A Note on Small Models of Relative Probability

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Abstract

In the axiomatic theory of relative probability expounded in appendices $*$iv and $*$v of *The Logic of Scientific Discovery*, the relation $a \sim c =_{\mathrm{Df}} \forall b[p(a,b) = p(c,b)]$ of probabilistic indistinguishability on a set S is demonstrably a congruence, and the quotient $\mathbb{S} = S/{\sim}$ is demonstrably a Boolean algebra. The two-element algebra $\{\mathbf{0}, \mathbf{1}\}$ satisfies the axioms if and only if $\mathfrak{p}(\mathbf{1} \mid \mathbf{1})$, $\mathfrak{p}(\mathbf{1} \mid \mathbf{0})$, and $\mathfrak{p}(\mathbf{0} \mid \mathbf{0})$ are assigned the value 1, and $\mathfrak{p}(\mathbf{0}, \mathbf{1})$ is assigned the value 0 (where $\mathfrak{p}$ is the interpretation of the quotient $p/{\sim}$). The four-element models are almost as straightforwardly described. This note sketches a method of construction and authentication that can, in principle, be applied to larger algebras, and identifies all the eight-element models of Popper's system.

0 Finite models of the theory of probability

An interest in the axiomatization of a deductive theory virtually enjoins an interest also in the models of the theory and of its subtheories, since these enable proofs of the consistency and independence of the axioms, and also proofs that specific sentences are not among the theory's theorems. From a practical, and also a pedagogic, point of view, finite models, are usually preferred when they are available. After a respectful nod towards the almost canonical axiomatization, given by Kolmogorov (1933), of the classical theory of probability, this paper investigates the existence and variety of small models of a stronger, and axiomatically much more engaging, alternative system, whose definitive expression was published by Karl Popper (1959).

Kolmogorov's axioms for the classical theory of probability are formulated in terms of a singular functor p defined on a Boolean algebra $\mathbb{S}$ (or some similar structure, such as a field of sets). In every finite model of the axioms, which will be stated in § 3 below, despite their being awfully familiar, $\mathbb{S}$ has cardinality 2^k, for some $k > 0$, and contains k *atoms*; that is, elements that, in the lattice ordering $\preceq$, *cover* the zero element $\perp$ of $\mathbb{S}$ (in other words, are immediately above $\perp$, without intervening elements). Every element of $\mathbb{S}$ is the join (or sum) of all the atoms below it in the ordering. If the real-valued function $\mathfrak{p}$ is appointed as the interpretation in the model of the functor p, then $\mathfrak{p}(\mathsf{a})$ is the sum of the values that it takes on the atoms

I am indebted to Philippe Mongin for impelling me to undertake this exercise. Some of the notation used differs from Popper's; in particular, I use the prime $'$ for complementation or negation, where Popper used the macron $\bar{\ }$; and, to separate arguments in expressions of relative probability, the pipe $|$, where Popper used a simple comma.

properly inferior to a. When $k = 1$, the unique atom is the unit element $\top$, which is the sum of the (empty) set of atoms properly beneath $\top$. The theory is usually extended by the partial definition, whenever $\mathsf{p}(\mathsf{c}) > 0$, of $\mathsf{p}(\mathsf{a} \mid \mathsf{c})$ as $\mathsf{p}(\mathsf{ac})/\mathsf{p}(\mathsf{c})$, where ac is the Boolean meet of a and c; and the singulary $\mathsf{p}(\mathsf{a})$ may then be identified with the binary $\mathsf{p}(\mathsf{a} \mid \top)$, where $\top$ is the unit element of the algebra $\mathbb{S}$. The *conditional distribution* of $\mathfrak{p}$, that is, the set of values of $\mathfrak{p}(\mathsf{a} \mid \mathsf{c})$, is determined by the (total) *marginal distribution* of $\mathfrak{p}$, the set of values of $\mathfrak{p}(\mathsf{a})$, which is itself determined by the values that $\mathfrak{p}$ takes on the atoms of $\mathbb{S}$. The conditional distribution $\mathfrak{p}(\mathsf{a} \mid \mathsf{c})$ can never be a total function, but if the marginal distribution $\mathfrak{p}$ is *strictly positive*, that is to say, if it takes a positive value on every atom of $\mathbb{S}$, then the only lacuna is where $\mathfrak{p}(\mathsf{a} \mid \bot)$ should be.

All this is well known. It is evident that there is nothing very fascinating to be said about the finite models of the classical theory of probability. Matters change, however, when we move to the theory of relative (or conditional) probability $\mathfrak{p}(\mathsf{a} \mid \mathsf{c})$ that was developed by Karl Popper in the 1950s. In this theory the distribution $\mathfrak{p}(\mathsf{a} \mid \mathsf{c})$ is a total function, so that even $\mathfrak{p}(\mathsf{a} \mid \bot)$ has a definite value (it is 1 for all a); and unless the marginal distribution $\mathfrak{p}(\mathsf{a})$ is strictly positive, there may be other values of $\mathfrak{p}(\mathsf{a} \mid \mathsf{c})$ that are not defined by the ratio $\mathfrak{p}(\mathsf{ac})/\mathfrak{p}(\mathsf{c})$ and are not determined by the marginal distribution. For Boolean algebras of cardinality 2 and 4, indeed, this cannot happen, but for larger algebras it can happen, and it does. It is the purpose of this paper to explore the range of possibilities admitted by the small finite models of Popper's theory.

The original form of Popper's axiomatization is sketched in § 1, and then reformulated so as to make it more easily comparable with the classical theory. In § 2 we get out of the way the two-element Boolean algebra $\{\mathbf{o}, \mathbf{1}\}$, which is anomalously simple in that $\mathbf{1}$ is its only atom, but ominously portends how much work may be needed, even in a small model, to check the truth of Popper's axioms, especially the general multiplication law B2. In § 3 we characterize the four-element models of Popper's theory, which extend in a pedestrian way the four-element models of the classical theory. We also develop a technique for checking that a purported model of the theory is indeed a model. Thereafter, in § 4, we consider models based on Boolean algebras of cardinality 8, which present some interesting features absent from smaller models. In § 5, in conclusion, we make a couple of applications, and venture several suggestions for further inquiry.

1 Popper's axiom systems for relative probability

Let S be a finite or denumerable set endowed with operations represented by concatenation and prime $'$: that is to say, for each a, b, c in S, ac and b' are in S. A set of axioms for a real-valued function $p(a, c)$ on S was presented, with an eye to an adequate axiomatization of the theory of probability, in appendices ∗iv and ∗v of *The Logic of Scientific Discovery* (Popper 1959) and, in an interestingly (but not drastically) different way, the 2nd and later editions of *Logik der Forschung* (Popper 1966–2008). Popper proved that the relation $a \sim c =_{\text{Df}} \forall b[p(a, b) = p(c, b)]$ of (probabilistic) *indistinguishability* is not only an equivalence relation on S but also a congruence, meaning that if $a \sim c$ then c may be substituted for a in all contexts *salva veritate*. A standard algebraic move then permits the replacement of each element of S by its equivalence class under $\sim$, yielding a set that we shall call $\mathbb{S}$, which contains elements $\mathsf{a}, \mathsf{b}, \mathsf{c}, \ldots$. At the same time, the operators of concatenation and prime, and the functor p, may be replaced by the resulting quotient operators, for which we shall use concatenation, prime $'$, and p, with little fear of ambiguity. For example, each b in $\mathbb{S}$ is an equivalence class of elements of S, and b' may be identified with the equivalence class of b' where b is any element of b. In the same way, when a

and c belong to $\mathbb{S}$, $\mathsf{p}(\mathsf{a} \mid \mathsf{c})$ is taken to equal $\mathsf{p}(a, c)$, where a and c belong to a and c respectively.

The axiom systems given by Popper in (1959) and (1966) are *autonomous* in the sense that no further assumptions are made involving the concatenation and prime operators. It is demonstrable, nonetheless, that the quotient $\mathbb{S} = S/\sim$ (the set of equivalence classes under $\sim$) has a rich algebraic structure. If the operator $\mathsf{a} \vee \mathsf{c}$ is defined from concatenation and prime by $(\mathsf{a}'\mathsf{c}')'$ (this is recognizably one of the De Morgan laws), then ac and $\mathsf{a} \vee \mathsf{c}$ separately obey the lattice laws of *idempotence*, *commutation*, and *association*, and together they obey the *distributive* law $\mathsf{b}(\mathsf{a} \vee \mathsf{c}) = \mathsf{ba} \vee \mathsf{bc}$. The relation $\mathsf{a} \preceq \mathsf{c}$, defined by $\forall \mathsf{b}[\mathsf{p}(\mathsf{a} \mid \mathsf{b}) \leq \mathsf{p}(\mathsf{c} \mid \mathsf{b})]$, is a lattice ordering on $\mathbb{S}$. We write $\mathsf{a} \prec \mathsf{c}$ when $\mathsf{a} \preceq \mathsf{c}$ and $\mathsf{a} \neq \mathsf{c}$. (Note that $\mathsf{a} \prec \mathsf{c}$ does not imply $\forall \mathsf{b}[\mathsf{p}(\mathsf{a} \mid \mathsf{b}) < \mathsf{p}(\mathsf{c} \mid \mathsf{b})]$.) For each a, c, the element ac is variously called the *meet*, *logical product*, or *conjunction* of a and c, and $\mathsf{a} \vee \mathsf{c}$ is variously called their *join*, *logical sum*, or *disjunction*. The lattice has a least (or *logically false*) element $\bot$ (which may be defined as bb', for any b in $\mathbb{S}$), and a greatest (or *logically true*) element $\top$ (which may be defined as $\mathsf{b} \vee \mathsf{b}'$, for any b in $\mathbb{S}$). In a nutshell, the set $\mathbb{S}$ has the structure of a Boolean algebra. If S is a collection of sentences, then $\mathbb{S}$ may be thought of as a collection of propositions (or statements) in which the ordering $\preceq$ represents derivability.

Given the definition of the congruence relation $\sim$, the axiom system $\mathbb{B}$ to be considered in the rest of this paper is derivable from, and parallel to, Popper's system. Among the new axioms are two (A1, A2, A3) that mention neither meet (concatenation) nor complement (prime), three (B1, B2) that mention only meet, and one (C) that mentions only complement. In Popper's original system the axiom called A2, which asserts that if a, c in S are intersubstitutable in the first argument of p, they are intersubstitutable also in the second argument, was needed to clinch the proof that the relation $\sim$ is a congruence. In the new regime, what is called A2 is a definition in terms of $\mathfrak{p}$ of the relation of identity (which is what the relation $\sim$ becomes when S is replaced by the quotient $\mathbb{S} = S/{\sim}$). Required additionally for a fully correct axiomatization (but here conspicuously omitted) are some assumptions for the arithmetic of the real numbers, strong enough to yield proofs that they are linearly ordered, and that if $k^2 \leq k$ then $0 \leq k \leq 1$.

A1 $$\exists \mathsf{c} \exists \mathsf{d}[\mathsf{p}(\mathsf{a} \mid \mathsf{b}) \neq \mathsf{p}(\mathsf{c} \mid \mathsf{d})]$$

A2 $$\forall \mathsf{c}[\mathsf{p}(\mathsf{a} \mid \mathsf{c}) = \mathsf{p}(\mathsf{b} \mid \mathsf{c})] \Rightarrow \mathsf{a} = \mathsf{b}$$

A3 $$\mathsf{p}(\mathsf{a} \mid \mathsf{a}) = \mathsf{p}(\mathsf{c} \mid \mathsf{c})$$

B1 $$\mathsf{p}(\mathsf{ab} \mid \mathsf{c}) \leq \mathsf{p}(\mathsf{a} \mid \mathsf{c})$$

B2 $$\mathsf{p}(\mathsf{ab} \mid \mathsf{c}) = \mathsf{p}(\mathsf{a} \mid \mathsf{bc})\mathsf{p}(\mathsf{b} \mid \mathsf{c})$$

C $$\mathsf{p}(\mathsf{c} \mid \mathsf{c}) \neq \mathsf{p}(\mathsf{b} \mid \mathsf{c}) \Rightarrow \mathsf{p}(\mathsf{c} \mid \mathsf{c}) = \mathsf{p}(\mathsf{a} \mid \mathsf{c}) + \mathsf{p}(\mathsf{a}' \mid \mathsf{c})$$

Among the consequences of these axioms that have not yet been declared, these must be recorded:

(meet) $$\mathsf{p}(\mathsf{ac} \mid \mathsf{c}) = \mathsf{p}(\mathsf{a} \mid \mathsf{c})$$

(bounds) $$0 \leq \mathsf{p}(\mathsf{a} \mid \mathsf{c}) \leq \mathsf{p}(\mathsf{a} \mid \bot) = \mathsf{p}(\top \mid \mathsf{c}) = \mathsf{p}(\mathsf{c} \mid \mathsf{c}) = 1$$

(addition) $$\mathsf{c} \neq \bot = \mathsf{ab} \Rightarrow \mathsf{p}(\mathsf{a} \vee \mathsf{b} \mid \mathsf{c}) = \mathsf{p}(\mathsf{a} \mid \mathsf{c}) + \mathsf{p}(\mathsf{b} \mid \mathsf{c})$$

(contrad) $$\mathsf{c} \neq \bot \Rightarrow \mathsf{p}(\bot \mid \mathsf{c}) = \mathsf{p}(\mathsf{c}' \mid \mathsf{c}) = 0$$

Appendix $*$v of Popper (1959) contains a wealth of results for the values taken by p on pairs of elements of the set S. The laws of idempotence, commutation, and association are proved for the concatenation operator in formulas (30), (40), and (62). (They were proved again, in most cases more transparently, in Popper & Miller 1994, § 1.) The corresponding laws for the disjunction operator $\vee$, can be extracted from (94), (93), and (92), by applying the definition $a \vee c = (a'c')'$. The distributive law emerges in formula (86). These proofs can all be repeated, with almost no changes, in the system $\mathbb{B}$ (whose domain $\boldsymbol{\mathcal{S}}$ is the quotient $S/\sim$). The formula (meet) can be extracted from (30′). Given the definitions of $\bot$ as bb' and of $\top$ as $\mathsf{b} \vee \mathsf{b}'$, proofs of (bounds) and (addition) may be pieced together from (18), (33′), (74), (25), (79), and other formulas already cited. The formula (contrad), which is not explicitly stated in Popper's appendix, may be obtained by first writing $\bot$ for both a and b in (addition), and thereafter writing c for a and c' for b in (addition), and deploying (bounds). (Note that lines (30′) and (33′) are to be found in footnote 1 of the appendix, and that the line numbering is slightly different in Popper 1966.)

2 Consistency of the axiom system $\mathbb{B}$

The proof that the system $\mathbb{B}$ is consistent is neither difficult nor unfamiliar, but we shall survey it carefully. It suffices to look at the smallest genuine Boolean algebra $\{\boldsymbol{0}, \boldsymbol{1}\}$, which is depicted in Figure 0. In this algebra, and in other algebras exhibited in the paper, the zero and unit elements $\boldsymbol{0}$ and $\boldsymbol{1}$ are the interpretations of the constants $\bot$ and $\top$ respectively, and meet and complement are the interpretations of the concatenation and prime operators respectively. To interpret the functor p, we must nominate a suitable numerical measure $\mathfrak{p}$. Since the formulas (bounds) and (addition) hold only if $\mathfrak{p}(\boldsymbol{1} \mid \boldsymbol{1}) = \mathfrak{p}(\boldsymbol{1} \mid \boldsymbol{0}) = \mathfrak{p}(\boldsymbol{0} \mid \boldsymbol{0}) = 1$ and $\mathfrak{p}(\boldsymbol{0}, \boldsymbol{1}) = 0$, the only possible two-element model of the axioms {A1, A3, B1, B2, C} is the one displayed in Table 0. The value of $\mathfrak{p}(\mathsf{a} \mid \mathsf{c})$ occupies the cell in the row and the column headed by a and c respectively.

Figure 0: *The two-element Boolean algebra*

$\mathfrak{p}$	$\boldsymbol{0}$	$\boldsymbol{1}$
$\boldsymbol{0}$	1	0
$\boldsymbol{1}$	1	1

Table 0: *A two-element model*

That this is indeed a model needs to be checked, by verifying each of the axioms. The truth of A1 is immediate, since $0 \neq 1$, and A3 is equally obvious. A2 is vacuously true in Table 0, since the rows (and also, as it happens, the columns) are different. The only pair $\{\mathsf{b}, \mathsf{c}\}$ for which $1 = \mathfrak{p}(\mathsf{c} \mid \mathsf{c}) \neq \mathfrak{p}(\mathsf{b} \mid \mathsf{c})$ is the pair $\{\boldsymbol{0}, \boldsymbol{1}\}$, and $\mathfrak{p}(\boldsymbol{1} \mid \boldsymbol{1}) = \mathfrak{p}(\mathsf{a} \mid \boldsymbol{1}) + \mathfrak{p}(\mathsf{a}' \mid \boldsymbol{1})$ whatever a is, as required by axiom C. Axiom B1 inevitably holds if $\mathfrak{p}(\mathsf{a} \mid \mathsf{c}) = 1$. But if $\mathfrak{p}(\mathsf{a} \mid \mathsf{c}) = 0$, then $\mathsf{a} = \mathbf{o}$, in which event $\mathsf{ab} = \mathbf{o}$. It follows that $\mathfrak{p}(\mathsf{ab} \mid \mathsf{c}) = 0$, and consequently B1 holds in this case too.

B2 is usually the trickiest axiom to check, but in the case of Table 0 the work is not too demanding. If the axiom fails, then at least one of the three terms has the value 0. We consider them in reverse order. If $\mathfrak{p}(\mathsf{b} \mid \mathsf{c}) = 0$ then $\mathsf{b} = \mathbf{o}$ and $\mathsf{c} = \mathbf{\imath}$, which implies that $\mathsf{ab} = \mathbf{o}$ and $\mathfrak{p}(\mathsf{ab} \mid \mathsf{c}) = 0$ (this also follows from B1 and (bounds)). If $\mathfrak{p}(\mathsf{a} \mid \mathsf{bc}) = 0$ then $\mathsf{a} = \mathbf{o}$ and $\mathsf{bc} = \mathbf{\imath}$, and so $\mathsf{ab} = \mathbf{o} \neq \mathsf{c}$, whence $\mathfrak{p}(\mathsf{ab} \mid \mathsf{c}) = 0$. In short, if the right-hand side of B2 has the value 0, then the left-hand side has the value 0. For the converse, suppose that $\mathfrak{p}(\mathsf{ab} \mid \mathsf{c}) = 0$, so that $\mathsf{ab} = \mathbf{o}$ and $\mathsf{c} = \mathbf{\imath}$. Then either $\mathsf{b} = \mathbf{\imath}$, in which case $\mathsf{a} = \mathbf{o}$ and $\mathsf{bc} = \mathbf{\imath}$, from which it follows that $\mathfrak{p}(\mathsf{a} \mid \mathsf{bc})$ is 0; or $\mathsf{b} = \mathbf{o}$, in which case $\mathfrak{p}(\mathsf{b} \mid \mathsf{c}) = 0$. Either way, the product $\mathfrak{p}(\mathsf{a} \mid \mathsf{bc})\mathfrak{p}(\mathsf{b} \mid \mathsf{c}) = 0$.

The claim that Table 0 is a model of the system $\mathbb{B}$ is in no way novel. Indeed Popper, who for this task identified $\mathbf{o}$ with 0 and $\mathbf{\imath}$ with 1, branded the proof trivial (*op. cit.*, appendix ∗iv). Note that if 0 and 1 are interpreted as the truth values *falsity* and *truth*, and $\mathbf{o}$ and $\mathbf{\imath}$ are any false and true statements, then $\mathfrak{p}(\mathsf{a} \mid \mathsf{c})$ is just the truth value of the material conditional $\mathsf{c} \rightarrow \mathsf{a}$.

3 Four-element models of the axiom system $\mathbb{B}$

The two-element algebra of Figure 0 contains only a single atom, namely $\mathbf{\imath}$, and has to be regarded as a special case, though not an especially delinquent one. In our next port of call, the four-element algebra of Figure 1, there are two atoms, each of which contradicts and complements the other: their meet is the zero element $\mathbf{o}$ and their join is the unit element $\mathbf{\imath}$. Since the law of double negation $\mathfrak{p}(\mathsf{a} \mid \mathsf{c}) = \mathfrak{p}(\mathsf{a}'' \mid \mathsf{c})$ (or $\mathsf{a} = \mathsf{a}''$) is an immediate consequence of C and (bounds), we can calmly refer to the atoms as $\mathfrak{y}$ and $\mathfrak{y}'$ without caring which of them is $\mathfrak{y}$ and which is $\mathfrak{y}'$.

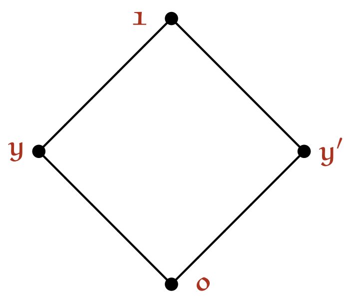

Figure 1: *The four-element Boolean algebra*

In algebras such as that depicted in Figure 1, a measure $\mathfrak{p}$ that satisfies the system $\mathbb{B}$ is wholly characterized by its marginal probability distribution, indeed by one or other of the two values assigned to $\mathfrak{p}(\mathfrak{y} \mid \mathbf{\imath})$ and $\mathfrak{p}(\mathfrak{y}' \mid \mathbf{\imath})$. (The marginal probability $\mathsf{p}(\mathsf{b} \mid \top)$ of an element b is variously called by philosophers and logicians the *absolute probability* of b, or its *prior probability*, or its *initial probability*, or its *unconditional probability*, and is usually abbreviated as $\mathsf{p}(\mathsf{b})$.) Reference to (bounds), (addition), and (contrad) enables us at once to enter the black numerals in Table 1. As anticipated, the remaining entries, $\mathfrak{p}(\mathfrak{y} \mid \mathbf{\imath})$ and $\mathfrak{p}(\mathfrak{y}' \mid \mathbf{\imath})$, which are here written in green, are constrained only by the equation $\mathsf{p}(\mathsf{b} \mid \top) + \mathsf{p}(\mathsf{b}' \mid \top) = 1$, which follows from (addition) and

$\mathfrak{p}$	$\mathfrak{o}$	$\mathfrak{y}$	$\mathfrak{y}'$	$\mathfrak{1}$
$\mathfrak{o}$	1	0	0	0
$\mathfrak{y}$	1	1	0	β
$\mathfrak{y}'$	1	0	1	1-β
$\mathfrak{1}$	1	1	1	1

Table 1: *A typical four-element model*

(bounds), and by the inequalities $0 \leq \mathsf{p}(\mathsf{b} \mid \top) \leq 1$, which are included in (bounds). (Although these constraints require the generality interpretation, as Kleene 1952, pp. 149f., called it, that is, both have to be read as universal statements, it is a fact that, if $\mathsf{p}(\mathsf{b} \mid \top) + \mathsf{p}(\mathsf{b}' \mid \top) = 1$ holds for all b then the two inequalities $0 \leq \mathsf{p}(\mathsf{b} \mid \top)$ and $\mathsf{p}(\mathsf{b}' \mid \top) \leq 1$ are equivalent for all b.)

This is unsurprising, since the *marginal distribution* $\mathfrak{p}(\mathsf{a} \mid \mathfrak{1})$, which occupies the rightmost (marginal) column of Table 1, satisfies Kolmogorov's axioms. Apart from the assumptions that p is real-valued and that $\mathcal{S}$ is a Boolean algebra (or something similar), these axioms may be deduced from (bounds) and (addition) by writing $\mathsf{p}(\mathsf{a})$ everywhere for $\mathsf{p}(\mathsf{a} \mid \top)$ and simplifying:

(bounds−) $$0 \leq \mathsf{p}(\mathsf{a}) \leq \mathsf{p}(\top) = 1$$

(addition−) $$\mathsf{ab} = \bot \Rightarrow \mathsf{p}(\mathsf{a} \vee \mathsf{b}) = \mathsf{p}(\mathsf{a}) + \mathsf{p}(\mathsf{b})$$

(It is not of much importance here, but it ought to be recorded that Kolmogorov did not state explicitly the inner inequality in (bounds−), which follows from the two outer inequalities together with (addition−): if $\mathsf{p}(\mathsf{a})$ is non-negative for every a, then $\mathsf{p}(\mathsf{a}) \leq \mathsf{p}(\mathsf{a} \vee \mathsf{a}') = \mathsf{p}(\top)$.) As already observed, in a finite model of these axioms (except Table 0, which boasts only a single atom), the full marginal distribution $\mathfrak{p}(\mathsf{a})$ is determined by the values of $\mathfrak{p}$ on the atoms: $\mathfrak{p}(\mathsf{a})$ is the sum of the values of $\mathfrak{p}$ on the atoms below a. Exactly the same is true in all models of $\mathbb{B}$.

Another respect in which the system $\mathbb{B}$ diverges from Kolmogorov's system concerns the possibility that atoms in algebras of cardinality greater than 2 receive zero (or unit) probability. Although Kolmogorov's system allows such an assignment, the assignment serves no discernible purpose. (Matters are different in algebras of infinite cardinality, where, indeed, it is common that every atom has zero marginal probability.) If β in Table 1 has the value 0, consideration of the marginal probabilities $\mathfrak{p}(\mathfrak{y}), \mathfrak{p}(\mathfrak{y}')$ does not help us to distinguish $\mathfrak{y}$ from $\mathfrak{o}$, or $\mathfrak{y}'$ from 1, and we may therefore identify $\mathfrak{y}$ with $\mathfrak{o}$ and $\mathfrak{y}'$ with $\mathfrak{1}$ without any danger. The same holds, *mutatis mutandis*, if $\beta = 1$. None of this is true when we focus attention on the conditional distribution $\mathfrak{p}(\mathsf{a} \mid \mathsf{c})$. Whatever value β may have, the four rows in Table 1 are all different, which implies that the four elements $\mathfrak{o}$, $\mathfrak{y}$, $\mathfrak{y}'$, and $\mathfrak{1}$ are probabilistically distinguishable from one another. It follows that in Popper's theory they cannot be casually identified. To be sure, the elements $\mathfrak{o}$ and $\mathfrak{y}$ may nonetheless be identical, in which case the elements $\mathfrak{y}'$ and $\mathfrak{1}$ are identical too, and Table 1 has only two rows and only two columns. It is therefore just Table 0 in disguise. The claim that a four-element model of $\mathbb{B}$ may contain an atom with zero probability is not blunted.

We still have to verify that, for all values of β in $[0, 1]$, Table 1 is indeed a model of the system $\mathbb{B}$. As in the previous section, axioms A1, A2, and A3 are true by inspection. The antecedent of axiom C holds unless c is identical with $\mathfrak{o}$. For every other element c it says that $\mathfrak{p}(\mathsf{a} \mid \mathsf{c}) + \mathfrak{p}(\mathsf{a}' \mid \mathsf{c}) = 1$ for all a; that is, in each of the columns headed $\mathfrak{y}$, $\mathfrak{y}'$, and $\mathfrak{1}$, the entries in the rows headed $\mathfrak{o}$ and $\mathfrak{1}$ sum to 1, as do the entries in the rows headed $\mathfrak{y}$ and $\mathfrak{y}'$. This is readily checked.

Axioms B1 and B2 are more onerous, and we shall approach them through a consequential reformulation, which we call $\mathbb{B}\star$, of the theory axiomatized by $\mathbb{B}$. The general strategy that is being followed here, and will be followed in § 4 too, is to use the axioms of $\mathbb{B}$ as a guide to constructing potential models, and then to test the correctness of these models by verifying their compliance with the axioms of $\mathbb{B}\star$. To forestall any misunderstanding, let me stress that the system $\mathbb{B}\star$ is not being put forward as a rival to Popper's original axiomatization. Three of its axioms, namely (chain), (bounds), and (complement), make use of lattice-theoretic terminology ($\prec$, $\perp$, $\top$) beyond the basic operators of concatenation and prime, and there is no reason to think that, if they were replaced by the expressions that define them, the system would imply, without further assumptions, any of the central laws of Boolean algebra. Two of the axioms of $\mathbb{B}\star$, moreover, unlike the axioms of $\mathbb{B}$, contain one or both of the numerals 0 and 1. $\mathbb{B}\star$ is being advanced solely as a way of easing the proofs that B1 and B2 hold in Table 1 and larger algebras.

Theorem 0 Let $\mathcal{S}$ be a Boolean algebra. The system $\mathbb{B}$ of axioms is equivalent to this set $\mathbb{B}\star$:

A1	$\exists c \exists d[p(a \mid b) \neq p(c \mid d)]$
A2	$\forall c[p(a \mid c) = p(b \mid c)] \Rightarrow a = b$
(meet)	$p(ac \mid c) = p(a \mid c)$
(chain)	$a \prec b \prec c \Rightarrow p(a \mid c) = p(a \mid b)p(b \mid c)$
(bounds)	$0 \leq p(a \mid c) \leq p(a \mid \perp) = p(\top \mid c) = p(c \mid c) = 1$
(complement)	$c \neq \perp \Rightarrow p(a \mid c) + p(a' \mid c) = 1$

Proof A1, A2 are axioms of $\mathbb{B}$, and (bounds) is a theorem of $\mathbb{B}$. Because (complement) is a consequence of axiom C and (bounds), it too is a theorem of $\mathbb{B}$. It was shown on p. 4 above that (meet) holds in $\mathbb{B}$ (it is enough to apply B2 and (bounds)). It implies at once that axiom B2 is equivalent in $\mathbb{B}$ to $p(abc \mid c) = p(abc \mid bc)p(bc \mid c)$. But if $a \prec b \prec c$ then $abc = a$ and $bc = b$, and so (chain) also holds in $\mathbb{B}$. (According to (bounds), the consequent of (chain) continues to be true when the antecedent is weakened to $a \preceq b \preceq c$. This need not be additionally postulated.)

For the converse, we prove B2 within $\mathbb{B}\star$, then B1, and finally A3 and C. It is a consequence of (chain) that $p(abc \mid c) = p(abc \mid bc)p(bc \mid c)$ when $abc \prec bc \prec c$. But this identity follows also if either $abc = bc$ or $bc = c$; for in the former case, it says that $p(bc \mid c) = p(bc \mid bc)p(bc \mid c)$, and in the latter case it says that $p(ac \mid c) = p(ac \mid c)p(c \mid c)$. Both identities are guaranteed by the final clause of (bounds). In short, $p(abc \mid c) = p(abc \mid bc)p(bc \mid c)$ whenever $abc \preceq bc \preceq c$, which means that it holds for all a, b, c. B2 now follows by three applications of (meet). But $0 \leq p(a \mid bc) \leq 1$ by (bounds), and in consequence the truth of B1 follows from the truth of B2.

Using (bounds), we at once derive axiom A3. We may derive also $a \neq \perp$ from $\exists b[p(a \mid a) \neq p(b \mid a)]$, and so, by (bounds) again, and (complement), $p(a \mid a) = p(c \mid a) + p(c' \mid a)$. But the conditional $\exists b[p(a \mid a) \neq p(b \mid a)] \Rightarrow p(a \mid a) = p(c \mid a) + p(c' \mid a)$ is equivalent to axiom C. ■

Corollary Let $\mathcal{S}$ be a Boolean algebra on which is defined a real-valued function $\mathfrak{p}$ that satisfies the system $\mathbb{B}$, and suppose that $a_0 \prec \ldots, \prec a_{k+2}$ is a finite increasing chain in $\mathcal{S}$ that contains at least three elements (that is to say, $k \geq 0$). Then $\mathfrak{p}(a_0 \mid a_{k+2}) = \prod\{\mathfrak{p}(a_j \mid a_{j+1}) \mid 0 \leq j \leq k+1\}$.

Proof If $k = 0$, the result is (chain). Mathematical induction delivers the proof for $k > 0$. ■

It should borne in mind that although (longchain) — that is, (chain) for chains of length greater than 3 — follows, in the manner indicated, from (chain) for chains of length 3, the converse does not hold in general. Checking whether (longchain) holds in a model can be useful, not as a proof that the function $\mathfrak{p}$ satisfies the system $\mathbb{B}$ (or, equivalently, $\mathbb{B}\star$), but as a proof that it does not.

The proof that Table 1 is a model of $\mathbb{B}$ is now rather easy, since it suffices to show that it is a model of $\mathbb{B}\star$. A1, A2, (bounds), and (complement) are, as before, true by construction. For (chain) there is hardly anything more to be done, since $\mathfrak{o} \prec \mathfrak{y} \prec \mathfrak{1}$ and $\mathfrak{o} \prec \mathfrak{y}' \prec \mathfrak{1}$ are the only two properly increasing chains in the algebra, and both $\mathfrak{p}(\mathfrak{o} \mid \mathfrak{y})\mathfrak{p}(\mathfrak{y} \mid \mathfrak{1})$ and $\mathfrak{p}(\mathfrak{o} \mid \mathfrak{y}')\mathfrak{p}(\mathfrak{y}' \mid \mathfrak{1})$ have the value 0, which is also the value of $\mathfrak{p}(\mathfrak{o} \mid \mathfrak{1})$. The proof of (meet), that is, the identity $\mathfrak{p}(\mathsf{ac} \mid \mathsf{c}) = \mathfrak{p}(\mathsf{a} \mid \mathsf{c})$, is a little more tiresome, but not much. It may be broken into three short steps: (a) if $\mathsf{a} \preceq \mathsf{c}$ then $\mathsf{ac} = \mathsf{a}$, whence (meet); (b) if $\mathsf{c} \prec \mathsf{a}$ then $\mathfrak{p}(\mathsf{ac} \mid \mathsf{c}) = 1$ by (bounds); while recourse to Figure 1 shows that either $\mathsf{c} = \mathfrak{o}$ or $\mathsf{a} = \mathfrak{1}$, and that $\mathfrak{p}(\mathsf{a} \mid \mathfrak{o}) = 1 = \mathfrak{p}(\mathfrak{1} \mid \mathsf{c})$; whence $\mathfrak{p}(\mathsf{a} \mid \mathsf{c}) = 1$, again by (bounds); and (c) if the pair a, c are incomparable by the ordering $\preceq$, then Figure 1 shows that one of them is $\mathfrak{y}$ and the other is $\mathfrak{y}'$, while Table 1 shows that $\mathfrak{p}(\mathfrak{y}\mathfrak{y}' \mid \mathfrak{y}') = \mathfrak{p}(\mathfrak{y} \mid \mathfrak{y}') = 0 = \mathfrak{p}(\mathfrak{y}' \mid \mathfrak{y}) = \mathfrak{p}(\mathfrak{y}'\mathfrak{y} \mid \mathfrak{y})$, so that in either case $\mathfrak{p}(\mathsf{ac} \mid \mathsf{c}) = \mathfrak{p}(\mathsf{a} \mid \mathsf{c})$.

4 Eight-element models of the axiom system $\mathbb{B}$

We turn our attention now to a more exciting example, the Boolean algebra that contains three distinct atoms $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$, and eight elements, which will be labelled $\mathfrak{o}$, $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$, $\mathfrak{z}'$, $\mathfrak{y}'$, $\mathfrak{x}'$, and $\mathfrak{1}$. It is illustrated in Figure 2. It should be clear that $\mathfrak{x}' = \mathfrak{y} \vee \mathfrak{z}$, that $\mathfrak{y}' = \mathfrak{z} \vee \mathfrak{x}$, and that $\mathfrak{z}' = \mathfrak{x} \vee \mathfrak{y}$.

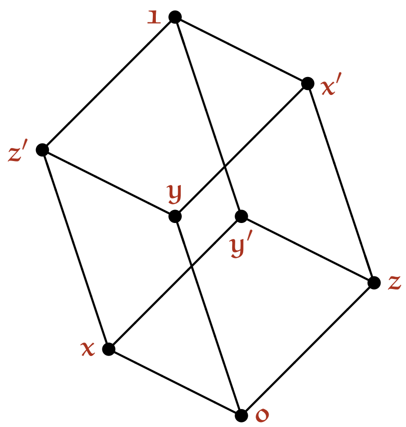

Figure 2: *The eight-element Boolean algebra*

As before, in any model that is based on this algebra, the marginal distribution $\mathfrak{p}(\mathsf{a} \mid \mathfrak{1})$ satisfies Kolmogorov's axioms. The three atoms, $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$, accordingly have marginal (or absolute) probabilities $\mathfrak{p}(\mathfrak{x} \mid \mathfrak{1})$, $\mathfrak{p}(\mathfrak{y} \mid \mathfrak{1})$, and $\mathfrak{p}(\mathfrak{z} \mid \mathfrak{1})$ that individually lie in the interval $[0, 1]$ and collectively sum to 1. These probabilities will be written for short as $\mathfrak{p}(\mathfrak{x})$, $\mathfrak{p}(\mathfrak{y})$, and $\mathfrak{p}(\mathfrak{z})$. There are now really three essentially different possibilities, one of them uninteresting and two of them potentially interesting. In the uninteresting case, each of these marginal probabilities is positive, so that, for every a, c in the algebra, except $\mathsf{c} = \mathfrak{o}$, the probability $\mathfrak{p}(\mathsf{a} \mid \mathsf{c})$ can be defined as $\mathfrak{p}(\mathsf{ac})/\mathfrak{p}(\mathsf{c})$ by the customary ratio formula, and for every a in the algebra, the probability $\mathfrak{p}(\mathsf{a} \mid \mathfrak{o})$ can be (indeed, must be) assigned the value 1. There is nothing new here. In the first of the interesting possibilities, exactly one of the three atoms, say $\mathfrak{x}$, is *insubstantial* or *null*, that is, is assigned zero marginal probability, which implies that there is some β such that $0 < \beta < 1$ and $\mathfrak{p}(\mathfrak{y}) = \beta$ and $\mathfrak{p}(\mathfrak{z}) = 1 - \beta$. In the second of the interesting possibilities, two of the three atoms, say $\mathfrak{x}$ and $\mathfrak{y}$, are insubstantial, and accordingly the third atom $\mathfrak{z}$ is assigned the marginal probability 1. In each case the values of $\mathfrak{p}(\mathfrak{o} \mid \mathsf{b}), \mathfrak{p}(\mathfrak{1} \mid \mathsf{b}), \mathfrak{p}(\mathsf{b} \mid \mathfrak{o})$, $\mathfrak{p}(\mathsf{b} \mid \mathsf{b})$, $\mathfrak{p}(\mathsf{b} \mid \mathsf{b}')$ (and, of course $\mathfrak{p}(\mathsf{b}' \mid \mathsf{b})$, since $\mathsf{b} = \mathsf{b}''$), which occupy the northern, southern, and western borders, and the diagonals, are settled by reference to (bounds) and (contrad). In both Table 2a (one insubstantial atom) and Table 2b (two insubstantial atoms) these values are inscribed in black.

One insubstantial atom For the time being we restrict attention to models in which for some β in the open interval $(0, 1)$, the three atoms, $\mathfrak{x}, \mathfrak{y}, \mathfrak{z}$ respectively have marginal probabilities $0, \beta, 1 - \beta$, here entered in Table 2a in green, and show how the entire conditional distribution $\mathfrak{p}(\mathsf{a} \mid \mathsf{c})$ is thereby fixed. The marginal probabilities $\mathfrak{p}(\mathfrak{x}')$, $\mathfrak{p}(\mathfrak{y}')$, and $\mathfrak{p}(\mathfrak{z}')$, which are obtained by use of (complement), are also entered in green. Now, apart from $\mathfrak{o}$ (whose performance as an argument of $\mathfrak{p}$ has been fully attended to), $\mathfrak{x}$ is the only element of the algebra with zero marginal probability, and only terms of the form $\mathfrak{p}(\mathsf{a} \mid \mathfrak{x})$ are going to require much thought. In short, for each element b, the values of $\mathfrak{p}(\mathsf{b} \mid \mathfrak{y})$, $\mathfrak{p}(\mathsf{b} \mid \mathfrak{z})$, $\mathfrak{p}(\mathsf{b} \mid \mathfrak{x}')$, $\mathfrak{p}(\mathsf{b} \mid \mathfrak{y}')$, and $\mathfrak{p}(\mathsf{b} \mid \mathfrak{z}')$ may be calculated by the classical ratio formula, and are entered in Table 2a in red. Moreover, since both $\mathfrak{x} \prec \mathfrak{y}'$ and $\mathfrak{x} \prec \mathfrak{z}'$, both $\mathfrak{p}(\mathfrak{y}' \mid \mathfrak{x})$ and $\mathfrak{p}(\mathfrak{z}' \mid \mathfrak{x})$ are equal to 1; and by (complement), $\mathfrak{p}(\mathfrak{y} \mid \mathfrak{x})$ and $\mathfrak{p}(\mathfrak{z} \mid \mathfrak{x})$ are equal to 0. These values are shown in blue in the $\mathfrak{x}$-column of Table 2a.

$\mathfrak{p}$	$\mathfrak{o}$	$\mathfrak{x}$	$\mathfrak{y}$	$\mathfrak{z}$	$\mathfrak{z}'$	$\mathfrak{y}'$	$\mathfrak{x}'$	$\mathfrak{1}$
$\mathfrak{o}$	1	0	0	0	0	0	0	0
$\mathfrak{x}$	1	1	0	0	0	0	0	0
$\mathfrak{y}$	1	0	1	0	1	0	β	β
$\mathfrak{z}$	1	0	0	1	0	1	1-β	1-β
$\mathfrak{z}'$	1	1	1	0	1	0	β	β
$\mathfrak{y}'$	1	1	0	0	0	1	1-β	1-β
$\mathfrak{x}'$	1	0	1	1	1	1	1	1
$\mathfrak{1}$	1	1	1	1	1	1	1	1

Table 2a: *An eight-element model with one insubstantial atom*

Table 2a shows that, if there exists a model of the theory $\mathbb{B}$ with three atoms, only one of which has a zero marginal probability, then its conditional distribution $\mathfrak{p}(\mathsf{a} \mid \mathsf{c})$ is fixed by the distribution of $\mathfrak{p}$ on the atoms. The proof that such a model exists; that is, that Table 2a is a model of $\mathbb{B}\star$, and therefore of $\mathbb{B}$, follows the path trodden in the case of Table 1. As before, what is asserted in A1, A2, (bounds), and (complement) is either true by construction or may be seen to be true by direct inspection. To verify (chain) in each model we need do nothing more with chains of the form $\mathfrak{o} \prec \mathsf{b} \prec \mathsf{c}$, since $\mathfrak{p}(\mathfrak{o} \mid \mathsf{b}) = 0$ whenever $\mathfrak{o} \prec \mathsf{b}$. There are six other relevant chains in Figure 2, all of which terminate at $\mathfrak{1}$. They are $\mathfrak{x} \prec \mathfrak{y}' \prec \mathfrak{1}$; $\mathfrak{x} \prec \mathfrak{z}' \prec \mathfrak{1}$;

$\mathfrak{y} \prec \mathfrak{x}' \prec \mathfrak{1}$; $\mathfrak{y} \prec \mathfrak{z}' \prec \mathfrak{1}$; $\mathfrak{z} \prec \mathfrak{y}' \prec \mathfrak{1}$, $\mathfrak{z} \prec \mathfrak{x}' \prec \mathfrak{1}$. It is easy to check that (chain) holds in Table 2a for each of these six chains. In fact, all that is required here is a proof of the identity $\mathfrak{p}(\mathsf{a} \mid \mathfrak{1}) = \mathfrak{p}(\mathsf{a} \mid \mathsf{c})\mathfrak{p}(\mathsf{c} \mid \mathfrak{1})$ when $\mathsf{a} \prec \mathsf{c}$. Since it holds when $\mathsf{a} = \mathsf{c}$, and $\mathsf{a} \preceq \mathsf{c}$ is equivalent to $\mathsf{ac} = \mathsf{a}$, what is at issue is the correctness of the *special multiplication law*, $\mathfrak{p}(\mathsf{ac}) = \mathfrak{p}(\mathsf{a} \mid \mathsf{c})\mathfrak{p}(\mathsf{c})$.

The verification of (meet) in Table 2a is not hard, but it is conveniently split into four steps.

(a) If $\mathsf{a} \preceq \mathsf{c}$ then $\mathsf{ac} = \mathsf{a}$, and it follows immediately that $\mathfrak{p}(\mathsf{ac} \mid \mathsf{c}) = \mathfrak{p}(\mathsf{a} \mid \mathsf{c})$ for all a and c.

(b) If $\mathsf{c} \prec \mathsf{a}$ then $\mathsf{ac} = \mathsf{c}$. Thanks to (bounds), we need prove for such a and c only that $\mathfrak{p}(\mathsf{a} \mid \mathsf{c}) = 1$. Figure 2 shows that, as before, this amounts to $\mathfrak{p}(\mathsf{a} \mid \bot) = 1 = \mathfrak{p}(\top \mid \mathsf{c})$, which is part of (bounds), together with $\mathfrak{p}(\mathsf{b}' \mid \mathsf{c}) = 1$ for each of the six (ordered) pairs c, b of distinct atoms. See the six coloured entries in the southwest quadrant of Table 2a.

(c) If the incomparable pair a, c are both atoms, or one is an atom and the other its complement, then $\mathsf{ac} = \mathfrak{o}$, and therefore $\mathfrak{p}(\mathsf{ac} \mid \mathsf{c}) = 0$. The corresponding values of $\mathfrak{p}(\mathsf{a} \mid \mathsf{c})$, which are also 0, appear as the six coloured entries in the northwest quadrant of Table 2a.

(d) Figure 2 shows that the one remaining possibility is that a, c are incomparable by the ordering $\preceq$ and that they are both the complements of atoms. Suppose, as an example, that $\mathsf{a} = \mathfrak{z}'$ and $\mathsf{c} = \mathfrak{y}'$. Then $\mathfrak{p}(\mathsf{a} \mid \mathsf{c}) = \mathfrak{p}(\mathfrak{z}' \mid \mathfrak{y}')$, while $\mathfrak{p}(\mathsf{ac} \mid \mathsf{c}) = \mathfrak{p}(\mathfrak{x} \mid \mathfrak{y}')$. Table 2a says that these probabilities both equal β. If $\mathsf{a} = \mathfrak{x}'$ and $\mathsf{c} = \mathfrak{y}'$, then $\mathfrak{p}(\mathsf{a} \mid \mathsf{c}) = \mathfrak{p}(\mathfrak{x}' \mid \mathfrak{y}')$, and $\mathfrak{p}(\mathsf{ac} \mid \mathsf{c}) = \mathfrak{p}(\mathfrak{z} \mid \mathfrak{y}')$, which both equal $1 - \beta$. There are four other ways of assigning different values of $\mathfrak{x}', \mathfrak{y}', \mathfrak{z}'$ to the variables a and c, and they all generate similar identities. The results are set out below and complete the proof that (meet) holds in Table 2a.

a	c	ac	$\mathfrak{p}(\mathsf{ac} \mid \mathsf{c})$	$\mathfrak{p}(\mathsf{a} \mid \mathsf{c})$
$\mathfrak{x}'$	$\mathfrak{y}'$	$\mathfrak{z}$	$\mathfrak{p}(\mathfrak{z} \mid \mathfrak{y}') = 1 - \beta$	$\mathfrak{p}(\mathfrak{x}' \mid \mathfrak{y}') = 1 - \beta$
$\mathfrak{x}'$	$\mathfrak{z}'$	$\mathfrak{y}$	$\mathfrak{p}(\mathfrak{y} \mid \mathfrak{z}') = 0$	$\mathfrak{p}(\mathfrak{x}' \mid \mathfrak{z}') = 0$
$\mathfrak{y}'$	$\mathfrak{x}'$	$\mathfrak{z}$	$\mathfrak{p}(\mathfrak{z} \mid \mathfrak{x}') = 1$	$\mathfrak{p}(\mathfrak{y}' \mid \mathfrak{x}') = 1$
$\mathfrak{y}'$	$\mathfrak{z}'$	$\mathfrak{x}$	$\mathfrak{p}(\mathfrak{x} \mid \mathfrak{z}') = 1$	$\mathfrak{p}(\mathfrak{y}' \mid \mathfrak{z}') = 1$
$\mathfrak{z}'$	$\mathfrak{x}'$	$\mathfrak{y}$	$\mathfrak{p}(\mathfrak{y} \mid \mathfrak{x}') = 0$	$\mathfrak{p}(\mathfrak{z}' \mid \mathfrak{x}') = 0$
$\mathfrak{z}'$	$\mathfrak{y}'$	$\mathfrak{x}$	$\mathfrak{p}(\mathfrak{x} \mid \mathfrak{y}') = \beta$	$\mathfrak{p}(\mathfrak{z}' \mid \mathfrak{y}') = \beta$

Two insubstantial atoms We now suppose that $\mathfrak{p}(\mathfrak{x}) = 0 = \mathfrak{p}(\mathfrak{z})$, and $\mathfrak{p}(\mathfrak{y}) = 1$, and enter these values in Table 2b in green. As before, the other non-trivial marginal probabilities, $\mathfrak{p}(\mathfrak{x}') = 1 = \mathfrak{p}(\mathfrak{z}')$, and $\mathfrak{p}(\mathfrak{y}') = 0$, also entered in green, are direct consequences of (complement).

For each element b, the values of $\mathfrak{p}(\mathsf{b} \mid \mathfrak{y})$, $\mathfrak{p}(\mathsf{b} \mid \mathfrak{x}')$, and $\mathfrak{p}(\mathsf{b} \mid \mathfrak{z}')$ may be calculated by the classical ratio formula, and are entered in Table 2b in red. Since $\mathfrak{x} \prec \mathfrak{y}'$, $\mathfrak{x} \prec \mathfrak{z}'$, $\mathfrak{y} \prec \mathfrak{z}'$, $\mathfrak{y} \prec \mathfrak{x}'$, $\mathfrak{z} \prec \mathfrak{x}'$, and $\mathfrak{z} \prec \mathfrak{y}'$ the value of each of $\mathfrak{p}(\mathfrak{y}' \mid \mathfrak{x})$, $\mathfrak{p}(\mathfrak{z}' \mid \mathfrak{x})$ $\mathfrak{p}(\mathfrak{z}' \mid \mathfrak{y})$, $\mathfrak{p}(\mathfrak{x}' \mid \mathfrak{y})$ $\mathfrak{p}(\mathfrak{x}' \mid \mathfrak{z})$, and $\mathfrak{p}(\mathfrak{y}' \mid \mathfrak{z})$ is 1, and is entered in the table in blue. Use of (complement) enables us to enter in yellow the values of two further probabilities in the upper half of the table, namely $\mathfrak{p}(\mathfrak{x} \mid \mathfrak{z})$ and $\mathfrak{p}(\mathfrak{z} \mid \mathfrak{x})$, which are both equal to 0, and four probabilities in the lower half, namely $\mathfrak{p}(\mathfrak{x}' \mid \mathfrak{z}')$, $\mathfrak{p}(\mathfrak{y}' \mid \mathfrak{z}')$, $\mathfrak{p}(\mathfrak{y}' \mid \mathfrak{x}')$, and $\mathfrak{p}(\mathfrak{z}' \mid \mathfrak{x}')$. But none of these elementary considerations helps us to complete the table. By (addition) and (bounds), $\mathfrak{p}(\mathfrak{x} \mid \mathfrak{y}') + \mathfrak{p}(\mathfrak{z} \mid \mathfrak{y}') = \mathfrak{p}(\mathfrak{x} \vee \mathfrak{z} \mid \mathfrak{y}') = \mathfrak{p}(\mathfrak{y}' \mid \mathfrak{y}') = 1$, and also $\mathfrak{p}(\mathfrak{x} \mid \mathfrak{y}') + \mathfrak{p}(\mathfrak{x}' \mid \mathfrak{y}') = \mathfrak{p}(\mathfrak{z} \mid \mathfrak{y}') + \mathfrak{p}(\mathfrak{z}' \mid \mathfrak{y}') = \mathfrak{p}(\mathfrak{1} \mid \mathfrak{y}') = 1$, so that each of the remaining probabilities fixes the other three, but one of them is left undetermined. We provisionally allocate the value β to $\mathfrak{p}(\mathfrak{x} \mid \mathfrak{y}')$, and enter the four values into Table 2b in purple.

$\mathfrak{p}$	$\mathfrak{o}$	$\mathfrak{x}$	$\mathfrak{y}$	$\mathfrak{z}$	$\mathfrak{z}'$	$\mathfrak{y}'$	$\mathfrak{x}'$	$\mathfrak{1}$
$\mathfrak{o}$	1	0	0	0	0	0	0	0
$\mathfrak{x}$	1	1	0	0	0	β	0	0
$\mathfrak{y}$	1	0	1	0	1	0	1	1
$\mathfrak{z}$	1	0	0	1	0	1-β	0	0
$\mathfrak{z}'$	1	1	1	0	1	β	1	1
$\mathfrak{y}'$	1	1	0	1	0	1	0	0
$\mathfrak{x}'$	1	0	1	1	1	1-β	1	1
$\mathfrak{1}$	1	1	1	1	1	1	1	1

Table 2b: *An eight-element model with two insubstantial atoms*

The proof that, whatever value β may take in the unit interval $[0, 1]$, Table 2b is a model of $\mathbb{B}\star$, and therefore of $\mathbb{B}$, is similar to the earlier proof for Table 2a. No more real work needs to be done to verify A1, A2, (bounds), (complement), and (chain), or to check steps (a), (b), and (c) in the verification of (meet), since the coloured cells in the southwest and northwest quadrants of Table 2b are exactly the same as the coloured cells in those quadrants of Table 2a (though not all the colours are the same). For step (d), we must check that the identity $\mathfrak{p}(\mathsf{ac} \mid \mathsf{c}) = \mathfrak{p}(\mathsf{a} \mid \mathsf{c})$ holds whenever a and c are the complements of distinct atoms in the algebra depicted in Figure 2. The results are set out below and complete the proof that (meet) holds in Table 2b.

a	c	ac	$\mathsf{p}(\mathsf{ac} \mid \mathsf{c})$	$\mathsf{p}(\mathsf{a} \mid \mathsf{c})$
$\mathfrak{x}'$	$\mathfrak{y}'$	$\mathfrak{z}$	$\mathfrak{p}(\mathfrak{z} \mid \mathfrak{y}') = 1 - \beta$	$\mathfrak{p}(\mathfrak{x}' \mid \mathfrak{y}') = 1 - \beta$
$\mathfrak{x}'$	$\mathfrak{z}'$	$\mathfrak{y}$	$\mathfrak{p}(\mathfrak{y} \mid \mathfrak{z}') = 1$	$\mathfrak{p}(\mathfrak{x}' \mid \mathfrak{z}') = 1$
$\mathfrak{y}'$	$\mathfrak{x}'$	$\mathfrak{z}$	$\mathfrak{p}(\mathfrak{z} \mid \mathfrak{x}') = 0$	$\mathfrak{p}(\mathfrak{y}' \mid \mathfrak{x}') = 0$
$\mathfrak{y}'$	$\mathfrak{z}'$	$\mathfrak{x}$	$\mathfrak{p}(\mathfrak{x} \mid \mathfrak{z}') = 0$	$\mathfrak{p}(\mathfrak{y}' \mid \mathfrak{z}') = 0$
$\mathfrak{z}'$	$\mathfrak{x}'$	$\mathfrak{y}$	$\mathfrak{p}(\mathfrak{y} \mid \mathfrak{x}') = 1$	$\mathfrak{p}(\mathfrak{z}' \mid \mathfrak{x}') = 1$
$\mathfrak{z}'$	$\mathfrak{y}'$	$\mathfrak{x}$	$\mathfrak{p}(\mathfrak{x} \mid \mathfrak{y}') = \beta$	$\mathfrak{p}(\mathfrak{z}' \mid \mathfrak{y}') = \beta$

5 Matters arising

Probabilistic independence In the classical theory of probability there coexist two definitions of probabilistic independence, one symmetrical, namely $\mathcal{U}(\mathsf{a}, \mathsf{c}) \leftrightarrow \mathsf{p}(\mathsf{ac}) = \mathsf{p}(\mathsf{a})\mathsf{p}(\mathsf{c})$, and one asymmetrical, namely $\mathcal{V}(\mathsf{a}, \mathsf{c}) \leftrightarrow \mathsf{p}(\mathsf{a} \mid \mathsf{c}) = \mathsf{p}(\mathsf{a})$. These definitions agree when $\mathsf{p}(\mathsf{c})$ is positive, but disagree when $\mathsf{p}(\mathsf{c}) = 0$. It is evident that $\mathcal{V}$ is the stronger relation, since multiplying its definiendum by $\mathsf{p}(\mathsf{c})$ yields the definiendum of $\mathcal{U}$. The last new appendix to *Logik der Forschung*, added to the 10th edition (1994), drew attention to some severe incongruities, within $\mathbb{B}$ and other theories in which zero and unit probabilities are widespread, that result from these definitions. In particular, every element a is independent, according to $\mathcal{U}$, of every element c (including perhaps itself) whose marginal probability is either 0 or 1; and also independent, according to $\mathcal{V}$, of every element c (including perhaps itself) whose marginal probability is 1. In response, Popper proposed a definition $\mathcal{W}(\mathsf{a},\mathsf{c}) \leftrightarrow \mathsf{p}(\mathsf{a} \mid \mathsf{c}) = \mathsf{p}(\mathsf{a} \mid \mathsf{c}')$ of the asymmetrical relation of *weak independence* and another, more innovative, of this symmetrical relation of mutual *independence*:

$$\text{DI} \qquad \mathcal{I}(\mathsf{a},\mathsf{c}) \leftrightarrow \mathcal{W}(\mathsf{a},\mathsf{c}) \;\&\; \mathcal{W}(\mathsf{a}',\mathsf{c}) \;\&\; \mathcal{W}(\mathsf{c},\mathsf{a}) \;\&\; \mathcal{W}(\mathsf{c}',\mathsf{a})$$

He showed that the relation $\mathcal{I}(\mathsf{a},\mathsf{c})$ never holds if either a or c is either $\bot$ or $\top$, and speculated that, nonetheless, some logically independent elements of marginal probability 0 or 1, for example some universal and existential statements, may be independent in the sense of $\mathcal{I}$. On slide 2-6 of my (2018) I offered an example to show that Popper's hunch here is correct; but the example is defective to the extent that the two elements chosen are not completely logically independent; and since the assignment of values to the function $\mathfrak{p}$ was not shown to satisfy the axioms of $\mathbb{B}$, it was also formally somewhat lazy. In addition I gave on slide 3-3 an example, much less welcome, that shows that probabilistic *independence* in the sense of $\mathcal{I}$ is compatible with simple logical *dependence* (a result announced in note 115 of Chapter 11 of Dorn 1997). This example is not defective, but it too was lazily laid out. The two examples may now be put on a firmer footing.

If we extract from Table 2b the columns headed $\mathfrak{x}$, $\mathfrak{z}$, $\mathfrak{z}'$, $\mathfrak{x}'$, and $\mathfrak{1}$, and the rows headed $\mathfrak{x}$, $\mathfrak{z}$, $\mathfrak{z}'$, and $\mathfrak{x}'$, we obtain the table below, from whose red, blue, and pink entries we may read off the truth of each of $\mathcal{W}(\mathfrak{x}',\mathfrak{z})$, $\mathcal{W}(\mathfrak{x},\mathfrak{z})$, $\mathcal{W}(\mathfrak{z},\mathfrak{x}')$, and $\mathcal{W}(\mathfrak{z}',\mathfrak{x}')$, which together imply $\mathcal{I}(\mathfrak{x}',\mathfrak{z})$. But in the algebra shown in Figure 2, the elements $\mathfrak{x}$ and $\mathfrak{z}$ are atoms, and $\mathfrak{z}$ logically implies $\mathfrak{x}'$. In sum, logically dependent elements may be probabilistically independent in the sense of $\mathcal{I}$.

$\mathfrak{p}$	$\mathfrak{x}$	$\mathfrak{z}$	$\mathfrak{z}'$	$\mathfrak{x}'$	$\mathfrak{1}$
$\mathfrak{x}$	1	0	0	0	0
$\mathfrak{z}$	0	1	0	0	0
$\mathfrak{z}'$	1	0	1	1	1
$\mathfrak{x}'$	0	1	1	1	1

Thanks to the symmetries inherent in the definition DI, which were pointed out in the discussion in Popper (1994), appendix $*XX$, it has hereby been shown also that elements that are mutually contrary, such as $\mathfrak{x}$ and $\mathfrak{z}$, or mutually subcontrary, such as $\mathfrak{x}'$ and $\mathfrak{z}'$, may be probabilistically independent. A concrete example is provided by taking $\mathfrak{1}$ to be the closed unit interval $[0,1]$, and $\mathfrak{x},\mathfrak{y},\mathfrak{z}$ to be respectively the singleton $\{0\}$, the open interval $(0,1)$, and the singleton $\{1\}$. These three subsets partition $[0,1]$ (that is, they are mutually exclusive and collectively exhaustive), and they may therefore serve as the three atoms of an eight-element model, such as Table 2b. In this model the value of β reflects the comparative probabilistic weight of the insubstantial atoms $\mathfrak{x}$ and $\mathfrak{z}$; that is to say, $\beta = \mathfrak{p}(\mathfrak{x} \mid \mathfrak{x} \vee \mathfrak{z}) = \mathfrak{p}(\mathfrak{x})/\mathfrak{p}(\mathfrak{x} \vee \mathfrak{z})$ and $1 - \beta = \mathfrak{p}(\mathfrak{z} \mid \mathfrak{x} \vee \mathfrak{z}) = \mathfrak{p}(\mathfrak{z})/\mathfrak{p}(\mathfrak{x} \vee \mathfrak{z})$.

This example does not fully vindicate Popper's expectation (*loc. cit.*) that 'it should … be possible [for] intuitively independent statements with the extreme probabilities 0 and 1 to be hypothetically independent', for the example that he gives, composed of the statements 'There exists a white raven' and 'There exists a golden mountain', suggests that by 'intuitively independent' he intended not just *simple logical independence* but something akin to *complete logical independence* in the sense of Moore (1910). But in the algebra being discussed $\mathfrak{x}$ and $\mathfrak{z}$ are atoms, and their complements $\mathfrak{x}'$ and $\mathfrak{z}'$ cannot both be false. Indeed, a glance is enough to show that in Figure 2 any two elements a,c stand in some logical relation: either *superalternation* $\mathsf{a} \vdash \mathsf{c}$, or *contrariety* $\mathsf{a} \vdash \mathsf{c}'$, or *subcontrariety* $\mathsf{a}' \vdash \mathsf{c}$, or *subalternation* $\mathsf{c} \vdash \mathsf{a}$. In short, in an eight-element algebra there do not exist two elements that are completely logically independent of each other.

Hájek's wallflower argument A curious result of Hájek (1989) states that if a classical probability function $p(a)$ is defined on a finite Boolean algebra with more than two atoms, then 'there are *more* distinct conditional probability values than distinct unconditional values' (p. 425). That is, the binary function $p(a \mid c)$ (which is defined, when $p(c) > 0$, by the ratio formula $p(ac)/p(c)$, and is undefined when $p(c) = 0$) has numerically more distinct values than does the singulary function $p(a)$. On p. 156 of his (2012) Hájek supplied a simple numerical example. In an algebra with three atoms $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$, each bearing marginal probability $1/3$, all the marginal probabilities are, by (addition) and (complement), in the quartet $\{0, 1/3, 2/3, 1\}$; but $\mathfrak{y}' = \mathfrak{x} \vee \mathfrak{z}$, and therefore $\mathfrak{p}(\mathfrak{x} \mid \mathfrak{y}') = \mathfrak{p}(\mathfrak{x}(\mathfrak{x} \vee \mathfrak{z}))/\mathfrak{p}(\mathfrak{x} \vee \mathfrak{z}) = \mathfrak{p}(\mathfrak{x})/(\mathfrak{p}(\mathfrak{x}) + \mathfrak{p}(\mathfrak{z})) = 1/2$, showing that $p(a \mid c)$ takes at least five values. It was presumably the recognition that some previously undefined values of $p(a \mid c)$ may become defined that led Hájek to claim, in the same paper, that 'the wallflower argument only has more bite . . . [if] . . . the conditional probabilities given by the usual ratio formula [are replaced by] . . . primitive conditional probabilities' (p. 159). This claim is not strictly true. In Table 2a, where β is equal neither to 0 nor to 1, both the singulary function $p(a)$ and the binary function $p(a \mid c)$ have four distinct values. In Table 2b, where β equals either 0 or 1, both the function $p(a)$ and the function $p(a \mid c)$ have two distinct values. It is an open question whether Hájek's theorem holds in all algebras with three or more substantial atoms. The falsity of Hájek's extended claim does not, however, significantly affect his main agendum, which is to subvert the project, variously called by him *Stalnaker's hypothesis*, *Adams' thesis*, and *the conditional construal of conditional probability* (CCCP), of somehow understanding the strength of the indicative conditional *if* a *then* c of ordinary language in terms of probabilistic conditionalization (or better, relativization). Note that this project, which has been pursued by a plethora of authors in a plethora of ways, takes on a new aspect when the function p is fully relativized, as it is in the system $\mathbb{B}$. Details of the new look are expounded in § 4 of Miller (2015).

Unfinished business The two previous subsections indicate three problems in need of solution: an unassailable definition of probabilistic independence; an example to show that the relation $\mathcal{I}$ (or its unassailable successor) can hold between completely independent elements of unit probability; and a delineation of the limits of validity of Hájek's argument. One possible solution to the first problem is to call two elements probabilistically independent if and only if in addition to standing in the relation $\mathcal{I}$, they are also logically independent (Miller 2018, slide 3-4); or even, in view of $\mathcal{I}$'s symmetries, completely independent. This latter refinement is disqualified by the uncomfortable fact that, in sufficiently large algebras, complete independence is compatible with probabilistic independence only if some of the probabilities involved are either 0 or 1 (*op. cit.*, slide 3-7). The second and third problems may be relatively straightforward. A related question asks for an exact characterization of the class of models of $\mathbb{B}$ (or at least of its finite models) in which the conditional distribution $\mathfrak{p}(a \mid c)$ is determined by the marginal distribution $\mathfrak{p}(a \mid \top)$. There are models of cardinality 8, such as the infinitely many models depicted in Table 2a, in which $\mathfrak{p}(a \mid c)$ is determined by $\mathfrak{p}(a \mid \top)$; and others, such as those depicted in Table 2b, where each of the three possible (but essentially identical) marginal distributions is compatible with infinitely many conditional distributions. It is an attractive conjecture, calling for proof or disproof, that in finite models of $\mathbb{B}$ something like this is true: the function $\mathfrak{p}(a \mid c)$ is determined by the function $\mathfrak{p}(a \mid \top)$ if and only if the model contains at most one null atom.

The multiplicative form of the identity (longchain) on p. 7 above suggests that it may be possible to understand the quantity $-\log \mathfrak{p}(\mathsf{a} \mid \mathsf{c})$ as a measure of the *directed (pseudo)-distance from* c *to* a. That would have the agreeable consequence (sometimes called *radial convexity*) that deductive chains are straight lines. On this, see the system of logical geometry of Miller (2010).

Another project to be commended is an investigation of the extent to which the multiplication law B2 can be replaced, in weaker theories of probability, by an axiom akin to (chain). In (1994), § 1, Popper & Miller presented a theory of probability $\mathbb{M}$, appropriate to lower semilattices rather than to Boolean algebras, in which meet (concatenation) is the only primitive operation. Axiom B2 (there called M2) is a fundamental component of the system $\mathbb{M}$, and it is crucial to the proof of the semilattice laws. It cannot be expected that a relative of (chain) will do anything like as much work, though by replacing the antecedent by $\forall \mathsf{d}[\mathsf{p}(\mathsf{a} \mid \mathsf{d}) \leq \mathsf{p}(\mathsf{b} \mid \mathsf{d}) \leq \mathsf{p}(\mathsf{c} \mid \mathsf{d})]$, it can be voided of semilattice terminology. Without doubt, other axioms, such as (meet), will be indispensable, but the relative ease with which (chain) can be checked does invite its further use.

References

Dorn, G. J. W. (1997). *Deductive, Probabilistic and Inductive Dependence. An Axiomatic Study in Probability Semantics.* Frankfurt am Main: Peter Lang.

Hájek, A. (1989). 'Probabilities of Conditionals — Revisited'. *Journal of Philosophical Logic* **18**, 4, pp. 423–428.

——— (2012). 'The Fall of "Adams' Thesis" '. *Journal of Logic, Language and Information* **21**, 2, pp. 145–161.

Kleene, S. C. (1952). *Introduction to Metamathematics*. Amsterdam: North-Holland Publishing Co., Groningen: P. Noordhoff N. V., and New York: D. van Nostrand Inc.

Kolmogorov, A. N. (1933). *Grundbegriffe der Wahrscheinlichkeitsrechnung*. Berlin: Springer. English translation 1950. *Foundations of the Theory of Probability*. New York: Chelsea Publishing Company.

Miller, D. W. (2010). 'A Refined Geometry of Logic'. *Principia* (Florianópolis) **13**, 3, pp. 339–356. `https://periodicos.ufsc.br/index.php/principia/article/view/1808-1711.2009v13n3p339/18041`.

——— (2015). 'Reconditioning the Conditional'. In J.-Y. Béziau, D. Krause, & J. R. B. Arenhart, editors (2015), pp. 205–215. *Conceptual Clarifications. Tributes to Patrick Suppes (1922–2014)*. TRIBUTES, 28. London: College Publications. Slightly modified reprint (2016). *Princípios* (Natal) **23**, 40, pp. 9–27.

——— (2018). 'Independence (Probabilistic) and Independence (Logical)' [slides]. In R. Neck, editor (2018), pp. 87–104. *Karl Popper and the Philosophy of Mathematics. Proceedings of the Conference held in Klagenfurt, 5–7 April, 2018.* Klagenfurt: Alpen-Adria-Universität Klagenfurt. `https://www.aau.at/wp-content/uploads/2018/06/KPF_NL-4_1_ Proceedings_final.pdf`.

Moore, E. H. (1910). 'Introduction to a Form of General Analysis'. In E. H. Moore, E. J. Wilczynski, & M. Mason, editors (1910), pp.1–150. *The New Haven Mathematical Colloquium (New Haven CT, 5–8 September 1906).* New Haven CT: Yale University Press.

Popper, K. R. (1935). *Logik der Forschung*. Vienna: Julius Springer Verlag. 2nd edition 1966. 10th edition 1994. Tübingen: J. C. B. Mohr (Paul Siebeck). 11th edition 2005. Tübingen: Mohr Siebeck.

——— (1959). *The Logic of Scientific Discovery*. Expanded English translation of Popper (1935). London: Hutchinson & Co.

Popper, K. R. & Miller, D. W. (1994). 'Contributions to the Formal Theory of Probability'. In P. W. Humphreys, editor (1994), pp. 3–21. *Patrick Suppes: Scientific Philosopher. Volume 1: Probability and Probabilistic Causality.* Dordrecht: Kluwer Academic Publishers. [The copyright in this paper belongs to the authors and their heirs and assigns, not (as stated) to the publisher.]