, mathematician and director of the ENS, for the defence to take place. Shortly after, Herbrand made a major contribution to the problem of consistency in arithmetic in which he used the finitistic methods of Hilbert (Herbrand 1931). In July 1931, this young mathematical genius was killed in a climbing accident. In the *Yearbook of*

the Association of Former Students of the Ecole Normale Supérieure (*Annuaire de l'Association des anciens élèves de l'ENS*) in 1931, Claude Chevalley and Albert Lautman, who co-sign the obituary, wrote: 'the sublime beauty of the paths indicated by this adored person' (Chevalley and Lautman 1931).

6. The Bourbaki group set out provide an encyclopedic treatment of the whole of mathematics based on a general theory of structure —Tr.

7. The street address of the ENS, Paris —Tr.

8. The nickname given to a sophomore (second year) in the preparatory literature class —Tr.

9. The philosopher Emile Chartier, also known as Alain, was head of the Lycée Henri IV from 1909 to 1933 —Tr.

10. See Sirinelli 1992. This is the richest published source on the life of Albert Lautman, who is mentioned on many occasions.

11. The National Science Fund —Tr.

12. *Centre National de la Recherche Scientifique* (National Center for Scientific Research) —Tr.

13. Those studying for the secondary level teaching qualification —Tr.

14. Up until 1969, the Doctorat d'Etat had two theses.

15. Among the witness accounts is that of the poet Patrice Tourdu La Pine.

16. A former student of the military school at Saint Cyr —Tr.

17. Tunisian soldiers who served in auxiliary units attached to the French Army —Tr.

18. *Société Nationale des Chemins de fer français*, or French National Railway —Tr.

19. The Pat O'Leary network recuperated and evacuated shot-down allied airmen —Tr.

20. The Spanish Maquis were resistance fighters who held several valleys and passes throughout the Pyrenees —Tr.

21. Another group of resistance fighters —Tr.

Albert Lautman and the Creative Dialectic of Modern Mathematics, by Fernando Zalamea

1. See the secondary bibliography on Lautman for additional references.

2. From now on all citations in quotes, without other indication, are from Lautman.

3. It is, therefore, necessary not to conflate the usual meaning of effective in mathematics (Brouwer's intuitionism or constructivism as exemplified by Markov) with the use made by Lautman.

4. Since the vast majority of mathematical examples studied by analytic philosophy can be reduced to trivial arithmetic or set theoretical cases (in so striking a way in Wittgenstein), one is forced to doubt that this approach oriented towards the *elements* can build a faithful image of mathematical activity.

5. See Deleuze 1994 in the *Secondary Bibliography*.

6. The Lautmanian 'structural schemas' anticipate (in their conception) the mathematical techniques of his time, and they can only be clearly defined in the later context of the mathematics of category theory. The schemas, the dialectic, the Same/Other pair, the ideas and Platonic–Lautmanian mixes acquire a notable technical *exactitude* through the notions of *diagram*, *free object*, *representable functor* and *adjoint pair* from category theory.

7. It would suffice to sample a random number of *Mathematical Reviews* to explain the very small space in mathematics occupied by the research into foundations, which contrasts surprisingly with the enormous space given to discussions about the foundations of mathematics in philosophy. In our opinion, the Lautmanian spectrum goes much further. In this regard, it is startling to browse the index of Lautman's work prepared for the Spanish edition of his writings (Lautman 2008): closely following the classification adopted by the *Mathematical Reviews* (MSC 2000) one observes the still very strong presence – 60 years later – of Lautmanian texts in an extraordinary quantity of ramifications in the MSC 2000. [The Mathematics Subject Classification (MSC 2000) is used to categorize items covered by the two reviewing databases, Mathematical Reviews (MR) and Zentralblatt MATH (Zbl) —Tr.].

8. The maps of contemporary mathematics are very similar to the maps of knowledge included in the *Enciclopedia Einaudi* (Romano 1977–1984). It is interesting to note that Jean Petitot has been one of the central experts of the *Enciclopedia*, and that his knowledge of Lautman perhaps influenced the remarkable tables and diagrams elaborated by Renato Betti and his collaborators (volume 15: *Sistematica*, Betti 1982). If one disreagrds the article by Bernays (1940) – unfortunately, little known and quickly forgotten – the text by Petitot (1982) in the *Enciclopedia Einaudi* was the first to have made Lautman known beyond the Francophone world. See the *Secondary Bibliography*.

9. Francastel's *relay* (1965) proportion – for the work of art – another mix of high value, in which perception is conjugated with the real and the imaginary. If we compare a definition of the work of art as a *form that signifies itself* (Focillon 1934), with a definition of mathematical work as a *structure that forms itself* (our extrapolation, motivated by Lautman), it is not difficult to point out – once again – the terrain underlying the aesthetic and mathematics. It happens to be a fundamental terrain, for Herbrand, as for Lautman, even if they didn't develop it. See the *Notice* on Herbrand (by Chevalley and Lautman 1931) and on Lautman (by his wife, 1946) published in the *Annuaire de l'Association amicale de secours des anciens élèves de l'Ecole Normale Supérieure*. For a recovery of the history of art that (unwittingly) follows Lautmanian lines, that combines the complex and the differentiated, and reconstructs them in a stratified and hierarchical dialogue attentive to the universal and the truth, see Thuillier 2003, 65 (Focillon's definition and subsequent discussion).

10. See Wiles 1995.

11. The great advantage that Alain Badiou has derived from his studies of Cohen's independence theorems in set theory is a kind of exception (Badiou declares himself a great admirer of Lautman, Badiou 2005 [1998]). Nevertheless, Cohen's forcing is a part of the logical (mathematical) rather than mathematics properly speaking. The philosophical ideas buried in the mathematical work of a Grothendieck, of a Langlands, or of a Gromov, to mention only the major examples, do not yet have the right of existence in the philosophical city.

12. Lautman does not appear to have been aware of the notes to the course on the *Sophist* given by Heidegger, in Marburg, during winter semester 1924–1925 (see Heidegger 1997). Even if, in 1928, Lautman had

attended a Franco–German encounter in Davos, and could, therefore, perhaps have heard talk of Heidegger, the first direct references to Heidegger only appears in *New research on the dialectical structure of mathematics*. Heidegger's course entangles his nascent hermeneutical method with a very careful reading of the *Sophist* (610 manuscript pages in German), and can thus provide exceptional philosophical support for Lautman's work, oriented specifically toward Plato and Heidegger. It is remarkable that Heidegger defines the fundamental work of the dialectic as that of '*renewing the relation between ontological structures each time to a unity, so that from this unity the whole ontological history of a being up to its concreteness can be followed*' (Heidegger 1997, 389, author's emphasis). If we leave aside the ontological terms, it is difficult to find so Lautmanian a resonance in another author.

13. It is the prerogative of the heirs of the later Wittgenstein, who converted the study of the world into a study of the linguistic constraints and blockages of knowledge: certainly a thriving academic industry, but unfortunately far from the reality in which science moves.

14. Lautman could not have know category theory, which emerged at the time of his death (Eilenberg and MacLane 1942; 1945). It is difficult to know to what extent the conversations with his friend Ehresmann – creator of the general theory of fiber bundles in the forties and proponent of category theory in France in the fifties – could have influenced the *resources* of a conception of mathematics as clearly categorical as that of Lautman. Nevertheless, in the session of the *Société Française de Philosophie* (4 February 1939) where Cavaillès and Lautman defended their theses, Ehresmann already signaled how the philosophical conceptions of Lautman merited in their turn to be treated technically and converted into a baggage *at the interior of mathematics itself* ('I think the general problems raised by Lautman can be expressed in mathematical terms, and, I would add, that one cannot help but express them in mathematical terms', Cavailles, 1994, 614). The rapid development of the mathematical theory of categories proves Ehresmann right.

15. Peter Freyd points out that the lemma didn't really appear in Nobuo Yoneda's original article (Yoneda 1958), but in a lecture given by Saunders MacLane (1998) on the treatment of Yoneda *Ext* functors of higher order (Freyd 2003, xii). It is amusing to know that the lemma, so close to the structural resources of Lautman's thought, in fact emerged in a lively discussion between Yoneda and MacLane in the Gare du Nord in Paris (See Biss 2003, 581).

16. See Dumitriu 1991, 28. For the original text in which Avicenna introduces his trimodal interpretation, see Avicenna 2005.

17. Cavaillès betrayed Lautman's 'ardor' (Ferrières 1950, 77) and Herbrand his 'exigency' (Chevalley 1987, 76).

Preface to the 1977 Edition, by Jean Dieudonné

1. The example chosen is Garland's thesis, 'A finiteness theorem for K_2 of a number field' (Garland 1971).

2. One can say without exaggeration that there have been more fundamental mathematical problems resolved since 1940 than from Thales to 1940.

Considerations on Mathematical Logic

1. Published by Rudolf Carnap and Hans Reichenbach, Leipzig: Felix Meiner Verlag, 1931–1940.

2. The number of logical operations is of little importance and varies with the operators. It is sufficient that one can deduce all other operations from the ones that have been chosen.

3. The work of Herbrand is inspired by the proofs given in Hilbert and Ackermann 1922.

Book I: Introduction

1. Some authors, like Helmut Hasse, clearly distinguish the notion of function 'as defined by analysis' from the notion of function 'as defined by algebra' (see Hasse 1933).

Chapter 1

1. The field $k(\sqrt{2})$ is generated by the 'adjunction' of $\sqrt{2}$ the field of rational numbers.

2. μ is here the measure of the space E, as defined by Lebesgue.

3. This means that there are the following relations between the functions:

$$\int_{-\pi}^{+\pi} \varphi_m(s)\varphi_n(s)ds = 0; m \neq n \quad \int_{-\pi}^{+\pi} |\varphi_m(s)|ds = 1$$

4. This whole paragraph is from Hellinger 1935, 106.

5. This theorem constitutes, under a particular form, the theorem of Riemann-Roch which makes the function $F(z)$ depend on $n - p + 1$ constants. See Picard 1896, 474.

6. The importance presented by 'the arithmetic theory of algebraic functions' for the problems encountered in this chapter must be noted here. The theory of algebraic functions and of their integrals is for Riemann a theory of analysis and is based on transcendent methods. Dedekind and Weber explained, in 1882, a purely arithmetic theory of algebraic functions with one variable, in which only finitist notions of arithmetic and algebra are called upon, even for the definition of a point. Thus, an algebraic fouction η is characterized by a symbolic quotient of two complex points called polygons:

$$\eta = \frac{\mathcal{A}}{\mathcal{B}}$$

Two polygons $\mathcal{A}$ and $\mathcal{B}$, likely to correspond as numerator and denominator to the same function, belong to the same class of polygons. The classes have a *dimension* defined by the number of elements of their base, and the theorem of Riemann-Roch is presented immediately in this theory as tied to the study of the dimension of the classes of polygons. The structural and dimensional point of view that only appears in second place in the ordinary theory thus appears prominently in the 'arithmetic' theory.

Chapter 2

1. We indicated in our principal thesis (Lautman 1938b) how, following the work of Elie Cartan, the metric broadly defined could, in certain cases, nevertheless be Riemannian.

2. See the introduction to this essay.

3. Poincaré 1928, xi. The summary of his work is by Poincaré himself.

4. For this whole paragraph, see Nevanlinna 1936, ch. 6.3.

Chapter 3

1. For all this, *cf*. de Broglie 1932, 199.

2. Recall briefly that a ring is a set of numbers in which two formal procedures of composition are defined: the addition and multiplication of two elements.

3. *Cf*. the lectures by von Neumann 1935 as well as van der Waerden 1931, ch. 16.

4. This whole paragraph is from Dubourdieu 1936; see also Kahler 1934.

5. For all that follows *cf*. Cartan 1937, 1311–1334; 1909, 1335–1384.

6. It is a matter here of infinite continuous groups, see the lecture by Cartan, 'Sur la structure des groupes infinis,' *Seminaire de l'Institut H. Poincare, annee 1936–1937* (Cartan 1909, 1335–1384).

Chapter 4

1. Here's the simplest definition of the norm: two numbers α and β are said to be congruent modulo $\mathcal{A}$ if their difference $\alpha - \beta$ is in $\mathcal{A}$; which is written $\alpha \equiv \beta (mod\,.\mathcal{A})$. This equivalence relation determines a division into classes in *K* and the number of these classes is called the norm of the ideal $\mathcal{A}$.

2. Recall that the two whole or fractional ideals $\mathcal{A}$ and $\mathcal{B}$ are equivalent, $\mathcal{A} \sim \mathcal{B}$, if their ratio $\mathcal{A}/\mathcal{B}$ is a principal ideal (formed from the multiples of a single whole number). This equivalence relation determines a division into classes of ideals of the field *K*.

3. For all this *cf*. Hecke 1923.

4. The work of Takagi, Artin, Herbrand and Chevalley have generally eliminated the analysis of existence theorems in class field theory, but in the case of the imaginary quadratic field, the function $J(z)$ gives more than the existence of the field of classes, it gives numbers belonging to this field.

Conclusion

1. *Cf*. the often-repeated adage of André Bloch: '*Nihil est in infinito quod non prius fuerit in finito*' (1926, 84). ['Nothing is in the infinite that was not first in the finite' —Tr.]

Book II: Introduction

1. *Cf.* this passage from Russell: 'They [the mathematical propositions] all have the characteristics which, a moment ago, we agreed to call tautology. This combined with the fact that they can be expressed wholly in terms of variables or logical constants . . . will give the definition of logic or pure mathematics', in Russell 1919 [1993, 204–205].

2. In our secondary thesis, *Essay on the unity of the mathematical sciences in their present development* (1938a), we present some aspects that seem to us to distinguish modern mathematics from classical mathematics.

Chapter 1

1. In fact, the global study and the local study do not lead to strictly equivalent results. Borel has in effect proved by the discovery of classes of quasi analytic functions that the class of Cauchy functions is more extended than the class of Weierstrass functions.

2. See mainly Cartan 1924, 294; 1925, 1; and especially 1927, 211.

3. Recall that a differential equation establishes a relation between a function of a single variable and a certain number of its successive derivatives, while a partial differential equation establishes a relation between a function of several variables and a certain number of its partial derivatives with respect to all or several of its variables.

4. An analytic function $f(z)$ is holomorphic in a connected region D of the plane of the complex variable z, if it *is* continuous in D, and if at any point z of D there correspond unique values for $f(z)$ and $f'(z)$.

5. For this whole paragraph, see Fehr 1936. This edition reproduces the lectures on the theory of partial differential equations held at the University of Geneva in June 1935. We are particularly inspired by Hadamard's lecture, from which the form of the equations cited in the text are also borrowed (Hadamard 1936).

6. This distinction, that will be discussed below, is found in Hopf's article (1932).

7. This is the case for set-theoretical topology, as well as combinatorial topology or algebraic topology.

8. The characteristic of a surface is a property of algebraic topology which will be returned to later (*cf.* Chapter 4).

9. Cartan 1930. Let us recall the meaning of some terms: one group being a set of transformations operating on the points of a space is said to be linear if the new coordinates x'_i of a point are expressed algebraically as a function of the old coordinates x_j by relations of the type $x'_i = a_{i_1}x_1 + a_{i_2}x_2 + \ldots a_{i_n}x_n$. The a_{ij} are the coefficients of the transformation. A quadratic form of two variables is, for example, the expression $Ax^2 + Bxy + Cy^2$. It is defined if its discriminant is negative.

10. γ is the sub-group of the linear adjoint group that corresponds to the largest sub-group g of G which leaves fixed a point in space.

11. The distance $\| f - g \|$ of two functions $f(x)$ and $g(x)$ is defined as the maximum absolute value, $| f(x) - g(x) |$, x belonging to the basic set ε.

12. These formulas of representation as well as those given for the polynomials of representation are valid only for analytic functions, in their circle of convergence. The formulas of approximate representation of continuous functions $f(x)$ by the polynomials $P(x)$ are, in the general case, more complicated.

Chapter 2

1. For this whole section, see Cartan 1924, 297.

2. Instead of envisaging the insertion of a polyhedron in space, topologists often consider the points belonging to the polyhedron to be taken from space.

3. For these definitions, see the topology texts already mentioned, as well as Lefschetz's *Topology* (1930).

4. Instead of a decomposition into simplices, a decomposition into cells obtained from the simplicial decomposition can sometimes be envisaged.

5. Given an n-dimensional simplex, the algebraic sum of its $(n - 1)$-dimensional faces is called the boundary of the simplex. An algebraic sum of k-dimensional simplices, each possibly multiplied by an integer coefficient, is called a k-chain. A closed chain is called a cycle, that is, a chain whose boundary is zero. Among these cycles, some are found at the same

time to be boundaries of (k + 1)-dimensional chains. Thus, for example, a circumference is not only a 1-dimensional chain, but a cycle, since it is closed, and a cycle boundary, since it is a boundary of a 2-dimensional simplex (the surface of the circle, topologically equivalent to the triangle). A cycle boundary is said to be homologous to zero. These preliminaries being posed, here is the definition of the independence of cycles: m k-dimensional cycles

$u_1\mathrm{k}, u_2\mathrm{k}, \ldots u_m\mathrm{k}$

are said to be independent if no linear combination

$t_1\mathrm{k} + t_2 u_2\mathrm{k} \ldots + t_m u_m\mathrm{k}$

of these m cycles exists, which is homologous to zero without all the coefficients t_i cancelling each other out. If m independent cycles can be found on a complex and m + 1 cannot be found on it, the Betti number of dimension k is m.

6. It is the extrinsic notion of *enlacement* that allows Alexander's theorem to be tied to Poincaré's theorem. It is said that there is *enlacement* between two closed curves without common points if there is *intersection* between the plane included at the interior of one these curves and the other curve. In the case which occupies us here, there is enlacement in R^n between the r-dimensional cycle boundaries belonging to Q and the $(n - r - 1)$-dimensional cycle boundaries belonging to $R^n - Q$. This means that there is intersection between the cycles boundaries of dimension r of Q and certain chains of dimension $n - r$ of $R^n - Q$. These cycles of dimension r and these chains of dimension $n - r$ can then be considered as belonging to the complexes in duality in R^n, and thus it is seen how the passage from the 'internal' case to the 'external' case is carried out.

7. Torsion groups are structural invariants determined at the same time as Betti numbers.

Chapter 3

1. Descartes 1976, 156; quoted and commented on by Étienne Gilson, Descartes 1925, 315.

2. *Cf.* in particular Winter 1911, 146–185.

3. *Cf.* van der Waerden 1930.

4. It goes without saying that we use the expression of imperfection here without any reference to the algebraic meaning of the term 'perfect field'.

5. *Cf.* for this theory Chevalley 1934 and Herbrand 1936.

6. Here are the definitions of notions used in this paragraph. Given an algebraic number field *k*, any set of numbers of the field is called the ideal of this field such that:

a) If α is part of this set, then so is $\lambda\alpha$, whatever the integer α;

b) If β is another element of this set, $\alpha + \beta$ is also included;

c) There is an integer μ such that for all α of the set $\alpha\mu$ is an integer.

Given any set of numbers of the field, $\alpha_1, \alpha_2 \ldots$, the set of $\gamma_1\alpha_1 + \lambda_2\alpha_2 + \ldots$, the λ_i being arbitrary integers of the field, forms an ideal, the ideal $(\alpha_1, \alpha_2 \ldots)$.

The ideal (α) given rise to by a single number $\alpha \neq 0$ is called principal. A product of two ideals can be defined, and an ideal integer **a** that cannot be put in the form **bc**, where **b** and **c** are integers different to (1), is called a prime ideal (Herbrand 1936, ch. 1.5).

7. What is interests us here is not so much the mathematical nature of this decomposition than the fact of knowing that it is easier for the prime ideals of the group of ideals *H* than for the ideals that do not belong to *H*. In fact, here is this law: given an ideal **A** of *K*, let **a** be the set of numbers of **A** that are in the base field *k*. Consider the conjugates of **A** in the Galois field conjugated of *K*. The product of all these conjugations is called the norm of the ideal **A**; and if **A** is prime in *K*, it can be demonstrated that Norm $\mathbf{A} = \mathbf{a}^f$, *f* is then said to be the relative degree of **A** with respect to *k*. In these conditions, only the prime ideals of *k* belonging to *H* decompose in *K* into a product of different prime ideals, of relative degree 1. If, on the other hand, **p** is a prime ideal of *k* not belonging to *H*, and if $\mathbf{p}^f$ is the smallest power of **p** situated in *H*, **p** is decomposed in *K* into a product of prime ideals of relative degree *f* with respect to *k*.

8. For this chapter, see Seifert and Threlfall 1934, ch. 7 and 8; Weyl 1913; Threlfall 1935.

9. According to the article by Threlfall (1935).

10. An algebraic function $\zeta = f(x)$ is a function which satisfies an algebraic equation of the type: $g_0(z)\zeta^n + g_1(z)\zeta^{n+1} + \ldots g_n(z) = 0$

11. Recall that a conformal correspondence between two domains is a biunivocal and bicontinuous representation of the two domains on one another and which is such that any analytic and regular function at one point of a domain is transformed into an analytic and regular function at the corresponding point.

Chapter 4

1. *Cf.* for this whole section Cavaillès 1938b.

2. For this whole part ,see Hilbert and Bernays 1939, and Bernays 1935, 196 ff.

3. We translate *'beweistheoretische Method'* by structural method and *'mengentheoretische Method'* by extensive method.

4. See Herbrand 1930, 118 [1971, 168] and particularly 1931, 3 [1971, 288].

5. It is said that an analytic function $\zeta = f(z)$ has, at a point z_0, a pole of order n, when the series expansion in the neighborhood of this point is presented in the form:

$$\zeta = \frac{a_1}{(z - z_0)^n} + \frac{a_2}{(z - z_0)^{n-1}} + \ldots \frac{a_n}{(z - z_0)} + a_{n+1} + \ldots \quad a_1 \neq 0$$

A function of the type $\zeta = \lg z = \lg r + i\varphi$ has at point $r = 0$ a logarithmic singularity, because at this point the expression *lg r* becomes infinite in negative values.

6. The first sentence is altered, and *Analysis situs* is replaced by 'topology', and the second sentence omitted in the completely revised third edition (1955) translated in 1964. See Weyl 1913 [1964, 152] —Tr.

7. Recall that a potential function of two variables follows the equation

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$$

Let there be an analytic function $\zeta = u + iv$. It is proved that the functions $u(x, y)$ and $v(x, y)$ that thus figure in the expression of an analytic function, when the real part is separated from the imaginary part, are potential functions.

8. *Cf.* for this proof Hurwitz and Courant 1925, 368.

9. See Hilbert 1897, reprinted in Hilbert 1935. The theorem in question is proved on p. 155. See also Hecke 1923, 154 [1981, 134].

10. $\overline{x_i}$ is the complex conjugate variable x_i.

11. Let s be an element of the group, $U(s)$ the transformation corresponding to s in a certain system of coordinates of the fundamental space E. Let A be the matrix of a change of coordinates of this space. The representation $s \rightarrow U(s)$ is transformed into an equivalent representation $s \rightarrow AU(s)A^{-1}$.

12. In the case of finite groups, any representation is equivalent to a unitary representation.

Chapter 5

1. De Broglie 1932, 43.

2. For all this, see Hellinger 1935, 94 ff.

3. We refer to the presentation of Andre Weil (1935).

4. In a general way $\overline{x}$ is the complex conjugated quantity of x.

5. For this whole section, see Montel 1927.

Chapter 6

1. Summary according to Julia 1938, 191.

2. For all of this *cf*. Seifert and Threlfall 1934, 290; Alexandroff and Hopf 1935, 532.

3. Or rather of the Riemann surface obtained by the superposition on the plane of z of an infinity of planes welded in crosses along the cuts $0 \rightarrow -\infty$ and $1 \rightarrow +\infty$.

4. For this whole paragraph, see Bieberbach, 227 ff.

Conclusion

1. The two terms in italics (the second by us) show that Boutroux brings together two different conceptions of mathematical reality.

2. Lautman replaces 'elementary number theory' with 'arithmetic' while maintaining the gist of the section from which the passage is cited —Tr.

3. For the work of Gödel and Gentzen, see Cavaillès 1938b.

4. We report here on the interpretation that Becker gives of the famous Aristotelian texts relating to Ideas–numbers (Becker 1931, 464 ff). Stenzel (1923) had proposed, for the generation of numbers from the one and the dyad, the following schema:

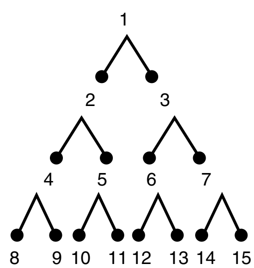

Figure 9

For Becker, this scheme has several major defects: first, only even numbers are truly generated by division, odd numbers resulting from the addition of a unit to the preceding even. In addition, the distinction of Ideas–numbers and ordinary numbers is not clearly explained.

Becker associates the following schemas to every Idea–number:

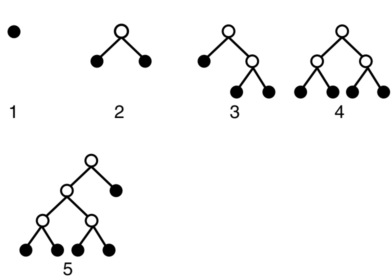

Figure 10

Each schema is composed of units. Some of these units are present in normal numbers (they are represented by black circles), others are hidden (the white circles) but all result from the dividing in two of a unit that is situated higher. The ideal number is thus the schema that generates an arithmetic number by means of present and hidden units (which would explain, for Becker, the mysterious text of the *Metaphysics*, M 7, 1081 A (Aristotle 1998): in the dyad, there is a third unit, in the triad a fourth and a fifth . . .).

5. For the application of this term to the philosophy of Plato, *cf.* Robin 1935, 149.

6. By translating *Zahlenmässige Gliederung* as numerical division, we believe that we have in no way betrayed the thought of the author.

7. This neo-positivism which thus appears to us to be unsustainable by excessive empiricism, is associated in an unexpected way to a rigorous 'tautological' and deductive conception of mathematics in what is called the physicalism of the Vienna Circle.

Book III

1. Perennial philosophy that is manifested in human action —Tr.

Chapter 1

1. As defined by Heidegger (*cf.* next note).

2. We will rely in what follows on Corbin's 1938 translation of Heidegger's *Vom Wesen des Grunde*, 1929 (Heidegger 1969, see Translator's Note).

Chapter 2

1. Hecke 1923, all that follows after paragraph 55 of chapter 8.

2. *Cf.* for these examples and for the whole theory presented in this paragraph, Ingham 1932, 8.

3. We have $\Gamma(s) = \int_0^\infty y^{s-1}e^{-y}dy \quad (\sigma > 0)$.

4. As in formula (XI), the expression $f(x) = 0(x)$ for $x \to \infty$, means that $\left|\frac{f(x)}{x}\right|$ is always less than a constant quantity.

Book IV

Chapter 1

1. From here on, all occurrences of the term 'structures', when relating to the theory of structures (or lattices), will be translated as 'lattices' — Tr.

Chapter 2

1. Throughout this chapter, we draw directly from the work of Louis de Broglie on the relations of relativity and its quanta: 1937a, 223–239; 1941, 183–204; 1937c, 301–307.

2. The statement of the first two properties is in de Broglie 1937a, 226.

3. Let us call q_i the coordinates of a particle in any space, $\dot{q}_i$ the derivatives of the coordinates with respect to time, and p_i the quantities:

$$\frac{\partial L}{\partial \dot{q}_i}.$$

In these conditions the energy H of the particle is given by the formula

$$H(q_i, p_i, t) = \sum p_i q_i - L(q_i, \dot{q}_i, t)$$

and from Hamilton's principle the canonical equations can be deduced:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}.$$

4. Summary according to de Broglie 1939, 14.

5. This term signifies here simply the state of things which are two, without reference to the more particular sense that it has in the previous chapter.

6. Lichnerowicz seems to be the first to have formulated in a precise mathematical way the necessity of considering two distinct times in physical theories (1939).

7. Summary according to Hilbert and Courant 1937.

8. The third equation in question is

$$\frac{du}{dt} = p_1 \frac{\partial F}{\partial p_1} + p_2 \frac{\partial F}{\partial p_2}$$

which, together with equations (I) and (II), defines a characteristic band, that is, a characteristic curve and a plane tangent to the curve at each point of the curve.

9. It was proved that this case can always be reduced to by introducing an additional independent variable.

10. *Cf.* for this section Bieberbach 1923.

11. A p-dimensional contact element in V_n is the set of one point of V and of a p-dimensional linear manifold tangent to this point V_n.

12. The Euler characteristic of a topological manifold is an invariant of global topology of which the following is an intuitive example: in the case of an arbitrary polyhedron of ordinary Euclidean space this invariant characteristic is equal to the sum of vertices plus the sum of sides and minus the sum of the edges. It is in this case always equal to 2.

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