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Revisiting Almost Second-Degree Stochastic Dominance

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Leshno and Levy [Leshno M, Levy H (2002) Preferred by "all" and preferred by "most" decision makers: Almost stochastic dominance. *Management Sci.* 48(8):1074–1085] established almost stochastic dominance to reveal preferences for most rather than all decision makers with an increasing and concave utility function. In this paper, we first provide a counterexample to the main theorem of Leshno and Levy related to almost second-degree stochastic dominance. We then redefine this dominance condition and show that the newly defined almost second-degree stochastic dominance is the necessary and sufficient condition to rank distributions for all decision makers excluding the pathological concave preferences. We further extend our results to almost higher-degree stochastic dominance.

Key words: stochastic dominance; almost stochastic dominance; risk aversion

1. Introduction

Stochastic dominance has long served as one of the main rules used to rank distributions. This rule can rank the distributions for *all* utility functions in a certain class. For example, second-degree stochastic dominance (SSD) ranks the distributions for all individuals with increasing and concave utility functions. Hundreds of papers have been devoted to this topic and have applied this rule to various fields of economics, finance, and statistics since the distinguished papers of Hadar and Russell (1969), Hanoch and Levy (1969), and Rothschild and Stiglitz (1970). See Levy (1992, 1998) for a useful survey of stochastic dominance and for further analyses.

Yet, in terms of holding for all decision makers with an increasing and concave utility, a small violation of the stochastic dominance rules makes the ranking invalid. Leshno and Levy (2002) (hereafter, LL) offered an example: A lottery X with a 0.01 probability of obtaining 0 and a 0.99 probability of obtaining one million dollars does not stochastically dominate another lottery Y that yields one dollar for sure, and vice versa. Yet it is not surprising that most individuals prefer X to Y. To complement the above drawback of stochastic dominance, LL provided an intriguing way of imposing restrictions on the first and second derivatives of utility so that the preferences that do not represent most decision makers are excluded. LL demonstrated that an individual

with utility $u(z) = z$ if $z \leq 1$ and $u(z) = 1$ if $z > 1$ would prefer lottery Y to X. Because this preference does not represent most decision makers, it is ruled out. Roughly speaking, decision makers with extreme preferences, e.g., zero and/or infinite marginal utility, are considered pathological and are eliminated in LL's set of decision makers. Moreover, LL further showed that almost first-degree stochastic dominance (AFSD) and almost second-degree stochastic dominance (ASSD) are the necessary and sufficient conditions to rank distributions for their defined set of decision makers, respectively.

Since Leshno and Levy's study, several papers have further applied their theorem. For example, Levy et al. (2010) constructed several experiments to show that the almost stochastic dominance (ASD) rule corresponds to sets of nonpathological preferences. Regarding investment strategies, Bali et al. (2009) used data from the United States to show that the ASD approach unambiguously supports the popular practice that suggests a higher stock-to-bond ratio for long investment horizons. Bali et al. (2011) further adopted the ASD rule to examine the practice of investing in stock market anomalies; they found that the ASD rule provides evidence for "the significance of size, short-term reversal, and momentum for short investment horizons and the significance of book-to-market and long-term reversal for longer term horizons" (p. 28).

Although LL's theorem has been widely applied, we find that the main theorem of LL related to ASSD is not valid. In this paper, we first provide a counterexample to the main theorem of LL related to ASSD. We then redefine ASSD and show that our defined ASSD is the necessary and sufficient condition for all decision makers excluding the "pathological concave preferences" (Bali et al. 2009, p. 819) used to rank distributions. Finally, we generalize our results to almost Nth-degree stochastic dominance (ANSD). We demonstrate the necessary and sufficient conditions on distributions for all individuals excluding the *pathological higher-order preferences* (defined in §4).

2. Discussion on Leshno and Levy's Characterization of ASSD

Let us first briefly describe the results of LL. They imposed the following restrictions on the utility function (see p. 1079):

$$U_1^*(\varepsilon) = \left\{ u \in U_1 \colon u'(x) \leq \inf\{u'(x)\}[1/\varepsilon - 1] \ \forall x \right\}, \quad \text{and} \quad (1)$$

$$U_2^*(\varepsilon) = \left\{ u \in U_2 \colon -u''(x) \leq \inf\{-u''(x)\}[1/\varepsilon - 1] \ \forall x \right\}, \quad (2)$$

where U_1 denotes the utility set with $u' \geq 0$, and U_2 denotes the utility set with $u' \geq 0$ and $u'' \leq 0$; ε is in the range of $(0, \frac{1}{2})$. Note that ε used in Equation (1) could be different from ε used in Equation (2). The random variable X is in the range of $[\underline{x}, \bar{x}]$. Furthermore, LL defined the AFSD and ASSD as follows (see p. 1080).

DEFINITION 1. For $0 < \varepsilon < \frac{1}{2}$,
(1) AFSD. F dominates G by ε-almost FSD ($F \succeq_1^{\text{almost}(\varepsilon)} G$) if and only if

$$\int_{S_1} [F(x) - G(x)] \, dx \leq \varepsilon \|F - G\|. \quad (3)$$

(2) ASSD. F dominates G by ε-almost SSD ($F \succeq_2^{\text{almost}(\varepsilon)} G$) if and only if

$$\int_{S_2} [F(x) - G(x)] \, dx \leq \varepsilon \|F - G\|, \quad (4)$$

and $E_F(X) \geq E_G(X)$, where

$$S_1(F, G) = \{x \in [\underline{x}, \bar{x}] \colon G(x) < F(x)\}, \quad (5)$$

$$S_2(F, G) = \left\{ x \in S_1(F, G) \colon \int_{\underline{x}}^x G(t) \, dt < \int_{\underline{x}}^x F(t) \, dt \right\}, \quad (6)$$

and

$$\|F - G\| = \int_{\underline{x}}^{\bar{x}} |F(x) - G(x)| \, dx.$$

Let $E_F(u)$ and $E_G(u)$ denote the expected utility under distributions F and G, respectively. LL further provided the following theorem.

ALLEGED THEOREM 1 (LL's THEOREM 1).
(1) *AFSD. F dominates G by ε-almost FSD ($F \succeq_1^{\text{almost}(\varepsilon)} G$) if and only if for all u in $U_1^*(\varepsilon)$, $E_F(u) \geq E_G(u)$.*
(2) *ASSD. F dominates G by ε-almost SSD ($F \succeq_2^{\text{almost}(\varepsilon)} G$) if and only if for all u in $U_2^*(\varepsilon)$, $E_F(u) \geq E_G(u)$.*

Although the first part of the above theorem is correct, the second part is not. We provide a counterexample to the second part of LL's theorem in Appendix A.

3. A Characterization of ASSD

In this section, we redefine ASSD and further provide the correct necessary and sufficient condition. Note that we do not change the definition of $U_2^*(\varepsilon)$ in LL. First, let us define the set of $\hat{S}_2$ as

$$\hat{S}_2(F, G) = \{x \in [\underline{x}, \bar{x}] \colon G^{(2)}(x) < F^{(2)}(x)\}, \quad (7)$$

where $F^{(2)}(x) = \int_{\underline{x}}^x F(t) \, dt$ and $G^{(2)}(x) = \int_{\underline{x}}^x G(t) \, dt$. It is obvious that $\hat{S}_2$ is not necessarily included in S_1 and $S_2(F, G) \subset \hat{S}_2(F, G)$. We can redefine ASSD as follows.

DEFINITION 2 (ASSD). For $0 < \varepsilon < \frac{1}{2}$, F dominates G by ε-almost SSD ($F \succeq_2^{\text{almost}(\varepsilon)} G$) if and only if

$$\int_{\hat{S}_2} [F^{(2)}(x) - G^{(2)}(x)] \, dx \leq \varepsilon \|F^{(2)} - G^{(2)}\|, \quad (8)$$

and $E_F(X) \geq E_G(X)$, where

$$\|F^{(2)} - G^{(2)}\| = \int_{\underline{x}}^{\bar{x}} |F^{(2)}(x) - G^{(2)}(x)| \, dx.$$

Now, with the new definition of ε-almost SSD, we can correct the second part of Theorem 2 in LL as follows.

THEOREM 1 (ASSD). *For all u in $U_2^*(\varepsilon)$, $E_F(u) \geq E_G(u)$ if and only if*

$$\int_{\hat{S}_2} [F^{(2)}(x) - G^{(2)}(x)] \, dx \leq \varepsilon \|F^{(2)} - G^{(2)}\|, \quad (9)$$

and $E_F(X) \geq E_G(X)$.

PROOF. See Appendix B. □

4. Almost Nth-Degree Stochastic Dominance

The previous section provides the distribution conditions for all decision makers excluding the pathological concave preferences. Recently, the literature has paid much attention to higher-order preferences (Eeckhoudt and Schlesinger 2006, Denuit and Eeckhoudt 2010). However, the conditions to rank

distributions in the sense of stochastic dominance for individuals with higher-order preferences still suffer the same critiques of LL; i.e., stochastic dominance rules cannot reveal most individuals' preferences even when there is a very small violation of these rules. This section will generalize our previous results for all individuals excluding the pathological higher-order preferences.[1]

Let us define

$$U_N = \left\{ u: (-1)^{n+1} u^{(n)} \geq 0, \ n = 1, 2, \ldots, N \right\}, \quad (10)$$

where $u^{(n)}$ denotes the nth derivative of the utility function u, and $N > 2$.[2] Furthermore, let

$$U_N^*(\varepsilon_N) = \big\{ u \in U_N: (-1)^{N+1} u^{(N)}(x)$$
$$\leq \inf\{(-1)^{N+1} u^{(N)}(x)\}[1/\varepsilon_N - 1] \ \forall x \big\}. \quad (11)$$

In other words, an individual with a utility function belonging to $U_N^*(\varepsilon_N)$ is the one whose nth derivative of the utility function alters in sign from $u' > 0$, $n = 1, 2, \ldots, N$, and the individual's Nth derivative is bounded. The preferences with extreme values of the Nth derivative are viewed as the pathological Nth-order preferences and are therefore excluded by Equation (11).

Let us define ε_N-almost NSD, $N > 2$, as follows.

DEFINITION 3 (ANSD). *For $0 < \varepsilon_N < \frac{1}{2}$, F dominates G by ε_N-almost NSD ($F \succeq_N^{\text{almost}(\varepsilon_N)} G$) if*

$$\int_{\hat{S}_N} [F^{(N)}(x) - G^{(N)}(x)] \, dx \leq \varepsilon_N \|F^{(N)} - G^{(N)}\|, \quad (12)$$

and $G^{(n)}(\bar{x}) - F^{(n)}(\bar{x}) \geq 0$, $n = 2, 3, \ldots, N$, $N > 2$, where

$$F^{(n)}(x) = \int_{\underline{x}}^{x} F^{(n-1)}(t) \, dt,$$

$$G^{(n)}(x) = \int_{\underline{x}}^{x} G^{(n-1)}(t) \, dt,$$

$$\hat{S}_N(F, G) = \{ x \in [\underline{x}, \bar{x}]: G^{(N)}(x) < F^{(N)}(x) \},$$

and

$$\|F^{(N)} - G^{(N)}\| = \int_{\underline{x}}^{\bar{x}} |F^{(N)}(x) - G^{(N)}(x)| \, dx.$$

Following the same argument as in §3, we obtain the following theorem.

THEOREM 2 (ANSD). *For all u in $U_N^*(\varepsilon_N)$, $N > 2$, $E_F(u) \geq E_G(u)$ if and only if*

$$\int_{\hat{S}_N} [F^{(N)}(x) - G^{(N)}(x)] \, dx \leq \varepsilon_N \|F^{(N)} - G^{(N)}\|, \quad (13)$$

and $G^{(n)}(\bar{x}) - F^{(n)}(\bar{x}) \geq 0$, $n = 2, 3, \ldots, N$.

PROOF. See Appendix C. □

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[1] The authors thank Professors Leshno and Levy for their suggestions on the generalization to higher-degree stochastic dominance.
[2] Because we have defined ASSD in §3, in this section, we start from $N > 2$.

Appendix A. A Counterexample to the Second Part of LL's Theorem 1

Let $x \in [0, 5]$. Assume that there are two payoff distributions where

$$F(x) = \begin{cases} 0 & \text{if } 0 \leq x < 2, \\ \frac{3}{4} & \text{if } 2 \leq x < 5, \\ 1 & \text{if } x = 5, \end{cases} \quad (A1)$$

and

$$G(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ \frac{1}{4} & \text{if } 1 \leq x < 3, \\ 1 & \text{if } 3 \leq x \leq 5, \end{cases} \quad (A2)$$

as shown in Figure A.1. In this example, we have $E_F(X) = 11/4 > E_G(X) = 5/2$, $\|F - G\| = 5/4$,

$$S_1(F, G) = \{x: G(x) < F(x)\} = \{x: x \in [2, 3]\}, \quad (A3)$$

and

$$S_2(F, G) = \left\{ x \in S_1(F, G): \int_0^x G(t) \, dt < \int_0^x F(t) \, dt \right\}$$
$$= \{x: x \in [\tfrac{5}{2}, 3]\}. \quad (A4)$$

Thus, according to the above definition of ASSD, it is obvious that F dominates G at ε-almost SSD, where

$$\varepsilon \geq \frac{\int_{S_2} [F(x) - G(x)] \, dx}{\|F - G\|} = \frac{1/4}{5/4} = \frac{1}{5}.$$

Theorem 2 predicts that all individuals with preferences $u \in U_2^*(\varepsilon^*)$, where $\varepsilon^* \in [\frac{1}{5}, \frac{1}{2})$ would prefer F to G; i.e., $E_F(u) \geq E_G(u)$. In the following, we will construct a utility function that belongs to $U_2^*(\varepsilon^*)$ and show that the decision maker would strictly prefer G to F; i.e., $E_F(u) < E_G(u)$.

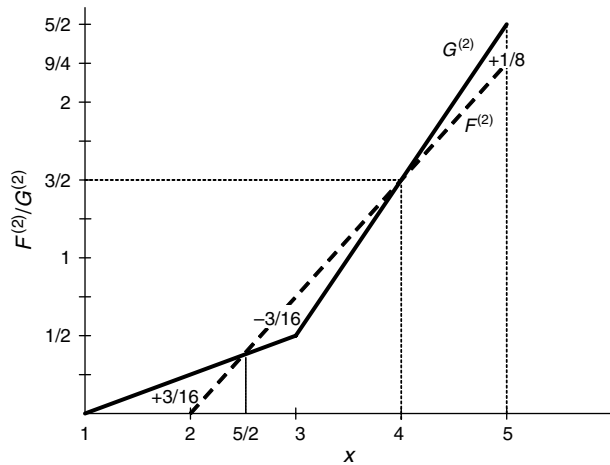Let a marginal utility function u' satisfy

$$u'(x) = \begin{cases} \frac{21}{2} - x & \text{if } 0 \leq x \leq \frac{5}{2}, \\ 18 - 4x & \text{if } \frac{5}{2} \leq x \leq 4, \\ 6 - x & \text{if } 4 \leq x \leq 5. \end{cases}$$

Figure A.1   The Cumulative Distribution of *F* and *G* in the Example
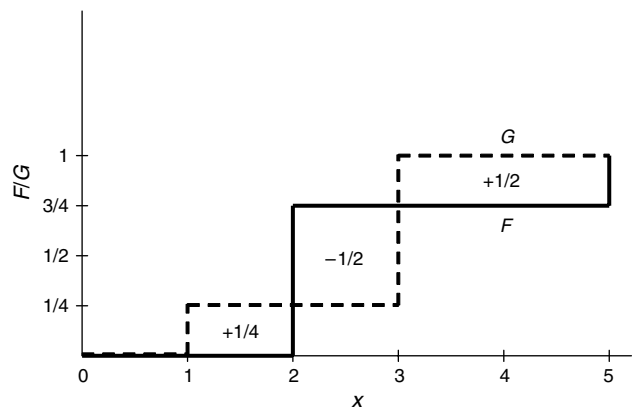
Figure A.2 $F^{(2)}$ and $G^{(2)}$ in the Example

Thus, we have $u' \geq 0$ and $u'' \leq 0$ and

$$-u''(x) = \begin{cases} 1 & \text{if } 0 \leq x < \frac{5}{2}, \\ 4 & \text{if } \frac{5}{2} \leq x < 4, \\ 1 & \text{if } 4 \leq x \leq 5. \end{cases}$$

It is obvious that $u \in U_2^*(\frac{1}{5})$, where

$$U_2^*\left(\frac{1}{5}\right) = \left\{ u: -u''(x) \leq \inf\{-u''(x)\}\left[\frac{1}{1/5} - 1\right], \forall x \right\}.$$

Let $F^{(2)}(x) = \int_0^x F(t)\,dt$ and $G^{(2)}(x) = \int_0^x G(t)\,dt$; $F^{(2)}$ and $G^{(2)}$ are shown in Figure A.2. Thus, we have

$$\begin{aligned} E_F(u) - E_G(u) &= \int_0^5 u'(x)[G(x) - F(x)]\,dx \\ &= u'(5)[G^{(2)}(5) - F^{(2)}(5)] \\ &\quad + \int_0^{5/2} [-u''(x)][G^{(2)}(x) - F^{(2)}(x)]\,dx \\ &\quad + \int_{5/2}^4 [-u''(x)][G^{(2)}(x) - F^{(2)}(x)]\,dx \\ &\quad + \int_4^5 [-u''(x)][G^{(2)}(x) - F^{(2)}(x)]\,dx \\ &= 1 \times (\tfrac{5}{2} - \tfrac{9}{4}) + 1 \times \tfrac{3}{16} + 4 \times (-\tfrac{3}{16}) + 1 \times \tfrac{1}{8} \\ &= -\tfrac{3}{16} \\ &< 0. \end{aligned}$$

This example illustrates that the necessary condition for ASSD is not valid.

Appendix B. Proof of Theorem 1
(1) "If" part: We show that if

$$\int_{\hat{S}_2} [F^{(2)}(x) - G^{(2)}(x)]\,dx \leq \varepsilon \|F^{(2)} - G^{(2)}\| \qquad (B1)$$

and

$$E_F(X) \geq E_G(X), \qquad (B2)$$

then $E_F(u) \geq E_G(u) \ \forall u \in U_2^*(\varepsilon)$. By integration by parts, we have

$$\begin{aligned} E_F(u) - E_G(u) &= u'(\bar{x})[G^{(2)}(\bar{x}) - F^{(2)}(\bar{x})] \\ &\quad + \int_{\underline{x}}^{\bar{x}} [-u''(x)][G^{(2)}(x) - F^{(2)}(x)]\,dx, \quad (B3) \end{aligned}$$

and $G^{(2)}(\bar{x}) - F^{(2)}(\bar{x}) = E_F(X) - E_G(X)$. Since $u' > 0$, according to (B2) and (B3),

$$\begin{aligned} E_F(u) - E_G(u) &\geq \int_{\underline{x}}^{\bar{x}} [-u''(x)][G^{(2)}(x) - F^{(2)}(x)]\,dx \\ &= \int_{\hat{S}_2} [-u''(x)][G^{(2)}(x) - F^{(2)}(x)]\,dx \\ &\quad + \int_{\hat{S}_2^C} [-u''(x)][G^{(2)}(x) - F^{(2)}(x)]\,dx, \end{aligned}$$

where $\hat{S}_2^C$ denotes the complement of $\hat{S}_2$ in $[\underline{x}, \bar{x}]$. Denote $\inf_{x \in [\underline{x}, \bar{x}]}\{-u''(x)\} = \underline{\theta}$ and $\sup_{x \in [\underline{x}, \bar{x}]}\{-u''(x)\} = \bar{\theta}$. Thus, we have

$$\begin{aligned} E_F(u) - E_G(u) &\geq \bar{\theta} \int_{\hat{S}_2} [G^{(2)}(x) - F^{(2)}(x)]\,dx \\ &\quad + \underline{\theta} \int_{\hat{S}_2^C} [G^{(2)}(x) - F^{(2)}(x)]\,dx \\ &= (\bar{\theta} + \underline{\theta}) \int_{\hat{S}_2} [G^{(2)}(x) - F^{(2)}(x)]\,dx \\ &\quad + \underline{\theta}\|F^{(2)} - G^{(2)}\|. \qquad (B4) \end{aligned}$$

Since $u \in U_2^*(\varepsilon)$, by definition, we have $\bar{\theta} \leq \underline{\theta}[1/\varepsilon - 1]$; i.e., $\varepsilon \leq \underline{\theta}/(\bar{\theta} + \underline{\theta})$. By (B1), we have

$$\begin{aligned} \int_{\hat{S}_2} [F^{(2)}(x) - G^{(2)}(x)]\,dx &\leq \varepsilon\|F^{(2)} - G^{(2)}\| \\ &\leq \frac{\underline{\theta}}{\bar{\theta} + \underline{\theta}}\|F^{(2)} - G^{(2)}\|. \qquad (B5) \end{aligned}$$

By (B4) and (B5), we prove that $E_F(u) - E_G(u) \geq 0 \ \forall u \in U_2^*(\varepsilon)$.

(2) "Only if" part: We show that if

$$\int_{\hat{S}_2} [F^{(2)}(x) - G^{(2)}(x)]\,dx > \varepsilon\|F^{(2)} - G^{(2)}\| \qquad (B6)$$

or

$$E_F(X) < E_G(X) \qquad (B7)$$

then there exists a $u \in U_2^*(\varepsilon)$ such that $E_F(u) - E_G(u) < 0$.

Let us first show that if (B6) holds, then $\exists u \in U_2^*(\varepsilon)$ such that $E_F(u) - E_G(u) < 0$. Assume that $\hat{S}_2 = [a, b]$, where $\underline{x} \leq a \leq b \leq \bar{x}$. Define a marginal utility function as follows:

$$u'(x) = \begin{cases} \underline{\theta}(\bar{x} - b) + \bar{\theta}(b - a) + \underline{\theta}(a - x) & \text{if } \underline{x} \leq x \leq a, \\ \underline{\theta}(\bar{x} - b) + \bar{\theta}(b - x) & \text{if } a \leq x \leq b, \\ \underline{\theta}(\bar{x} - x) & \text{if } b \leq x \leq \bar{x}. \end{cases}$$

It is obvious that $u \in U_2^*(\varepsilon)$, $\varepsilon = \underline{\theta}/(\bar{\theta} + \underline{\theta})$. Since $u'(\bar{x}) = 0$, from (B3),

$$\begin{aligned} E_F(u) - E_G(u) &= \int_{\underline{x}}^{\bar{x}} [-u''(x)][G^{(2)}(x) - F^{(2)}(x)]\,dx \\ &= \bar{\theta} \int_a^b [G^{(2)}(x) - F^{(2)}(x)]\,dx \\ &\quad + \underline{\theta} \int_{\hat{S}_2^C} [G^{(2)}(x) - F^{(2)}(x)]\,dx \\ &= (\bar{\theta} + \underline{\theta}) \int_a^b [G^{(2)}(x) - F^{(2)}(x)]\,dx \\ &\quad + \underline{\theta}\|F^{(2)} - G^{(2)}\|. \qquad (B8) \end{aligned}$$

Since $\varepsilon = \underline{\theta}/(\bar{\theta} + \underline{\theta})$, (B6) and (B8) imply that the above defined u exhibits $E_F(u) - E_G(u) < 0$.

Next, we show that if (B7) holds, then $\exists u \in U_2^*(\varepsilon)$ such that $E_F(u) - E_G(u) < 0$. Define a marginal utility function

as follows:

$$u'(x) = \begin{cases} c - \eta_1 x & \text{if } x \leq x_0, \\ c + (\eta_2 - \eta_1)x_0 - \eta_2 x & \text{if } x_0 \leq x, \end{cases}$$

where $x_0 \in (\underline{x}, \bar{x})$; and c, η_1, and η_2 are positive constants such that $c > \eta_1 x_0$, $\eta_2 > \eta_1$, and $c > \eta_2 \bar{x} - (\eta_2 - \eta_1)x_0$ to guarantee $u \in U_2^*(\varepsilon)$, $\varepsilon = \eta_1/(\eta_1 + \eta_2)$. From Equation (B3), we have

$$\begin{aligned} E_F(u) - E_G(u) &= [c + (\eta_2 - \eta_1)x_0 - \eta_2 \bar{x}][G^{(2)}(\bar{x}) - F^{(2)}(\bar{x})] \\ &\quad + \eta_1 \int_{\underline{x}}^{x_0} [G^{(2)}(x) - F^{(2)}(x)]\, dx \\ &\quad + \eta_2 \int_{x_0}^{\bar{x}} [G^{(2)}(x) - F^{(2)}(x)]\, dx \\ &\leq [c + (\eta_2 - \eta_1)x_0 - \eta_2 \bar{x}][G^{(2)}(\bar{x}) - F^{(2)}(\bar{x})] \\ &\quad + \eta_2 \|F^{(2)} - G^{(2)}\|. \end{aligned} \quad (B9)$$

Since $G^{(2)}(\bar{x}) - F^{(2)}(\bar{x}) = E_F(X) - E_G(X) < 0$, if

$$c > \eta_2 \bar{x} - (\eta_2 - \eta_1)x_0 + \frac{\eta_2 \|F^{(2)} - G^{(2)}\|}{E_G(X) - E_F(X)}, \quad (B10)$$

then $E_F(u) - E_G(u) < 0$, which completes the proof.

Appendix C. Proof of Theorem 2
The proof is similar to the proof of Theorem 1. Integrating $E_F(u) - E_G(u)$ by parts yields

$$\begin{aligned} &E_F(u) - E_G(u) \\ &= \int_{\underline{x}}^{\bar{x}} u'(x)[G(x) - F(x)]\, dx \\ &= u'(\bar{x})[G^{(2)}(\bar{x}) - F^{(2)}(\bar{x})] + \int_{\underline{x}}^{\bar{x}} [-u''(x)][G^{(2)}(x) - F^{(2)}(x)]\, dx \\ &= u'(\bar{x})[G^{(2)}(\bar{x}) - F^{(2)}(\bar{x})] + [-u''(\bar{x})][G^{(3)}(\bar{x}) - F^{(3)}(\bar{x})] \\ &\quad + \int_{\underline{x}}^{\bar{x}} [u'''(x)][G^{(3)}(x) - F^{(3)}(x)]\, dx \\ &= \cdots \\ &= \sum_{n=2}^{N} (-1)^n u^{(n-1)}(\bar{x})[G^{(n)}(\bar{x}) - F^{(n)}(\bar{x})] \\ &\quad + \int_{\underline{x}}^{\bar{x}} [(-1)^{N+1} u^{(N)}(x)][G^{(N)}(x) - F^{(N)}(x)]\, dx. \quad (C1) \end{aligned}$$

Since $u \in U_N^*(\varepsilon_N)$, we have $(-1)^n u^{(n-1)} \geq 0$, $n = 2, 3, \ldots, N$. Thus, if $G^{(n)}(\bar{x}) - F^{(n)}(\bar{x}) \geq 0, n = 2, 3, \ldots, N$, then $\forall u \in U_N^*(\varepsilon_N)$, the first term of Equation (C1) is positive. Furthermore, if $\int_{\hat{S}_N} [F^{(N)}(x) - G^{(N)}(x)]\, dx \leq \varepsilon_N \|F^{(N)} - G^{(N)}\|$,

then following the proof of Theorem 1, we can obtain

$$\int_{\underline{x}}^{\bar{x}} [(-1)^{N+1} u^{(N)}(x)][G^{(N)}(x) - F^{(N)}(x)]\, dx \geq 0 \quad \forall u \in U_N^*(\varepsilon_N).$$

Thus, the above concludes the sufficiency part.

For the necessity part, if

$$\int_{\hat{S}_N} [F^{(N)}(x) - G^{(N)}(x)]\, dx > \varepsilon_N \|F^{(N)} - G^{(N)}\|,$$

then we can easily find a utility function $u \in U_N^*(\varepsilon_N)$ and follow a similar process to the proof of Theorem 1 to show that $E_F(u) < E_G(u)$. The utility function satisfies the following conditions: (1) $u^{(N-1)}(x)$ is a piecewise linear function; and (2) $u^{(n-1)}(\bar{x}) = 0$, $n = 2, 3, \ldots, N$. On the other hand, if there exists an integer k, $2 \leq k \leq N$, such that $G^{(k)}(\bar{x}) - F^{(k)}(\bar{x}) < 0$, then, similar to the proof of Theorem 1, we can construct a utility function $u \in U_N^*(\varepsilon_N)$ such that $E_F(u) < E_G(u)$. The constructed utility function satisfies the following conditions: (1) $u^{(N-1)}(x)$ is a piecewise linear function; (2) $u^{(n-1)}(\bar{x}) = 0$ for $n \neq k$; (3) $(-1)^k u^{(k-1)}(\bar{x})$ is relatively large; and (4) $\forall x \in [\underline{x}, \bar{x}]$, $(-1)^{N+1} u^{(N)}(x)$ is small enough.

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CORRECTION

In this article, "Revisiting Almost Second-Degree Stochastic Dominance" by Larry Y. Tzeng, Rachel J. Huang, and Pai-Ta Shih (first published in *Articles in Advance*, November 28, 2012, *Management Science*, DOI:10.1287/mnsc.1120.1616), the *x*-axis labels in Figure A.2 have been corrected to read as follows: 1, 2, 5/2, 3, 4, 5; and the sixth, seventh, and ninth displayed equations in Appendix A have been corrected to read as follows:

$$u'(x) = \begin{cases} \frac{21}{2} - x & \text{if } 0 \leq x \leq \frac{5}{2}, \\ 18 - 4x & \text{if } \frac{5}{2} \leq x \leq 4, \\ 6 - x & \text{if } 4 \leq x \leq 5. \end{cases} \qquad -u''(x) = \begin{cases} 1 & \text{if } 0 \leq x < \frac{5}{2}, \\ 4 & \text{if } \frac{5}{2} \leq x < 4, \\ 1 & \text{if } 4 \leq x \leq 5. \end{cases}$$

$$\begin{aligned} E_F(u) - E_G(u) &= \int_0^5 u'(x)[G(x) - F(x)]\, dx \\ &= u'(5)[G^{(2)}(5) - F^{(2)}(5)] \\ &\quad + \int_0^{5/2} [-u''(x)][G^{(2)}(x) - F^{(2)}(x)]\, dx \\ &\quad + \int_{5/2}^4 [-u''(x)][G^{(2)}(x) - F^{(2)}(x)]\, dx \\ &\quad + \int_4^5 [-u''(x)][G^{(2)}(x) - F^{(2)}(x)]\, dx \\ &= 1 \times (\tfrac{5}{2} - \tfrac{9}{4}) + 1 \times \tfrac{3}{16} + 4 \times (-\tfrac{3}{16}) + 1 \times \tfrac{1}{8} \\ &= -\tfrac{3}{16} \\ &< 0. \end{aligned}$$