Some Restricted Lindenbaum Theorems Equivalent to the Axiom of Choice*

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Abstract

Dzik (1981) gives a direct proof of the axiom of choice from the generalized Lindenbaum extension theorem *LET*. The converse is part of every decent logical education. Inspection of Dzik's proof shows that its premise *let* attributes a very special version of the Lindenbaum extension property to a very special class of deductive systems. The problem therefore arises of giving a direct proof, not using the axiom of choice, of the conditional *let* $\Rightarrow$ *LET*. A partial solution is provided.

0 Preliminaries

Let S be any set. An operation $\mathbf{Cn}$: $\wp(S) \mapsto \wp(S)$ is called a *consequence operation* if it satisfies the following postulates:

(0)
(a) if $X \subseteq S$ then $X \subseteq \mathbf{Cn}(X) \subseteq S$
(b) if $X \subseteq S$ then $\mathbf{Cn}(\mathbf{Cn}(X)) = \mathbf{Cn}(X)$
(c) if $X \subseteq S$ then $\mathbf{Cn}(X) = \bigcup\{\mathbf{Cn}(Z) \mid Z \subseteq X \text{ and } |Z| < \aleph_0\}$.

These are the postulates of Tarski (1930), pp. 63f., with the omission of a postulate stating that S is denumerable. This restriction is not wanted here. We shall however ignore systems in which S is finite. (All the results of this paper hold for consequence operations based on a finite S, even the empty set $\emptyset$.) Postulate (0c) is set-theoretically equivalent to the conjunction of the principles of *monotony* ($0c_0$) and *finitariness* ($0c_1$):

(0)
(c_0) if $Z \subseteq X$ then $\mathbf{Cn}(Z) \subseteq \mathbf{Cn}(X)$
(c_1) if $x \in \mathbf{Cn}(X)$ then $x \in \mathbf{Cn}(Z)$ for some finite $Z \subseteq X$.

A consequence of (0) that we shall appeal to in § 1 is a form of *transitivity* or *cut*.

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LEMMA 0 If $\mathbf{Cn}$ is a consequence operation then

(0) (d) if $y \in \mathbf{Cn}(Y)$ then $\mathbf{Cn}(Y \cup \{y\}) = \mathbf{Cn}(Y)$.

PROOF: If $y \in \mathbf{Cn}(Y)$ then $\{y\} \subseteq \mathbf{Cn}(Y)$, so $Y \cup \{y\} \subseteq Y \cup \mathbf{Cn}(Y) = \mathbf{Cn}(Y)$, by (0a). Twice applying monotony ($0c_0$), and then (0b), we may infer that $\mathbf{Cn}(Y) \subseteq \mathbf{Cn}(Y \cup \{y\}) \subseteq \mathbf{Cn}(\mathbf{Cn}(Y)) = \mathbf{Cn}(Y)$. ■

We call a pair $\langle S, \mathbf{Cn}\rangle$ satisfying (0) a deductive *system*. Reference to S is usually omitted. If $y \in \mathbf{Cn}(Y)$ we may say that Y *implies* y. It is easily checked that the operations $\mathbf{Cn}_{\min}$ ($= Y$ for all $Y \subseteq S$) and $\mathbf{Cn}_{\max}$ ($= S$ for all $Y \subseteq S$) satisfy (0) whatever set S is.

The set $Y \subseteq S$ is a (deductive) *theory* if $\mathbf{Cn}(Y) = Y$ and is *inconsistent* if $\mathbf{Cn}(Y) = S$. When only one consequence operation $\mathbf{Cn}$ is under discussion, boldface uppercase letters are used to denote theories. $S = \mathbf{Cn}(S) = \mathbf{S}$ is always a theory. The theory $\mathbf{Y}$ is *maximal* if it is consistent and no consistent theory properly extends it; in symbols, if $\mathbf{Y} \neq \mathbf{S}$, and $\mathbf{X} = \mathbf{S}$ whenever $\mathbf{Y} \subset \mathbf{X}$. An operation $\mathbf{Cn}$ satisfying ($0a,b,c_0$) is *compact* if the following condition holds.

(1) if $\mathbf{Cn}(Y) = \mathbf{S}$ then $\mathbf{Cn}(X) = \mathbf{S}$ for some finite $X \subseteq Y$.

Though finitariness ($0c_1$) is often confused with compactness (1), they are independent properties.

The classic extension theorem *LT* of Lindenbaum (Tarski 1930, pp. 98f.) states that if $\mathbf{Cn}$ is compact then every consistent $Y \subseteq S$ has a maximal extension. Note that the consequence operation $\mathbf{Cn}_{\min}$ defined above is not compact when S is infinite, yet Lindenbaum's theorem holds: if $y \notin Y$ then $S \setminus \{y\}$ is a maximal extension of Y under $\mathbf{Cn}_{\min}$.

1 The Equivalence of Lindenbaum's Theorem and the Axiom of Choice

In this section we report proofs (due to others) that an important but neglected generalization of the Lindenbaum extension theorem, here called *LET*, is set-theoretically equivalent to the axiom of choice *AC* (given some background set theory such as **ZF**).

The main business of the paper, which follows, is the investigation of a variety of restricted versions of *LET* that are also equivalent to *AC*. It will be shown that some of the equivalences among these different variants of *LET* (not one of which is mentioned in the catalogue of Howard & Rubin 1998), are susceptible of pleasingly elementary proofs.

Although *LET* is an apparently far-reaching generalization of Lindenbaum's original theorem (Béziau 1995 discusses thoroughly the various forms that the theorem takes), the proofs of the two results are almost identical. One of the first formulations and proofs of *LET* appears in the Lemma (pp. 238f.) of Łoś (1951), and the theorem has been formulated also by Asser, da Costa, and others. The axiom of choice *AC* is assumed.

We first define a *y-saturated set* (in the system $\langle S, \mathbf{Cn}\rangle$) as a set that does not imply y (according to $\mathbf{Cn}$) and cannot be extended further without implying y.

(2) $$Y \text{ is } y\text{-saturated} \iff_{\mathrm{Df}} \begin{cases} \text{(a)} & y \notin \mathbf{Cn}(Y) \\ \text{(b)} & y \in \mathbf{Cn}(Y \cup \{x\}) \text{ for every } x \notin Y. \end{cases}$$

LEMMA 1 If Y is a y-saturated set, then Y is a deductive theory.

PROOF: Suppose that $x \in \mathbf{Cn}(Y)$, yet $x \notin Y$. By (0d), $\mathbf{Cn}(Y \cup \{x\}) = \mathbf{Cn}(Y)$, and hence by (2) y both is and is not an element of $\mathbf{Cn}(Y)$. ■

If $\mathbf{Y}$ is y-saturated for some $y \in S$ then it is *saturated*. Thanks to Lemma 1 we shall assume without further comment that every saturated set is a deductive theory.

It is fairly obvious that $\mathbf{Y}$ is maximal if & only if $\mathbf{Y}$ is y-saturated for every $y \notin \mathbf{Y}$ (Béziau, *op.cit.*, p. 11). This result and related results are collected in the appendix below.

THEOREM 2 (*LET*) If $\langle S, \mathbf{Cn}\rangle$ is a deductive system, and y is an element of S that does not belong to $\mathbf{Cn}(Y)$, then Y has a y-saturated extension $\mathbf{Y}$.

PROOF: Let the elements of S be well ordered in the sequence $\{y_\nu \mid \nu < \lambda\}$, where λ is a limit ordinal. (This is where *AC* is used.) We define the following sequence $\{Y_\nu \mid \nu \leq \lambda\}$ of extensions of Y.

$$\begin{aligned} Y_0 &= Y \\ Y_{\nu+1} &= \begin{cases} Y_\nu \cup \{y_\nu\} & \text{if } y \notin \mathbf{Cn}(Y_\nu \cup \{y_\nu\}) \\ Y_\nu & \text{otherwise} \end{cases} \\ Y_\kappa &= \bigcup \{Y_\nu \mid \nu < \kappa\} \text{ for limit } \kappa. \end{aligned}$$

$\mathbf{Cn}(Y_\lambda)$ is the required theory $\mathbf{Y}$, which is obviously an extension of Y. To prove (2b), suppose that $x \notin \mathbf{Y}$. By (0a), $x \notin Y_\lambda$. Since x must be y_ν for some $\nu < \lambda$, it is clear that $x \notin Y_{\nu+1}$. In other words, $y \in \mathbf{Cn}(Y_\nu \cup \{x\}) \subseteq \mathbf{Cn}(Y_\lambda \cup \{x\}) \subseteq \mathbf{Cn}(\mathbf{Y} \cup \{x\})$.

To prove (2a), we appeal to the finitariness of $\mathbf{Cn}$. Suppose that $y \in \mathbf{Y}$. Then $y \in \mathbf{Cn}(X)$ where X is a finite subset of Y_λ. In the construction of Y_λ there must have been a stage, say the move from Y_ν to $Y_{\nu+1}$ when the final element of X qualified for inclusion (it may already have been present, as an element of Y). That is, $y \notin \mathbf{Cn}(Y_\nu \cup \{y_\nu\})$, and $X \subseteq Y_{\nu+1} = Y_\nu \cup \{y_\nu\}$, which by monotony ($0c_0$) implies that $y \in \mathbf{Cn}(X) \subseteq \mathbf{Cn}(Y_\nu \cup \{y_\nu\})$. This is a contradiction. ■

This theorem would be of little interest if it said only that every Y not containing y can be extended to a set (rather than a theory) satisfying (2a-b). But note that *AC* is not needed for the proof that $\mathbf{Cn}_{\min}$ satisfies *LET*. The set $S \setminus \{y\}$ always obliges.

The prominent role given, in the proof of Theorem 2, to *AC* (in the form of the well-ordering theorem) does not show that it is a necessary premise for the proof of *LET*. But it is. Dzik (1981) gives a direct proof of *AC* from *LET*, which may be presented in the following slightly simplified way.

In the terminology of Russell (1919), pp. 119f., a *selection* from a family $\mathfrak{J}$ of sets a set containing exactly one element from each element of $\mathfrak{J}$. It is well known (and easily proved) that *AC* is equivalent to the statement that every family of pairwise disjoint non-empty sets admits a selection (Jech 1973, p. 6).

THEOREM 3 If *LET* (Theorem 2) holds, then every family of pairwise disjoint non-empty sets admits a selection (*AC*).

PROOF: Let $\mathfrak{J}$ be a family of disjoint non-empty sets, and put $S = \bigcup \mathfrak{J}$. We define a consequence operation $\mathbf{Cn}$ on subsets Y of S as follows:

$$\mathbf{Cn}(Y) = \begin{cases} Y & \text{if } Y \text{ is a selection from a subfamily of } \mathfrak{J} \\ S & \text{otherwise.} \end{cases} \tag{3}$$

That is to say, $\mathbf{Cn}(Y) = Y$ if Y contains at most one element from each set in $\mathfrak{J}$; otherwise $\mathbf{Cn}(Y) = S$. It is simple to show that $\mathbf{Cn}$ is a consequence operation. It is obvious that it is extremal.

If every element of the family $\mathfrak{J}$ is a unit set, the set S is itself a selection from $\mathfrak{J}$, and nothing more needs to be proved. In this case, every $Y \subseteq S$ is a selection from some subfamily of $\mathfrak{J}$, and hence $\mathbf{Cn} = \mathbf{Cn}_{\min}$. Otherwise, there exists some $Y \in \mathfrak{J}$ with distinct elements y, y' in S. Now $y \notin \mathbf{Cn}(\emptyset) = \emptyset$, and so by Theorem 2 there is a y-saturated theory $\mathbf{Y}$ that extends $\emptyset$; that is to say,

(i) $\emptyset \subseteq \mathbf{Y}$
(ii) $y \notin \mathbf{Y} = \mathbf{Cn}(\mathbf{Y})$
(iii) if $x \notin \mathbf{Y}$ then $y \in \mathbf{Cn}(\mathbf{Y} \cup \{x\})$.

By clause (ii), the theory $\mathbf{Y} \neq S$. By (3), $\mathbf{Y}$ is a selection from a subfamily of $\mathfrak{J}$. We prove that $\mathbf{Y}$ is a selection from the whole of $\mathfrak{J}$.

The first task is to show that Y has a representative in $\mathbf{Y}$. Let y' be any element of Y distinct from y. If $y' \notin \mathbf{Y}$ then by (iii) $y \in \mathbf{Cn}(\mathbf{Y} \cup \{y'\})$. But by (ii), $y \notin \mathbf{Y} \cup \{y'\}$, which means that $\mathbf{Y} \cup \{y'\} \neq \mathbf{Cn}(\mathbf{Y} \cup \{y'\})$. It follows from (3) that $\mathbf{Y} \cup \{y'\}$ is not a selection from a subfamily of $\mathfrak{J}$. But since $\mathbf{Y}$ is such a selection, and $\mathfrak{J}$ is a disjoint family, this means that there is already in $\mathbf{Y}$ a representative of the set Y. We have shown that if $\mathbf{Y}$ contains no representative of Y, then it contains a representative of Y. It follows that $\mathbf{Y}$ contains a representative of Y.

The proof is much the same for a set $X \in \mathfrak{J}$ that is distinct from Y. If no $x \in X$ is selected by $\mathbf{Y}$, then $y \in \mathbf{Cn}(\mathbf{Y} \cup \{x\})$ for every $x \in X$, by (iii). By (ii), and the fact that Y and X are disjoint, $\mathbf{Cn}(\mathbf{Y} \cup \{x\}) \neq \mathbf{Y} \cup \{x\}$. But then by (3), $\mathbf{Y} \cup \{x\}$ is not a selection from a subfamily of $\mathfrak{J}$. It follows, in the same way as before, that there is already in $\mathbf{Y}$ a representative of the set X. ■

Corollary If *LT* holds, then every family of pairwise disjoint non-empty sets admits a selection (*AC*).

Proof: Lindenbaum's theorem *LT*, mentioned above, says that if $\mathbf{Cn}$ is compact then every consistent $Y \subseteq S$ has a maximal extension. Now the consequence operation $\mathbf{Cn}$ defined in (3) is plainly compact except in the case, which need not detain us, in which $\mathfrak{J}$ is a family of unit sets. We may therefore assume that $\emptyset$ has a maximal extension $\mathbf{Y}$. We may choose y to be any element of S that is excluded from $\mathbf{Y}$. Since clauses (i)-(iii) all hold, the proof that $\mathbf{Y}$ is a selection from $\mathfrak{J}$ proceeds as before. ■

These two theorems, *LT* and *LET*, are therefore each set-theoretically equivalent to *AC* (relative to some background set theory such as **ZF**).

2 Untapped Depths in Dzik's Proof

The proof above yields more than a pretty demonstration of what is, as Dzik himself admits, an unsurprising result. Inspection reveals that much less than the full intuitive power of *LET* is used in defining the selection $\mathbf{Y}$ for the family $\mathfrak{J}$. For unless $\mathfrak{J}$ is a family composed entirely of unit sets (in which case the existence of a selection is trite), the operation $\mathbf{Cn}$ has four special properties:

(4)
(a) $\mathbf{Cn}(Y)$ takes only the values Y and S
(b) there is no $y \in S$ for which $\mathbf{Cn}(\{y\}) = S$
(c) if $\mathbf{Cn}(Y) = S$ then $\mathbf{Cn}(\{x, z\}) = S$ for some $x, z \in Y$
(d) if $\mathbf{Cn}(\{x, y\}) = S = \mathbf{Cn}(\{y, z\})$, and $x \neq z$, then $\mathbf{Cn}(\{x, z\}) = S$.

What is more, what need be assumed about $\mathbf{Cn}$ is only that, for at least one $y \in \bigcup \mathfrak{J}$ there exists a y-saturated theory. (Dzik's own proof, unlike the one given above, does not assume that if $y \in Y \in \mathfrak{J}$ then $|Y| > 1$. It is readily checked that if $|Y| = 1$, and $\mathbf{Y}$ is y-saturated, then $\mathbf{Y} \cup \{y\}$ is a selection from $\mathfrak{J}$. See also Lemma 22.) This is intuitively weaker than the assumption that, for every $y \in S$, there exists a y-saturated theory, and intuitively much weaker than the assumption that for every $y \in S$, and every Y not implying y, there exists a y-saturated theory extending Y. Indeed, it is a misnomer to call the hypothesis of Dzik's proof a version of the Lindenbaum *extension* theorem at all.

For brevity we shall call a family $\mathfrak{J}$ of disjoint non-empty sets not all of which are unit sets an *active* family. It has already been made clear that *AC* is equivalent to the statement that every active family of sets admits a selection, and that we need not concern ourselves further with inactive families.

If the operations $\mathbf{Cn}_{\min}$ and $\mathbf{Cn}_{\max}$ defined above are called *extreme*, the more extensive class of operations identified in (4a) may be called *extremal*. Consequence operations that satisfy (4b) will be called *punctilious*. If consequence operations that satisfy (1) are called compact, as they are, then those that satisfy the stronger condition

(5) if $\mathbf{Cn}(Y) = S$ then $\mathbf{Cn}(X) = S$ for some $X \subseteq Y$ for which $|X| = k$

may be called k-*compact*. Unless S is empty, a compact consequence operation **Cn** that incorporates the usual rules for conjunction is 1-compact (and therefore not punctilious). The consequence operations identified in (4c) are 2-compact. We shall call consequence operations that satisfy (4d) *equilateral*, for the following reason. The set $\mathbf{Cn}(\{x, z\})$ can be regarded as a lattice-valued metric operation, with greatest value S, that measures the distance between x and z. (4c) then says that if x, y, z are the vertices of a triangle with two distinct sides of length S, then they are the vertices of an equilateral triangle. (For more on lattice-valued metrics see Miller 1984.) These terms will be applied to systems as well as to consequence operations.

Let *LETa* be the restriction of *LET* to systems $\langle S, \mathbf{Cn}\rangle$ in which **Cn** is extremal; and *LETabcd* be the restriction of *LET* to systems in which **Cn** satisfies all four clauses of (4). Similar expressions will be used in an obvious way. Theorems 2 and 3 demonstrate the truth of all the conditionals *LET*ß $\Rightarrow$ *LET* where ß is some subset of {*a*, *b*, *c*, *d*}. Furthermore, let *let* say that if $\langle S, \mathbf{Cn}\rangle$ satisfies (4a-d) then there exists a saturated theory **Y** in $\langle S, \mathbf{Cn}\rangle$. The conditional *let* $\Rightarrow$ *LET* has also been demonstrated.

Within **ZF**, of course, the antecedents and consequents of all these conditionals are equivalent to *AC*, and so all the conditionals can be proved without appeal to *AC*. But the question quite naturally arises whether they can be proved directly, without (in effect) first proving *AC* from the antecedent, and then proving the consequent from *AC*. To provide some answers to these vague but mathematically clear questions is the principal purpose of this paper. In § 3 we give a positive answer for the conditional *LETa* $\Rightarrow$ *LET*. In § 4 we give a positive answer also for *LETab* $\Rightarrow$ *LET*, and discuss briefly the difficulties encountered in giving a proof of *LETac* $\Rightarrow$ *LET*. We shall show in § 5 that all the four properties mentioned in (4) play a role in Dzik's proof, and in § 6 that the conditional *let* $\Rightarrow$ *LETabcd* holds (again without recourse to *AC*). In § 7 we summarize what still needs to be achieved. The paper concludes with an appendix, in which are gathered a number of useful results about saturated and maximal theories.

3 Extremal Consequence Operations

Throughout this section, S is a fixed (but arbitrary) set. We shall, however, have to consider simultaneously more than one consequence operation on $\wp(S)$. The use of boldface characters for theories is therefore suspended.

Let $\langle S, \mathbf{Cn}\rangle$ be a system and y an element of S. We write $\mathcal{L}(\mathbf{Cn}, y)$ for the statement that every $Y \subseteq S$ not implying y can be extended in $\langle S, \mathbf{Cn}\rangle$ to a y-saturated theory:

(6) $$\mathcal{L}(\mathbf{Cn}, y) \iff_{\mathrm{Df}} \begin{cases} \text{for every } Y, \text{ if } y \notin \mathbf{Cn}(Y) \text{ then there is } X \supseteq Y \text{ such that} \\ \text{(a)} \quad y \notin \mathbf{Cn}(X) = X \\ \text{(b)} \quad y \in \mathbf{Cn}(X \cup \{x\}) \text{ for every } x \notin X. \end{cases}$$

In these terms, what *LET* asserts is that $\mathcal{L}(\mathbf{Cn}, y)$ holds for every consequence operation **Cn** on $\wp(S)$ and every element y of S.

What will be proved, without appeal to the axiom of choice *AC*, is that every consequence operation **Cn** may be associated with a family of explicitly defined extremal consequence operations $\{\mathbf{Cn}^y \mid y \in S\}$ such that for every y, $\mathcal{L}(\mathbf{Cn}^y, y)$ if & only if $\mathcal{L}(\mathbf{Cn}, y)$. It follows that *LET* holds if $\mathcal{L}(\mathbf{Cn}^y, y)$ holds for every y; and hence that *LET* holds generally if it holds for every extremal consequence operation.

Let **Cn** be a (finitary) consequence operation and y an element of S. For each y the consequence operation $\mathbf{Cn}^y$ is defined as follows.

(7) $$\mathbf{Cn}^y(Y) =_{\mathrm{Df}} \begin{cases} Y & \text{if } y \notin \mathbf{Cn}(Y) \\ S & \text{otherwise.} \end{cases}$$

Note that if S contains a minimal element $\bot$ or a maximal element $\top$ under **Cn** then $\mathbf{Cn}^{\bot} = \mathbf{Cn}_{\min}$ and $\mathbf{Cn}^{\top} = \mathbf{Cn}_{\max}$. See also Lemma 24 in the appendix below.

Lemma 4 For each y the operation Cn^y defined in (7) is an extremal consequence operation.

Proof: It is obvious that $Y \subseteq \mathsf{Cn}^y(Y) = \mathsf{Cn}^y(\mathsf{Cn}^y(Y))$, and that Cn^y is extremal. Assume that $X \subseteq Z$ and that $x \in \mathsf{Cn}^y(X)$. This latter set is either X or S. If $\mathsf{Cn}^y(X) = X$, we may infer from $X \subseteq Z$ and $x \in \mathsf{Cn}^y(X)$ that $x \in Z$, and so $x \in \mathsf{Cn}^y(Z)$. If $\mathsf{Cn}^y(X) = S$, then by (7), $y \in \mathsf{Cn}(X)$. Since Cn obeys ($0c_0$), it follows that $y \in \mathsf{Cn}(Z)$, so by (7) again, $y \in \mathsf{Cn}^y(Z)$. This proves that Cn^y obeys the monotony law ($0c_0$).

To show that Cn^y is finitary, note first that it holds generally that $x \in \mathsf{Cn}^y(\{x\})$. Assume that $x \in \mathsf{Cn}^y(Y)$. This latter set is either Y or S. If $\mathsf{Cn}^y(Y) = Y$, we may infer that $\{x\} \subseteq Y$, and hence that $x \in \mathsf{Cn}^y(X)$ where $X = \{x\}$ is a finite subset of Y. If $\mathsf{Cn}^y(Y) = S$, in which case $y \in \mathsf{Cn}(Y)$ (even if $\mathsf{Cn}^y(Y) = Y$ too). From the finitariness ($0c_1$) of Cn it follows that $y \in \mathsf{Cn}(X)$, where X is a finite subset of Y, which implies that $\mathsf{Cn}^y(X) = S$. Hence $x \in \mathsf{Cn}^y(X)$ where X is a finite subset of Y. ■

Corollary 0 For each y the operation Cn^y defined in (7) is compact.

Proof: Omitted. ■

Corollary 1 The following three conditions are equivalent.

$$(8)\qquad \begin{array}{ll} \text{(a)} & y \in \mathsf{Cn}^y(Y) \\ \text{(b)} & y \in \mathsf{Cn}(Y) \\ \text{(c)} & \mathsf{Cn}^y(Y) = S. \end{array}$$

Proof: If y is in $\mathsf{Cn}^y(Y)$ and not in $\mathsf{Cn}(Y)$, then $\mathsf{Cn}^y(Y) = Y$, by (7). Hence $y \in Y$, and so $y \in \mathsf{Cn}(Y)$ by (0a). It follows that if (a) $y \in \mathsf{Cn}^y(Y)$ then (b) $y \in \mathsf{Cn}(Y)$. Hence (c) $\mathsf{Cn}^y(Y) = S$, from which (a) is an immediate consequence. ■

Corollary 2 The operation $\mathsf{C}^y(Y)$ defined on $\wp(S)$ by

$$(9)\qquad \mathsf{C}^y(Y) =_{\mathrm{Df}} \begin{cases} Y & \text{if } y \notin Y \\ S & \text{otherwise} \end{cases}$$

is an extremal consequence operation.

Proof: Take for Cn in the lemma the extreme consequence operation $\mathsf{Cn}_{\min}$. ■

Lemma 5 X is a y-saturated theory in $\langle S, \mathsf{Cn}^y \rangle$ if & only if it is a y-saturated theory in $\langle S, \mathsf{Cn} \rangle$.

Proof: Suppose that $x \notin X$. It follows immediately from the equivalence of (8a) and (8b) that $y \notin \mathsf{Cn}^y(X)$ if & only if $y \notin \mathsf{Cn}(X)$, and that $y \in \mathsf{Cn}^y(X \cup \{x\})$ if & only if $y \in \mathsf{Cn}(X \cup \{x\})$. It follows that X satisfies (2a-b) in $\langle S, \mathsf{Cn}^y \rangle$ if & only if it satisfies (2a-b) in $\langle S, \mathsf{Cn} \rangle$. It remains to be shown that X is a theory in the one system if & only if it is a theory in the other.

Suppose then that $X = \mathsf{Cn}^y(X)$, and that X is y-saturated in $\langle S, \mathsf{Cn}^y \rangle$. Suppose further that there is some $x \in \mathsf{Cn}(X)$ such that $x \notin X$. We have just shown that X is y-saturated in $\langle S, \mathsf{Cn} \rangle$, and so $y \in \mathsf{Cn}(X \cup \{x\})$, which, since $x \in \mathsf{Cn}(x)$, is identical with $\mathsf{Cn}(X)$. By the equivalence of (8a) and (8b), $y \in \mathsf{Cn}^y(X)$. Since $X = \mathsf{Cn}^y(X)$, by assumption, $y \in X$, contradicting the supposition that X is y-saturated in $\langle S, \mathsf{Cn}^y \rangle$. We may conclude that if $x \in \mathsf{Cn}(X)$ then $x \in X$, which means that X is a theory in $\langle S, \mathsf{Cn} \rangle$.

The converse is more straightforward. Assume that $X = \mathsf{Cn}(X)$ and that $y \notin \mathsf{Cn}(X)$. By (7), $\mathsf{Cn}^y(X) = X$. That is, X is a theory in $\langle S, \mathsf{Cn}^y \rangle$. ■

Lemma 6 $\mathcal{L}(\mathsf{Cn}^y, y)$ if & only if $\mathcal{L}(\mathsf{Cn}, y)$.

Proof: Immediate. ■

Theorem 7 If *LET* (Theorem 2) holda for every extremal consequence operation **Cn**, then it holds for every consequence operation.

Proof: The result follows from Lemmas 4 and 6. To allay any suspicion that there is at any point an appeal to *AC*, the argument is set out more explicitly in (10), where $\mathcal{C}(\mathbf{f})$ means that $\mathbf{f}$ is a consequence operation and $\mathcal{E}(\mathbf{f})$ means that $\mathbf{f}$ is an extremal consequence operation:

	(a)	Lemma 4	for every $\mathbf{f}$ and every y: if $\mathcal{C}(\mathbf{f})$ then $\mathcal{E}(\mathbf{f}^y)$
	(b)	Lemma 6	for every $\mathbf{f}$ and every y: $\mathcal{L}(\mathbf{f}^y, y)$ if & only if $\mathcal{L}(\mathbf{f}, y)$
(10)	(c)	Assumption	for every $\mathbf{f}$: if $\mathcal{E}(\mathbf{f})$ then for every y, $\mathcal{L}(\mathbf{f}, y)$
	(d)	Conclusion	for every $\mathbf{f}$: if $\mathcal{C}(\mathbf{f})$ then for every y, $\mathcal{L}(\mathbf{f}, y)$.

Although rather long-winded if spelt out in detail, the argument is unquestionably valid in a two-sorted elementary logic (with function symbols). The assumption (10c) is *LETa*, the restriction of *LET* to extremal consequence operations. The conclusion (10d) is *LET*. ■

4 Punctilious, 2-compact, and Equilateral Consequence Operations

We have now shown that *LETa* $\Rightarrow$ *LET*. Although we shall not achieve the aim of demonstrating (without recourse to *AC*) the stronger conditional *LETabcd* $\Rightarrow$ *LET*, we shall in this section make some progress with regard to punctilious consequence operations (4b). It will be proved, without calling on *AC*, that every extremal consequence operation **Cn** can be associated with an extremal punctilious $\mathbf{Cn}^\star$ such that if $\mathbf{Cn}^\star$ satisfies *LET* then so does **Cn**. This will establish the conditional *LETab* $\Rightarrow$ *LETa*.

For this task we shall alter not only the consequence operation **Cn** but also the class S. It will therefore be necessary to be more explicit than before about which deductive system we are discussing. We start with a general lemma.

Let $\langle S, \mathbf{Cn}\rangle$ be any deductive system. We make the fairly obvious definition

$$(11)\quad \begin{array}{lrcl} \text{(a)} & S^\star & =_{\text{Df}} & S \setminus \{y \mid \mathbf{Cn}(\{y\}) = S\} \\ \text{(b)} & \mathbf{Cn}^\star(Y) & =_{\text{Df}} & \mathbf{Cn}(Y) \cap S^\star \text{ for all } Y \subseteq S^\star. \end{array}$$

Lemma 8 Let $\langle S, \mathbf{Cn}\rangle$ be an extremal deductive system with at least three consistent theories. The system $\langle S^\star, \mathbf{Cn}^\star\rangle$ is an extremal punctilious deductive system.

Proof: That $\langle S^\star, \mathbf{Cn}^\star\rangle$ is an extremal system is a direct consequence of Lemma 23, which is in the appendix. Now suppose that $\mathbf{Cn}^\star(\{y\}) = S^\star$ for some $y \in S^\star$. By (11), $\mathbf{Cn}(\{y\}) \cap S^\star = S^\star$, which implies that $S^\star \subseteq \mathbf{Cn}(\{y\})$. Unless $S^\star = \{y\}$, the extremality of $\langle S, \mathbf{Cn}\rangle$ requires that $\mathbf{Cn}(\{y\}) = S$. But then by (11), $y \notin S^\star$, contrary to supposition.

$S^\star = \{y\}$, however, is not possible unless $\langle S, \mathbf{Cn}\rangle$ contains only two consistent theories, namely Ø and $\{y\}$. If there is any other theory $\mathbf{Y} \subset S$, then $\mathbf{Y}$ must have an element u distinct from y for which $\mathbf{Cn}(\{u\}) \neq S$. But then $u \in S^\star$. ■

Let $\langle S, \mathbf{Cn}\rangle$ be any deductive system. Refining (6), we shall write

$$(12)\quad \mathcal{L}(\langle S, \mathbf{Cn}\rangle) \iff_{\text{Df}} \begin{cases} \text{for every } Y \subseteq S \text{ and } y \in S \setminus \mathbf{Cn}(Y) \text{ there is } X \subseteq S \text{ such that} \\ \text{(a)} \quad y \notin \mathbf{Cn}(X) = X \supseteq Y \\ \text{(b)} \quad y \in \mathbf{Cn}(X \cup \{x\}) \text{ for every } x \in S \setminus X. \end{cases}$$

$\mathcal{L}(\langle S, \mathbf{Cn}\rangle)$ says no more than that $\mathcal{L}(\mathbf{Cn}, y)$ holds for every y, with the complication that not only the operation **Cn** but also the set S is made explicit. *LET* says that $\mathcal{L}(\langle S, \mathbf{Cn}\rangle)$ holds for every system $\langle S, \mathbf{Cn}\rangle$.

THEOREM 9 Let $\langle S, \mathsf{Cn}\rangle$ be an extremal deductive system containing at least three consistent theories, and let $\langle S^\star, \mathsf{Cn}^\star\rangle$ be the system defined in (11). If $\mathcal{L}(\langle S^\star, \mathsf{Cn}^\star\rangle)$ then $\mathcal{L}(\langle S, \mathsf{Cn}\rangle)$.

PROOF: Thanks to Lemma 8, we may assume that $\langle S^\star, \mathsf{Cn}^\star\rangle$ is both extremal and punctilious. We are given that $\mathcal{L}(\langle S^\star, \mathsf{Cn}^\star\rangle)$:

(13) for every $Y \subseteq S^\star$ and $y \in S^\star \setminus \mathsf{Cn}^\star(Y)$ there is $X \subseteq S^\star$ such that
(a) $y \notin \mathsf{Cn}^\star(X) = X \supseteq Y$
(b) $y \in \mathsf{Cn}^\star(X \cup \{x\})$ for every $x \in S^\star \setminus X$.

Suppose that $Y \subseteq S$, and that $y \in S \setminus \mathsf{Cn}(Y)$. Initially we suppose also that $\mathsf{Cn}(\{y\}) \neq S$; that is, that $y \in S^\star$. Since $\mathsf{Cn}(Y)$ does not contain y, the set Y is consistent, and hence $Y \subseteq S^\star$; in addition, $y \notin \mathsf{Cn}^\star(Y) = \mathsf{Cn}(Y) \cap S^\star$. It therefore follows that there exists a set $X \subseteq S^\star$ that satisfies (13a) and (13b). We shall show that in the system $\langle S, \mathsf{Cn}\rangle$ this set X is a y-saturated extension of the set Y; that is to say, that (12a) and (12b) are satisfied.

To show first that $X = \mathsf{Cn}(X)$, let us suppose that $x \in \mathsf{Cn}(X) \setminus X$. Since $X = \mathsf{Cn}^\star(X) = \mathsf{Cn}(X) \cap S^\star$, x cannot belong to $S^\star$, and so $\mathsf{Cn}(\{x\}) = S$ by (11a). Since $x \in \mathsf{Cn}(X)$, by assumption, $S = \mathsf{Cn}(\{x\}) \subseteq \mathsf{Cn}(\mathsf{Cn}(X)) = \mathsf{Cn}(X)$, by (0b,$\text{c}_0$), from which follows the identity $\mathsf{Cn}^\star(X) = S^\star$, by (11b). This is not possible, since $y \notin \mathsf{Cn}^\star(X)$, by (13a), and $y \in S^\star$ by assumption. We may conclude that there is no x in $\mathsf{Cn}(X) \setminus X$.

It is immediate that $Y \subseteq X$. Since y is in $S^\star$ but not in $\mathsf{Cn}^\star(X)$, it follows that $y \notin \mathsf{Cn}(X)$. This proves (12a).

Now suppose that $x \in S \setminus X$. By (11a), either $x \in S^\star \setminus X$ or $\mathsf{Cn}(\{x\}) = S$. In the former case, by (13b) and (11b), $y \in \mathsf{Cn}(X \cup \{x\})$. In the latter case, by monotony (0c_0), $y \in \mathsf{Cn}(X \cup \{x\})$. This proves (12b).

We must finally consider the possibility that at the outset we chose an element y for which $\mathsf{Cn}(\{y\}) = S$. If $\mathsf{Cn}(Y) = S^\star$, then manifestly in $\langle S, \mathsf{Cn}\rangle$ it is a maximal theory not containing y, for the only elements of S that it lacks are those that are individually inconsistent; hence Y has a y-saturated extension in $\langle S, \mathsf{Cn}\rangle$. The alternative is that there is in $\mathsf{Cn}(Y)$ some element u for which $\mathsf{Cn}(\{u\}) \neq S$. By what we have already proved, Y has in $\langle S, \mathsf{Cn}\rangle$ a u-saturated extension X. By Lemma 22, which is to be found in the appendix below, either X or $X \cup \{u\}$ is a maximal theory in $\langle S, \mathsf{Cn}\rangle$. Whichever one it is, it is plainly a y-saturated extension of Y. ■

COROLLARY If $\mathcal{L}(\langle S, \mathsf{Cn}\rangle)$ holds for every extremal punctilious system $\langle S, \mathsf{Cn}\rangle$ then it holds for every extremal system.

PROOF: The only systems to which the Theorem does not provide the proof are those in which S has only two consistent theories, Ø and $\{y\}$. Since Ø has Ø as a y-saturated extension, every theory that does not imply y has a y-saturated extension, and hence $\mathcal{L}(\langle S, \mathsf{Cn}\rangle)$ holds. ■

Can we get any further towards proving *LETabcd* ⇒ *LET*? At this point matters become much trickier. How, for example, might we hope to show the conditional *LETabc* ⇒ *LETab*, that if *LET* holds for all extremal punctilious 2-compact consequence operations then it holds for all that are extremal and punctilious? A natural approach to this problem might be to adjust the values of an arbitrary extremal punctilious consequence operation **Cn** in this way:

$$(14)\qquad \mathsf{C}(Y) =_{\text{Df}} \begin{cases} Y & \text{if } \mathsf{Cn}(Y) = Y \\ Y & \text{if } \mathsf{Cn}(Y) = S \\ & \text{and there are no } x, z \in Y \text{ such that } \mathsf{Cn}(\{x, z\}) = S \\ S & \text{otherwise.} \end{cases}$$

LEMMA 10 Let $\langle S, \mathsf{Cn}\rangle$ be an extremal punctilious deductive system. The system $\langle S, \mathsf{C}\rangle$ defined in (14) is an extremal punctilious 2-compact deductive system. If $\langle S, \mathsf{Cn}\rangle$ is equilateral, then so is $\langle S, \mathsf{C}\rangle$.

PROOF: Omitted. ■

Unfortunately it does not seem to be possible (without assuming *AC*) to show that whenever $\langle S, \mathsf{C}\rangle$ satisfies *LET* then so does $\langle S, \mathsf{Cn}\rangle$. Even the problem of proving (without assuming *AC*) the conditional *LETac* $\Rightarrow$ *LET* remains open. But it is worth recording that if this result could be proved, there is one direction in which it could not be improved.

THEOREM 11 Without assuming *AC* it is possible to show that *LET* holds for all extremal 1-compact consequence operations.

PROOF: Let Cn be a 1-compact extremal consequence operation. Suppose that $y \notin \mathsf{Cn}(Y)$, and let $X = Y \cup \{x \mid \mathsf{Cn}(\{x\}) \neq S\}$. If $\mathsf{Cn}(X) = S$, then by 1-compactness there is $x \in Y \cup \{x \mid \mathsf{Cn}(\{x\}) \neq S\}$ for which $\mathsf{Cn}(\{x\}) = S$. This implies that $\mathsf{Cn}(Y) = S$, which is impossible if $y \notin \mathsf{Cn}(Y)$. Hence $\mathsf{Cn}(X) = X$ (since Cn is extremal).

If $x \notin X$ then $\mathsf{Cn}(\{x\}) = S$, and hence $\mathsf{Cn}(X \cup \{x\}) = S$. Thus X is a maximal theory. In other words, even in the absence of *AC*, both *LET* and Lindenbaum's theorem *LT* hold when restricted to 1-compact extremal consequence operations. ■

5 A Representation Theorem

In this section we turn to the task of establishing that the four properties detailed in (4) exactly characterize the consequence operations that emerge from Dzik's construction. We begin with an overdue definition

(15) (a) $x \sim z =_{\mathrm{Df}} x = z \vee \mathsf{Cn}(\{x, z\}) = S$

(b) $\tilde{x} =_{\mathrm{Df}} \{z \in S \mid x \sim z\}$.

LEMMA 12 If $\langle S, \mathsf{Cn}\rangle$ is a punctilious and equilateral system then the relation $\sim$ is an equivalence relation on S, and for each x, $\tilde{x}$ is the equivalence class of x.

PROOF: (15a) is straightforward. (15b) is immediate. ■

LEMMA 13 Let $\langle S, \mathsf{Cn}\rangle$ be an extremal, punctilious, 2-compact, and equilateral system. There exists an active family $\mathfrak{J}$ of sets such that $S = \bigcup \mathfrak{J}$ and (3) holds.

PROOF: Put $\mathfrak{J} = \{\tilde{u} \mid u \in S\}$. It is immediate that $S = \bigcup \mathfrak{J}$. Now suppose that $\mathsf{Cn}(Y) = S$. Because Cn is 2-compact (4c) and punctilious (4b), $\mathsf{Cn}(\{x, z\}) = S$ for distinct $x, z \in Y$, from which it follows that both x and z belong to $\tilde{x}$. In other words, Y is not a selection from any subfamily of $\mathfrak{J}$. The converse is even easier: if $\mathsf{Cn}(Y) \neq S$, then Y is a selection from some subfamily of $\mathfrak{J}$. Note that no inactive family $\mathfrak{J}$ could fill the bill here; for $\mathsf{Cn}(S) = S$, and 2-compactness would fail. ■

LEMMA 14 Let $\mathfrak{J}$ be an active family of sets. Let Cn be defined as in (3), where $S = \bigcup \mathfrak{J}$. Then $\langle S, \mathsf{Cn}\rangle$ is an extremal, punctilious, 2-compact, and equilateral system.

PROOF: It is immediate that Cn is extremal. If $\mathsf{Cn}(Y) = S$, then either $Y = S$, or Y is not a selection. Since $\mathfrak{J}$ is not a family of unit sets, in either case Y contains two distinct elements x, z from the same element of $\mathfrak{J}$. Hence Cn is punctilious. By (3), moroever, $\mathsf{Cn}(\{x, z\}) = S$, since the elements of $\mathfrak{J}$ are disjoint. Hence Cn is 2-compact. It is then straightforward to check that Cn is also equilateral. ■

LEMMA 15 Let $\langle S, \mathsf{Cn}\rangle$ be an extremal, punctilious, 2-compact, and equilateral system, and let $\mathfrak{J} = \{\tilde{u} \mid u \in S\}$. Suppose that $y \notin \mathbf{Y}$. Then $\mathbf{Y}$ is a y-saturated theory in $\langle S, \mathsf{Cn}\rangle$ if & only if either $\mathbf{Y}$ is a selection from $\mathfrak{J}$ or $\tilde{y} = \{y\}$ and $\mathbf{Y}$ is a selection from $\mathfrak{J} \setminus \{\tilde{y}\}$.

Proof: We assume throughout the proof that $y \notin \mathbf{Y}$. Let $\mathbf{Y}$ be a y-saturated theory. It is obvious that $\mathbf{Y}$ cannot contain two elements of any $\tilde{u} \in \mathfrak{J}$. We must show in addition that $\mathbf{Y}$ contains an element of each $\tilde{u} \in \mathfrak{J}$. If not, then $u \notin \mathbf{Y}$ and so $y \in \mathbf{Cn}(\mathbf{Y} \cup \{u\})$. If this last set is S, then since $\mathbf{Cn}$ is punctilious and 2-compact, $\mathbf{Cn}(\{x, u\}) = S$ for some $x \in \mathbf{Y}$, which means that $\mathbf{Y}$ contains an element of $\tilde{u}$ after all. Hence $\mathbf{Cn}(\mathbf{Y} \cup \{u\}) = \mathbf{Y} \cup \{u\}$, since $\mathbf{Cn}$ is extremal. We may conclude, since $y \notin \mathbf{Y}$, that $y = u$. In other words, the only $\tilde{u}$ that $\mathbf{Y}$ can fail to contain an element of is $\tilde{y}$.

Suppose first that $\tilde{y} \neq \{y\}$. We shall show that $\mathbf{Y}$ is a selection from $\mathfrak{J}$. By assumption there is an element x distinct from y in $\tilde{y}$. Since $x \notin \mathbf{Y}$, we must have $y \in \mathbf{Cn}(\mathbf{Y} \cup \{x\})$, from which it follows by (4a) that $\mathbf{Cn}(\mathbf{Y} \cup \{x\}) = S$. By 2-compactness and punctiliousness, there exists in $\mathbf{Y}$ after all an element of $\tilde{y}$. If, alternatively, $\tilde{y} = \{y\}$, then it is immediate that $\mathbf{Y}$ is a selection from $\mathfrak{J} \setminus \{\tilde{y}\}$.

For the converse, it is plain that $y \in \mathbf{Cn}(\mathbf{Y} \cup \{y\})$. So choose any other $u \notin \mathbf{Y}$. It makes no difference whether $\mathbf{Y}$ is a selection from $\mathfrak{J}$ or a selection from $\mathfrak{J} \setminus \{\tilde{y}\}$; in each case $\mathbf{Y}$ contains an element of $\tilde{u}$, and hence the set $\mathbf{Y} \cup \{u\}$ is inconsistent. We conclude that $y \in \mathbf{Cn}(\mathbf{Y} \cup \{u\})$. ■

Corollary 0 Let $\langle S, \mathbf{Cn}\rangle$ be an extremal, punctilious, 2-compact, and equilateral system, and let $\mathfrak{J} = \{\tilde{u} \mid u \in S\}$. Then $\mathbf{Y}$ is a maximal theory in $\langle S, \mathbf{Cn}\rangle$ if & only if $\mathbf{Y}$ is a selection from $\mathfrak{J}$.

Proof: If $\mathbf{Y}$ is maximal then it is y-saturated for each $y \notin \mathbf{Y}$ (Lemma 19). If $\tilde{x} = \tilde{y}$ whenever x, distinct from y, does not belong to $\mathbf{Y}$, then, since $\mathfrak{J}$ is active, $\tilde{y}$ must contain at least two elements, and it follows that $\mathbf{Y}$ is a selection from $\mathfrak{J}$. On the other hand, $\mathbf{Y}$ cannot be a selection from both $\mathfrak{J} \setminus \{\tilde{x}\}$ and $\mathfrak{J} \setminus \{\tilde{y}\}$ if $\tilde{x} \neq \tilde{y}$, and hence $\mathbf{Y}$ is again compelled to be a selection from $\mathfrak{J}$. The converse is immediate. ■

Corollary 1 If $\mathbf{Y}$ is a saturated theory in an extremal, punctilious, 2-compact, and equilateral system $\langle S, \mathbf{Cn}\rangle$ then $\mathbf{Y}$ is a selection from an active family $\mathfrak{J}$ satisfying (3).

Proof: All that needs to be shown is that if $\tilde{y} = \{y\}$ and $\mathfrak{J}$ is an active family, then $\mathfrak{J} \setminus \{\tilde{y}\}$ is an active family. Once stated, this is obvious. ■

Lemma 16 If $\mathbf{Y}$ is a selection from an active family $\mathfrak{J}$ of sets, then there exists an extremal, punctilious, 2-compact, and equilateral system satisfying (3) in which $\mathbf{Y}$ is saturated.

Proof: Put $S = \bigcup \mathfrak{J}$, and define $\mathbf{Cn}$ as in (3). By Lemma 14, $\mathbf{Cn}$ has all the listed properties. It is a simple matter to check that $\mathbf{Y}$ is a saturated theory. ■

Theorem 17 $\mathbf{Y}$ is a saturated theory in an extremal, punctilious, 2-compact, and equilateral system $\langle S, \mathbf{Cn}\rangle$ if & only if it is a selection from an active family $\mathfrak{J}$ of sets satisfying (3).

Proof: Combine the corollary to Lemma 13 with Lemma 16. ■

6 One Saturated Theory Is Enough

The aim of this section is to prove (without using *AC*) the truth of the conditional *let* $\Rightarrow$ *LETabcd*: if in every system satisfying (4a-d) there exists a saturated theory $\mathbf{Y}$, then in every such system every set Y not implying y has a saturated extension not implying y. As usual we prove this conditional of the form $\forall y \Phi y \to \forall y \Psi y$ by proving the stronger universalized conditional $\forall y(\Phi y \to \Psi y)$.

Theorem 18 Let $\langle S, \mathbf{Cn}\rangle$ be an extremal punctilious 2-compact equilateral system in which there is a saturated theory. If $x \notin \mathbf{Cn}(X)$, then X has an x-saturated extension.

Proof: By Lemma 22 there exists a maximal theory $\mathbf{Y}$ in $\langle S, \mathbf{Cn}\rangle$. By corollary 0 to Lemma 15, $\mathbf{Y}$ is a selection from $\mathfrak{J} = \{\tilde{u} \mid u \in S\}$.

Suppose that $x \notin \mathbf{Cn}(X)$. We define

$$(16) \qquad W \quad =_{\text{Df}} \quad X \cup (\{u \in \mathbf{Y} \mid \mathbf{Cn}(X \cup \{u\}) \neq S\} \setminus \{x\}).$$

The idea is that in supplementing X by $\mathbf{Y}$, elements of $\mathbf{Y}$ that individually contradict elements of X are deleted, as is x (should it happen to be an element of $\mathbf{Y}$), in the hope that other elements of X will more or less make up for the loss of the deleted elements, so that W will remain a selection from either $\mathfrak{J}$ or $\mathfrak{J} \setminus \{x\}$ (Lemma 15). It will turn out, however, that (16) is not quite enough. We shall see that W omits x, that it extends X, and that it is a consistent theory, but that it is sometimes necessary to supplement it to obtain a theory that is x-saturated.

It is immediate that W does not contain x. Since X is a consistent set, it can contain no u for which $\mathbf{Cn}(X \cup \{u\}) = S$. In other words, no element of X is removed from $X \cup \mathbf{Y}$ by the subtraction of $\{u \in \mathbf{Y} \mid \mathbf{Cn}(X \cup \{u\}) = S\}$. Hence $X \subseteq W$.

The 2-compactness of $\mathbf{Cn}$ guarantees that W is consistent. Were it not, it would have some inconsistent two-element subset, which could not consist of two elements of $\mathbf{Y}$, since $\mathbf{Y}$ is maximal, and so would have to consist of an element of X and an element of $\{u \in \mathbf{Y} \mid \mathbf{Cn}(X \cup \{u\}) \neq S\}$. This is impossible. From the consistency of W it follows by extremality that W is a theory.

To show that W is x-saturated, there are four cases to consider, each splitting off from the previous one. In the first case, $x \notin \mathbf{Y}$. Hence if u belongs to $\mathbf{Y}$ but not to W, then $\mathbf{Cn}(X \cup \{u\}) = S$, and so an element of $\tilde{u}$ belongs to W. In other words, since W is consistent, and $\mathbf{Y}$ is a selection from $\mathfrak{J}$ by assumption, W too is a selection from $\mathfrak{J}$. In the second case, $x \in \mathbf{Y}$, but $\tilde{x} = \{x\}$. Here W contains no element of $\tilde{x}$, but otherwise matches $\mathbf{Y}$; it is therefore a selection from $\mathfrak{J} \setminus \{x\}$. In the third case $x \in \mathbf{Y}$ and X contains an element, not x of course, of $\tilde{x}$. Here again W is a selection from $\mathfrak{J}$.

In the fourth and final case, $x \in \mathbf{Y}$ and $\tilde{x} \neq \{x\}$, but X contains no element of $\tilde{x}$. Here we have to replace W by $W^\dagger = W \cup \{z\}$ where z is any such element of $\tilde{x}$ that is distinct from x. It is plain that $W^\dagger$ is consistent, for if it were not then there would be some $u \in W$ such that $u \in \tilde{z} = \tilde{x}$. But since $u \notin \mathbf{Y}$ (because $x \in \mathbf{Y}$, and $\mathbf{Y}$ is consistent), it would follow that $u \in X$, contrary to hypothesis. Because $\mathbf{Cn}$ is extremal, $W^\dagger$ is a theory, as before. Previous considerations show that $W^\dagger$ is a selection from $\mathfrak{J}$. ■

Note that although in defining $W^\dagger$ it is necessary to select an element z from $\tilde{x}$, there is no appeal to *AC*. For each X and x at most one choice is made, and nowhere do we collect together the choices made for different X and x.

7 Summary: What Remains to be Done

The above considerations have made three positive steps towards the goal of establishing (without appeal to *AC*) the conditional *let* $\Rightarrow$ *LET*. We have proved the conditional *let* $\Rightarrow$ *LETabcd* (Theorem 18), the conditional *LETab* $\Rightarrow$ *LETa* (the corollary to Theorem 9), and the conditional *LETa* $\Rightarrow$ *LET* (Theorem 7). The unbridged gap is the conditional *LETabcd* $\Rightarrow$ *LETab*. Whether this conditional is amenable to a simple proof is for the moment an open question.

It may be mentioned in conclusion that quite analogous problems arise with respect to the original Lindenbaum theorem *LT*. These must await another occasion.

Appendix: Further Results

This section contains a few useful results concerning maximal and saturated theories in extremal and more general systems, and also some modest generalizations. Lemma 22 has already been used in Theorem 9 and in Lemma 18, and Lemma 23 was used in Lemma 8. Lemmas 19 and 20 are stated without proof by Béziau, *op.cit.*, p. 11.

Lemma 19 Let $\langle S, \mathbf{Cn} \rangle$ be any system. If the theory $\mathbf{Y}$ is maximal in $\langle S, \mathbf{Cn} \rangle$, then it is y-saturated for every $y \notin \mathbf{Y}$.

Proof: Suppose that $y \notin \mathbf{Y}$. For every x, if $x \notin \mathbf{Y}$ then $\mathbf{Cn}(\mathbf{Y} \cup \{x\})$ is a proper extension of $\mathbf{Y}$, and so by hypothesis $\mathbf{Cn}(\mathbf{Y} \cup \{x\}) = S$. This means that if $x \notin \mathbf{Y}$ then $y \in \mathbf{Cn}(\mathbf{Y} \cup \{x\})$. That is, $\mathbf{Y}$ is y-saturated. ■

Lemma 20 Let $\langle S, \mathbf{Cn} \rangle$ be any system. If $\mathbf{Y}$ is y-saturated for every $y \notin \mathbf{Y}$ in $\langle S, \mathbf{Cn} \rangle$, then it is maximal.

Proof: If $\mathbf{Y} = S \setminus \{y\}$ then it is obviously maximal. So choose any two elements x, z outside $\mathbf{Y}$. Since $\mathbf{Y}$ is z-saturated, $z \in \mathbf{Cn}(\mathbf{Y} \cup \{x\})$. This holds for every x, z, which means that every proper extension $\mathbf{Y} \cup \{x\}$ of $\mathbf{Y}$ implies every $z \notin \mathbf{Y}$, as well as every $z \in \mathbf{Y}$. In other words, $\mathbf{Y}$ is maximal. ■

Lemma 21 Let $\langle S, \mathbf{Cn} \rangle$ be any extremal system. If $\mathbf{Y}$ is y-saturated for two distinct $y \notin \mathbf{Y}$ in $\langle S, \mathbf{Cn} \rangle$, then it is maximal.

Proof: Suppose that $\mathbf{Y}$ is both x-saturated and z-saturated, where $x \neq z$. Then both $x \notin \mathbf{Y}$ and $z \notin \mathbf{Y}$. Choose any $y \notin \mathbf{Y}$. Then $y = z$ or $y \neq z$. In the first case, $x \in \mathbf{Cn}(\mathbf{Y} \cup \{y\})$, since $\mathbf{Y}$ is x-saturated, and hence $\mathbf{Cn}(\mathbf{Y} \cup \{y\}) = S$, since $\langle S, \mathbf{Cn} \rangle$ is extremal and $y \neq x$. In the second case, $z \in \mathbf{Cn}(\mathbf{Y} \cup \{y\})$, since $\mathbf{Y}$ is z-saturated, and hence $\mathbf{Cn}(\mathbf{Y} \cup \{y\}) = S$, since $\langle S, \mathbf{Cn} \rangle$ is extremal and $y \neq z$. In either case, therefore, if $y \notin \mathbf{Y}$ then $\mathbf{Cn}(\mathbf{Y} \cup \{y\}) = S$. This means that $\mathbf{Y}$ is maximal. ■

Note that Lemma 21 cannot be generally improved. It is possible for $\mathbf{Y}$ to be y-saturated in an extremal system $\langle S, \mathbf{Cn} \rangle$, but not to be maximal. For a simple example, let $S = \{a, c\}$ and $\mathbf{Cn}(Y) = Y$ for every $Y \subseteq S$ except for $\{c\}$. Then the empty set $\emptyset$ is a-saturated but not maximal. On the other hand, we do have the following result.

Lemma 22 Let $\langle S, \mathbf{Cn} \rangle$ be any extremal system. If $\mathbf{Y}$ is y-saturated in $\langle S, \mathbf{Cn} \rangle$, then either $\mathbf{Y}$ is maximal or $\mathbf{Y} \cup \{y\}$ is maximal.

Proof: If $x \neq y$ and $x \notin \mathbf{Y}$, then $y \in \mathbf{Cn}(\mathbf{Y} \cup \{x\})$. Since $y \notin \mathbf{Y}$ and $\mathbf{Cn}$ is extremal, it follows that $\mathbf{Cn}(\mathbf{Y} \cup \{x\}) = S$. If also $\mathbf{Cn}(\mathbf{Y} \cup \{y\}) = S$, then $\mathbf{Y}$ is maximal. If not, then $\mathbf{Cn}(\mathbf{Y} \cup \{y\} \cup \{x\}) = S$ whenever $x \notin \mathbf{Y} \cup \{y\}$. In other words, $\mathbf{Y} \cup \{y\}$ is maximal. ■

Corollary Let $\langle S, \mathbf{Cn} \rangle$ be any extremal system. If $\mathbf{Y}$ is y-saturated in $\langle S, \mathbf{Cn} \rangle$, and $\mathbf{Cn}(\{y\}) = S$, then $\mathbf{Y}$ is maximal.

Proof: Immediate. ■

Lemma 23 Let $\langle S, \mathbf{Cn} \rangle$ be any deductive system satisfying (0). For each $Y \subseteq U \subseteq S$ we put $\mathbf{C}^U(Y) = \mathbf{Cn}(Y) \cap U$. Then $\langle U, \mathbf{C}^U \rangle$ is a deductive system.

Proof: If $Y \subseteq U$ then $Y \subseteq \mathbf{Cn}(Y) \cap U$ by (0a) applied to $\mathbf{Cn}$. Hence (0a) holds for $\mathbf{C}^U$. Since $\mathbf{Cn}(Y) \cap U \subseteq \mathbf{Cn}(Y)$, it follows by ($0c_0$) applied to $\mathbf{Cn}$ that $\mathbf{Cn}(\mathbf{Cn}(Y) \cap U) \subseteq \mathbf{Cn}(\mathbf{Cn}(Y)) = \mathbf{Cn}(Y)$. That implies that for $Y \subseteq U$, we have $\mathbf{C}^U(\mathbf{C}^U(Y)) = \mathbf{Cn}(\mathbf{Cn}(Y) \cap U) \cap U \subseteq \mathbf{Cn}(Y) \cap U = \mathbf{C}^U(Y)$, which is (0b). Moreover, if $X \subseteq Z \subseteq U$ then $\mathbf{C}^U(X) \subseteq \mathbf{C}^U(Z)$, which is ($0c_0$).

Now suppose that $Y \subseteq U$, and that $y \in \mathbf{C}^U(Y) = \mathbf{Cn}(Y) \cap U$. Since $y \in \mathbf{Cn}(Y)$, there is a finite subset X of Y for which $y \in \mathbf{Cn}(X)$ (by ($0c_1$) applied to $\mathbf{Cn}$); and so $y \in \mathbf{C}^U(X)$ where X is a finite subset of Y. That is, ($0c_1$) holds for $\mathbf{C}^U$. ■

Corollary If $\langle S, \mathbf{Cn} \rangle$ is an extremal system then so is $\langle U, \mathbf{C}^U \rangle$.

Proof: Take any $Y \subseteq S$. If $\mathbf{Cn}(Y) = Y$ then $\mathbf{C}^U(Y) = Y \cap U = Y$, and if $\mathbf{Cn}(Y) = S$ then $\mathbf{C}^U(Y) = S \cap U = U$. ■

Note that although the Lemma holds for every $U \subseteq S$, the pair $\langle S, \mathbf{C}^U \rangle$ is a deductive system only if $U = S$. Note also that the converse of the corollary is not generally valid.

We conclude with some generalizations of Definition 7, and some related results (whose proofs are omitted). The first variant is rather obvious. Let $\mathbf{Cn}$ be a (finitary) consequence operation and X a subset of S. For each X the consequence operation $\mathbf{Cn}^X$ is defined as follows.

(17)
$$\mathbf{Cn}^X(Y) =_{\mathrm{Df}} \begin{cases} Y & \text{if } Y \nvdash X \\ S & \text{otherwise.} \end{cases}$$

According to this definition, $\mathbf{Cn}^S = \mathbf{Cn}_{\min}$ and $\mathbf{Cn}^{\emptyset} = \mathbf{Cn}_{\max}$. We can also generalize Lemma 4 above.

LEMMA 24 For each finite X the operation $\mathbf{Cn}^X$ defined in (17) is an extremal consequence operation.

PROOF: Omitted. ■

It is straightforward to show that if X is infinite the operation $\mathbf{Cn}^X$ may fail to satisfy finitariness ($0c_1$).

We may write $\mathbf{Cn}^{zx}$ (and similar expressions) as abbreviations for $(\mathbf{Cn}^z)^x$ (and similar expressions). In other words

(18)
$$\mathbf{Cn}^{zx}(Y) =_{\mathrm{Df}} \begin{cases} Y & \text{if } x \notin \mathbf{Cn}^z(Y) \\ S & \text{otherwise.} \end{cases}$$

LEMMA 25 $\mathbf{Cn}^{zx}(Y) = \mathbf{Cn}^{xz}(Y)$ for all x, z, Y if and only if $\mathbf{Cn}$ is extremal.

PROOF: Omitted. ■

LEMMA 26 $\mathbf{Cn}^{yy}(Y) = \mathbf{Cn}^y(Y)$ for all y, Y.

PROOF: Omitted. ■

LEMMA 27 $\mathbf{Cn}^{(xz)y}(Y) = \mathbf{Cn}^{x(zy)}(Y)$ for all x, y, z, Y.

PROOF: Omitted. ■

We state finally a simple characterization of extremal consequence operations. For any function $\psi(Y)$, define $\mathbf{c}^\psi$ as follows.

(19)
$$\mathbf{c}^\psi(Y) =_{\mathrm{Df}} \begin{cases} Y & \text{if } \psi(Y) = \emptyset \\ S & \text{otherwise.} \end{cases}$$

THEOREM 28 The operation $\mathbf{Cn}$: $\wp(S) \mapsto \wp(S)$ is an extremal consequence operation if and only if $\mathbf{Cn} = \mathbf{c}^\psi$ for some function ψ that satisfies (0c); that is to say,

(20)
$$\psi(X) = \bigcup\{\psi(Z) \mid Z \subseteq X \text{and } |Z| < \aleph_0\}.$$

PROOF: Omitted. ■

The function $\mathbf{Cn}^y$ defined from $\mathbf{Cn}$ in (7) is obtained by setting $\psi(Y) = \mathbf{Cn}(Y) \cap \{y\}$.

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