Lattice-valued Probability

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Abstract

A theory of probability is outlined that permits the values of the probability function to lie in any Brouwerian algebra.

0 Introduction

The axioms $\mathbb{K}$ for the elementary (numerical) theory of probability introduced by Kolmogorov (1933), Chapter I, § 1, can be written:

$$(0) \qquad 0 \leq \mathfrak{p}(x) \leq \mathfrak{p}(1) = 1$$

$$(1) \qquad x \cdot z = 0 \Longrightarrow \mathfrak{p}(x+z) = \mathfrak{p}(x) + \mathfrak{p}(z),$$

where $x, z, \ldots$ belong to some Boolean algebra $\mathfrak{A}$ with zero 0, unit 1, meet $\cdot$, and join $+$, and $\mathfrak{p}$ is a real-valued function on $\mathfrak{A}$; that is, $\mathfrak{p} : \mathfrak{A} \mapsto [0,1]$. The duplicate use of $0, 1$, and $+$ (as operations on both $\mathfrak{A}$ and $\Re$) is deliberate. An alternative self-dual set $\mathbb{K}^\star$ of axioms is

$$(2) \qquad 0 = \mathfrak{p}(0) \leq \mathfrak{p}(x) \leq \mathfrak{p}(1) = 1$$

$$(3) \qquad \mathfrak{p}(x \cdot z) + \mathfrak{p}(x+z) = \mathfrak{p}(x) + \mathfrak{p}(z).$$

Note that, when (3) replaces (1), it is necessary to state, as (2) does, that $\mathfrak{p}(0) = 0$, so as to exclude the function $\mathfrak{p}$ whose range is $\{1\}$. A function $\mathfrak{p}$ whose range is $\{0,1\}$ can, however, provide an acceptable interpretation of the theory $\mathbb{K}$, however little it merits the name of probability.

The following lemma is standard.

Lemma 0: $\mathbb{K}$ is equivalent to $\mathbb{K}^\star$, and it follows from each that

$$(4) \qquad x \leq z \Longrightarrow \mathfrak{p}(x) \leq \mathfrak{p}(z),$$

where $\leq$ stands for the ordering on the Boolean algebra $\mathfrak{A}$ (as well as the ordering on $\Re$).

Sections 0 and 1 of this paper were originally written during the 1980s, probably in 1984 or in 1985. Although sections 2 and 3 are new, the message of this rewritten and digitally remastered version is substantially unchanged.

Proof: To establish equivalence, it suffices to prove (3) from (1). If $\sim z$ is the Boolean complement of the element z, then both $x + \sim x \cdot z$, which is identical with $x + z$, and $x \cdot z + \sim x \cdot z$, which is identical with z, are sums of disjoint elements, and so

$$\mathfrak{p}(x \cdot z) + \mathfrak{p}(x + z) = \mathfrak{p}(x \cdot z) + \mathfrak{p}(x) + \mathfrak{p}(\sim x \cdot z) \qquad 1 \tag{5}$$

$$= \mathfrak{p}(x) + \mathfrak{p}(x \cdot z) + \mathfrak{p}(\sim x \cdot z) \qquad 5 \tag{6}$$

$$= \mathfrak{p}(x) + \mathfrak{p}(z) \qquad 1, 6 \tag{7}$$

$$\mathfrak{p}(x) \leq \mathfrak{p}(x + z) \qquad 0, 5 \tag{8}$$

Since $x + z = z$ when $x \leq z$, this proves (4). ■

Each of the axioms of $\mathbb{K}$ and $\mathbb{K}^\star$ makes perfectly good sense if $0, 1, +$, and $\leq$ are given lattice-theoretical meanings instead of arithmetical ones, and if we replace the real-valued function $\mathfrak{p}(x)$ by a function $|x|$ (which may be called a *norm*, or *valuation*) that takes values in any Boolean algebra or any other lattice $\mathfrak{L}$. Unlike the theory of lattice-valued metrics, which has been studied for more than half a century (see Blumenthal & Menger 1970, or Miller 1977 for references), the theory of lattice-valued norms is, to my knowledge, almost non-existent; though it should be mentioned that in (1959), appendix $*$iv, p. 341 (Classics edition, p. 347), Popper used a simple example in order to give an independence proof for one of the postulates in his axiomatization of numerical probability. It is the aim of this paper to launch the theory of lattice-valued norms (or lattice-valued probability).

The task is not without some interest. The algebraic theory $\mathbb{L}$ whose axioms are

$$0 \leq |x| \leq |1| = 1, \tag{9}$$

$$x \cdot z = 0 \implies |x + z| = |x| + |z|, \tag{10}$$

despite being a straight transcription of the Kolmogorov axioms (0) and (1), is much weaker than its archetype. Axiom (9) says only that $|1| = 1$, and $\mathbb{L}$ itself is too weak to exclude the possibility that $|y| = 1$ for all $y \in \mathfrak{A}$. If (9) is strengthened in the natural way, by adding $|0| = 0$, then these axioms can be proved equivalent to the transcription $\mathbb{L}^\star$ of the alternative axioms,

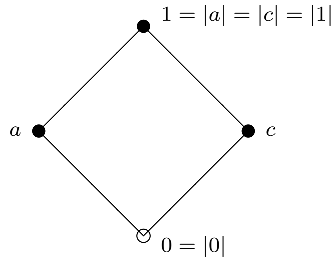

Figure 0: 4-element Boolean algebra

$$0 = |0| \leq |x| \leq |1| = 1 \tag{11}$$

$$|x \cdot z| + |x + z| = |x| + |z|, \tag{12}$$

exactly as Lemma 0 was proved. But (11), which of course says only that $|0| = 0$ and $|1| = 1$, and (12) are still together too weak to exclude the possibility that $|y| = 1$ for all non-zero $y \in \mathfrak{A}$. This may be seen by inspection of the 4-element Boolean algebra $\mathfrak{A}$ depicted in Figure 0, in which the norm $|\ | : \mathfrak{A} \mapsto \mathfrak{A}$ takes the value 1 for of each of the three solid elements (including the top element 1); $|0| = 0$..

A more specific problem is that although

(13) $$|y| + |{\sim}y| \;=\; 1$$

is a theorem of $\mathbb{L}^\star$, it is not an adequate replica of the familiar law of complementation

(14) $$\mathfrak{p}(y) + \mathfrak{p}({\sim}y) \;=\; 1.$$

For it follows from (14) that if the probability of an element y decreases, then the probability of ${\sim}y$ increases; that is, that

(15) $$\mathfrak{p}(x) < \mathfrak{p}(z) \implies \mathfrak{p}({\sim}z) < \mathfrak{p}({\sim}x).$$

Nothing like this can be derived from (13), or from $\mathbb{L}^\star$, even if $<$ is replaced by $\leq$. These axioms are together compatible with existence of elements x, z for which

(16) $$|x| < |z| \;\;\&\;\; |{\sim}x| < |{\sim}z|.$$

This is illustrated by the 16-element Boolean algebra depicted as a regular tesseract in Figure 1. The norm of each of the nine solid elements (including the top element 1) is 1; $|0| = 0$; the norm of the three elements marked $\star$ is some non-extreme element b of the algebra, and the norm of the three elements marked $\diamond$ is ${\sim}b$. The four elements marked $a, c, {\sim}a, {\sim}c$ provide an example satisfying (16): the elements a and ${\sim}a$ have norms b and ${\sim}b$ respectively, while each of c and ${\sim}c$ has norm 1, with the result that $|a| < |c|$ and $|{\sim}a| < |{\sim}c|$.

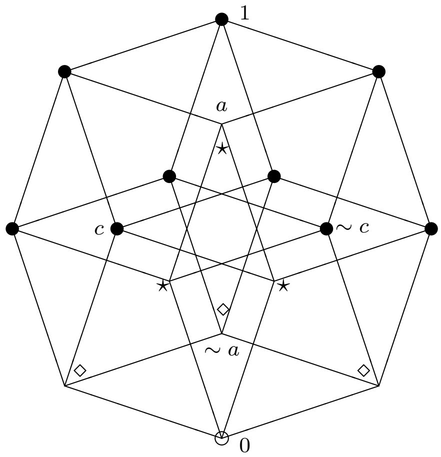

Figure 1: 16-element Boolean algebra

It is immediate that (11) holds. To prove (12), note that there are four kinds of lattice quadrangle with fringe elements x and z, bottom element $x \cdot z$, and top element $x + z$.

(i) Degenerate quadrangles, in which $x \leq z$ or $z \leq x$. For these quadrangles (12) holds trivially.

(ii) Non-degenerate quadrangles for which $|x + z| \neq 1$. There are only two. Since $|0| + |b| = |b| + |b|$, the quadrangle whose top element is a satisfies (12). The same holds when a and b are replaced by ${\sim}a$ and ${\sim}b$.

(iii) Non-degenerate quadrangles for which $|x + z| = 1$, and either $|x| = 1$ or $|z| = 1$. For these quadrangles (12) holds trivially.

(iv) Non-degenerate quadrangles for which $|x + z| = 1$, and $|x| \neq 1 \neq |z|$. Then one of x and z is a $\star$ element, and the other a $\diamond$ element (since those quadrangles in which both x and z are $\star$ elements, or both $\diamond$ elements, have been dealt with in clause (ii)). Since $b + {\sim}b = 1$, (12) once more holds.

1 Resolution

We obtain something much more recognizable if we first rewrite (1) and (3) as

(17) $$x \cdot z = 0 \implies \mathfrak{p}(x+z) - \mathfrak{p}(x) \;=\; \mathfrak{p}(z),$$

(18) $$\mathfrak{p}(x+z) - \mathfrak{p}(x) \;=\; \mathfrak{p}(z) - \mathfrak{p}(x \cdot z),$$

in which the binary arithmetical operation of subtraction appears. (Note that the singulary operation *minus*, which is also represented by $-$, has not been used above.) Now there is a well known method of specifying an operation of lattice subtraction $-$ that to some extent behaves like an inverse of addition; and though (like complementation $\sim$) it is not defined in every lattice, it is defined in every Boolean algebra. In general the remainder $z - x$ may be defined as the least element y for which $z \leq x + y$; that is to say,

(19) $$z - x \leq y \;\leftrightarrow\; z \leq x + y.$$

Bounded lattices that contain a remainder operation that satisfies (19) constitute the variety of *Brouwerian algebras*; Boolean algebras constitute a proper subvariety. In a Brouwerian algebra, the *authocomplement* $\sim y$ is identified with $1 - y$, and it is immediate that

(20) $$1 \;\leq\; y + \sim y$$ 19

(21) $$y + \sim y \;=\; 1,$$ 20

the *tertium non datur*. Note that the law of non-contradiction $y \cdot \sim y = 0$ is not generally true.

Some care must be taken, since even in Boolean algebras the remainder operation $-$ does not behave exactly like its arithmetical counterpart. For example, $z - x$ is quite different from $z + \sim x$ (since $z - z = 0$ but $z + \sim z = 1$), and there is no equivalence between $y - z = x$ and $y - x = z$ (since $0 - z = 0$ for all z, but $0 - 0 = z$ only if $z = 0$). As a consequence, although the inequalities in (19) are equivalent, there is no implication either way between the equations $z - x = y$ and $z = x + y$. Inequalities involving the remainder are duals of intuitionistic implications involving the conditional. Table 0 lists the main inequalities that we shall call on, together with their duals ($\top$ is an intuitionistic tautology, and $\bot$ a contradiction).

(22)	$\sim 1 = 0$	$\bot \dashv\vdash \neg\top$
(23)	$y - 0 = y$	$q \dashv\vdash \top \to q$
(24)	$y - y = 0$	$\top \dashv\vdash q \to q$
(25)	$y - 1 = 0$	$\top \dashv\vdash \bot \to q$
(26)	$x - z \leq x$	$p \vdash r \to p$
(27)	$x \cdot z + (x - z) = x$	$p \dashv\vdash (p \vee r) \wedge (r \to p)$
(28)	$(x - y) + z \leq x + z$	$p \wedge r \vdash (q \to p) \wedge r$
(29)	$x + (z - x) = x + z$	$p \wedge r \dashv\vdash p \wedge (p \to r)$
(30)	$(x + z) - z = x - z$	$p \to r \dashv\vdash p \to (p \wedge r)$

Table 0: *Inequalities in Brouwerian algebra and their intuitionistic duals*

Whenever the values of the function $|\;| : \mathfrak{A} \mapsto \mathfrak{B}$ lie in a Brouwerian algebra $\mathfrak{B}$, the lattice transcriptions of (17) and (18),

(31) $$x \cdot z = 0 \implies |x+z| - |x| = |z|,$$

(32) $$|x+z| - |x| \;=\; |z| - |x \cdot z|$$

are meaningful, whether or not $\mathfrak{B}$ is identical with the domain $\mathfrak{A}$ of $|\ |$. What is interesting is that the theory $\mathbb{M}$, which may be axiomatized either by (9) and (31), or by (11) and (32), is substantially stronger than $\mathbb{L}$.

Theorem 1: Let $|\ |$ be a function from a Boolean algebra $\mathfrak{A}$ into a Brouwerian algebra $\mathfrak{B}$. Axioms (9) and (31) are together equivalent to axioms (11) and (32). Only if $\mathfrak{A}$ is the two-element Boolean algebra can $|y| = 1$ for every non-zero element y of $\mathfrak{A}$.

Proof: Thanks to the identity (23), (31) follows from (11) and (32). We therefore begin with (9) and (31). Equation (35) repeats equation (13).

(33)	$y \cdot \sim y = 0$	$\Longrightarrow$	$1 - \|y\| = \|\sim y\|$	$9, 31, 21$
(34)	$\|\sim y\|$	$=$	$\sim \|y\|$	33
(35)	$\|y\| + \|\sim y\|$	$=$	1	$34, 21$
(36)	$\sim\sim \|y\|$	$=$	$\|y\|$	34
(37)	$x \cdot (\sim x \cdot z) = 0$	$\Longrightarrow$	$\|x + z\| - \|x\| = \|\sim x \cdot z\|$	31
(38)	$(x \cdot z) \cdot (\sim x \cdot z) = 0$	$\Longrightarrow$	$\|z\| - \|x \cdot z\| = \|\sim x \cdot z\|$	31
(39)	$\|x + z\| - \|x\|$	$=$	$\|z\| - \|x \cdot z\|.$	$37, 38$

The Boolean law $y \cdot \sim y = 0$ is silently applied at lines (34) and (39). According to line (34), $|y|$ and $|\sim y|$ are neither both equal to 0 nor both equal to 1; it follows that unless $\mathfrak{A} = \{0, 1\}$, it contains a non-zero element y for which $|y| \neq 1$. Line (39) is the same as (32), and the proof is complete. ■

Theorem 2: Let $|\ |$ be a function from a Boolean algebra $\mathfrak{A}$ into a Brouwerian algebra $\mathfrak{B}$ that satisfies the theory $\mathbb{M}$. Then $|\ |$ is a monotone function (42), and the law $|x + z| = |x| + |z|$ of additivity holds generally.

Proof: Either (31) or (32) yields (40). The Boolean law $y \cdot \sim y = 0$ is applied at line (41).

(40)	$(x \cdot \sim z) \cdot (x \cdot z) = 0$	$\Longrightarrow$	$\|x\| - \|x \cdot \sim z\| = \|x \cdot z\|$	31
(41)	$\|x \cdot z\|$	$\leq$	$\|x\|$	$40, 26$
(42)	$z \leq x$	$\Longrightarrow$	$\|z\| \leq \|x\|$	41
(43)	$\|x\| + \|z\|$	$\leq$	$\|x + z\|$	42
(44)	$\|x + z\|$	$\leq$	$(\|z\| - \|x \cdot z\|) + \|x\|$	$19, 39$
(45)	$\|x + z\|$	$\leq$	$\|x\| + \|z\|$	$28, 44$
(46)	$\|x + z\|$	$=$	$\|x\| + \|z\|.$	$43, 45$

This concludes the proof. ■

Corollary: Equation (12) is a theorem of $\mathbb{M}$.

Proof: (12) follows immediately from (42) and (46). ■

Theorem 3: If the range (the set of values) of the function $|\ |$ is also a Boolean algebra, then $|\ |$ is a Boolean homomorphism.

Proof: Lines (34) and (46) provide the necessary information. ■

Corollary: When $\mathfrak{A}$ and $\mathfrak{B}$ are both Boolean algebras, every homomorphism $|\ | : \mathfrak{A} \mapsto \mathfrak{B}$ satisfies $\mathbb{M}$.

Proof: This is obvious for (11). The proof of (32) is an easy exercise in Boolean algebra. ■

Every bounded chain is a Brouwerian algebra, since $z - x$ is identical with z when $x < z$, and is identical with 0 otherwise. The remainder $1 - x$, in particular, takes only the values 0 and 1.

Lemma 4: If the function $|\ | : \mathfrak{A} \mapsto \mathfrak{B}$ satisfies $\mathbb{M}$, and $\mathfrak{B}$ is a chain, then $|\ |$ takes only the values 0 and 1.

Proof: Let x be an element of the Boolean algebra $\mathfrak{A}$. By (32),

$$|x| = |x \cdot \sim x| + (|x + \sim x| - |\sim x|);$$

and so by (11), $|x| = 0 + (1 - |\sim x|)$. If $|\sim x| = 1$ then $|x| = 0$, and if $|\sim x| \neq 1$, then $|x| = 1$, and hence $|\sim x| = 0$ in accordance with (34). ■

An important example is the closed unit interval of the real line, with the standard ordering, but the theory $\mathbb{M}$ does not include as a special case the theory $\mathbb{K}$ (in which $\mathfrak{p}$ may take any number of distinct values). The arithmetical operation + cannot be understood as a join operation (since, for example, $x \leq z$ does not imply that $x + z = z$). Equation (36) states that every term of the form $|y|$ satisfies the law of double negation, a law that does not hold generally in Brouwerian algebras (any more than it holds in intuitionistic logic). But this does not imply that the range of $|\ |$ is a Boolean subalgebra of $\mathfrak{B}$. In particular, it is not a theorem of $\mathbb{M}$ that $|x \cdot z| = |x| \cdot |z|$ (which would hold if $|\ |$ were a homomorphism). This is illustrated in Figure 2, in which the norm $|\ |$ defined on the 4-element Boolean algebra $\mathfrak{A}$ on the left takes values in the 5-element Brouwerian algebra $\mathfrak{B}$ on the right. Using (31), it is easily shown that the theory $\mathbb{M}$ holds. But $|a \cdot c| = |0| = 0 \neq \beta = |a| \cdot |c|$.

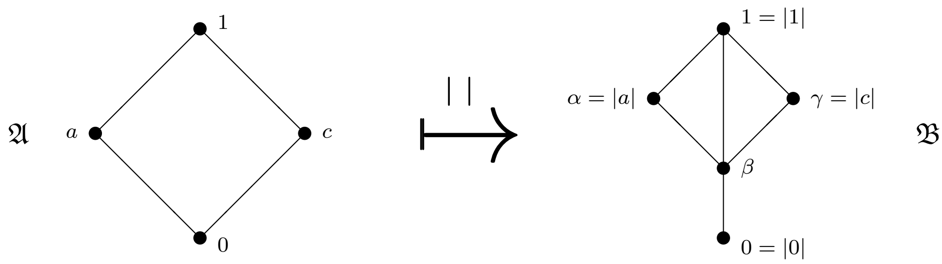

Figure 2: Failure of $|x \cdot z| = |x| \cdot |z|$ in $\mathbb{M}$

2 Alternatives

The arithmetical equation (3) can be reorganized in many equivalent ways by moving terms from side to side. These equivalences are not preserved in the algebraic transcriptions, as we have seen: (32) is much richer in consequences than (12) is. The question arises whether similar results are obtainable from algebraic transcriptions of other reorganizations of (3). In making these transformations it must be remembered that $x - z$ is not in general equivalent to $x + u$ for any u, and certainly not to $x + \sim z$. The equation $\mathfrak{p}(x \cdot z) - (\mathfrak{p}(x) + \mathfrak{p}(z)) = -\mathfrak{p}(x + z)$, for example, cannot sensibly be transformed into $|x \cdot z| - (|x| + |z|) = \sim |x + z|$.

Equations (32) and equations (47)–(58) below essentially exhaust the cases where each side of the equation contains at least one of the four terms (trivial reletterings are excluded). For the enumeration to be complete, each equation (j) $y = w$ has to be supplemented by two further equations, (jl) $y - w = 0$, which is equivalent to $y \leq w$, and (jr) $w - y = 0$, which is equivalent to $w \leq y$. The equation (j) is obviously equivalent to the conjunction of (jl) and (jr).

$$\begin{array}{lrcl}
(32) & |x+z| - |x| & = & |z| - |x \cdot z| \\
(47) & |x \cdot z| - |x| & = & |z| - |x+z| \\
(48) & |x+z| & = & |x| - (|x \cdot z| - |z|) \\
(49) & |x+z| & = & |x| + (|z| - |x \cdot z|) \\
(50) & |x+z| & = & (|x| + |z|) - |x \cdot z| \\
(51) & |x \cdot z| & = & |x| - (|x+z| - |z|) \\
(52) & |x \cdot z| & = & |x| + (|z| - |x+z|) \\
(53) & |x \cdot z| & = & (|x| + |z|) - |x+z| \\
(54) & |x| & = & (|x \cdot z| + |x+z|) - |z| \\
(55) & |x| & = & |x+z| - (|z| - |x \cdot z|) \\
(56) & |x| & = & |x \cdot z| - (|z| - |x+z|) \\
(57) & |x| & = & |x+z| + (|x \cdot z| - |z|) \\
(58) & |x| & = & |x \cdot z| + (|x+z| - |z|)
\end{array}$$

To these we may add

$$(12) \qquad |x \cdot z| + |x+z| \quad = \quad |x| + |z|,$$

though we have already established its unacceptable weakness.

When (j) is one of the 42 equations enumerated, the name $\mathbb{M}^{\mathrm{j}}$ will be used to denote the theory axiomatized by (11) and (j). The theory $\mathbb{M}^{32}$ and the theory $\mathbb{M}$ are therefore the same theory. We begin by showing that the set of 42 equations just enumerated contains some redundancies.

Lemma 5: In every Brouwerian algebra, each equation in the following list is equivalent to those with which it is grouped: {(32r), (51r), (58l)}; {(47r), (48r), (57l)}; {(47l), (52l), (56r)}; {(32l), (49l), (55r)}; {(12l), (54r)}; {(12r), (50r), (53r)}.

Proof: Write the equations as inequalities, and apply (19), interchanging x and z where necessary. The slight imbalance in the list (five sets of three, and one of two) results from the identity of the two versions of (54r) that emerge from symmetrical applications of (19) to (12l). ■

Lemma 6: The theory $\mathbb{M}^{58}$ is equivalent to $\mathbb{M}$.

Proof: It is evident that, given (11), the principal axiom (32) of $\mathbb{M}$ follows from (58). The following lines show how (58) may be derived within $\mathbb{M}$.

$$\begin{array}{lrcll}
(59) & |x+z| - |z| & \leq & |x| & 19, 46 \\
(60) & |x \cdot z| + (|x+z| - |z|) & \leq & |x \cdot z| + |x| \;=\; |x| & 59, 42 \\
(61) & |x| & \leq & |x \cdot z| + (|x+z| - |z|) & 32, 19 \\
(58) & |x| & = & |x \cdot z| + (|x+z| - |z|). & 60, 61
\end{array}$$

It must be remembered that (46) and (42) have been derived from (32) in Theorem 2. ■

Corollary The theory $\mathbb{M}^{55}$ is consistent and implies $\mathbb{M}$.

Proof: If $|\;|$ is the identity function on the Boolean algebra $\mathfrak{A}$, then $\mathbb{M}^{55}$ reduces to a simple Boolean identity. It is evident too that, given (11) and (23), the principal axiom (32) of $\mathbb{M}$ follows from (55). A proof is wanting, but $\mathbb{M}^{55}$ appears to be strictly stronger than $\mathbb{M}$. ■

We now eliminate from 30 distinct equations that remain those that contradict (11) and are therefore of little further interest. None of them is the equivalent of any other equation in the original list.

Lemma 7: Each of the theories $\mathbb{M}^{48l}$, $\mathbb{M}^{50l}$, $\mathbb{M}^{52r}$, $\mathbb{M}^{53l}$, $\mathbb{M}^{54l}$, $\mathbb{M}^{56l}$, and $\mathbb{M}^{57r}$, is inconsistent.

Proof: Put $x = 0$ and $z = 1$ in ($48l$) and in ($57r$). Put $x = 1$ and $z = 0$ in ($52r$) and in ($56l$). Put $x = 1 = z$ in ($50l$), in ($53l$), and in ($54l$). In each case the identities $|0| = 0$ and $|1| = 1$ yield the inequality $1 \leq 0$. ■

Corollary: Each of the theories $\mathbb{M}^{48}$, $\mathbb{M}^{50}$, $\mathbb{M}^{52}$, $\mathbb{M}^{53}$, $\mathbb{M}^{54}$, $\mathbb{M}^{56}$, and $\mathbb{M}^{57}$ contradicts $\mathbb{M}^{11}$.

Proof: Immediate. ■

This leaves 16 distinct equations. We now eliminate some that are too weak to be useful.

Lemma 8: None of the theories $\mathbb{M}^{12}$, $\mathbb{M}^{47}$, $\mathbb{M}^{49}$, $\mathbb{M}^{51l}$, $\mathbb{M}^{53r}$, $\mathbb{M}^{54r}$, $\mathbb{M}^{55r}$, $\mathbb{M}^{56r}$, $\mathbb{M}^{57l}$, and $\mathbb{M}^{58r}$, excludes the norm depicted in Figure 0 on p. 2 above.

Proof: It is necessary to check only that the equation holds when $x \leq z$, when $z \leq x$, and for a and c (in either order). These proofs are all straightforward. A proof in the same style is spelt out in Lemma 9 below. ■

Corollary: None of the theories $\mathbb{M}^{12l}$, $\mathbb{M}^{12r}$, $\mathbb{M}^{32l}$, $\mathbb{M}^{47l}$, $\mathbb{M}^{47r}$, $\mathbb{M}^{48r}$, $\mathbb{M}^{49l}$, $\mathbb{M}^{49r}$, $\mathbb{M}^{50r}$, and $\mathbb{M}^{52l}$ excludes the norm depicted in Figure 0.

Proof: For ($12l$) and ($12r$), ($47l$), ($47r$), ($49l$), and ($49r$), the proof is immediate, since they are consequences of equations already shown to be too weak. According to Lemma 5, the others (and some of those just mentioned) are equivalent to equations that are too weak. ■

We now eliminate four theories that show weaknesses that stop short of triviality.

Lemma 9: The theory $\mathbb{M}^{51}$ implies that $|\sim y| \leq \sim |y|$ and also the double negation law (36), but does not imply the identity (34).

Proof: Substitute $\sim y$ for x and y for z in (51), and then use the Boolean law $y \cdot \sim y = 0$ and (11). The result is $0 = |\sim y| - (1 - |y|)$, from which, by application of (19), the desired inequality follows. Writing 1 for x and y for z in (51) establishes that $|y| = \sim\sim |y|$, which is (36).

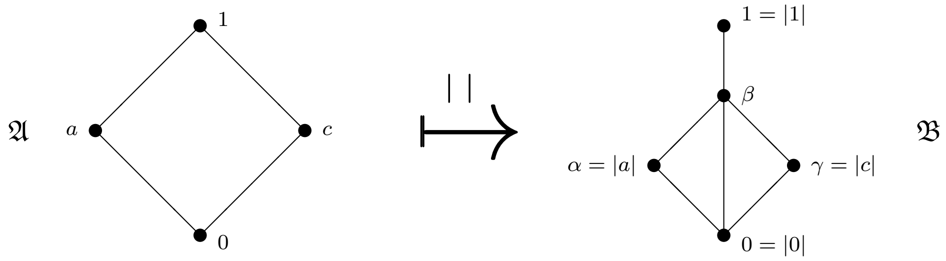

Figure 3: Failure of $|\sim y| = \sim |y|$ in $\mathbb{M}^{51}$

Figure 3 provides a counterexample to (34). That $\mathbb{M}^{51}$ holds may be checked in four stages. The proof of (11) requires no work. If $x = a$ and $z = c$ in $\mathfrak{A}$, then (51) reduces to

$$|0| = |a| - (|1| - |c|) = \alpha - (1 - \gamma) = \alpha - 1 = 0;$$

and likewise if $x = c$ and $z = a$ in $\mathfrak{A}$. If $x \leq z$ in $\mathfrak{A}$ then (51) reduces to $|x| = |x| - (|z| - |z|)$, which is trivial. If $z < x$ in $\mathfrak{A}$ then (51) reduces to $|z| = |x| - (|x| - |z|)$. This holds when $x = 1$, by the double negation law (36) just proved. It holds also when $x < 1$, since in that case $z = 0$.

It is immediate that $\gamma = |c| = |\sim a| \neq \sim|a| = \sim\alpha = 1$, and that (34) fails. ■

Corollary 0: The theories $\mathbb{M}^{32r}$, $\mathbb{M}^{51r}$, and $\mathbb{M}^{58l}$ do not imply the identity (34).

Proof: Use the lemma and Lemma 5. ■

Corollary 1: The theories $\mathbb{M}^{32l}$, $\mathbb{M}^{49l}$, and $\mathbb{M}^{55l}$ do not imply the identity (34).

Proof: The proof for $\mathbb{M}^{55l}$ is similar to that just given. Lemma 5 completes the proof. ■

The theories mentioned in Corollary 0 to Lemma 9 also fail to enforce the additivity law (46).

Lemma 10: The theories $\mathbb{M}^{32r}$, $\mathbb{M}^{51r}$ and $\mathbb{M}^{58l}$ do not imply the identity (46).

Proof: It is easily checked that $\mathbb{M}^{32r}$ holds for the norm $|\ |$ displayed in Figure 4. But (46)

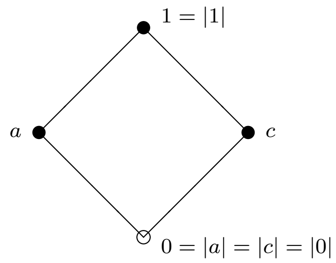

Figure 4: Failure of $|x + z| = |x| + |z|$ in $\mathbb{M}^{32r}$, $\mathbb{M}^{51r}$ and $\mathbb{M}^{58l}$

fails, since $|a| + |c| = 0 \neq 1 = |a + c|$. ■

These results are summarized in Table 1, where $\circ$ indicates that the equation is too weak to rule out the norm $|\ |$ for which $|y| = 1$ whenever $y \neq 0$ (Figure 0), and $\star$ indicates that the equation excludes this norm but is too weak to yield (34). Equations marked $\times$ contradict (11).

	32	47	48	49	50	51	52	53	54	55	56	57	58	12
$l\&r$	$\mathbb{M}$	$\circ$	$\times$	$\circ$	$\times$	$\star$	$\times$	$\times$	$\times$		$\times$	$\times$	$\mathbb{M}$	$\circ$
l	$\circ$	$\circ$	$\times$	$\circ$	$\times$	$\circ$	$\circ$	$\times$	$\times$	$\star$	$\times$	$\circ$	$\star$	$\circ$
r	$\star$	$\circ$	$\circ$	$\circ$	$\circ$	$\star$	$\times$	$\circ$	$\circ$	$\circ$	$\circ$	$\times$	$\circ$	$\circ$

Table 1: *Summary of Lemmas* 5–10

If we dismiss all those theories marked with a $\times$ or a $\circ$ or a $\star$, we are left with one alternative to the theory $\mathbb{M}$, namely $\mathbb{M}^{55}$, which has been shown in the corollary to Lemma 6 to be at least as strong as $\mathbb{M}$. Some light may be thrown on this theory, and on some of the others marked with a $\star$, by identifying $|\ |$ throughout with the identity function, and considering the duality between each of the 42 enumerated equations and a corresponding statement of logical implication. Each of the inequalities or equations indicted in Lemma 8, in particular, and accordingly each of those mentioned in its corollary, is dual to a valid implication or bi-implication of intuitionistic logic. Equation (58r), for example, corresponds to $p \vdash (r \rightarrow (p \wedge r)) \wedge (p \vee r)$; while (57$l$)

corresponds to $(r \rightarrow (p \vee r)) \wedge (p \wedge r) \vdash p$. In the same way, none of the inequalities or equations indicted in Lemma 7, and accordingly none of those mentioned in its corollary, corresponds to a valid implication of classical logic. Equation (52r), for example, corresponds to $p \vee r \vdash p \wedge ((p \wedge r) \rightarrow r)$, while equation (48$l$) corresponds to $(r \rightarrow (p \vee r)) \rightarrow p \vdash p \wedge r$. All this seems sensible. But some of the inequalities not indicted in Lemma 8 also correspond to valid intuitionistic implications; for example, (58l) corresponds to $(r \rightarrow (p \wedge r)) \wedge (p \vee r) \vdash p$, and (32$r$) and (51$r$), which are logically equivalent to it (Lemma 5), of course also correspond to valid formulas. These are three of the four inequalities for which Table 1 shows a star. There are just two inequalities that correspond to implications that are classically valid but intuitionistically invalid: (55l), which corresponds to $((p \vee r) \rightarrow r) \rightarrow (p \wedge r) \vdash p$, and (51$l$), which corresponds to $(r \rightarrow (p \wedge r)) \rightarrow p \vdash p \vee r$. They are not equivalent, and indeed the latter is too weak to exclude the norm depicted in Figure 0, but the former is not.

It is to be hoped that further investigation of the theory $\mathbb{M}^{55}$ will either establish its identity with $\mathbb{M}$ or will indicate in what way it is too strong (or, perhaps, in what way $\mathbb{M}$ is too weak).

3 Closing Remarks

A standard extension to the theory $\mathbb{K}$ incorporates also Kolmogorov's *axiom of continuity* (1933, Chapter II, § 1), which states that if $\mathbf{y} = \{y_j \mid j \in \mathcal{N}\}$ is a decreasing sequence of elements of $\mathfrak{A}$, that is, one for which $y_{j+1} \leq y_j$ for every $j \in \mathcal{N}$, then

(62) $$\prod \mathbf{y} = 0 \implies \lim_{n \Longrightarrow \infty} \mathfrak{p}(y_j) = 0,$$

where $\prod \mathbf{y}$ is the (infinite) meet of the elements in the sequence $\mathbf{y}$. This axiom (62) is equivalent to the *generalized addition theorem* (ibidem): if $\mathbf{y} \subset \mathfrak{A}$ is a countable collection of pairwise disjoint sets, then

(63) $$\mathfrak{p}(\sum \mathbf{y}) = \sum_{y \in \mathbf{y}} \mathfrak{p}(y),$$

provided that $\mathfrak{A}$ contains the infinite join $\sum \mathbf{y}$ of $\mathbf{y}$. These laws can be rendered lattice-theoretically in the obvious way, by writing $|\,|$ instead of $\mathfrak{p}$ in their consequents. It is clear that the same problem arises for

(64) $$\forall x \in \mathbf{y}\, \forall z \in \mathbf{y}\, (x \neq z \Longrightarrow x \cap z = 0) \implies |\sum \mathbf{y}| = \sum_{y \in \mathbf{y}} |y|,$$

as arose for (10), that it is powerless on its own to prevent $|y|$ from being equal to 1 for all $y \in \mathfrak{A}$. It remains to be seen whether the incorporation of the law (64) into the theory $\mathbb{M}$ yields a theory of much mathematical interest.

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