How Inefficient is the $1/N$ Asset-Allocation Strategy?*

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Abstract

In this paper, we compare the out-of-sample performance of the rule allocating $1/N$ to each of the N available assets to several static and dynamic models of optimal asset-allocation for ten datasets. We devote particular attention to models the literature has proposed to account for estimation and model error. We find that the $1/N$ asset-allocation rule typically has a higher out-of-sample Sharpe ratio, a higher certainty-equivalent return, and a lower turnover than optimal asset allocation policies. The intuition for the poor performance of the policies from the optimizing models is that the gain from optimal diversification relative to naïve diversification under the $1/N$ rule is typically smaller than the loss arising from having to use as inputs for the optimizing models parameters that are estimated with error rather than known precisely. Simulations show that the performance of optimal strategies relative to the $1/N$ rule improves with the length of the estimation window, which reduces estimation error. For instance, for the case where wealth can be allocated across four risky assets with an average cross-sectional annual idiosyncratic volatility of 20%, it takes an estimation window of 50 years in order for the classical mean-variance policy implemented using maximum-likelihood estimates of the moments to outperform $1/N$. But if the average idiosyncratic volatility drops to 10%, the length of the required estimation window increases to 500 years; and, when the number of assets increases to 100 while average idiosyncratic volatility is 20%, the length of the required estimation window is more than 1,000 years.

Keywords: Portfolio choice, asset allocation, investment management.
JEL Classification: G11.

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1 Introduction

In about the 4th century, Rabbi Issac bar Aha proposed the following rule for asset allocation:[1] "One should always divide his wealth into three parts: a third in land, a third in merchandise, and a third ready to hand." After a brief lull in the literature on asset allocation, there have been considerable advances in the last fifty years.[2] Markowitz (1952) identified the optimal rule for allocating wealth across risky assets in a static setting. Tobin (1958) showed how the optimal portfolio would consist of only two funds if the investor could hold a riskfree asset in addition to risky assets, while Sharpe (1964) and Lintner (1965) explained that in equilibrium the risky-asset portfolio would be the market portfolio. Samuelson (1969) and Merton (1969) showed that these portfolio rules would be optimal even in a multiperiod setting if the investment opportunity set is constant. Merton (1971) determined the optimal portfolio policies when the investment opportunity set is stochastic.

Implementing empirically the portfolio policies suggested by the theoretical models described above requires one to first estimate the parameters of the model[3] for a particular set of asset returns and then solve for the optimal portfolio weights.[4] Traditionally, this estimation has been done using methods from classical statistics such as maximum likelihood, ordinary least squares, and generalized methods of moments. But portfolios constructed using point estimates from classical econometric methods have extreme weights that fluctuate substantially over time.[5] One approach adopted in the literature to deal with these problems has been to use Bayesian "shrinkage" estimators that incorporate a prior.[6] A second approach, proposed by Black and Litterman (1990, 1992), combines two sets of priors—one based on an equilibrium asset pricing model and the other based on the subjective views of the investor. A third approach imposes portfolio constraints prohibiting shortsales (Frost and Savarino (1988) and Chopra (1993)), which Jagannathan and Ma (2003) show can be interpreted as shrinking the extreme elements of the covariance matrix. Recently, Michaud (1998) has advocated the use of resampling methods.[7]

Our objective in this paper is to evaluate how inefficient it is to use the *naïve* $1/N$ asset-allocation rule, where $1/N$ is allocated to each of the N assets available for investment, rather than static and dynamic models of *optimal* asset allocation. We consider two versions of the $1/N$ rule: (i) the investor constantly rebalances the portfolio in order to maintain the $1/N$ allocation over

[1] Babylonian Talmud: Tractate Baba Mezi'a, folio 42a.

[2] For a detailed survey of this literature, see Campbell and Viceira (2002) and Brandt (2004).

[3] The parameters to be estimated in the case of static portfolio models are the expected returns vector and the variance-covariance of returns matrix, while for models of dynamic portfolio choice one needs to estimate the parameters for the processes driving the riskfree interest rate, expected returns on the risky assets, and the volatilities and correlations of the risky asset returns.

[4] Recent work has focused on either deriving explicit analytic expressions for the optimal portfolio policies for particular specifications of the stochastic investment environment opportunity set, as in Brennan and Xia (2000, 2002) Campbell and Viceira (1999, 2001), Campbell, Chan, and Viceira (2003), Campbell, Cocco, Gomes, and Viceira (2001), Chacko and Viceira (2004), Kim and Omberg (1996), Liu (2001), Skiadas and Schroder (1999), Wachter (2002), and Xia (2001) or using numerical methods to solve these models, as in Balduzzi and Lynch (1999), Bansal, Dahlquist, and Harvey (2004), Brennan, Schwartz, and Lagnado (1997), Lynch (2001), and Lynch and Balduzzi (2000).

[5] For a discussion of the problems entailed in implementing mean-variance optimal portfolios, see Hodges and Brealey (1978), Michaud (1989), Best and Grauer (1991), and Litterman (2003).

[6] A detailed description of the Bayesian models is provided in Sections 2.2.3 and 2.2.4.

[7] Scherer (2002) describes the resampling approach in detail and discusses some of its limitations, while Harvey, Liechty, Liechty, and Müller (2003) discuss other limitations and provide an estimate of the loss incurred by an investor who chooses a portfolio based on this approach.

time; and, (ii) the investor allocates $1/N$ at the initial date and then holds this portfolio until the terminal date ("buy-and-hold").

There are several reasons for studying the $1/N$ asset-allocation rule. One, it is easy to implement because it does not require any estimation or optimization. Two, despite the sophisticated theoretical models developed in the last fifty years and the advances in methods for estimating the parameters for these models, investors continue to use such simple allocation rules for allocating their wealth across assets. For instance, Benartzi and Thaler (2001) and Liang and Weisbenner (2002) document that investors allocate their wealth across assets using the naïve $1/N$-rule.[8] There is evidence also that investors often take the "path of least resistance" and exhibit inertia when making investment and rebalancing decisions. For instance, in allocating their wealth in pension schemes, employees often accept the default asset allocation decision made by employers (Madrian and Shea (2000) and Choi, Laibson, Madrian, and Metrick (2001)), and in contrast to what optimal asset-allocation models would suggest, many employees never rebalance these initial allocations (Choi, Laibson, Madrian, and Metrick (2004)). Finally, MacKinlay and Pastor (2000) show that when a risk factor is missing from an asset pricing model, the resulting mispricing affects the covariance matrix of residuals, and if one exploits this, then under the assumption that all assets have the same expected return, one gets the $1/N$ rule.[9]

We compare the out-of-sample performance of the $1/N$ rule to that of strategies from about ten models of static and dynamic asset allocation across ten datasets using three performance criteria. These three criteria are: (i) the out-of-sample Sharpe ratio of each asset-allocation strategy; (ii) the certainty equivalent (CEQ), which for the static models is obtained from the expected utility for a mean-variance investor and for the dynamic models is obtained from the lifetime expected utility of an investor with Epstein and Zin (1989) utility;[10] and, (iii) the turnover (trading volume) for each portfolio strategy.

In order to do a comprehensive study that analyzes the out-of sample performance of a wide class of asset allocation strategies accounting for both changes in the opportunity set (dynamic strategies) as well as model and estimation error (Bayesian and shrinkage methods), we consider over ten asset-allocation models.[11] The various models we consider are listed in Table 1. The *static* models of optimal asset allocation that we study include the Markowitz (1952) mean-variance optimal portfolio where the expected returns and variance-covariance matrix are estimated by their sample counterparts, the minimum-variance portfolio, the Bayes-Stein shrinkage portfolio studied by Jorion (1985, 1986), the "Data-and-Model" approach in Pástor (2000), Pástor and Stambaugh

[8]Huberman and Jiang (2004) find, based on a study of half a million participants in more than six hundred 401(k) pension plans, that participants tend to invest in only a small number of funds of the total funds offered to them, and that participants tend to allocate their contributions evenly across the funds that they use, with the tendency weakening with the number of funds used.

[9]Another reason to consider the $1/N$ strategy is that in their study of an evolutionary market to examine which rules dominate the market in the long run, that is, which rules are evolutionary stable, Evstigneev, Hens, and Schenk-Hopp (2004) find that the $1/N$ strategy does quite well.

[10]The CEQ and the Sharpe ratio differ because the CEQ for the static models is a linear function of variance, whereas the Sharpe ratio is a nonlinear function of standard deviation. For dynamic models, where agents maximize the expected utility of lifetime consumption, the CEQ depends also on higher moments of the return distribution, and so that is another reason for the two measures to differ.

[11]In earlier work, Bloomfield, Leftwich, and Long (1977) show that mean-variance optimal portfolios do not outperform an equally-weighted portfolio and Jorion (1991) finds that the equally-weighted and value-weighted indices have an out-of-sample performance similar to that of the minimum-variance portfolio and to the tangency portfolio obtained with shrinkage methods. More recently, Carlson, Chapman, Kaniel, and Yan (2004), using a general equilibrium model with heterogeneous agents, analyze the role of predictability in asset returns and conclude that simple unconditional consumption and portfolio rules outperform, in terms of utility cost, conditional optimal policies.

(2000) and Wang (2004), the "Three-Fund" approach of Kan and Zhou (2005), and the short-sale-constrained portfolios of Frost and Savarino (1988), Chopra (1993), and Jagannathan and Ma (2003). The two models of *dynamic* asset allocation that we consider allow for stochastic interest rates (Campbell and Viceira, 2001) and for time-varying expected returns on assets (Campbell and Viceira, 1999; Campbell, Chan, and Viceira, 2003).[12]

The ten datasets for which we evaluate the performance of the different asset allocation models are listed in Table 2. There are several reasons why we decided to consider more than a single dataset. One, we did not want our results to be limited to a particular dataset; this was specially important given the nature of our findings. Two, when comparing the performance of the $1/N$ strategy to that of strategies from the various optimal asset allocation models developed in other papers, we wished to use the same dataset as that used in the original paper proposing a particular asset-allocation model. Finally, to understand better the results that we find based on the empirical datasets we also perform our analysis on simulated data.

Our main finding is that the $1/N$ allocation rule (with or without rebalancing at each trading date) is far from inefficient. In fact, the $1/N$ strategy performs quite well out-of-sample: it often has a higher Sharpe ratio and lower turnover than the policies suggested by both the static and the dynamic models of optimal asset allocation. The reason for this result is that the gain from optimal diversification relative to naïve diversification under the $1/N$ rule is typically smaller than the loss arising from using inputs for the optimizing models that have been estimated with error. That is, the optimizing models do have a higher Sharpe ratio than the $1/N$ rule *in-sample*, but out-of-sample the significant estimation error offsets the gains from using a model that delivers optimal diversification as opposed to the naïve diversification offered by the $1/N$ rule. Dynamic asset allocation strategies, which are optimal when the investment opportunity set is stochastic, also do not out-perform the $1/N$ policy because of the difficulty in estimating precisely the processes driving returns. Moreover, the $1/N$ strategy has several other attractive properties. For instance, it is a balanced strategy that always has a positive amount invested in each asset. This is in contrast (a) to unconstrained strategies that often entail very large short positions that may be difficult and costly to implement, and (b) to constrained strategies, which typically entail a zero investment in some of the assets.

Our simulations for the static models indicate that if one knew the *true* value of the moments, then obviously the strategy with the highest out-of-sample Sharpe ratio would be the one suggested by the mean-variance model. But if the moments have to be estimated, then the relative performance of the $1/N$ and the optimal asset-allocation strategies depends on the length of the estimation window. As expected, the longer the estimation window, the better the performance of optimal asset-allocation strategies. Moreover, the estimation window length necessary for the optimal strategies to outperform the $1/N$ strategy increases with: (a) a decrease in the idiosyncratic volatility or (b) an increase in the number of assets. For instance, if wealth can be allocated across four risky assets with an average cross sectional annual idiosyncratic volatility of 20%, we need an estimation window of 50 years in order for the classical mean-variance policy implemented using maximum-likelihood estimates of the moments to outperform $1/N$. But with four assets and

[12]Several models of optimal dynamic asset allocation have been studied in the literature; a partial list is provided in Footnote 4. In order to limit the length of this paper, we restrict attention to two models considered in the book by Campbell and Viceira (2002). These two models allow for stochastic interest rates and time-varying expected returns. The key reason for choosing these two models is that they cover the two important factors of the opportunity set that could be stochastic—the interest rate and expected stock returns. The third factor that could be stochastic, the volatility, has been found to have a much smaller impact on dynamic asset allocation (Chacko and Viceira, 2004).

a lower idiosyncratic volatility of 10%, we need 500 years, and with an increase in assets to 100 and with an average idiosyncratic volatility of 20%, we need more than 1000 years. Finally, we also find that for longer estimation windows, because the estimation problem is less severe, the turnover of the optimal strategies decreases and the imposition of constraints has a smaller effect in improving performance.

To understand the intuition for our findings, observe that the vector of optimal weights under the mean-variance model is given by the product of three terms: the risk tolerance of the investor, the inverse of the variance-covariance matrix of returns, and the vector of expected excess returns over the riskfree rate. To implement this portfolio, one needs to estimate both the variance-covariance matrix and also the vector of expected returns. It is well known (see Merton (1980) and the simulation results described above) that a very long time series of data is required in order to estimate expected returns precisely. Similarly, the estimate of the variance-covariance matrix is poorly behaved (see Green and Hollifield (1992) and Jagannathan and Ma (2003)). So, in computing the mean-variance optimal portfolio weights using the sample moments, one ends up multiplying the poorly estimated expected returns by the inverse of the sample covariance matrix, which also contains estimation error. The resulting portfolio weights often consist of extreme positive and negative positions that are far from optimal. Extensions of the mean-variance model are designed to reduce the error in estimating expected returns and to improve the behavior of the variance-covariance matrix (by shrinking the extreme observations). But, the effect of the estimation error is so large that the level of shrinkage proposed by the various optimizing models is not sufficient. The $1/N$ rule performs well because by ignoring the data, it essentially shrinks all moments completely, and this level of shrinkage turns out to be superior.[13] A second reason why the $1/N$ rule performs well in the datasets we consider is that we are using it to allocate wealth across portfolios of assets rather than individual stocks, and these portfolios already contain a certain level of diversification. Thus, the loss from naïve as opposed to optimal diversification is much smaller in the context of asset allocation than it would be for allocating wealth across individual assets.

The rest of the paper is organized as follows. We describe the $1/N$ and optimal asset-allocation strategies (both static and dynamic) in Section 2. Our methodology for comparing the performance of different asset-allocation policies and the datasets for which this comparison is made are described in Section 3. The results from the comparison of the performance of $1/N$ strategy to those from the optimizing models are given in Section 4. Insights from the analysis of simulated data are reported in Section 5. Robustness checks and details of other experiments that we carried out but do not report in the paper are listed in Section 6. We conclude in Section 7. Detailed descriptions of the optimal static and dynamic asset-allocation strategies we considered are contained in Appendix A and B, respectively.

2 Description of asset-allocation models considered

In this section, we describe the three kinds of "models" that we consider: (i) $1/N$ asset-allocation rule; (ii) static or myopic models for optimal asset allocation; and, (iii) dynamic models for optimal asset allocation. These strategies are listed in Table 1, and readers familiar with the literature on static and dynamic models of optimal asset allocation can skip Sections 2.2 and 2.3.

[13]Another reason why the $1/N$ strategy does well, at least for domestic data, is that it implicitly loads on small size and momentum stocks, which have been shown by Brandt, Santa-Clara, and Valkanov (2005) to improve performance.

2.1 The $1/N$ strategy for asset allocation

Suppose that there are N risky assets and one riskless asset. The naïve $1/N$ diversification rule allocates $1/N$ to each of the N risky assets available for investment. This is the equally-weighted portfolio, where the weight in each risky asset is set equal to $\mathrm{w}_j = 1/N$.[14]

We consider two versions of the $1/N$ asset-allocation strategy – with and without rebalancing. In the case with rebalancing, at each trading date the asset allocation is revised so that after rebalancing the weights are such that the amount invested in each of the assets is again $1/N$. In the case without rebalancing, the strategy is a "buy-and-hold" portfolio where the $1/N$ allocation is made at the beginning of the investment horizon and this position is never rebalanced thereafter.

While these two trading strategies are quite simple, they are less naïve than they may appear. First, they do embody some diversification because wealth is invested across all N assets. So, while they may not be diversified optimally, they do enjoy the benefits of (suboptimal) diversification. Second, the two strategies can be interpreted as ones where the investor actually has a view about future returns. In the case without rebalancing, the portfolio over time assigns higher weight to assets that have done well in the past; so, this can be interpreted as a "momentum" strategy. On the other hand, the strategy with rebalancing requires a reallocation of wealth away from past winners and toward past losers; hence, this can be interpreted as a "contrarian" strategy.

In addition to these two naïve strategies, we also consider the strategy of investing in the value-weighted market portfolio, which we label as the "single-asset strategy."

2.2 Optimal strategies for myopic asset allocation

In order to understand the relation between the various myopic (static) strategies discussed below, we can think of the mean-variance portfolios obtained with sample moments as being at one extreme, where estimation error is ignored all together. On the other extreme is the $1/N$ strategy, which ignores the data all together and "shrinks" all expected returns, variances and covariances on all risky assets to common values. The empirical Bayes-Stein strategy is in-between these two extremes because it shrinks only the expected returns, as shown in equation (7) below. The unconstrained minimum-variance strategy is one that shrinks completely the expected returns to a common value while not adjusting at all the variance-covariance matrix. Imposing constraints on shortselling is equivalent to shrinking partially the variance-covariance matrix, as shown by Jagannathan and Ma (2003); thus, imposing shortselling constraints on the minimum-variance portfolio is equivalent to shrinking the variance-covariance matrix while setting all the expected returns equal to a common value. And, the Bayesian "Data-and-Model" approach of Pástor (2000) shrinks both the expected returns and the variance-covariance matrix, as demonstrated in Wang (2004), and shown in equations (A5) and (A6) in Appendix A. Finally, Kan and Zhou (2005) propose a "three-fund" portfolio rule to deal with estimation error. Their rule belongs to the class of shrinkage rules with a particular choice of shrinkage estimators for means and variances.

[14]One could choose to evaluate the performance of the portfolio of *all* assets, including the investment in the riskfree asset, or one could evaluate the performance of the fund with *only* risky assets. We considered both cases but report the results for only the fund with just risky assets. The main reason for doing so is that the performance of the fund of just risky assets depends only on the asset-allocation decision, while the performance of the portfolio with all assets depends also on market-timing ability. However, the qualitative results for the two cases are similar and the results for the case with investment in all assets are discussed in Section 6.5.

In what follows, we use R_t to denote the N vector of excess returns (over the risk free asset) on the N risky assets at date t and $\mathbf{1}_N$ to denote a N vector of ones.

2.2.1 Sample-based mean-variance optimal portfolios

Mean-variance optimal portfolios are typically derived by assuming that investors have preferences over only the mean and variance of the returns from a chosen portfolio. At each time t, the investor chooses a portfolio w_t to maximize the following

$$\mathrm{w}_t^\top \mu_t - \frac{\gamma}{2}\mathrm{w}_t^\top \Sigma_t \mathrm{w}_t, \tag{1}$$

where μ_t is the N vector of expected *excess* returns over the risk-free asset, Σ_t is the corresponding $N \times N$ variance-covariance matrix, and γ is the investor's risk aversion.

The moments μ_t and Σ_t are estimated through the following sample analogues, where M is used to denote the length of the sample:

$$\hat{\mu}_t = \frac{1}{M}\sum_{s=t-M+1}^{t} R_s, \text{ and} \tag{2}$$

$$\hat{\Sigma}_t = \frac{1}{M-N-2}\sum_{s=t-M+1}^{t} (R_s - \hat{\mu}_t)(R_s - \hat{\mu}_t)^\top. \tag{3}$$

The choice of the estimator $\hat{\Sigma}_t$ is motivated by the fact that $\hat{\Sigma}_t^{-1}$ is an unbiased estimator of Σ_t^{-1}.

The optimal mean-variance (MV) portfolio chosen at each time t is given by:

$$\hat{\mathrm{w}}_t^{\mathrm{MV}} = \frac{1}{\gamma}\hat{\Sigma}_t^{-1}\hat{\mu}_t. \tag{4}$$

In the presence of a constraint on short selling, the investor solves the same problem as in (1), but with the addition of a constraint that prohibits short sales and/or borrowing.

In the rest of the paper, we refer to the sample-based mean-variance strategy as the "mean-variance" strategy.

2.2.2 Minimum-variance portfolios

Even though the minimum-variance portfolio is not an optimal asset-allocation strategy (except in the limit where the expected returns on all risky assets are assumed to be equal), we consider this strategy because of the attention it has received in Chan, Karceski, and Lakonishok (1999) and Jagannathan and Ma (2003).

The weights allocated to the risky assets at any time t in the minimum-variance portfolio are obtained by minimizing the variance of the portfolio:

$$\mathrm{w}_t^\top \Sigma_t \mathrm{w}_t, \tag{5}$$

subject to $\mathrm{w}_t^\top \mathbf{1}_N = 1$, and implementing the solution to the above minimization problem in the data leads to a portfolio given by:

$$\hat{\mathrm{w}}_t^{\mathrm{MIN}} = \frac{1}{\mathbf{1}_N^\top \hat{\Sigma}_t^{-1}\mathbf{1}_N} \times \hat{\Sigma}_t^{-1}\mathbf{1}_N. \tag{6}$$

In the presence of short-sale constraints, one minimizes the expression in (5) while imposing a short-sale constraint. Jagannathan and Ma (2003) show that constraining short sales is equivalent to "shrinking" the extreme values in the variance-covariance matrix. They also find that with a constraint on short-sales, "the sample covariance matrix performs as well as covariance matrix estimates based on factor models, shrinkage estimators and daily data" (p. 1651). Because of this finding, we do not include in our comparison the analysis of the performance of other models, such as Best and Grauer (1992), Ledoit (1996), Chan, Karceski, and Lakonishok (1999), and Ledoit and Wolf (2003), that have been developed to deal explicitly with the problems in estimating the covariance matrix in the context of portfolio optimization.[15]

2.2.3 Bayes-Stein portfolios

In the literature, the Bayesian approach to estimation error has been implemented in different ways. Barry (1974), and Bawa, Brown, and Klein (1979), use either a non-informative diffuse prior or a predictive distribution obtained by integrating over the unknown parameter. In a second implementation, Jobson and Korkie (1980), Jorion (1985, 1986), Frost and Savarino (1986), and Dumas and Jacquillat (1990), use empirical Bayes estimators, which shrink estimated returns closer to a common value and move the portfolio weights closer to the global minimum-variance portfolio. In a third implementation, Pástor (2000), and Pástor and Stambaugh (2000) use the equilibrium implications of an asset pricing model to establish a prior; thus, in the case where one uses the CAPM to establish the prior, the resulting weights move closer to those for a value-weighted portfolio.[16] We do not report the performance of the Bayes-diffuse-prior portfolio because it is virtually indistinguishable from the mean-variance case.[17] We discuss below the empirical Bayes-Stein approach for dealing with estimation error, and in Section 2.2.4, the approach where the data is supplemented with a belief that returns are generated by a particular asset-pricing model.

The Bayes-Stein (BS) portfolio is obtained by solving the problem in (1), but where instead of the sample estimates for μ and Σ in (2) and (3), the investor uses *shrinkage* estimators, defined as a convex combination of the sample mean $\hat{\mu}$ and a common (global) mean $\overline{\mu}$.[18]

In our implementation of shrinkage estimation, we rely on the Bayesian interpretation of the shrinkage estimator (hence the name "Bayes-Stein" portfolios) and take the grand mean $\overline{\mu}$ to be the mean of the minimum-variance portfolio, μ^{MIN}. More specifically, following Jorion (1986), we use the following shrinkage estimator for the expected return and covariance matrix

$$\hat{\mu}_t^{\mathrm{BS}} = (1-\hat{\phi})\,\hat{\mu}_t + \hat{\phi}\,\hat{\mu}_t^{\mathrm{MIN}}, \tag{7}$$

$$\hat{\Sigma}_t^{\mathrm{BS}} = \hat{\Sigma}_t\left(1+\frac{1}{M+\hat{\lambda}}\right) + \frac{\hat{\lambda}}{M(M+1+\hat{\lambda})}\frac{\mathbf{1}_N\mathbf{1}_N^\top}{\mathbf{1}_N^\top\hat{\Sigma}_t^{-1}\mathbf{1}_N}, \tag{8}$$

[15]See Sections III.B and III.C of Jagannathan and Ma (2003) for an extensive discussion of the performance of other models of estimating the sample covariance matrix.

[16]The models discussed above use Bayesian estimation methods in the context of static asset allocation. Bayesian methods have also been used in the context of dynamic asset allocation, where the investment opportunity set is stochastic; see, for instance, Avramov (2004), Barberis (2000), Johannes, Polson, and Stroud (2002), and Kandel and Stambaugh (1996).

[17]The reason for this is that for the case of the Bayesian diffuse-prior portfolio, parameter uncertainty is incorporated by inflating the variance-covariance matrix by the factor $1+\frac{1}{M}$ (see Bawa, Brown, and Klein (1979)), while still using the historical mean as a predictor of expected returns. For large enough M, 120 in our case, this correction to the variance-covariance matrix has only a negligible effect on performance of the mean-variance portfolio.

[18]The concept of shrinkage estimation was developed by James and Stein (1961).

where $\hat{\Sigma}_t$ is defined in (3), $\hat{\mu}_t^{\text{MIN}}$ is the average excess return on the sample global minimum-variance portfolio, $\hat{\mu}_t^{\text{MIN}} = \frac{\hat{\mu}_t \hat{\Sigma}_t^{-1} \mathbf{1}_N}{\mathbf{1}_N^\top \hat{\Sigma}_t^{-1} \mathbf{1}_N}$, $\hat{\phi} = \frac{\hat{\lambda}}{M+\hat{\lambda}}$, and $\hat{\lambda} = \frac{N+2}{(\hat{\mu}-\mu^{\text{MIN}})^\top \hat{\Sigma}^{-1}(\hat{\mu}-\mu^{\text{MIN}})}$.

Under the Bayes-Stein approach, the portfolio weights at each date t is given by

$$\hat{\text{w}}_t^{\text{BS}} = \frac{1}{\gamma} \left(\hat{\Sigma}_t^{\text{BS}}\right)^{-1} \hat{\mu}_t^{\text{BS}}. \tag{9}$$

In the presence of a constraint on short selling, the investor solves the same problem as in (1), where the expected returns and variance-covariance matrix are given by the Bayes-Stein estimators, $\hat{\mu}_t^{\text{BS}}$ and $\hat{\Sigma}_t^{\text{BS}}$, and with the additional constraint that prohibits short sales and/or borrowing.

2.2.4 Bayesian "Data-and-Model" portfolios

Under the "Data-and-Model" (DM) approach developed in Pástor (2000), and Pástor and Stambaugh (2000), estimation of the moments of asset returns is done using not just the data but also using the belief that the asset returns are generated by a particular asset-pricing model. We describe the main features of this approach in Appendix A.1.

Under the Bayesian "Data-and-Model" approach, the resulting portfolio weights at each time t are, given by

$$\hat{\text{w}}_t^{\text{DM}} = \frac{1}{\gamma} \left(\hat{\Sigma}_t^{\text{DM}}\right)^{-1} \hat{\mu}_t^{\text{DM}}, \tag{10}$$

where $\hat{\mu}_t^{\text{DM}}$ and $\hat{\Sigma}_t^{\text{DM}}$ are estimators of the expected return and variance covariance matrix that incorporate the belief of a Bayesian investor about the validity of a particular asset pricing model; the expressions for these estimators are given in equations (A5) and (A6) of the appendix, and in our empirical analysis, we consider the case where the investor believes in the model with a subjective probability of 50%. The equations for the estimators, derived in Wang (2004), show that the Bayesian "Data-and-Model" approach implies a linear shrinkage for expected returns and a quadratic shrinkage for the variance-covariance matrix.

2.2.5 Optimal "Three-Fund" portfolios

To improve on the models that use Bayesian shrinkage estimators, Kan and Zhou (2005) propose a "three-fund" portfolio rule, where the role of the third fund is to minimize "estimation risk." The intuition underlying their result is that, because estimation risk cannot be diversified away by holding just a combination of the tangency portfolio and of the risk-free asset, an investor will benefit from holding also some other risky-asset portfolio, that is, a third fund. Kan and Zhou search for this optimal three-fund portfolio rule in the class of portfolios that can be expressed as a combination of the sample tangency portfolio and the minimum-variance portfolio; that is,

$$\hat{\text{w}}_t^{\text{KZ}} = \frac{1}{\gamma}(c\,\hat{\Sigma}_t^{-1}\hat{\mu}_t + d\,\hat{\Sigma}_t^{-1}\mathbf{1}_N), \tag{11}$$

where c and d are chosen optimally to maximize the expected utility of a mean-variance investor.[19]

[19]The expression for the optimal portfolio weight, with the optimal expressions for c and d substituted in, is reported in equation (A14) of Appendix A.2.

2.3 Optimal strategies for dynamic asset allocation

In this section, we describe optimal *dynamic* asset allocation policies that allow for changes in the investment opportunity set. In particular, we consider two cases: the first allows for time-variation in the short-term real interest rate (keeping fixed the moments of the excess returns on the remaining risky assets) and the second allows for time variation in the risk-premia demanded by risky assets. We rely on the approach in Campbell and Viceira (1999, 2001, 2002) and Campbell, Chan, and Viceira (2003) to estimate the parameters of the model and to obtain the optimal consumption and portfolio policies. Below we summarize the structure of the optimal consumption and portfolio rules for the two cases we consider, with the details of their derivation provided in Appendix B.

2.3.1 Stochastic interest rates

Campbell and Viceira (2001) study intertemporal asset allocation in the presence of stochastic real interest rates. Their goal is to develop a theory of long-term portfolio choice when the investment opportunity set is changing over time because of changes in *only* the short-term real interest rate; all the moments of *excess* returns (over the short-rate) on all risky assets are assumed to be constant.[20]

In order to obtain bond returns that allow empirically reasonable predictions about an investor's holding of short-term bonds, long-term bonds and equities, Campbell and Viceira (2001) develop a two-factor model of the term structure, where the two factors are (i) the expected log return on the short-term indexed bond and (ii) the expected log rate of inflation. For this model, Campbell and Viceira (2001) obtain the following linear specification of the portfolio policy

$$\hat{\mathrm{w}}_t^{\mathrm{CV}} = \frac{1}{\gamma}\hat{\boldsymbol{\Sigma}}_t^{-1}\hat{\mathbf{a}}, \tag{12}$$

where $\hat{\boldsymbol{\Sigma}}_t$ is the time t estimate of the variance-covariance matrix for excess returns over the reference (short-term) asset and $\hat{\mathbf{a}}$ is a vector containing the mean excess return over the reference asset (*myopic component*) and conditional covariances with the consumption-wealth ratio (*hedging component*). The optimal log-consumption-wealth ratio ($c_t - w_t$) is shown to be a linear function of the expected log return x_t on the one-period indexed bond,

$$c_t - w_t = b_0 + b_1 x_t, \tag{13}$$

where the quantities b_0 and b_1 along with other details about the model are given in Appendix B.1. In our implementation, we use equations (12) and (13) as the optimal portfolio and consumption rules in the presence of stochastic interest rates.

2.3.2 Time-varying expected returns

Campbell, Chan, and Viceira (2003) study dynamic asset allocation with time-varying expected returns. To model the predictability of asset returns, they use a vector autoregressive (VAR) system of the form

$$z_{t+1} = \boldsymbol{\Phi}_0 + \boldsymbol{\Phi}_1 \mathbf{z}_t + \mathbf{v}_{t+1}, \tag{14}$$

where $\mathbf{z}_t$ is a vector containing both the returns on the assets and the state variables chosen to describe their dynamics, $\boldsymbol{\Phi}_0$ and $\boldsymbol{\Phi}_1$ are coefficients and $\mathbf{v}_{t+1}$ a vector of normal i.i.d. shocks.

[20] This assumption of constant risk premium implies that the expected returns on risky assets can change over time but only in parallel with the riskless interest rate.

Campbell, Chan, and Viceira (2003) then show that the optimal portfolio, $\hat{\mathrm{w}}_t^{\mathrm{CCV}}$, can be expressed as a linear function of the state variables $\mathbf{z}_t$

$$\hat{\mathrm{w}}_t^{\mathrm{CCV}} = \hat{\mathbf{A}}_0 + \hat{\mathbf{A}}_1\,\mathbf{z}_t, \tag{15}$$

where the definitions of $\hat{\mathbf{A}}_0$ and $\hat{\mathbf{A}}_1$, along with other details of the model, are given in Appendix B.2. They also show that the optimal log consumption-wealth ratio $(c_t - w_t)$ can be expressed as a *quadratic* function of the state variables $\mathbf{z}_t$

$$c_t - w_t = b_0 + \mathbf{B}_1^{\top}\mathbf{z}_t + \mathbf{z}_t^{\top}\mathbf{B}_2\mathbf{z}_t, \tag{16}$$

where b_0, $\mathbf{B}_1$ and $\mathbf{B}_2$ are defined in Appendix B.2. In our implementation, we use equations (15) and (16) as the optimal portfolio and consumption rules when expected returns are time-varying.

3 Description of methodology, performance measures & datasets

This section is divided into three parts. In the first part, we explain our methodology for comparing the performance of the $1/N$ asset-allocation rule to the performance of strategies from various optimizing models. In the second part, we describe the various measures used to evaluate and compare the performance of the different asset-allocation strategies. In the third part, we describe the various datasets that we use to compare the performance of the $1/N$ and optimal asset-allocation strategies. The results of this comparison are reported in Section 4, and the results based on simulated data are given in Section 5.

3.1 Methodology: The steps in comparing portfolio performance

Below, we provide the details of each step in our comparison of the performance of the $1/N$ asset-allocation rule to that of strategies from the various static and dynamic models of optimal asset allocation. There were many choices one could have made in undertaking each step of this comparison. We considered several alternatives for each step, but below we report the results for only one set of choices. The main reason for this is that the qualitative results are not very different for the different alternatives we considered. The results we report are the ones that are typically *least favorable* for the $1/N$ strategy; for instance, we considered different lengths for the estimation windows, but report the case only for the longest estimation window for which the effect of estimation error is the smallest. The results for the others specifications we considered but decided not to report are summarized in Section 6.

The analysis of each optimal strategy consists of the following six steps. One, we choose a window over which to estimate the parameters. We denote the length of the estimation window by $M \leq T$, where T denotes the total number of observations. We choose an estimation window of $M = 120$ data points; for monthly datasets, this corresponds to ten years, while for quarterly data this corresponds to thirty years.[21] Two, we estimate the parameter values over the estimation window for the particular asset-allocation model being considered. Three, using these estimated parameters to form forecasts of future moments of returns (expected returns, variances, etc.), we solve the model for the optimal portfolio weights. Four, we measure the return from holding the

[21] To check the robustness of our results we also considered an estimation window of $M = 60$, which corresponds to five years for monthly data and fifteen years for quarterly data. For the simulated data, we consider estimation windows of 10, 100, 250, and 500 years.

portfolio with these weights over the next period, that is, *out-of-sample.* For monthly data, the out-of-sample holding period is one month, while for quarterly data it is one quarter.[22] Five, we repeat this "rolling-window" procedure for the next period, by including the data for the new date and dropping the data for the earliest period. We continue doing this until the end of the dataset is reached. Finally, for each portfolio strategy we compute the quantities we wish to report. As explained in Footnote 14, we report these quantities for the fund with investment in just the risky assets.

In the section below, we describe each of the quantities that is reported and explain how this quantity is computed.

3.2 Summary statistics reported for each asset-allocation strategy

In the context of the experiment we are considering, *in-sample* refers to the case where the statistics are based on the *entire* time-series of excess returns; that is, the estimation window is equal to the length of the data series: $M = T$. In contrast, the *out-of-sample* statistics refer to the case where the window for estimating the parameters of the model and the resulting portfolio weights at each date is limited to the last $M = 120 < T$ data points, and the return on this portfolio depends on the returns on each asset in the *next* period. The model parameters and weights are then re-estimated using a rolling-window approach.

3.2.1 In-sample mean, variance, and Sharpe ratio of returns

Let k denote a particular asset-allocation model and w^k denote the portfolio weights recommended by such a model. In general, these weights will be a function of the moments of asset returns estimated according to the methodology suggested by approach k (for instance, maximum-likelihood for the classical mean-variance approach, and Bayesian estimation for the Bayes-Stein approach). Let $\hat{\mu}^k$ and $\hat{\Sigma}^k$ denote the estimate of such moments obtained by using the *entire* time series of excess returns. Then, for a particular strategy k, the *in-sample* mean return of the portfolio is $\hat{\mu}^{k^\top}\, \hat{\mathrm{w}}^k$, the in-sample variance of the portfolio is $\hat{\mathrm{w}}^{k^\top}\, \hat{\Sigma}^k\, \hat{\mathrm{w}}^k$, and the in-sample estimate of the Sharpe ratio, $\widehat{\mathrm{SR}}_{\mathrm{IS}}$, is

$$\widehat{\mathrm{SR}}^k_{\mathrm{IS}} = \frac{\hat{\mu}^{k^\top}\, \hat{\mathrm{w}}^k}{\sqrt{\hat{\mathrm{w}}^{k^\top}\, \hat{\Sigma}^k\, \hat{\mathrm{w}}^k}}. \tag{17}$$

3.2.2 Out-of-sample mean, variance and Sharpe ratio of returns

To compute the out-of-sample returns on the portfolio, we follow the "rolling window" scheme described above to estimate the moments for each individual asset at each date t, and determine the vector of portfolio weights $\hat{\mathrm{w}}^k_t$ for the asset-allocation model k. Holding the portfolio $\hat{\mathrm{w}}^k_t$ for one period gives the following *out-of-sample* excess return at time $t+1$: $\hat{R}^k_{t+1} = \hat{\mathrm{w}}^{k^\top}_t R_{t+1}$, where R_{t+1} denotes the returns in excess of the benchmark (risk-free) rate at time $t+1$. After collecting the time series of $T - M$ excess returns R^k_t, the out-of-sample mean, variance, and Sharpe ratio of

[22] As one of the robustness checks, we considered also a holding period of one year; this does not have a qualitative effect on the results.

returns are,

$$\hat{\mu}^k = \frac{1}{T-M}\sum_{s=1}^{T-M}\hat{R}_s^k, \tag{18}$$

$$(\hat{\sigma}^k)^2 = \frac{1}{T-M-1}\sum_{s=1}^{T-M}(\hat{R}_s^k-\hat{\mu}^k)^2, \tag{19}$$

$$\widehat{\mathrm{SR}}_{\mathrm{OS}}^k = \frac{\hat{\mu}_k}{\hat{\sigma}_k}. \tag{20}$$

In order to measure the statistical significance of the difference in the Sharpe ratio of a particular strategy from that of the two $1/N$ strategies (one with rebalancing and the other without rebalancing) that serve as benchmarks, we also report the P-value for the Sharpe ratio of each strategy relative to these two strategies. These P-values are computed using the same approach as in Jobson and Korkie (1981), with the correction pointed out in Memmel (2003).[23]

3.2.3 Certainty Equivalent (CEQ) for static models

Each point of the time series of excess returns R_t^k generated by implementing strategy k can also be interpreted as an independent realization of a random variable. Following this logic, which is typical of the literature on static models of optimal asset allocation, we can define the concept of *certainty equivalent* return (CEQ) for a given strategy k. The CEQ is hence the return that makes an investor indifferent between getting CEQ for sure or a gamble with random realization $\tilde{X}$. Formally:

$$u(\mathrm{CEQ}) = E[u(\tilde{X})], \tag{21}$$

where $u(\cdot)$ is the utility function. If the investor's preferences can be summarized by the first two moments of the distribution of $\tilde{x}$, then, the CEQ is typically defined as

$$\mathrm{CEQ}_{\mathrm{static}}^k = \hat{\mu}^k - \frac{\gamma}{2}(\hat{\sigma}^k)^2, \tag{22}$$

where $\hat{\mu}^k$ and $(\hat{\sigma}^k)^2$ refer to the first two moments of the realized excess returns under strategy k as given in equations (18) and (19), and γ is the risk aversion of the investor.[24]

Just as we did for the Sharpe ratios, in order to measure the statistical significance of the difference in the CEQ of a particular strategy from that of the two $1/N$ strategies that serve as benchmarks (with and without rebalancing), we also report the P-value for the CEQ of each strategy relative to these two strategies. These P-values are computed by relying on the asymptotic properties of functional forms of the estimators of means and variance.[25]

[23] Specifically, consider two portfolios i and n, with $\hat{\mu}_i$, $\hat{\mu}_n$, $\hat{\sigma}_i$, $\hat{\sigma}_n$, $\hat{\sigma}_{i,n}$ being their respective estimated means, variances and covariances over a sample of size $T-M$. We wish to test the hypothesis that the Sharpe ratios of these two portfolios are identical, i.e., $\mathrm{H}_0 : \frac{\hat{\mu}_i}{\hat{\sigma}_i} - \frac{\hat{\mu}_n}{\hat{\sigma}_n} = 0$. The Jobson and Korkie test statistic is defined as follows, $\hat{z}_{\mathrm{JK}} = \frac{\hat{\sigma}_n\hat{\mu}_i - \hat{\sigma}_i\hat{\mu}_n}{\sqrt{\hat{\vartheta}}}$, where $\hat{\vartheta} = \frac{1}{T-M}(2\hat{\sigma}_i^2\hat{\sigma}_n^2 - 2\hat{\sigma}_i\hat{\sigma}_n\hat{\sigma}_{i,n} + \frac{1}{2}\hat{\mu}_i^2\hat{\sigma}_n^2 + \frac{1}{2}\hat{\mu}_n^2\hat{\sigma}_i^2 - \frac{\hat{\mu}_i\hat{\mu}_n}{\hat{\sigma}_i\hat{\sigma}_n}\hat{\sigma}_{i,n}^2)$. The test statistic $\hat{z}_{\mathrm{JK}}$ is asymptotically distributed as a standard normal.

[24] To be precise, the definition in equation (22) refers to the level of expected utility of a mean-variance investor, and it can be shown that this is approximately the CEQ of an investor with quadratic utility. Notwithstanding this caveat, and following common practice, we interpret it as the certainty equivalent for strategy k.

[25] If we define v to be the vector of moments $v = (\mu_i, \mu_n, \sigma_i, \sigma_n)$, and $\hat{v}$ its empirical counterpart obtained from a sample of size $T-M$, and $f(\cdot)$ is the expression for the difference in the certainty equivalent of the two strategies i

3.2.4 Certainty Equivalent (CEQ) for dynamic models

For dynamic models of asset allocation, we define the certainty equivalent as the level of initial wealth that would generate the same utility level as that under the optimal consumption and portfolio rules. Formally, if we assume an investor starts at $t = 0$ with one dollar of wealth and follows the optimal consumption and investment strategy (C_t^k, w_t^k), the dynamics of wealth under this strategy are given by

$$W_{t+1}^k = (W_t^k - C_t^k) R_p^k, \tag{24}$$

where R_p^k is the gross return on the optimal portfolio under strategy k. Denoting by $U(\cdot)$ the investor's utility function and by J_0^k the indirect utility derived by using the consumption strategy C^k, the certainty equivalent at time $t = 0$ for strategy k is given by,[26]

$$\mathrm{CEQ}_{\text{dynamic}}^k = U^{-1}(J_0^k). \tag{25}$$

When comparing dynamic models of asset allocation to static models (including $1/N$), we need to decide what consumption policy to use for the static models. To isolate the effect of portfolio choice, we assume that the finite-horizon static models ($1/N$ and also the strategies from the optimizing models) have the *same consumption-wealth ratio* as that under the optimal dynamic strategy. However, because the portfolio policies are different, the *level* of wealth at each date will be different under the static and dynamic models (see equation (24)), leading to differences in consumption, and therefore, differences in lifetime utility. In addition, note that the dynamic portfolio policies we consider are optimal for infinite horizon but the experiment that we consider is by its very nature for a finite horizon. We have implicitly assumed that the optimal infinite-horizon dynamic policies provide a good approximation to the optimal finite-horizon dynamic policies.

3.2.5 Portfolio turnover

To get a sense of the trading volume, we also compute the portfolio turnover for each asset allocation strategy. This measure is related to the level of transactions costs incurred in the implementation of the portfolio strategy, but it is important to realize that in the presence of transactions costs it would no longer be optimal to implement the same portfolio strategy as that from the model without transactions costs. In order to define turnover, let $\hat{\mathrm{w}}_{j,t}^k$ denote the portfolio weight in asset j chosen at time t under strategy k, $\hat{\mathrm{w}}_{j,t+}^k$ the portfolio weight *before* rebalancing but at $t + 1$, and $\hat{\mathrm{w}}_{j,t+1}^k$ the desired portfolio weight at time $t + 1$ (after rebalancing). Then, turnover is defined as the sum of the absolute value of the rebalancing trades across the N available assets and over the

and n, $f(v) = (\mu_i - \frac{\gamma}{2}\sigma_i^2) - (\mu_n - \frac{\gamma}{2}\sigma_n^2)$, then (from, for example, Greene (2002)) the asymptotic distribution of $f(v)$ is $\sqrt{T}(f(\hat{v}) - f(v)) \to \mathcal{N}\left(0, \frac{\partial f}{\partial v}^\top \Theta \frac{\partial f}{\partial v}\right)$, where

$$\Theta = \begin{pmatrix} \sigma_i^2 & \sigma_{i,n} & 0 & 0 \\ \sigma_{in} & \sigma_n^2 & 0 & 0 \\ 0 & 0 & 2\sigma_i^4 & 2\sigma_{i,n}^2 \\ 0 & 0 & 2\sigma_{i,n}^2 & 2\sigma_n^4 \end{pmatrix}. \tag{23}$$

[26]In all our applications we use the Epstein and Zin (1989) preferences described in equation (B1) of Appendix B with elasticity of intertemporal substitution $\psi = 1$.

$T-M$ trading dates, normalized by the total number of trading dates, $T-M$:

$$\text{Turnover} = \frac{1}{T-M}\sum_{t=1}^{T-M}\sum_{j=1}^{N}\left(\left|\hat{\mathrm{w}}_{j,t+1}^{k}-\hat{\mathrm{w}}_{j,t^{+}}^{k}\right|\right). \tag{26}$$

For example, in the case of the $1/N$ strategy, $\mathrm{w}_{j,t}^{k}=\mathrm{w}_{j,t+1}^{k}=1/N$ but $\mathrm{w}_{j,t^{+}}^{k}$ is different, because of the change in asset prices between t and $t+1$, which causes a change in the relative weights in the portfolio.[27] The turnover quantity defined above can be interpreted as the average percentage of wealth traded in each period.[28]

3.2.6 Min, max, average and standard deviation of portfolio weights

So far, all the measures we have considered are at the level of the overall portfolio. To get a sense of the holding in each particular asset over time, we also provide some summary statistics for the weight invested in each asset under the different portfolio strategies. These summary statistics are the time-series minimum, maximum, average, and the standard deviation of the portfolio weight allocated to each asset in the portfolio.

3.3 Details of the various datasets considered

We considered nine empirical datasets of returns. To understand how our results depend on the properties of the data, we also considered a tenth dataset, where the data is generated via simulations; results based on this simulated data are described in Section 5.

A list of the datasets considered is given in Table 2. Note that in addition to the returns on the risky assets described in the datasets, each dataset also includes the rate of return on the 90-day T-bill, obtained from Kenneth French's website, which is used as a benchmark asset (that is, returns are expressed in excess of this short rate). Observe also that Datasets #8 and 9 are quarterly, while the rest are all monthly. Details of each dataset and the motivation for considering it are provided below.

3.3.1 Ten sector portfolios

This dataset consists of monthly excess returns on ten portfolios tracking the sectors composing the S&P500 index.[29] The data span from January, 1981 to December, 2002. The choice of this dataset was motivated by the fact that if one wished to consider an allocation across just domestic equities, then one logical class of assets to consider would be sector portfolios.

[27] Note that we denote the weights for the $1/N$ strategy using w rather than ŵ because for the $1/N$ strategy the weights do not depend on estimated moments.

[28] Our measure of turnover is different from the one commonly used in the mutual fund industry because, instead of considering the *minimum* between the amount sold or purchased, we consider the *sum* of the purchases and sales in each period. If there are no inflows or outflows from the portfolio (that is, the portfolio is self-financing), then the amount sold must be equal to the amount purchased; in this case, our measure of turnover would be exactly double that of the industry definition. Because one is interested in the relative turnover across different strategies, this difference in definition does not affect any inference about relative magnitudes.

[29] We are grateful to Roberto Wessels for creating this dataset and for making it available to us.

3.3.2 Ten industry portfolios

In the second dataset we consider, the risky assets that the investor can hold are ten industry portfolios in the US. The ten industries are: Consumer-Discretionary, Consumer-Staples, Manufacturing, Energy, High-Tech, Telecommunication, Wholesale and Retail, Health, Utilities, and Other. The monthly returns range from July 1963 to November 2004 and were obtained from Kenneth French's website. The motivation for considering this dataset is similar to that for the dataset considered above: if one wished to consider an allocation across domestic equities only, then one way to classify assets would be by industry.

3.3.3 Nine international equity indexes

The next dataset we consider consists of nine international equity indexes whose returns are computed from the month-end US-dollar value of the country equity index for the period January 1970 to July 2001. The equity indices are for Canada, France, Germany, Italy, Japan, Switzerland, United Kingdom, United States and the World. Data are from MSCI (Morgan Stanley Capital International).

There are two motivations for considering this data. One, it does not limit the investor to just domestic assets. Two, this data is similar to that used in Jorion (1985, 1986), where the Bayes-Stein shrinkage model is proposed.

3.3.4 US market, SMB and HML portfolios

We consider an updated version of the dataset used by Pástor (2000) in the paper describing the Bayesian "Data-and-Model" approach to asset allocation. To implement this approach, it is assumed that a factor model dominates the structure of asset returns. The factor is represented by the excess return on the market, defined as the value-weighted return on all NYSE, AMEX and NASDAQ stocks (from CRSP) minus the one-month Treasury bill rate. The other risky assets considered are the Fama-French portfolios, HML and SMB. The former is a zero-cost portfolio that is long in high book-to-market stocks and short in low book-to-market stocks. The latter is a zero-cost portfolio that is long in small stocks and short in big stocks. The data consists of monthly returns from July 1963 to November 2004. The data are taken from Kenneth French's website.

3.3.5 US Market, HML, SMB and Size- and Book-to-Market-sorted portfolio assuming a single-factor model

The next dataset is the one used by Wang (2004) to study the Data-and-Model approach of Pástor (2000) and Pástor and Stambaugh (2000). The data consist of returns on the twenty Fama and French portfolios of firms sorted by size and book to market, in addition to the Fama-French portfolios HML and SMB, and also the market portfolio for a total of 23 risky assets.[30] Returns are monthly and the data span from July 1926 to December 2002. In our first use of this data, we assume that returns are generated by a single-factor model, and we assume the market to be this factor; that is, we assume that the CAPM holds.

[30] As in Wang (2004), we exclude the five portfolio containing the largest firm since the market, SMB and HML are almost a linear combination of the 25 Fama-French portfolios.

3.3.6 US Market, HML, SMB, and Size- and Book-to-Market-sorted portfolio assuming a three-factor model

In this specification, we use the same data just described above, but instead of assuming a single-factor return-generating model, we assume a three-factor model for the return-generating process. The three factors are the market, HML and SMB portfolios (essentially, this is an APT setting).

The motivation for considering this dataset is the finding that the CAPM, which is based on the market portfolio, is not a very good pricing model, and that HML and SMB are factor portfolios that are useful for pricing assets. Thus, it is possible that when we use these additional factor portfolios the Data-and-Model approach will perform better.

3.3.7 US Market, HML, SMB, MOM, and Size- and Book-to-Market-sorted portfolio

In this specification, we use data similar to that described above, but with the addition of a fourth factor portfolio, where the fourth factor is a momentum (MOM) portfolio. We use the methodology of Pástor (2000) to construct Bayesian portfolios, and here the market, HML, SMB, and MOM portfolios serve as the four factor portfolios. There are a total of 497 monthly observations from July 1963 to November 2004. The data are taken from Kenneth French's website.

The motivation for considering this dataset, and in particular, the Data-and-Model strategy based on a return-generating model where the MOM portfolio is one of the factors, is because the empirical literature on asset returns finds some evidence of momentum (see, for instance, Jegadeesh and Titman (1993)). Recall that the $1/N$ rule (without rebalancing) can be viewed as a momentum portfolio. Hence, considering this dataset allows us to examine whether the $1/N$ strategy outperforms the Data-and-Model strategy when the latter explicitly allows for a momentum factor.

3.3.8 Market portfolio and 10-year nominal bond with stochastic interest rates

The motivation for considering this dataset is that it is the one that is used in Campbell and Viceira (2001) to examine the performance of a dynamic model of asset allocation with stochastic interest rates. The assets available for investment are the equity market portfolio, a three-month nominal bond, and a 10-year nominal bond. The two-factor term structure model is estimated by applying a Kalman filter to data on US nominal interest rates, equities, and inflation. In particular, we use nominal zero-coupon yields at maturities three months, one year, three years, and ten years. For equity, we use the value-weighted return, including dividends, on the NYSE, AMEX, and NASDAQ markets. For inflation, we use a consumer price index. The data is quarterly from January 1954 to September 1996.

3.3.9 Market portfolio and 5-year bond with time-varying expected returns

The motivation for considering this dataset is that it is the one used in Campbell and Viceira (1999) and Campbell, Chan, and Viceira (2003) to study a dynamic model for asset allocation with predictability in stock returns. The data consists of returns on three assets, two of which are risky and one is riskless. The riskless rate is represented by the real Treasury bill rate, constructed as the difference between the yield on a 90-day T-bill and log inflation. The two risky assets are, respectively, the value-weighted excess return (including dividends) on the market, which includes all stocks on NYSE, NASDAQ and AMEX, and the excess return on the 5-year bond. The data are quarterly and span from September 1952 to September 1999. This data is obtained from Campbell, Chan, and Viceira (2003) and further details about the data construction are available in their paper.

4 Insights from the nine empirical datasets considered

In this section, we compare the performance of the $1/N$, myopic, and dynamic asset-allocation strategies that are explained in Section 2, using the methodology described in Sections 3.1 and 3.2, for the nine datasets that are described in Section 3.3. Results based on simulated data are described in Section 5.

For each dataset, the results are reported in a separate table. Only the myopic (that is, static) asset-allocation strategies are considered in Tables 3–8, while both static and dynamic strategies are considered in Tables 9 and 10. The format of each table is similar. Column 1 of each table lists the particular statistic that is being reported for each strategy. The rest of the columns report the performance for the various asset-allocation strategies being considered; in order to highlight the statistics for the $1/N$ strategy with rebalancing (in Column 4), these are italicized. The statistics for each strategy are reported in three panels: Panel A gives statistics about the *in-sample* performance; Panel B gives statistics about the *out-of-sample* performance; and, Panel C, gives statistics about the out-of-sample portfolio weights for each risky asset.

Observe that the difference between the in-sample Sharpe ratio of the mean-variance strategy and the $1/N$ strategy gives us a measure of the loss from naïve rather than optimal diversification. This also gives us a measure of how inefficient the $1/N$ portfolio is relative to the mean-variance optimal portfolio, which is on the efficiency frontier. Similarly, the difference between the in-sample and out-of-sample Sharpe ratio for the mean-variance strategy gives us a measure of the effect of estimation error.

Before discussing the results for each particular dataset, we present an overall summary of the findings from the nine empirical datasets we examine.

4.1 Overall summary of findings across all datasets

We find that *in-sample* of all the static models it is the classical mean-variance strategy that has the highest Sharpe ratio, as one would expect by construction. But, *out-of-sample*, the Sharpe ratio is typically lowest for the classical mean-variance strategy because this strategy makes no adjustment at all for estimation risk. The strategies that typically do well in terms of the out-of-sample Sharpe ratio and CEQ are the $1/N$ policies. Of the strategies from the optimizing models, there is no single strategy that always dominates the others: sometimes, it is the Bayes-Stein-constrained strategy that has the highest Sharpe ratio, sometimes it is the minimum-variance strategy without constraints, and sometimes it is the minimum-variance-constrained strategy. Comparing the $1/N$ policy (with or without rebalancing) to the minimum-variance-constrained strategy, which is the focus of a recent paper by Jagannathan and Ma (2003), we find that the Sharpe ratio of the minimum-variance-constrained strategy is significantly superior to that of the $1/N$ strategy for only one dataset, and even in this case the minimum-variance-constrained strategy has a lower CEQ, its turnover is 2-6 times higher, and it has zero investment in several assets leading to an unbalanced portfolio. Dynamic asset allocation strategies, which are optimal when the investment opportunity set is stochastic, also do not out-perform the $1/N$ policy; moreover, the dynamic policies often entail taking very extreme short and large positions in assets.

4.2 Results for ten sector portfolios

The results for the fist dataset, consisting of returns on 10 sector portfolios of the S&P500, are reported in Table 3. Panel A of this tables gives the in-sample Sharpe ratio for different static allocation strategies. From this panel, we see that in-sample it is the mean-variance portfolio that has the highest Sharpe ratio of 0.38481. The Sharpe ratio for the $1/N$ strategies is only 0.18762.[31]

Panel B of Table 3 reports the out-of-sample statistics about the performance of different static allocation strategies. From this panel we see that the strategies with the highest Sharpe ratio are the two $1/N$ strategies: the Sharpe ratio for the case with rebalancing is 0.18762—just as it was in-sample. The Sharpe ratio for the case without rebalancing is 0.17245. The single-asset strategy where one invests entirely in the S&P 500 also performs quite well; it has the third-highest Sharpe ratio, which is 0.14436. The strategy with the lowest Sharpe ratio is the three-fund strategy, followed by the mean-variance strategy. The strategies with constraints perform better than the unconstrained strategies. The Data-and-Model approach in which the investor places 50% confidence in the CAPM ($\omega = 0.5$) also does better than the unconstrained strategies. Looking at the P-values with respect to the $1/N$ rule with rebalancing, we see that out-of-sample its performance is not significantly different from that of the $1/N$ rule without rebalancing (P-value of 0.29), but the P-value for the Data-and-Model strategy is 0.11, for the mean-variance-constrained strategy is 0.09, and for the minimum-variance-constrained strategy is 0.01.

The CEQ of each strategy is also reported in Panel B of Table 3. The ranking of the strategies by CEQ is very similar to that by Sharpe ratios: The $1/N$ strategy with rebalancing has the highest CEQ followed by the $1/N$ strategy without rebalancing, and then the single-asset strategy of holding the market portfolio.

The turnover of each strategy is reported in Panel B of Table 3. The single-asset strategy and the $1/N$ strategy with no rebalancing have zero turnover. The $1/N$ strategy with rebalancing has a turnover of 0.03051 (that is, on average 3.051% of wealth is traded every month), while the turnover of the minimum-variance-constrained strategy is 0.07526. The unconstrained strategies, such as the mean-variance, minimum-variance, Data-and-Model, and three-fund, have much higher turnover than the constrained strategies; for instance, the minimum-variance strategy has a turnover of 0.19964, while the mean-variance strategy has a turnover of 1.1894, which is the highest of all strategies considered.

Panel C of Table 3 gives statistics about the time-series of the weights allocated to each asset under each strategy. From this panel we see that the $1/N$ strategy (with rebalancing) has 1/11 invested in the eleven risky assets available (the 10 sector portfolios and the S&P500) with zero variation, which is measured by the standard deviation. The unconstrained policies have substantial variation in the portfolio weights. The constrained strategies, on the other hand, have lower variation, but typically do not invest at all in some of the assets. For example, the mean-variance-constrained strategy never invests in Energy, Material, Industrials, Consumer-Discretionary, Telecom, Utilities and the S&P500.

[31] In-sample, there is no rebalancing and so the two versions of the $1/N$ strategy are identical.

4.3 Results for ten industry portfolios

The results of this dataset are reported in Table 4. Because the results are very similar to those for the dataset on ten sector portfolios described above, we only discuss the differences.

Panel A of Table 4 shows that in-sample the Sharpe ratio is highest for the mean-variance strategy, 0.21239, which is substantially higher than that for the $1/N$ strategies (without or with rebalancing) that have a Sharpe ratio of only 0.13531. From Panel B, which gives the out-of-sample performance measures, we see that the Sharpe ratio for the mean-variance strategy has dropped to -0.03634 from its in-sample value of 0.21239. A similar drop in performance is noticed for all the other unconstrained strategies: the Sharpe ratio of the three-fund and Bayes-Stein strategies is also negative, while that for the Data-and-Model strategy is 0.01881. The presence of constraints raises the Sharpe ratio substantially relative to that of the unconstrained policies; for instance, the Sharpe ratio for the minimum-variance constrained strategy is 0.14253. The highest Sharpe, however, is for the unconstrained minimum-variance strategy, 0.15536. The Sharpe ratio for the $1/N$ policy with rebalancing, of course, is the same as it was in sample, 0.13531, and the difference with the Sharpe ratio of the minimum-variance strategy has a P-value of 0.29. Even the $1/N$ policy without rebalancing has a Sharpe ratio whose difference from that of the Sharpe ratio for the minimum-variance strategy has a P-value of 0.24. Thus, even though the minimum-variance portfolio has the highest Sharpe ratio, the difference between this and the Sharpe ratio for the $1/N$ strategies is not statistically significant. Similarly, the CEQ of the minimum-variance-constrained strategy is 0.00517, while the CEQ for the $1/N$ policy with rebalancing is 0.00498 and for $1/N$ without rebalancing is 0.00466. Again, these differences are not significant: the P-value is 0.39 for the case with rebalancing and 0.49 for the case without rebalancing.

Turnover for each strategy is reported in the last row of Panel B of Table 4 and is 0.02162 for the $1/N$-with-rebalancing strategy, while the turnover for the strategies from the optimizing models is much higher: for instance, the turnover for the minimum-variance-constrained strategy is 0.0557, for the minimum-variance strategy is 0.46800, and for the Bayes-Stein, Data-and-Model, and mean-variance strategies is more than 100.

Panel C of Table 4 shows that the unconstrained policies, such as the mean-variance, minimum-variance, Bayes-Stein, three-fund, and Data-and-Model, all entail very unreasonable positions that vary from very large negative to very large positive holdings. The constrained policies, on the other hand, have much more reasonable positions.

4.4 Results for nine international equity indexes

The results for asset allocation across international equity indices are presented in Table 5. The in-sample Sharpe ratio for the mean-variance strategy is 0.20902, while that for the $1/N$ strategies is 0.12767 and for the single-asset-strategy (investing in the World index) is 0.12392. From Panel B, we see that out-of-sample it is the minimum-variance portfolio that has the highest Sharpe ratio, but now it is the constrained minimum-variance strategy that has the higher Sharpe ratio of 0.15014 while the unconstrained policy has a Sharpe ratio of 0.14896. The minimum-variance strategies are followed by the $1/N$-with-rebalancing strategy, which has a Sharpe ratio of 0.12767. The P-value for the difference in the Sharpe ratio of the minimum-variance-constrained and the $1/N$-with-rebalancing strategy is 0.15, while the P-value relative to the $1/N$-without-rebalancing strategy is 0.07. The highest CEQ is also for the unconstrained minimum-variance strategy, 0.00543, with the CEQ of the constrained minimum-variance strategy being slightly lower, 0.00537. The CEQ for the $1/N$-with-rebalancing strategy is 0.00461 and for the case without rebalancing is 0.00401; their difference with the CEQ of the minimum-variance strategy is statistically insignificant (P-values are 0.23 and 0.12).

Again, turnover is substantially different for the $1/N$ strategies and the strategies from the optimization models. The single-asset and the $1/N$-without-rebalancing strategies have zero turnover, while that of the $1/N$-with-rebalancing strategy is 0.02931. The turnover of the minimum-variance-constrained strategy is 0.06647, with the other strategies having still higher turnover; for instance, the turnover of the unconstrained minimum-variance strategy is 0.214, the three-fund strategy is 24.01, the Data-and-Model is 40.89, the Bayes-Stein strategy is 51.59, and the mean-variance strategy is 124.14.

The insights from Panel C of Table 5, which summarizes the behavior of the weights in each asset over time for the different strategies, are similar to those for the earlier datasets: the weights for the unconstrained strategies, with the exception of the minimum-variance portfolio, range from very large negative to very large positive positions.

4.5 Results for US market, SMB, and HML portfolios

Results for this dataset are contained in Table 6. As before, Panel A shows that in-sample the mean-variance strategy has the highest Sharpe ratio, 0.28507. The Sharpe ratio for the three-fund strategy is 0.27123. The $1/N$ strategies have a Sharpe ratio of 0.22403, and the only strategy with an even lower in-sample Sharpe ratio is the single-asset strategy of holding the market portfolio, which has a Sharpe ratio of 0.11377.

From Panel B, we see that the strategy with the highest out-of-sample Sharpe ratio of 0.25462 is the three-fund strategy of Kan and Zhou (2005). Quite close to this are the Sharpe ratios for the Bayes-Stein strategy, 0.25363 and the minimum-variance strategies (both with and without constraints), 0.24925. The Sharpe-ratio for the $1/N$-with-rebalancing strategy is, of course, the same as it was in-sample, 0.22403, while that of the $1/N$-without-rebalancing strategy is 0.22711. Again, the difference between the Sharpe ratio of the best strategy from the optimizing asset-allocation models and the $1/N$ strategies is not statistically significant; the P-value of the three-fund strategy against the $1/N$-with-rebalancing strategy is 0.22 and against the $1/N$-without-rebalancing strategy is 0.17.

Panel B of Table 6 also gives the CEQ for the different strategies. The strategy with the highest CEQ is the mean-variance strategy (0.00448) followed by the Data-and-Model strategy (0.00438), the three-fund strategy (0.00437), the Bayes-Stein model (0.00430) and the single-asset strategy (0.00423). The CEQ for the $1/N$-with-rebalancing strategy is 0.00394 (with a P-value of 0.30 against the mean-variance strategy) and the CEQ for the $1/N$-without-rebalancing strategy is 0.00381 (with a P-value of 0.21 against the mean-variance strategy).

Turnover for the different strategies is reported in the last row of Panel B of Table 6. The turnover for the $1/N$-with-rebalancing strategy is 0.0237, while that of the three-fund strategy (which has the highest Sharpe ratio) is 0.06175 and that of the mean-variance strategy (which has the highest CEQ) is 0.06720.

Panel C of Table 6 provides statistics about the evolution of the portfolio weights over time for the different strategies. We see that both the three-fund strategy and the mean-variance strategy entail taking short positions in the SMB portfolio, with the mean-variance portfolio also taking short positions in the market portfolio. Compared to the other datasets studied above, the weights in each asset show substantially less variation for this dataset. This is not surprising, given that the investable assets in this experiments are represented by broad portfolio of stocks.

The reason why the strategies from the Bayes-Stein, minimum-variance, and mean-variance models perform quite well for this dataset, even in the absence of constraints, is that the three factor portfolios (SMB, HML, and MKT) are almost statistically independent from each other. Consequently, the variance-covariance matrix is well-conditioned (nonsingular), and hence, the policies obtained from the optimizing models are less extreme and fluctuate less over time because the error in estimating expected returns is not magnified when the vector of mean excess returns is multiplied by the inverse of the well-conditioned covariance matrix.[32]

4.6 Results for US Market, HML, SMB and Twenty Size- and Book-to-Market-sorted portfolios assuming a single-factor model

Results for this dataset are presented in Table 7. Panel A of this table shows that in-sample the mean-variance strategy has the highest Sharpe ratio (0.50882), while the Sharpe ratio for the $1/N$ strategy is only 0.16744, and the Sharpe ratio for the single-asset strategy of holding the market portfolio is 0.11377.

Panel B shows that the out-of-sample Sharpe ratio of the mean-variance strategy drops steeply to -0.00467. Now, it is the minimum-variance-constrained strategy that has the highest out-of-sample Sharpe ratio, 0.25574, followed by the other constrained strategies (Bayes-Stein-constrained and mean-variance-constrained). The $1/N$-without-rebalancing strategy has a Sharpe ratio of 0.18906, while the $1/N$-with-rebalancing strategy has a Sharpe ratio of 0.16744. The Sharpe ratio for the single-asset strategy (investing in the market portfolio) is only 0.11377, but even this is higher than that of the unconstrained strategies (Data-and-Model, mean-variance, Bayes-Stein, three-fund, and minimum-variance). In contrast to all the other datasets, in this dataset the difference in the Sharpe ratio for the best-performing strategy (minimum-variance-constrained) and the $1/N$ strategies is statistically significant: the P-value for the $1/N$-without-rebalancing strategy is 0.07 and that for the $1/N$-with-rebalancing strategy is 0.04. The reason why the minimum-variance-constrained strategy performs so well is that the investable assets are large portfolios whose second-moments can be estimated with higher precision, and as a consequence the portfolio weights under this strategy do not fluctuate as much (see Panel C).

Studying the CEQ of the different strategies, we see that it is the $1/N$ strategy (without rebalancing) that has the highest CEQ, 0.008, followed by the mean-variance constrained that has a CEQ of 0.00745. The CEQ of the $1/N$-with-rebalancing strategy is 0.00705. The minimum-variance-constrained strategy has a CEQ that is much lower, only 0.00398; the difference in this CEQ and that for the $1/N$-without-rebalancing strategy has a P-value of 0.03 and the difference with that for the $1/N$-with-rebalancing strategy has a P-value of 0.08. So even though the minimum-

[32]The condition number of a symmetric positive definite matrix is the ratio of its largest to its smallest eigenvalue. The larger the condition number of a covariance matrix, the closer it is to being singular. Moreover, when computing portfolio policies as $w = \frac{1}{\gamma}\hat{\Sigma}^{-1}\hat{\mu}$, the impact of the estimation error in $\hat{\mu}$ and $\hat{\Sigma}$ on the quality of the computed policy w is proportional to the condition number. See Golub and Loan (1996, Chapter 2). For the SMB, HML, and MKT dataset, the condition number of the sample covariance matrix ranges between 3.6 and 8. This is a much smaller condition number than those we have observed for the other empirical datasets, which range between 8 and 378,235.

variance constrained strategy has a higher Sharpe ratio than that for the $1/N$ strategies, its CEQ is lower.

Turnover of the different strategies is given in the last row of Panel B of Table 7. Turnover for the $1/N$-without-rebalancing strategy is zero and that for the $1/N$-with-rebalancing strategy is 0.0185. The turnover for the minimum-variance constrained strategy (the one with the highest Sharpe ratio) is 0.03235. The turnover values for the mean-variance-constrained and Bayes-Stein policies are 8 to 10 times higher, and the turnover for the unconstrained strategies (Data-and-Model, mean-variance, Bayes-Stein and three-fund) is even higher, ranging from 47 to 151.

Panel C of Table 7 gives some information about the properties of the portfolio weights—to conserve space, we report only the weights in SMB, HML and the market portfolio. From this panel, we see that $1/N$-with-rebalancing strategy has $1/23 = 0.043$ invested in each risky asset; the unconstrained policies, on the other hand, have very unreasonable portfolio weights that range from very large negative positions to very large positions. The minimum-variance-constrained strategy has zero invested in 19 out of the 20 size- and book-to-market-sorted portfolios, with most of the wealth being invested in the HML portfolio (0.554 on average), the SMB portfolio (0.249 on average), and the market portfolio (0.114 on average).

The reason why the unconstrained strategies from the optimizing models perform poorly relative to the constrained strategies for this dataset is that the size- and book-to-market-sorted portfolios have high betas (between 0.85 and 1.5). Thus, the covariance matrix is closer to singular. Moreover, for this dataset we need to estimate the parameters of 23 different portfolios. So estimation error is much more important than for the previous dataset. As a result, the *unconstrained* mean-variance, minimum-variance, and Bayes-Stein policies perform poorly. Of course, in this case imposing constraints reduces the effect of estimation error, and hence, the *constrained* policies perform much better.

Also, note that the reason that the $1/N$ policy does not perform relatively well in this dataset is because the size- and book-to-market-sorted portfolios are not a good investment; that is, they have a low Sharpe ratio out of sample, and one would be much better off investing 1/3 in SMB, HML, and MKT (Table 6 shows that this would give a Sharpe ratio of 0.22403 for the case with rebalancing), instead of investing 1/23 in each of the 23 portfolios (which, from Table 7, has a Sharpe ratio of only 0.16744). Moreover, note that *only* the constrained-minimum-variance policy does better in this dataset than investing 1/3 in each of the three factors. The reason why the constrained minimum-variance policy does so well is because the constraints, along with ignoring expected returns, neutralize the impact of estimation error.

4.7 Results for US Market, HML, SMB, and Twenty Size- and Book-to-Market-sorted portfolio assuming a three-factor model

The results of this dataset are not discussed in detail because they are very similar to those for the dataset just described, where instead of a three-factor asset pricing model there was a single-factor asset pricing model. As one might expect, the only strategy for which there is some difference in this dataset is the Data-and-Model approach but even for this strategy the change is not substantial.

4.8 Results for US Market, HML, SMB, MOM, and Twenty Size- and Book-to-Market-sorted portfolio

For this dataset, Table 8 shows that in-sample the highest Sharpe ratio is attained by the mean-variance strategy, 0.52372. Imposing constraints reduces the Sharpe ratio—for the mean-variance-constrained policy it is only 0.21060. The Sharpe ratio for the $1/N$ strategies is even lower, 0.16749.

The out-of-sample Sharpe ratios in Panel B of Table 8 tell a different story. The highest Sharpe ratio is for the Bayes-Stein-constrained strategy (0.19664), followed by the mean-variance-constrained strategy (0.18921), and then the $1/N$-without-rebalancing strategy (0.18701) and the $1/N$-without-rebalancing strategy (0.16749). The minimum-variance-constrained strategy has a lower Sharpe ratio of 0.14768. The difference in the Sharpe ratios for the Bayes-Stein-constrained strategy and the $1/N$ strategies (without or with rebalancing) are not significant: the P-values are 0.37 and 0.18, respectively. The unconstrained strategies (Bayes-Stein, mean-variance, minimum-variance and Data-and-Model) perform quite poorly.

In terms of CEQ, we see from Panel B of Table 8 that it is the $1/N$-without-rebalancing strategy that has the highest CEQ, 0.00789, followed by the mean-variance-constrained strategy (0.00745), Bayes-Stein-constrained (0.00730), the $1/N$-with-rebalancing strategy (0.00681) and the single-asset strategy (0.00423). The unconstrained strategies do quite poorly in terms of CEQ, just as they did for the Sharpe ratio.

Turnover, which is given in the last row of Panel B of Table 8, is 0.01938 for the $1/N$-without-rebalancing strategy but 0.25821 for the Bayes-Stein-constrained strategy (which had the highest Sharpe ratio). The unconstrained strategies have substantially higher turnover: 4.55 for the three-fund strategy, 6.71 for the Data-and-Model strategy, 12.39 for the mean-variance strategy, and 23.99 for the Bayes-Stein strategy.

Panel C of Table 8 gives, for the various strategies, the holdings in SMB, HML, MOM and the market portfolio (holdings in the other 20 assets are not reported). From this panel, we see that the $1/N$-with-rebalancing strategy has $1/24 = 0.42$ invested in each asset. We also see that the unconstrained strategies (mean-variance, Bayes-Stein, three-fund, and Data-and-Model) have substantial variation in weights—the positions vary from large negative to large positive quantities. The minimum-variance portfolio behaves better. The constrained policies, of course, show even less variation, but they involve a zero investment in several assets; for instance, the mean-variance-constrained strategy has a zero investment in 12 of the 24 risky assets, while the minimum-variance-constrained strategy has almost the entire wealth invested in the MOM portfolio with almost zero in all the other assets.

Overall, the interpretation of the results for this dataset is similar to that for the dataset discussed in Section 4.6. The main difference is the presence of the MOM portfolio, which has a very low variance. As a result, the unconstrained policies do a bit better because they invest in the MOM portfolio. The performance of the constrained policies is similar to what it was for the dataset discussed in Section 4.6, except that now the constrained minimum-variance policy does much worse in terms of Sharpe ratio. This is because the minimum-variance policy invests almost everything in the MOM, which has a very low variance. While this reduces variance, it also substantially reduces the mean because the MOM portfolio also has a very low mean return.

4.9 Results for US Market and 10-year nominal bond with stochastic interest rates

There are two important limitations of the analysis conducted so far: (1) We have assumed that the moments of asset returns follow very simple processes (i.i.d.) so that the investment opportunity is constant over time, and (2) that the investor is myopic (i.e. chooses a portfolio without considering the gains from intertemporal hedging). We address both these limitations by considering investors who are maximizing utility of lifetime consumption by choosing *dynamic* asset-allocation strategies that are designed to take advantage of changes in the investment opportunity set. In this section, we consider the case of stochastic interest rates, and in the next we examine the case where expected returns on assets are time varying. Note that in contrast to the datasets considered so far, which consisted of returns on a monthly frequency, the two datasets for the two dynamic models are on a quarterly frequency.

The results for the case with stochastic interest rate are given in Tables 9. The optimal asset allocation strategy for this case is analyzed in Campbell and Viceira (2001) and has been reviewed in Section 2.3.1 with the data described in Section 3.3.8. The results for the dynamic portfolio strategy is given in the very last column of Tables 9.[33] In the table we consider the case where risk aversion is three ($\gamma = 3$), because the CEQ values for the case where $\gamma = 1$ are too small (of the order 10^{-70}) for meaningful comparisons (for completeness, the CEQs for the case of $\gamma = 1$ are reported in Footnote 34). Note also that because the dataset is quarterly, so the statistics we report in the tables are quarterly rather than monthly.

From Panel A of Table 9 we see that in-sample, by construction the mean-variance strategy has the highest Sharpe ratio, 0.41952. The $1/N$ strategies have an in-sample Sharpe ratio of 0.41105. Note that the dynamic strategy does not have the highest Sharpe ratio because the objective function for the dynamic strategy is not the optimization of mean-variance utility but instead the expected utility from lifetime consumption, with the utility function being the one in Epstein and Zin (1989).

From Panel B of Table 9 we see that out-of-sample the strategy with the highest Sharpe ratio is the $1/N$-with-rebalancing strategy which has a Sharpe ratio of 0.41105, while the dynamic strategy has a Sharpe ratio of 0.35284, and the difference between the two has a P-value of 0.10. We do not report the dynamic strategy for the case with short-selling constraints because for the case of $\gamma = 3$, the constraints are not binding (this can be seen by examining the statistics for the portfolio policies reported in Panel C). The minimum-variance strategies have Sharpe ratios of 0.35229 and the Bayes-Stein constrained of 0.31188. The mean-variance constrained strategy has a Sharpe ratio of 0.29979. The unconstrained mean-variance and Bayes-Stein policies have lower Sharpe ratios, which are statistically different from that of the $1/N$ policy.

As discussed above, the appropriate metric for these policies is the expected utility from lifetime consumption. We report this in terms of the CEQ value, and we now find that the relative performance of the dynamic strategy improves: its CEQ is 0.00223. However, the single-asset strategy (that is, holding the market portfolio) has a slightly higher CEQ of 0.00229, while the CEQ of the $1/N$ strategies is, 0.00219, which is only slightly smaller than that for the dynamic strategy.[34]

[33] We do not consider the Data-and-Model strategy for this dataset because it did not make sense to view the returns on the 10-year bond as being driven by the market factor or the market, HML, SMB, and MOM factors.

[34] For the case of unit risk aversion, the CEQ of the dynamic strategy is 5.7×10^{-73}, for the strategy of holding just the market it is 2.3×10^{-72}, and for the $1/N$ (with-rebalancing) strategy it is 1×10^{-73}.

Looking at the turnover reported in the last row of Panel B of Table 9, we see that the turnover for the $1/N$-with-rebalancing strategy is 0.02464 while that for the dynamic strategy is 0.04225.

Panel C of Table 9 gives us a sense of the portfolio weights for the various strategies described above. We see that the only strategies that invest very little in the market are the minimum-variance policies (on average, only 0.032 is invested in the market). The other strategies have a substantial investment in the market portfolio.

4.10 Results for 5-year bond and US Market with time-varying expected returns

We now turn to the results for the analysis of dynamic asset allocations when expected asset returns are time-varying. The optimal asset allocation strategy for this case is analyzed in Campbell and Viceira (1999) and Campbell, Chan, and Viceira (2003), and summarized in Section 2.3.2. Just as for the previous dataset, we consider the case where risk aversion is three ($\gamma = 3$). Note also that when examining the performance of the dynamic model out-of-sample, we consider two cases: one, where the estimation is done each quarter (labeled "re-est") and the other, where the estimation is done only once (labeled "one-est"). We also report the results for the case where the weights are constrained to be between 0 and 1.

From Panel A of Table 10, we see that in-sample the model with the highest Sharpe ratio, 0.39357, is the dynamic model. The minimum-variance and the three-fund strategies have similar Sharpe ratios in-sample. The Sharpe-ratio for the $1/N$ strategies is 0.35383. The single-asset strategy has a Sharpe ratio of 0.30517, which is the lowest of all strategies.

Panel B of Table 10 shows that out-of-sample the $1/N$-without-rebalancing strategy has the highest Sharpe ratio, 0.39886. The dynamic strategy where the estimation is done only once has the next-highest Sharpe ratio of 0.36337, while the Sharpe ratio of the $1/N$-with-rebalancing strategy is 0.35383, with the difference against the dynamic strategy with estimation done only once being statistically different. The dynamic strategy where estimation is done each quarter has the lowest Sharpe ratio, 0.12151. Imposing short-selling constraints raises the Sharpe ratio to 0.25696.

As mentioned in the previous section, the appropriate metric for these policies is not the Sharpe ratio but rather the expected utility of lifetime consumption, which we express in terms of the CEQ. According to this metric, the best strategy is investing in a single asset, the market, which has a CEQ of 0.00349. The $1/N$-without-rebalancing strategy has the next-highest CEQ of 0.00324, and the CEQ of the $1/N$-with-rebalancing strategy is 0.00311. The dynamic strategies (estimation done only once and estimation done each quarter) have the lowest CEQ of all policies, 0.00201 and 0.00191, respectively.[35] Imposing short-selling constraints improves the CEQ marginally to 0.00277.

The turnover for the dynamic strategies is also quite high, especially when compared to the turnover for the $1/N$ strategies. Short-selling constraints lead to a substantial reduction in turnover. Panel C of Table 10 gives some useful insights about the investment in each asset under the different strategies. For instance, we see that the dynamic policies entail taking extreme long and short positions, especially in the case where the estimation is done only once. The same is true of the other unconstrained policies.

[35] For the case of unit risk aversion, the CEQ of the single-asset strategy of investing in the market is 2.58×10^{-59}, the CEQ of the $1/N$-with-rebalancing strategy is 2.74×10^{-61}, and the CEQ of the dynamic strategy (with rebalancing) is 1.04×10^{-64}.

5 Simulations

In addition to the nine empirical datasets analyzed in the previous section, we also consider a dataset generated through simulation. Our objective in undertaking this exercise is to examine whether the results from the empirical datasets are confirmed by a dataset of known statistical properties and also to gain some insights into the nature of the estimation error.

5.1 Details of data simulations

For simulating returns, we adopt the following approach.[36] We assume that the market is composed of N risky assets and one risk-free asset. The N risky assets include K factors. The excess returns of the remaining $N-K$ risky assets are generated by the following model

$$R_{a,t} = \alpha + BR_{b,t} + \epsilon_t, \tag{27}$$

where $R_{a,t}$ is the $(N-K)$ vector of excess asset returns, α is the $(N-K)$ vector of mispricing coefficients, B is the $(N-K)\times K$ matrix of factor loadings, $R_{b,t}$ is the K vector of excess returns on the factor ("benchmark") portfolios, and ϵ_t is the $(N-K)$ vector of noises. We assume that factor returns and noises follow a multivariate normal distribution with the following moments for all t

$$\begin{aligned} E[\epsilon_t] &= 0, \\ E[\epsilon_t\, \epsilon_t^\top] &= \Sigma_\epsilon, \\ E[R_{b,t}] &= \mu_b, \\ E[(R_{b,t}-\mu_b)\,(R_{b,t}-\mu_b)^\top] &= \Sigma_b, \\ cov(R_{b,t}, \epsilon_t^\top) &= 0. \end{aligned}$$

Finally, the risk-free rate follows a normal distribution with mean r_f and variance $\sigma_{r_f}^2$.

To initialize the simulation we need to choose values for (i) the average risk-free rate r_f and its variance $\sigma_{r_f}^2$, (ii) the mispricing α, (iii) the factor loading B, (iv) the mean μ_b of the factors and their variance-covariance matrix Σ_b and (v) the variance Σ_ϵ of the noise.

In our *base case*, we assume there are four risky assets ($N=4$) and a single factor ($K=1$). We choose the risk-free rate to have annual average of 2% and standard deviation of 2%. The factor return has an annual average of 8% and a standard deviation of 16%. This amounts to an annual Sharpe ratio of 0.5 for the factor. The mispricing α is set to zero and the factor loadings B for each of the other three risky assets are evenly spread between 0.5 and 1.5. Finally, the variance-covariance matrix of noise Σ_ϵ is assumed to be diagonal with each of the three elements of the diagonal drawn from a uniform distribution with support $[0.15, 0.25]$, that is, the cross-sectional average annual idiosyncratic volatility is 20%. We use Monte-Carlo sampling to generate monthly return data for 2000 years ($T = 24,000$).[37] We use an estimation window length of 10 years ($M = 120$), which matches our choice when analyzing the empirical datasets.

[36] This is similar to the approach adopted in MacKinlay and Pastor (2000).

[37] The longer the time series of simulated data, the more statistically significant the differences between the out-of-sample performances of different policies. For an infinitely long time series, the P-values of these differences would all be zero. A time series of $T = 24,000$ offers a good trade-off between accuracy and computation time. In particular, using MATLAB we can perform all out-of-sample calculations for the case with $N=4$ in around 30 minutes. However, because the factor return has a monthly mean of 0.64% and a monthly volatility of 4.6%, we know that the mean of the out-of-sample factor return will lie with 95% confidence in the interval $0.64\% \pm 2\sigma_c$, where $\sigma_c = 4.6\%/\sqrt{24000}$.

After studying the base case, we perform sensitivity analysis with respect to the following: the length of the estimation window, $M = \{120, 1200, 3000, 6000\}$, the number of risky assets, $N = \{4, 10, 100\}$, and the cross-sectional average annual idiosyncratic volatility (10%, 20%, 30%).

5.2 Results for the base case

Table 11 reports the Sharpe ratio for different static portfolio strategies for data simulated using the base case parameters.[38] There is an additional strategy considered in these tables: the last column in these tables titled "mean-variance-true" reports the Sharpe ratio of the portfolio strategy where the investor is assumed to know the parameters generating returns.

From Panel A of this table, we see that in sample the mean-variance portfolio has the highest Sharpe ratio, 0.14794. The $1/N$ strategy has a Sharpe ratio of 0.13045. Finally, note that the mean-variance-true strategy has a Sharpe ratio of 0.14771, which is slightly smaller than that of the mean-variance portfolio. This is not surprising because in-sample, the mean-variance policy computed from the *sample* mean and variances must always achieve the best performance.

Panel B shows that out-of-sample, the single-asset and mean-variance-true strategies perform the best, with a Sharpe ratio of 0.14773. This is because the data was generated assuming a single-factor structure, and both the single-asset and the mean-variance-true policies invest all wealth in the factor portfolio. On the other hand, when the investor does not know the true moments and needs to estimate them, then the strategy with the highest point estimate of the Sharpe ratio is the minimum-variance-constrained, with a Sharpe ratio of 0.13299. The $1/N$-with-rebalancing strategy follows closely with a Sharpe ratio of 0.13046, with the difference between the two Sharpe ratios not being statistically significant (P-value of 0.17). The Sharpe ratios of the mean-variance, Bayes-Stein, three-fund, and Data-and-Model policies are much lower and their corresponding P-values demonstrate that their difference with the Sharpe ratio of the $1/N$-with-rebalancing strategy is statistically significant. As in the empirical datasets, the Sharpe ratio of the mean-variance, minimum variance, and Bayes-Stein strategies improve with the imposition of constraints, which alleviate the effect of estimation error.

In terms of CEQ, the $1/N$-with-rebalancing strategy performs better than the strategies from the optimal models of myopic asset allocation. The P-values show that the difference in the CEQs is statistically significant.

5.3 Increasing the estimation window length

We now consider the case where the length of the estimation window is 100 years ($M = 1200$) instead of 10 years. The results for this case are reported in Table 12.

The in-sample performance of all strategies for this case, reported in Panel A, is identical to that for the case with ($M = 120$). This is because for both cases, the in-sample data (that is, all 24000 observations) are the same. Panel B shows that the out-of-sample performance of the single-asset, $1/N$, and mean-variance-true policies is the same as in Table 11, because these policies do not use any estimated parameters. The Sharpe ratio of all other policies improves when the estimation window length is increased from 10 to a 100 years. As a result, the Data-and-Model, Bayes-Stein, three-fund, and mean-variance policies outperform the $1/N$-with-rebalancing policy. Moreover, the P-values of the differences between their Sharpe ratios and that of the $1/N$-with-rebalancing policy

[38] In the simulations, we consider only the static portfolio strategies and not the dynamic policies.

are significant. Unlike for shorter estimation window lengths, the imposition of constraints does *not* increase the Sharpe ratios of the Bayes-Stein and mean-variance policies.[39]

In terms of CEQ, the $1/N$-with-rebalancing policy is also outperformed by the mean-variance and Data-and-Model policies, and the P-values of the differences are significant. Contrary to the Sharpe ratios, the CEQs of the mean-variance and Bayes-Stein policies increase with the imposition of constraints. This is because the CEQ is a linear function of the variance, whereas the Sharpe ratio is a nonlinear function of the *standard deviation*. As a result, changes in the mean and variance impact Sharpe ratio and CEQ differently.

Finally, comparing the turnover reported in Panel B of Table 11 and Table 12, we see that with a longer estimation window, the turnover of the optimizing models also becomes more reasonable, but even with an estimation window of 100 years, turnover for strategies from optimizing models is higher than it is for the $1/N$-with-rebalancing strategy.

The above results indicate that, for our the dataset simulated using the base-case parameter values, an estimation window of about 100 years is needed before the mean-variance and other optimizing policies beat the $1/N$ policy. We now study how the length of the estimation window needed for the mean-variance policy to outperform the $1/N$ policy in terms of Sharpe ratio varies with the level of idiosyncratic volatility and the number of risky assets.

5.4 Varying the magnitude of idiosyncratic volatility

In our simulations, we find that the estimation window length necessary for the mean-variance policy to beat the $1/N$ policy in terms of Sharpe ratio is longer whenever the idiosyncratic volatility decreases relative to the volatility of the factor portfolio. For instance, we find that with a cross-sectional average annual idiosyncratic volatility of 10% (as opposed to the 20% used in the base case), the mean-variance strategy outperforms the $1/N$-with-rebalancing policy only for an estimation window length of 500 years (the table with the results for this case is not reported).

The reason for this is that the smaller the idiosyncratic volatility, the closer is the covariance matrix of asset returns to being singular. In particular, when the idiosyncratic volatility is zero and the returns on all risky assets are perfectly correlated, the covariance matrix has rank one. Because the inverse of the covariance matrix of asset returns is used to compute the mean-variance portfolio weights, the impact of estimation error is larger whenever idiosyncratic volatility is smaller relative to factor volatility.

A good measure of the singularity of a covariance matrix is its condition number; that is, the ratio of its largest to its smallest eigenvalue. The larger the condition number of a covariance matrix, the closest it is to being singular. In our simulated data set, the condition number of the covariance matrix of asset returns is 6.99 for idiosyncratic volatility of 10%, 2.22 for idiosyncratic volatility of 20%, and 1.64 for idiosyncratic volatility of 30%. This roughly implies that when we compute a portfolio policy as $w = \frac{1}{\gamma}\hat{\Sigma}^{-1}\hat{\mu}$, the impact of the estimation error in $\hat{\mu}$ and $\hat{\Sigma}$ on the quality of the computed policy w is proportional to 1.64, 2.22, and 6.99 when the idiosyncratic volatility is 30%, 20%, and 10%, respectively. See Golub and Loan (1996, Chapter 2) for a discussion on condition numbers and their impact in the numerical solution of linear systems.

[39]The Sharpe ratio of the minimum-variance policy, on the other hand, does increase with the imposition of constraints.

5.5 Increasing the number of risky assets

Table 13 gives the results for the case with 100 risky assets ($N = 100$) and estimation window of 100 years ($M = 1200$).[40] Panel A gives the in-sample performance of all policies. As in the base case, the mean-variance strategy has the highest Sharpe ratio, 0.15868. The $1/N$ strategies have a Sharpe ratio of 0.14609. Note that the in-sample Sharpe ratios of both the mean-variance and $1/N$ policies grow as the number of risky assets grows. This is because the investor can choose from a larger number of risky assets and thus can achieve a better Sharpe ratio. But the in-sample Sharpe ratio of the minimum-variance policy decreases dramatically. This is because the minimum-variance policy ignores information about the mean returns and focuses in minimizing variance. As a result, although the in-sample variance of the min-variance policy decreases substantially when there are more risky assets, the in-sample mean also decreases substantially[41] and the combined effect yields a decrease in the Sharpe ratio. Observe that this effect is not as marked for the constrained minimum-variance policy because of the regularizing effect of the constraints.

Panel B gives the out-of-sample performance of the policies. The single-asset and the mean-variance-true strategies have the highest Sharpe ratio, 0.14773. As mentioned before, this is not surprising because the data is generated assuming a single-factor structure. When the investor does not know the true moments and needs to estimate them, then the strategy with the highest point estimate of the Sharpe ratio is the $1/N$-with-rebalancing policy with a Sharpe ratio of 0.14640. The out-of-sample Sharpe ratios of the mean-variance, minimum-variance, Bayes-Stein, three-fund, and Data-and-Model policies dramatically decrease with an increase in the number of risky assets. The reason for this is that the estimation error is larger when there are many assets. To see this, note that the number of parameters to be estimated grows quadratically with the number of assets, whereas the amount of data available grows only linearly with the number of assets. This is confirmed by the fact that the imposition of constraints helps the performance of the mean-variance, minimum-variance, and Bayes-Stein policies. Also, note that the difference between the Sharpe ratio of the $1/N$-with-rebalancing policy and all optimizing policies is statistically significant.

Finally, the turnover of all static optimal policies increases dramatically when the number of assets increases.

5.6 Summary of insights from simulations

Our simulations indicate that if one knew the *true* value of the moments, then obviously the strategy with the highest out-of-sample Sharpe ratio would be the one suggested by the mean-variance model. But if the moments have to be estimated, then the relative performance of the $1/N$ and the optimal asset-allocation strategies depends on the length of the estimation window. As one would expect, the longer the estimation window, the better the performance of optimal asset-allocation strategies. Moreover, we find that the estimation window length necessary for the optimal strategies to outperform the $1/N$ strategy is larger whenever: (a) the idiosyncratic risk is smaller or (b) the number of assets is larger. For instance, for the case where wealth can be allocated across four risky assets with an idiosyncratic volatility of 20%, we need an estimation

[40] We have chosen to keep the estimation window of 100 years because the optimizing policies are more stable for longer estimation windows and this facilitates the sensitivity analysis.

[41] The mean return on the portfolio decreases because the unconstrained minimum-variance strategy involves being long assets with low betas and short assets with high beta. Consequently, the mean return is low because the lower the beta, the lower the mean return of a stock. As the number of assets increases, and the more spread out are the betas, the stronger is this effect.

window of 50 years in order for the mean-variance policy to outperform $1/N$. But with four assets and idiosyncratic volatility of 10%, we need 500 years, and with 100 assets and idiosyncratic volatility of 20%, we need more than 1000 years. Finally, we also observe that for longer estimation windows the turnover of the optimal strategies decreases and the imposition of constraints has a smaller effect in improving performance.

6 Robustness checks: Results for other specifications

In this manuscript, we have reported only a small subset of the experiments we considered. (In all, we had more than 360 tables.) As mentioned before, in choosing which set of results to present, we tried to pick the setting that would be least favorable to the $1/N$ strategy. Below, we mention some of the other experiments we studied.

6.1 Results for different levels of risk aversion

For all the static asset allocation policies, we have reported results for only the case where risk aversion equals unity. For the dynamic asset allocation models, we report results only for the case where risk aversion equals three, with the CEQ for the case of unit risk aversion reported in footnotes. In general, we considered the following levels of risk aversion: $\gamma = \{1, 2, 3, 4, 5, 10, 20\}$. We found that the results were not very different across risk aversion levels. In particular, the results for the unconstrained static policies were not affected at all by the level of risk aversion, which is a consequence of two-fund separation (and because we are looking at the performance of the fund of just risky assets). There was a small effect of the level of risk aversion on the performance of the constrained policies (for which two-fund separation does not hold) because the level of risk aversion determines whether the constraints will be binding or not. The only quantity that is influenced significantly by the choice of risk aversion is the CEQ. But, even in this case, the qualitative results for different levels of risk aversion are similar to those reported for the case of unit risk aversion.[42]

6.2 Results for different lengths of the estimation window

In the paper, we have reported results only for the case where the length of the estimation window is $M = 120$, which corresponds to 10 years for monthly data and to 30 years for quarterly data. For monthly data, we considered also the case of $M = 60$, while for quarterly data we considered the cases of $M = 40$ and $M = 60$. Having a smaller estimation window only reduces the performance of the optimal models of asset allocation. This effect is discussed in the simulations reported above.

6.3 Results for different holding periods

The results we present in the paper are for a holding period of one period, which is one month for the dataset with monthly returns and one quarter for the dataset with quarterly returns. Because Chan, Karceski, and Lakonishok (1999) and Jagannathan and Ma (2003) consider a holding period of one year, we also considered a holding period of one year and found that it does not affect our conclusions.

[42]Given that we report both the mean and variance of portfolio returns, the CEQ of the static policies for values of risk aversion other than unity can be computed from the information provided in the tables.

6.4 Results for Sharpe ratio computed over the full holding period

The Sharpe ratio we report for the myopic portfolio choice models is a one-period Sharpe ratio; that is, it is computed for the case where agents care about wealth next period. In principle, one might be interested also in the long-horizon Sharpe ratio, that is, the Sharpe ratio computed over the entire holding period of the portfolio. We address this concern in our simulations by considering also the experiment where we simulate many paths of returns, and compute the Sharpe ratio for terminal wealth across these paths (instead of across realizations for a single path); we find that the qualitative insights from these long-horizon Sharpe ratios are similar to the ones that are reported in the paper.

6.5 Results for all assets, not the portfolio with just risky assets

In the current version of the paper, we report results only for the performance of the fund of just risky assets, rather than the performance of the overall portfolio (consisting of both the riskless asset and the risky assets). The reason for making this choice was that we wanted to focus on the effect of asset allocation alone, and if one considered the performance of the overall portfolio, then that would depend also on market-timing ability. But, even if one considers the performance of the overall portfolio that includes the riskfree asset, the results are very similar to those reported in the manuscript.

6.6 Results for other asset classes

Finally, most of the asset classes we have considered contain only financial assets, such as bonds and equities. One might wonder whether the results would be different if one included other asset classes, such as commodities and real estate, which have a lower correlation with equities. Note that if one were allocating wealth across assets that were relatively uncorrelated, then keeping all else fixed, the loss from naive rather than optimal diversification when using the $1/N$ rule would be smaller rather than larger.

7 Conclusions

We have compared the performance of the $1/N$ allocation rule (without and with rebalancing over time) to the optimal policies from static and dynamic models of asset allocation. We consider both constrained and unconstrained strategies. The comparison with the static allocation rules is done for nine different datasets. The comparison with the dynamic asset allocation rules is done using the two datasets used in the original studies. In addition to these datasets, we also consider simulated data.

Our main finding from the empirical datasets is that the $1/N$ allocation rule (with or without rebalancing at each trading date) performs quite well out-of-sample: it often has a higher Sharpe ratio and certainty-equivalent value than the policies suggested by both the static and the dynamic models of optimal asset allocation, and it always has a lower turnover than that for the strategies from the optimizing models. Our simulation analysis suggests that if one knew the *true* moments of asset returns, then it would be best to use the portfolio weights suggested by an optimizing model. But if the moments have to be estimated, then the relative performance of the $1/N$ and the optimal asset-allocation strategies depends on the length of the estimation window. As one would expect, the longer the estimation window, the better the performance of optimal asset-allocation strategies.

But we find that the estimation window length necessary for the optimal strategies to outperform the $1/N$ strategy may be very large: For instance, for the case where wealth can be allocated across four risky assets with an idiosyncratic volatility of 20%, we need an estimation window of 50 years in order for the mean-variance policy to outperform $1/N$. But with four assets and idiosyncratic volatility of 10%, we need 500 years, and with 100 assets and idiosyncratic volatility of 20%, we need more than a 1000 years.

Hence, the main finding of our analysis is that for many asset allocation problems, the large error in forecasting moments of asset returns may overwhelm the gains from optimization, and so the $1/N$ rule may outperform the strategies from optimizing models. This finding has three implications. One, when evaluating the performance of a particular strategy for optimal asset allocation, the $1/N$ naïve-diversification rule should serve at least as a first obvious benchmark. Two, while there has been considerable progress in the design of optimal portfolios, more energy needs to be devoted to improving the estimation of parameters for the moments of asset returns or the processes driving these moments. Three, in addition to using just the statistical moments of returns, it may be useful to include other conditioning variables in constructing portfolios. This includes both firm-specific information and macroeconomic variables; see, for example, Brandt, Santa-Clara, and Valkanov (2005) and Brandt and Santa-Clara (2005).

A Details for myopic asset allocation strategies

A.1 Bayesian "Data-and-Model" portfolios

We assume the market is composed of N risky assets and one risk-free asset. The N risky assets include K factors. The excess returns of the remaining $N-K$ risky assets follow the factor model defined in (27). Then, the "true" mean vector and variance-covariance matrix of returns are:

$$\mu = \begin{pmatrix} \mu_a \\ \mu_b \end{pmatrix}, \quad \Sigma = \begin{pmatrix} B\Sigma_{bb}B^\top + \Sigma_\epsilon & B\Sigma_{bb} \\ \Sigma_{bb}B^\top & \Sigma_{bb} \end{pmatrix}, \tag{A1}$$

where Σ_{bb} is the variance-covariance matrix of the returns on the K factors. If the asset pricing model is true, then $\mu_a = B\mu_b$ and the mispricing term α is zero.

Let $\hat{B}$ and $\hat{\Sigma}_\epsilon$ be the maximum likelihood estimators of the factor loadings B and of the variance-covariance matrix of residuals, Σ_ϵ, obtained by estimating the unrestricted regression in (27). These will be the estimators chosen by an investor who completely ignores the prediction of the asset pricing model. Similarly, let $\overline{B}$ and $\overline{\Sigma}_\epsilon$ be the quantities obtained by estimating the model (27) with the restriction that $\alpha = 0$. These would be the estimators chosen by an investor who dogmatically believes in the asset pricing model.

A Bayesian investor expresses his belief about the validity of the asset pricing model by postulating a *prior* belief on the mispricing term α. The prior on α, conditional on Σ_ϵ, is assumed to have a Normal mean zero and variance $\tau\Sigma_\epsilon$:

$$p(\alpha|\Sigma_\epsilon) = \mathcal{N}(0, \tau\Sigma_\epsilon), \tag{A2}$$

with τ determining the precision of the prior belief over the validity of the asset pricing model.[43] Finally, let $\hat{\mu}_a$, $\hat{\mu}_b$ and $\hat{\Sigma}_{bb}$ be, respectively, the sample mean of the assets' return R_{at}, and the sample mean and variance of the factors' returns R_{bt}.

Under the assumptions described above, Wang (2004) shows how to obtain estimators for the expected return and variance covariance matrix that account for the belief of a Bayesian investor over the validity of a particular asset pricing model. More specifically, (see Wang (2004, Theorem 1) if one denotes by

$$\widehat{SR}^2 = \hat{\mu}_b^\top \hat{\Sigma}_{bb}^{-1} \hat{\mu}_b \tag{A3}$$

the square of the highest Sharpe ratio of the efficient frontier spanned by the mean and variance of the factor portfolios, and by

$$\omega = \frac{1}{1 + M\tau/(1 + \widehat{SR}^2)} \tag{A4}$$

the degree of confidence a Bayesian investor places in the asset pricing model (that is, $\omega = 1$ implies dogmatic belief in the model), then a Bayesian "Data-and-Model" (DM) investor with a degree of confidence ω in the model will use the following *shrinkage* estimators of the expected return and

[43] The priors on the factor loadings, B, the variance-covariance of the residuals, Σ_ϵ, as well as on the expected returns and variance-covariance matrix of the factors are assumed to be non-informative because the asset-pricing model does not impose any restrictions on these parameters.

variance-covariance matrix of the investable assets:

$$\hat{\mu}^{\text{DM}} = \omega \begin{pmatrix} \overline{B}\hat{\mu}_a \\ \hat{\mu}_b \end{pmatrix} + (1-\omega) \begin{pmatrix} \hat{\mu}_a \\ \hat{\mu}_b \end{pmatrix} \tag{A5}$$

$$\hat{\Sigma}^{\text{DM}} = \begin{pmatrix} \hat{\Sigma}_{aa}^{\text{DM}}(\omega) & \hat{\Sigma}_{ab}^{\text{DM}}(\omega) \\ \hat{\Sigma}_{ab}^{\text{DM}}(\omega)^\top & \hat{\Sigma}_{bb}^{\text{DM}}(\omega) \end{pmatrix}, \tag{A6}$$

where

$$\begin{aligned} \hat{\Sigma}_{aa}^{\text{DM}}(\omega) &= \kappa\,(\omega\overline{B} + (1-\omega)\hat{B})\,\hat{\Sigma}_{bb}\,(\omega\overline{B} + (1-\omega)\hat{B})^\top \\ &\quad + h\,(\omega\overline{\delta} + (1-\omega)\hat{\delta})\,(\omega\overline{\Sigma}_\epsilon + (1-\omega)\hat{\Sigma}_\epsilon), && \text{(A7)} \\ \hat{\Sigma}_{ab}^{\text{DM}}(\omega) &= \kappa\,(\omega\overline{B} + (1-\omega)\hat{B})\,\hat{\Sigma}_{bb}, && \text{(A8)} \\ \hat{\Sigma}_{bb}^{\text{DM}}(\omega) &= \kappa\hat{\Sigma}_{bb}, && \text{(A9)} \end{aligned}$$

and

$$\begin{aligned} \overline{\delta} &= \frac{M(M-2)+K}{M(M-K-2)} - \frac{k+3}{M(M-K-2)} \cdot \frac{\hat{S}^2}{1+\hat{S}^2}, && \text{(A10)} \\ \hat{\delta} &= \frac{(M-2)(M+1)}{M(M-K-2)}, && \text{(A11)} \\ \kappa &= \frac{M+1}{M-K-2}, && \text{(A12)} \\ h &= \frac{M}{M-N-1}. && \text{(A13)} \end{aligned}$$

Observe, from equations (A5) and (A6), that the parameter ω acts as a *linear* shrinkage factor for the means and a *quadratic* shrinkage factor for the variance.

A.2 Optimal "Three-Fund" portfolios

After choosing the parameters c and d in (11) that maximize expected utility of a mean-variance investor, Kan and Zhou (KZ), propose the following "three-fund" optimal portfolio weights:

$$\hat{\mathrm{w}}^{\text{KZ}} = \frac{(M-N-1)(M-N-4)}{\gamma\,M\,(M-2)} \left[\left(\frac{\hat{\psi}_a^2}{\hat{\psi}_a^2 + \frac{N}{M}} \right) \hat{\Sigma}^{-1}\hat{\mu} + \left(\frac{\frac{N}{M}}{\hat{\psi}_a^2 + \frac{N}{M}} \right) \hat{\mu}^{\text{MIN}} \hat{\Sigma}^{-1} \mathbf{1}_N \right], \tag{A14}$$

where

$$\begin{aligned} \hat{\psi}_a^2 &= \frac{(M-N-1)\hat{\psi}^2 - (N-1)}{M} + \frac{2(\hat{\psi}^2)^{\frac{N-1}{2}}(1+\hat{\psi}^2)^{-\frac{M-2}{2}}}{M\,B_{\hat{\psi}^2/(1+\hat{\psi}^2)}((N-1)/2,(M-N+1)/2)}, && \text{(A15)} \\ \hat{\psi}^2 &= (\hat{\mu} - \hat{\mu}^{\text{MIN}})^\top \hat{\Sigma}^{-1} (\hat{\mu} - \hat{\mu}^{\text{MIN}}), && \text{(A16)} \end{aligned}$$

and where the Incomplete Beta function is given by

$$B_x(a,b) = \int_0^x y^{a-1}(1-y)^{b-1}dy. \tag{A17}$$

B Details for dynamic asset allocation strategies

This appendix summarizes the methodology used in Campbell and Viceira (1999, 2001, 2002) and Campbell, Chan, and Viceira (2003) to obtain solutions for the consumption and portfolio choice of an infinitely lived investor for the case of stochastic interest rates and predictable asset returns.

Consider a discrete-time portfolio and consumption problem with an infinitely-lived investor. There are N assets available for investment, one of which is taken to be a reference asset, usually a short-term instrument (which need not be riskfree).

To allow separation between intertemporal rate of substitution and risk aversion, investors are assumed to have recursive preferences as in Epstein and Zin (1989),

$$U(C_t, E_t(U_{t+1})) = \left[(1-\delta)C_t^{\frac{1-\gamma}{\theta}} + \delta\left(E_t\left(U_{t+1}^{1-\gamma}\right)\right)^{\frac{1}{\theta}}\right]^{\frac{\theta}{1-\gamma}}, \quad \theta \equiv \frac{1-\gamma}{1-\psi^{-1}}, \tag{B1}$$

where C_t represents consumption at time t, $\gamma > 0$ is the coefficient of risk-aversion, ψ is the elasticity of intertemporal substitution, $0 < \delta < 1$ a time discount factor, and $E_t(\cdot)$ the conditional expectation operator.[44] At each time t, the investor allocates his wealth after consumption across the N available assets in order to maximize lifetime expected utility under the budget constraint

$$W_{t+1} = (W_t - C_t)R_{p,t+1}, \tag{B2}$$

where W_t is the wealth at time t and $R_{p,t}$ is the (gross) return on a portfolio of the N available assets,

$$R_{p,t} = R_{1,t} + \sum_{i=2}^{N} \mathrm{w}_{i,t}(R_{i,t} - R_{1,t}), \tag{B3}$$

with $\mathrm{w}_{i,t}$ being the portfolio weight on asset i. Notice that, unlike the previous section, we denote by $R_{i,t}$ the gross return on asset i and that, from expression (B3), Asset 1 is taken to be the reference asset. Under these conditions, Epstein and Zin (1989) show that the first-order optimality condition (Euler equation) for consumption is

$$E_t\left\{\left[\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\psi}}\right]^{\theta} R_{p,t+1}^{\theta-1}R_{i,t+1}\right\} = 1, \quad \text{for all } i = 1,\ldots,N. \tag{B4}$$

Campbell and Viceira obtain the approximate portfolio and consumption policies after undertaking the following steps. First, apply a *first-order* Taylor expansion to the log of the budget constraint (B2) around the unconditional mean of the log consumption-wealth ratio.[45] This yields the approximate budget constraint

$$\Delta w_{t+1} \approx r_{p,t+1} + \left(1 - \frac{1}{\rho}\right)(c_t - w_t) + k, \quad \rho = 1 - \exp\{E(c_t - w_t)\}, \tag{B5}$$

where $k = \log(\rho) + (1-\rho)\log(1-\rho)/\rho$, ρ is the log-linearization parameter and $w_t = \log(W_t)$. Second, apply a *second-order* Taylor expansion to the log of the Euler equation (B4) around the conditional means of $\Delta c_{t+1} \equiv \log(C_{t+1}/C_t)$, $r_{p,t+1} \equiv \log(R_{p,t+1})$ and $r_{i,t+1} \equiv \log(R_{i,t+1})$.

[44] The special case of time-separable utility obtains when $\gamma = \psi^{-1}$.

[45] The approximation will be exact only if the consumption-wealth ratio is constant over-time, as it happens when the elasticity of intertemporal substitution ψ is equal to one.

The outcome of these approximation is the following pair of Euler equations,

$$\begin{aligned} E_t(\Delta c_{t+1}) &= \psi \log \delta + \psi E_t(r_{p,t+1}) + \\ &\quad \frac{\theta}{2\psi} Var_t(\Delta c_{t+1} - \psi r_{p,t+1}) \qquad \text{(B6)} \\ E_t(r_{i,t+1} - r_{1,i}) + \frac{1}{2} Var_t(r_{i,t+1} - r_{1,i}) &= \frac{\theta}{\psi} Cov_t(r_{i,t+1}, \Delta c_{t+1}) + \qquad \text{(B7)} \\ &\quad (1-\theta) Cov_t(r_{i,t+1}, r_{p,t+1}) - Cov_t(r_{i,t}, r_{1,t+1}). \end{aligned}$$

Equation (B6) is the Euler equation for the optimal log consumption-wealth ratio $c_t - w_t \equiv \log(C_t/W_t)$ and equation (B7) is the Euler equation characterizing optimal portfolios w_t.

In the next two subsections, we specialize the above general framework to the case of portfolio choice with stochastic interest rates and with predictable asset returns.

B.1 Optimal portfolios with stochastic interest rates

Campbell and Viceira (2001) consider a discrete-time, two-factor model of the term structure where the two factors are (i) the expected log return on the short-term indexed bond and (ii) the expected log rate of inflation. Each factor is assumed to follow a normal first order auto-regressive AR(1) process with constant variance, which implies that log bond yields are linear in the factors.[46]

The *real* factor is chosen to be the expected log return on the one-period real (indexed) bond. Let the log return on the short-term indexed bond be $r_{1,t+1}$. Modeling this return is equivalent to modeling the negative of the log of the stochastic discount factor, M_{t+1}, which prices all the asset in the economy, because $r_{1,t+1} = -\log E_t(M_{t+1}) \equiv -m_{t+1}$. Using a discrete-time version of Vasicek (1977), it is assumed that $-m_{t+1}$ obeys the following AR(1) process

$$\begin{aligned} -m_{t+1} &= x_t + v_{m,t+1}, \qquad \text{(B8)} \\ x_{t+1} &= (1-\phi_x)\mu_x + \phi_x x_t + \varepsilon_{x,t+1}, \qquad \text{(B9)} \\ v_{m,t+1} &= \beta_{mx}\varepsilon_{x,t+1} + \varepsilon_{m,t+1}, \qquad \text{(B10)} \end{aligned}$$

where the quantities $\varepsilon_{x,t+1}$ and $\varepsilon_{m,t+1}$ are normally distributed white noise shocks with variances σ_x^2 and σ_m^2. Note that x_t is the real factor and denotes the conditional expectation of the negative of the log of the pricing kernel.

The *nominal* factor is taken to be the expected log of inflation rate. Denote by π_{t+1} the realized log-inflation. As above, using a discrete-time version of Langetieg (1980), the model assumes that π_{t+1} follows a normal AR(1) process of the form

$$\begin{aligned} \pi_{t+1} &= z_t + v_{\pi,t+1}, \qquad \text{(B11)} \\ z_{t+1} &= (1-\phi_z)\mu_z + \phi_z z_t + v_{z,t+1}, \qquad \text{(B12)} \\ v_{z,t+1} &= \beta_{zx}\varepsilon_{x,t+1} + \beta_{zm}\varepsilon_{m,t+1} + \varepsilon_{z,t+1}, \qquad \text{(B13)} \\ v_{\pi,t+1} &= \beta_{\pi x}\varepsilon_{x,t+1} + \beta_{\pi m}\varepsilon_{m,t+1} + \beta_{\pi z}\varepsilon_{z,t+1} + \varepsilon_{\pi,t+1}, \qquad \text{(B14)} \end{aligned}$$

where $\varepsilon_{z,t+1}$ and $\varepsilon_{\pi,t+1}$ are normally distributed white noise shocks with variances σ_z^2 and σ_π^2, and z_t is the nominal factor, which denotes the conditional expectation of the log inflation. Notice that

[46] The model belongs to the class of "affine" term structure models (see Dai and Singleton (2000) and Duffie and Kan (1996)).

the model allows for the shocks to realized inflation, $v_{\pi,t+1}$, and to expected inflation, $v_{z,t+1}$, to be correlated with each other and with the real shocks to the model. In particular, that expected inflation rate z_{t+1} can be correlated with the stochastic discount factor (i.e., the short-term real rate) implies that the Fisher hypothesis does not hold in this model and nominal interest rates do not move one-for-one with expected inflation.[47]

The above term-structure model belongs to the class of affine models which admits as a solution log yields that are linear in the state variables. The log return on the one-period indexed bond can be expressed as

$$r_{1,t+1} = x_t - \frac{1}{2}\left(\beta_{m\ x}^2 \sigma_x^2 + \sigma_m^2\right), \tag{B15}$$

and the one-period log return on an n-maturity indexed bond in excess of the one-period log interest rate is given by

$$r_{n,t+1} - r_{1,t+1} = \frac{1}{2} B_{n-1}^2 \sigma_x^2 - \beta_{mx} B_{n-1} \sigma_x^2 - B_{n-1} \varepsilon_{x,t+1}, \tag{B16}$$

where B_n is a function only of maturity n defined as $B_n = \frac{1-\phi_x^n}{1-\phi_x}$. From the fundamental pricing equation $1 = E_t(M_{t+1} R_{t+1})$ and the above result, one obtains that the risk premium on an n-period bond is given by

$$\begin{aligned} E_t(r_{n,t+1} - r_{1,t+1}) + \frac{1}{2} Var_t(r_{n,t+1} - r_{1,t+1}) &= Cov_t(r_{n,t+1} - r_{1,t+1}, -m_{t+1}) \\ &= -\beta_{mx} B_{n-1} \sigma_x^2. \end{aligned} \tag{B17}$$

Note that the conditional covariance is constant over time and depends only on bond maturity.

To evaluate the effect of stochastic interest rates on practical asset allocation problems, Campbell and Viceira (2001) introduce also equity in the model as an alternative long-term investment asset. They assume that the unexpected log return on equities is affected by the same shocks influencing the expected and unexpected part of the log stochastic discount factor. Specifically, letting $r_{e,t+1}$ denote the log return on equity, the model considered implies that

$$r_{e,t+1} - E_t(r_{e,t+1}) = \beta_{ex}\ \varepsilon_{x,t+1} + \beta_{em}\ \varepsilon_{m,t+1}. \tag{B18}$$

This specification is consistent with a representative-agent Lucas economy where expected aggregate consumption growth follows an AR(1) process (see Campbell (1999)). The risk premium on equities over a one-period real return is

$$\begin{aligned} E_t(r_{e,t+1} - r_{1,t+1}) + \frac{1}{2} Var_t(r_{e,t+1} - r_{1,t+1}) &= Cov_t(r_{e,t+1} - r_{1,t+1}, -m_{t+1}) \\ &= -\beta_{mx}\beta_{ex}\sigma_x^2 + \beta_{em}\sigma_m^2. \end{aligned} \tag{B19}$$

As in the case of n-period bond, the risk premium on equity is constant over time.

Campbell and Viceira (2001) obtain the approximate consumption and portfolio policy by (i) guessing a functional form for the log consumption-wealth ratio, (ii) using this guess, as well as, the expression of the risk-premia for the long term bond (B17) and for equity (B19) implied by the term-structure model in the log-linearized Euler equation (B7), and (iii) verifying the guess and solving for the coefficients characterizing the optimal consumption-wealth ratio.

[47] In other words, this means that nominal bond prices can include an inflation risk premium as well as a real term premium.

The outcome of this approach is a linear specification for the log-consumption-wealth ratio

$$c_t - w_t = b_0 + b_1 x_t, \tag{B20}$$

where

$$\begin{aligned} b_0 &= \frac{\rho}{1-\rho}\left[\frac{(1-\psi)(1-\gamma)}{2\gamma}\left(\beta_m x + \frac{\rho}{1-\rho\phi_x}\right)^2 \sigma_x^2 - \frac{1}{2}(1-\psi)\sigma_m^2 \right. \\ &\quad \left. -\psi \log\delta + k + \mu_x(1-\phi_x)\frac{\rho(1-\psi)}{1-\rho\phi_x}\right] \qquad \text{(B21)} \\ b_1 &= (1-\psi)\frac{\rho}{1-\rho\phi_x}, \qquad \text{(B22)} \end{aligned}$$

and $\rho = 1 - \exp\{b_0 + b_1\mu_x\}$. The solution is analytical, given the log-linearization parameter, ρ.

Denoting by $\mathbf{r}_t$ the log return on the N risky assets, and using as a reference asset the short-term indexed bond with log return $r_{1,t}$, the optimal portfolio is given by

$$\hat{\mathrm{w}}^{\mathrm{CV}} = \frac{1}{\gamma}\hat{\mathbf{\Sigma}}_t^{-1}\hat{\mathbf{a}}, \tag{B23}$$

where $\hat{\mathbf{\Sigma}}_t$ is the time t estimate of the variance-covariance matrix for excess returns over the reference (short-term) asset and

$$\hat{\mathbf{a}} = \hat{\mathbf{m}} + \gamma\hat{\mathbf{p}} + \frac{1-\gamma}{1-\psi}\hat{\mathbf{h}}. \tag{B24}$$

The quantity $\hat{\mathbf{m}}$ denotes a vector of Jensen's inequality adjusted mean excess returns, and captures the myopic part of the portfolio,

$$\hat{\mathbf{m}}_i = E_t(\mathbf{r}_{i,t+1} - r_{1,t+1}) + \frac{1}{2}Var_t(\mathbf{r}_{i,t+1} - r_{1,t+1}). \tag{B25}$$

The term $\hat{\mathbf{p}}$ is a vector of conditional covariances with the real return on the short-term bond,

$$\hat{\mathbf{p}}_i = Cov_t[\mathbf{r}_{i,t+1} - r_{1,t+1}, r_{1,t+1}]. \tag{B26}$$

The term $\hat{\mathbf{h}}_t$ is a vector of conditional covariances with the consumption-wealth ratio and represents the intertemporal hedging component of the optimal portfolio,

$$\hat{\mathbf{h}}_t = Cov_t(\mathbf{r}_{t+1} - r_{1,t+1}, c_{t+1} - w_{t+1}) = \frac{\rho}{1-\rho\phi_x}(1-\psi)Cov_t(\mathbf{r}_{t+1} - r_{1,t+1}, x_{t+1}). \tag{B27}$$

In our implementation, we use equations (B23) and (B20) as the optimal portfolio and consumption rules in the presence of stochastic interest rates.

B.2 Optimal portfolios with time-varying expected returns

We now consider the model of dynamic asset allocation with time-varying expected returns. Let $\mathbf{z}_t$ denote the $m \times 1$ dimensional vector containing (1) the log return $r_{1,t}$ on reference short-term asset; (2) the log return $\mathbf{x}_t$ on the remaining N asset, *in excess* of asset 1 (the reference asset) and (3) a $(m - N - 1) \times 1$ vector $\mathbf{s}_t$ of state variables, i.e.

$$\mathbf{z}_t = \begin{bmatrix} r_{1,t} \\ \mathbf{x}_t \\ \mathbf{s}_t \end{bmatrix} = \begin{bmatrix} r_{1,t} \\ r_{2,t} - r_{1,t} \\ \vdots \\ r_{n,t} - r_{1,t} \\ \mathbf{s}_t \end{bmatrix}. \tag{B28}$$

The dynamics of $\mathbf{z}_t$ are modeled as a first-order vector autoregressive (VAR) process of the form

$$z_{t+1} = \mathbf{\Phi}_0 + \mathbf{\Phi}_1 \mathbf{z}_t + \mathbf{v}_{t+1}, \tag{B29}$$

where $\mathbf{\Phi}_0$ is a $m \times 1$ vector, $\mathbf{\Phi}_1$ is a $m \times m$ matrix of coefficients, and $\mathbf{v}_{t+1}$ is a $m \times 1$ vector of normal i.i.d. shocks with mean zero and covariance $\mathbf{\Sigma}_v$, that is,

$$\mathbf{v}_{t+1} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \mathbf{\Sigma}_v) \tag{B30}$$

$$\mathbf{\Sigma}_v = \begin{bmatrix} \sigma_1^2 & \boldsymbol{\sigma}_{1x}^\top & \boldsymbol{\sigma}_{1s}^\top \\ \boldsymbol{\sigma}_{1x} & \hat{\mathbf{\Sigma}}_{xx} & \mathbf{\Sigma}_{xs}^\top \\ \boldsymbol{\sigma}_{1s} & \mathbf{\Sigma}_{xs} & \Sigma_{ss} \end{bmatrix}. \tag{B31}$$

As is clear from its specification, the above model does not allow for stochastic volatility but instead assumes that the shocks are homoscedastic, independently distributed over-time but potentially correlated in the cross-section.

Imposing this VAR structure on the evolution of state variables, Campbell, Chan, and Viceira (2003) show that the optimal portfolios $\mathrm{w}_t^{\text{CCV}}$ satisfying the log-linear Euler equation (B7) can be expressed as a linear function of the state variables $\mathbf{z}_t$,

$$\hat{\mathrm{w}}_t^{\text{CCV}} = \hat{\mathbf{A}}_0 + \hat{\mathbf{A}}_1 \, \mathbf{z}_t, \tag{B32}$$

with

$$\hat{\mathbf{A}}_0 = \left(\frac{1}{\gamma}\right) \hat{\mathbf{\Sigma}}_{xx}^{-1} \left(\mathbf{H}_x \mathbf{\Phi}_0 + \frac{1}{2}\boldsymbol{\sigma}_x^2 + (1-\gamma)\boldsymbol{\sigma}_{1x}\right) + \left(1 - \frac{1}{\gamma}\right) \hat{\mathbf{\Sigma}}_{xx}^{-1} \left(\frac{-\mathbf{\Lambda}_0}{1-\psi}\right) \tag{B33}$$

$$\hat{\mathbf{A}}_1 = \left(\frac{1}{\gamma}\right) \hat{\mathbf{\Sigma}}_{xx}^{-1} \mathbf{\Phi}_1 + \left(1 - \frac{1}{\gamma}\right) \hat{\mathbf{\Sigma}}_{xx}^{-1} \left(\frac{-\mathbf{\Lambda}_1}{1-\psi}\right), \tag{B34}$$

where $\mathbf{H}_x$ is a $N \times m$ selection matrix that select the vector of excess returns from the full state vector $\mathbf{z}_t$, as defined in (B28) and where the quantities $\mathbf{\Lambda}_0$ and $\mathbf{\Lambda}_1$ are defined in the appendix of Campbell, Chan, and Viceira (2003). A closer look at the $N \times 1$ vector $\hat{\mathbf{A}}_0$ in (B33) and $N \times m$ matrix $\hat{\mathbf{A}}_1$ in (B34) reveals the intuitive nature of this results. The terms premultiplied by $(1-1/\gamma)$ in the above equation represent the effect of intertemporal hedging on the portfolio decision. If, for example, the investor has log utility ($\gamma = 1$), then the portfolio holdings will not be affected by the elasticity of intertemporal substitution ψ and the optimal portfolio is the myopic mean-variance portfolio. Since the estimation of the moments for this portfolio is performed assuming a VAR structure in the data, in general, the weights in this portfolio will not correspond to the weights obtained by using an *unconditional* estimate of the moment, unless the VAR system includes only a constant as an explanatory variable. This distinction is relevant when we compare dynamic (conditional) portfolio policies to static policies.

Finally, the log consumption/wealth ratio $(c_t - w_t)$ satisfying Euler equations (B7)-(B6) can be expressed as a quadratic function of the state variables $\mathbf{z}_t$

$$c_t - w_t = b_0 + \mathbf{B}_1^\top \mathbf{z}_t + \mathbf{z}_t^\top \mathbf{B}_2 \mathbf{z}_t, \tag{B35}$$

where b_0, the $m \times 1$ dimensional vector $\mathbf{B}_1$ and the $m \times m$ matrix $\mathbf{B}_2$ are described in the appendix of Campbell, Chan, and Viceira (2003). From the derivation there, it can also be seen that the coefficient $\hat{\mathbf{A}}_0$ and $\hat{\mathbf{A}}_1$ in the optimal portfolio strategy depends on the parameters $\mathbf{B}_1$ and $\mathbf{B}_2$ of the consumption-wealth ratio equation through the coefficient matrices $\mathbf{\Lambda}_0$ and $\mathbf{\Lambda}_1$.

In our implementation, we use equations (B32) and (B35) as the optimal portfolio and consumption rules in the presence of predictability in asset returns.

Table 1: List of asset-allocation models considered

This table lists the various asset-allocation models we consider. The first eleven models that are listed are all static, while the last two are dynamic.

#	Model
1	Single-asset portfolio (holding the market or benchmark portfolio)
2	Simple portfolio: $1/N$ without rebalancing
3	Simple portfolio: $1/N$ with rebalancing (*this is our benchmark strategy*)
4	Mean-variance portfolio (moments estimated using maximum-likelihood)
5	Mean-variance portfolio with borrowing and short-sale constraints
6	Minimum-variance portfolio
7	Minimum-variance portfolio with borrowing and short-sale constraints
8	Bayes-Stein shrinkage portfolio
9	Bayes-Stein shrinkage portfolio with borrowing and short-sale constraints
10	Data-and-Model approach (Pástor, 2000)
11	Three-fund portfolio (Kan and Zhou, 2005)
12	Dynamic model with stochastic interest rates
13	Dynamic model with time-varying asset returns

Table 2: List of datasets considered

This table lists the various datasets we consider, along with the period spanned by the data. Each dataset also has a short-term asset which is assumed to be the 90-day nominal US T-bill. The first seven datasets are all at a monthly frequency. The eighth and ninth datasets are at a quarterly frequency. The last set of data are based on simulations of monthly returns.

#	Dataset	Period spanned
1	Ten sector portfolio that track sectors composing the S&P500	Jan. 1981–Dec. 2002
2	Ten industry portfolios	Jul. 1963–Nov. 2004
3	Nine international equity indexes	Jan. 1970–Jul. 2001
4	US market, SMB and HML portfolios	Jul. 1963–Nov. 2004
5	US market, SMB and HML portfolios and twenty Size and Book-to-Market sorted portfolios (using a one-factor model)	Jul. 1963–Nov. 2004
6	US market, SMB and HML portfolios and twenty Size and Book-to-Market sorted portfolios (using a three-factor model)	Jul. 1963–Nov. 2004
7	US market, SMB, HML and MOM portfolios and twenty Size and Book-to-Market sorted portfolios	Jul. 1963–Nov. 2004
8	US market portfolio and 10-year bond, with stochastic interest rates	Jan. 1954–Sep. 1996
9	US market portfolio and 5-year bond, with time-varying returns	Sep. 1952–Sep. 1999
10	Simulated data	

Table 3: Ten S&P sector portfolios

The table reports a variety of statistics for eleven different static asset allocation strategies. Panel A gives the in-sample mean, variance and Sharpe ratio of excess returns; Panel B gives the out-of-sample mean, variance, Sharpe ratio and Certainty Equivalent (CEQ) of excess returns, along with the P-values that the Sharpe ratio and CEQ are different from those for the $1/N$ strategy (without and with rebalancing); Panel C gives the minimum, maximum, average and standard deviation over time of the portfolio weight allocated to a particular asset. The investor has available for investment a US T-bill and ten S&P sector portfolios. The dataset consists of monthly excess returns on 10 portfolios tracking the sectors composing the S&P500 index. There are a total of 264 monthly observations from January 1981 to December 2002. The risk-free rate is the 90-day T-bill, obtained from CRSP.

Statistic \ Strategy	Single asset	$1/N$ (no rebal)	$1/N$ (*with rebal*)	Mean-var	Mean-var constr.	Min-var	Min-var constr.	Bayes-Stein	Bayes-Stein constr.	3-fund	Data&model $\omega = 0.50$
Panel A: Statistics about in-sample performance											
Mean	0.00629	0.00779	*0.00779*	0.05995	0.01410	0.00278	0.00646	0.02411	0.00938	0.01980	0.01837
Variance	0.00190	0.00172	*0.00172*	0.02427	0.00406	0.00113	0.00140	0.00983	0.00333	0.00667	0.00647
Sharpe Ratio	0.14436	0.18762	*0.18762*	0.38481	0.22136	0.08292	0.17258	0.24320	0.16242	0.24246	0.22836
Panel B: Statistics about out-of-sample performance											
Mean	0.00629	0.00786	*0.00779*	0.00653	0.00733	0.00304	0.00315	0.00454	0.00780	0.00327	0.00592
Variance	0.00190	0.00208	*0.00172*	0.00677	0.00675	0.00138	0.00142	0.00314	0.00527	0.00229	0.00327
Sharpe Ratio	0.14436	0.17245	*0.18762*	0.07942	0.08924	0.08200	0.08344	0.08107	0.10752	0.06829	0.10362
pVal.-($1/N$-rebal.)	0.09067	0.29577	*0.50000*	0.11912	0.09219	0.04578	0.01298	0.09122	0.13804	0.05113	0.11898
pVal.-($1/N$-no rebal.)	0.14583	0.50000	*0.29577*	0.12786	0.05336	0.09573	0.06908	0.10131	0.09964	0.05910	0.12990
CEQ	0.00534	0.00682	*0.00693*	0.00315	0.00396	0.00235	0.00244	0.00297	0.00517	0.00212	0.00429
pVal.-($1/N$-rebal.)	0.12245	0.46522	*0.50000*	0.27515	0.29371	0.03143	0.00761	0.16190	0.35622	0.07094	0.02313
pVal.-($1/N$-no rebal.)	0.10564	0.50000	*0.46522*	0.26540	0.25659	0.06260	0.04288	0.14928	0.32627	0.06462	0.02203
Turnover	0.00000	0.00000	*0.03051*	1.18940	0.13813	0.19964	0.07526	0.68375	0.11093	0.60460	0.62193
Panel C: Statistics about portfolio weights											
Energy											
Min	0.000	0.083	*0.091*	-0.165	0.000	-0.060	0.000	-0.124	0.000	-0.111	-0.089
Max	0.000	0.107	*0.091*	0.698	0.000	0.176	0.061	0.425	0.000	0.319	0.416
Avg	0.000	0.091	*0.091*	0.254	0.000	0.056	0.005	0.165	0.000	0.126	0.152
StdDev	0.000	0.004	*0.000*	0.188	0.000	0.069	0.011	0.121	0.000	0.096	0.111
Material											
Min	0.000	0.082	*0.091*	-0.776	0.000	0.149	0.047	-0.370	0.000	-0.241	-0.375
Max	0.000	0.105	*0.091*	1.296	0.000	0.539	0.268	0.949	0.000	0.811	0.876
Avg	0.000	0.091	*0.091*	0.325	0.000	0.344	0.141	0.324	0.000	0.314	0.245
StdDev	0.000	0.003	*0.000*	0.645	0.000	0.135	0.055	0.419	0.000	0.324	0.399
Industrials											
Min	0.000	0.078	*0.091*	-1.432	0.000	-0.314	0.000	-0.954	0.000	-0.768	-0.691
Max	0.000	0.098	*0.091*	0.073	0.000	0.059	0.000	0.001	0.000	-0.016	0.048
Avg	0.000	0.091	*0.091*	-0.498	0.000	-0.151	0.000	-0.348	0.000	-0.286	-0.279
StdDev	0.000	0.003	*0.000*	0.327	0.000	0.092	0.000	0.218	0.000	0.179	0.175

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Table 3 (cont.): Ten S&P sector portfolios

Statistic / Strategy	Single asset	$1/N$ (no rebal)	$1/N$ (*with rebal*)	Mean-var	Mean-var constr.	Min-var	Min-var constr.	Bayes-Stein	Bayes-Stein constr.	3-fund	Data&model $\omega = 0.50$
Panel C (cont.): Statistics about portfolio weights											
Cons-Discr.											
Min	0.000	0.081	*0.091*	-0.810	0.000	-0.064	0.000	-0.466	0.000	-0.337	-0.434
Max	0.000	0.100	*0.091*	1.268	0.000	0.300	0.131	0.824	0.000	0.648	0.613
Avg	0.000	0.091	*0.091*	0.089	0.000	0.117	0.019	0.101	0.000	0.106	0.059
StdDev	0.000	0.003	*0.000*	0.446	0.000	0.100	0.035	0.267	0.000	0.197	0.262
Cons-Staples											
Min	0.000	0.082	*0.091*	-0.209	0.000	-0.051	0.000	-0.108	0.000	-0.082	-0.108
Max	0.000	0.101	*0.091*	3.053	1.000	0.448	0.325	1.886	1.000	1.455	2.027
Avg	0.000	0.091	*0.091*	1.541	0.493	0.187	0.043	0.930	0.534	0.651	0.970
StdDev	0.000	0.003	*0.000*	0.948	0.477	0.123	0.081	0.547	0.462	0.381	0.661
Helth-Care											
Min	0.000	0.081	*0.091*	-0.605	0.000	-0.087	0.000	-0.248	0.000	-0.116	-0.402
Max	0.000	0.101	*0.091*	2.935	0.834	0.498	0.197	1.687	0.779	1.193	1.419
Avg	0.000	0.091	*0.091*	0.437	0.071	0.174	0.067	0.329	0.118	0.289	0.217
StdDev	0.000	0.004	*0.000*	0.571	0.181	0.175	0.046	0.275	0.213	0.166	0.296
Financials											
Min	0.000	0.080	*0.091*	0.125	0.000	-0.388	0.000	-0.079	0.000	-0.179	0.079
Max	0.000	0.103	*0.091*	2.046	0.436	0.094	0.000	1.026	0.429	0.621	0.989
Avg	0.000	0.091	*0.091*	0.650	0.036	-0.122	0.000	0.311	0.048	0.166	0.371
StdDev	0.000	0.003	*0.000*	0.321	0.089	0.164	0.000	0.181	0.102	0.142	0.172
Inf-Tech.											
Min	0.000	0.066	*0.091*	-0.373	0.000	-0.127	0.000	-0.214	0.000	-0.171	-0.300
Max	0.000	0.123	*0.091*	1.837	1.000	0.173	0.089	1.005	1.000	0.675	0.888
Avg	0.000	0.091	*0.091*	0.473	0.399	0.054	0.008	0.293	0.299	0.220	0.242
StdDev	0.000	0.007	*0.000*	0.465	0.410	0.072	0.022	0.245	0.329	0.160	0.251
Telecom											
Min	0.000	0.077	*0.091*	-0.603	0.000	-0.020	0.000	-0.354	0.000	-0.260	-0.324
Max	0.000	0.117	*0.091*	1.269	0.000	0.354	0.208	0.861	0.000	0.698	0.857
Avg	0.000	0.091	*0.091*	0.546	0.000	0.149	0.100	0.368	0.000	0.286	0.332
StdDev	0.000	0.005	*0.000*	0.456	0.000	0.120	0.051	0.287	0.000	0.214	0.283
Utilities											
Min	0.000	0.077	*0.091*	-1.092	0.000	0.267	0.188	-0.511	0.000	-0.281	-0.528
Max	0.000	0.104	*0.091*	0.528	0.000	0.805	0.765	0.569	0.000	0.652	0.299
Avg	0.000	0.091	*0.091*	-0.119	0.000	0.661	0.582	0.228	0.000	0.381	-0.064
StdDev	0.000	0.004	*0.000*	0.350	0.000	0.123	0.129	0.205	0.000	0.170	0.194
S&P500											
Min	1.000	0.085	*0.091*	-5.166	0.000	-1.470	0.000	-3.463	0.000	-2.785	-3.158
Max	1.000	0.095	*0.091*	-0.130	0.000	0.699	0.209	0.153	0.000	0.293	0.459
Avg	1.000	0.091	*0.091*	-2.697	0.000	-0.469	0.034	-1.701	0.000	-1.255	-1.246
StdDev	0.000	0.001	*0.000*	1.470	0.000	0.823	0.064	1.068	0.000	0.900	1.066

Table 4: Ten industry portfolios

The table reports a variety of statistics for eleven different static asset allocation strategies. Panel A gives the in-sample mean, variance and Sharpe ratio of excess returns; Panel B gives the out-of-sample mean, variance, Sharpe ratio and Certainty Equivalent (CEQ) of excess returns, along with the P-values that the Sharpe ratio and CEQ are different from those for the $1/N$ strategy (without and with rebalancing); Panel C gives the minimum, maximum, average and standard deviation over time of the portfolio weight allocated to a particular asset. The investor has available for investment a US T-bill, the US market portfolio, and ten industry portfolios (Consumer Non Durables, Consumer Durables, Manufacturing, Energy, High-Tech, Telecommunication, Wholesale and Retail, Health, Utilities and Other). There are a total of 497 monthly observations from July 1963 to November 2004. This data was obtained from Kenneth French's website.

Statistic / Strategy	Single asset	$1/N$ (no rebal)	$1/N$ (*with rebal*)	Mean-var	Mean-var constr.	Min-var	Min-var constr.	Bayes-Stein	Bayes-Stein constr.	3-fund	Data&model $\omega = 0.50$
Panel A: Statistics about in-sample performance											
Mean	0.00532	0.00595	*0.00595*	0.01225	0.00722	0.00494	0.00555	0.00635	0.00591	0.00499	0.00686
Variance	0.00219	0.00193	*0.00193*	0.00332	0.00204	0.00134	0.00145	0.00173	0.00196	0.00135	0.00225
Sharpe Ratio	0.11377	0.13531	*0.13531*	0.21239	0.15995	0.13490	0.14599	0.15269	0.13357	0.13606	0.14463
Panel B: Statistics about out-of-sample performance											
Mean	0.00532	0.00561	*0.00595*	-0.04805	0.00421	0.00589	0.00536	-0.02781	0.00478	-0.02050	0.06387
Variance	0.00219	0.00189	*0.00193*	1.74820	0.00386	0.00144	0.00141	0.75968	0.00341	0.58878	11.52589
Sharpe Ratio	0.11377	0.12895	*0.13531*	-0.03634	0.06780	0.15536	0.14253	-0.03190	0.08193	-0.02672	0.01881
pVal.-($1/N$-rebal.)	0.00920	0.07918	*0.50000*	0.01150	0.02864	0.29838	0.40615	0.01305	0.05794	0.01542	0.05883
pVal.-($1/N$-no rebal.)	0.03783	0.50000	*0.07918*	0.01427	0.03456	0.24106	0.32810	0.01613	0.07028	0.01893	0.06961
CEQ	0.00423	0.00466	*0.00498*	-0.92215	0.00228	0.00517	0.00465	-0.40765	0.00308	-0.31489	-5.69908
pVal.-($1/N$-rebal.)	0.04070	0.05303	*0.50000*	0.00000	0.09600	0.45238	0.39747	0.00000	0.15378	0.00000	0.00000
pVal.-($1/N$-no rebal.)	0.14635	0.50000	*0.05303*	0.00000	0.11532	0.37145	0.49619	0.00000	0.18583	0.00000	0.00000
Turnover	0.00000	0.00000	*0.02162*	13132.02866	0.15501	0.46800	0.05570	218.16550	0.15616	214.96018	756.47789
Panel C: Statistics about portfolio weights											
NoDur											
Min	0.000	0.083	*0.091*	-613.975	0.000	-0.544	0.000	-408.794	0.000	-363.317	-1860.074
Max	0.000	0.101	*0.091*	768.705	1.000	0.874	0.273	504.758	1.000	441.886	1846.662
Avg	0.000	0.091	*0.091*	6.082	0.313	0.209	0.027	3.421	0.314	2.019	10.884
StdDev	0.000	0.002	*0.000*	55.652	0.441	0.317	0.066	36.655	0.429	32.244	166.887
Durbl											
Min	0.000	0.082	*0.091*	-365.659	0.000	-0.387	0.000	-243.551	0.000	-216.487	-1070.886
Max	0.000	0.102	*0.091*	516.656	0.871	0.336	0.070	339.134	0.749	296.849	1229.482
Avg	0.000	0.091	*0.091*	3.744	0.035	0.012	0.004	2.111	0.034	1.317	6.596
StdDev	0.000	0.003	*0.000*	35.855	0.149	0.170	0.013	23.597	0.130	20.737	100.786
Manuf											
Min	0.000	0.084	*0.091*	-1909.952	0.000	-0.684	0.000	-1272.310	0.000	-1130.981	-3848.560
Max	0.000	0.098	*0.091*	2541.457	0.000	0.952	0.189	1668.071	0.000	1460.032	5704.717
Avg	0.000	0.091	*0.091*	11.969	0.000	0.097	0.008	6.754	0.000	4.178	26.174
StdDev	0.000	0.002	*0.000*	180.381	0.000	0.349	0.027	118.952	0.000	104.695	407.804

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Table 4 (cont.): Ten industry portfolios

Statistic	Strategy	Single asset	1/N (no rebal)	1/N (*with rebal*)	Mean-var	Mean-var constr.	Min-var	Min-var constr.	Bayes-Stein	Bayes-Stein constr.	3-fund	Data&model $\omega = 0.50$
		Panel C (cont.): Statistics about portfolio weights										
Enrgy												
Min		0.000	0.081	*0.091*	-1178.712	0.000	-0.401	0.000	-785.203	0.000	-697.984	-2469.391
Max		0.000	0.105	*0.091*	1547.427	1.000	0.625	0.282	1015.615	1.000	888.939	3521.404
Avg		0.000	0.091	*0.091*	8.603	0.179	0.141	0.115	4.743	0.157	2.691	17.293
StdDev		0.000	0.004	*0.000*	110.142	0.337	0.232	0.062	72.622	0.280	63.920	256.319
HiTec												
Min		0.000	0.072	*0.091*	-675.870	0.000	-0.461	0.000	-450.217	0.000	-400.202	-1520.678
Max		0.000	0.110	*0.091*	841.154	1.000	0.658	0.127	552.095	1.000	483.242	1942.393
Avg		0.000	0.091	*0.091*	5.396	0.078	-0.000	0.024	2.927	0.056	1.614	10.399
StdDev		0.000	0.004	*0.000*	61.335	0.223	0.282	0.036	40.437	0.184	35.595	152.631
Telcm												
Min		0.000	0.079	*0.091*	-452.838	0.000	-0.256	0.000	-301.523	0.000	-267.985	-1348.404
Max		0.000	0.106	*0.091*	529.063	1.000	0.779	0.862	347.406	1.000	304.135	1233.869
Avg		0.000	0.091	*0.091*	3.520	0.107	0.274	0.318	2.043	0.151	1.268	6.648
StdDev		0.000	0.003	*0.000*	39.617	0.268	0.306	0.302	26.104	0.281	22.958	113.339
Shops												
Min		0.000	0.083	*0.091*	-684.525	0.000	-0.539	0.000	-456.170	0.000	-405.557	-859.009
Max		0.000	0.100	*0.091*	857.381	0.786	0.275	0.216	562.542	0.702	492.312	1872.461
Avg		0.000	0.091	*0.091*	3.726	0.012	-0.094	0.032	2.092	0.011	1.346	7.482
StdDev		0.000	0.002	*0.000*	62.288	0.073	0.222	0.063	41.129	0.063	36.239	112.705
Hlth												
Min		0.000	0.079	*0.091*	-617.269	0.000	-0.361	0.000	-411.198	0.000	-365.524	-1175.831
Max		0.000	0.102	*0.091*	748.076	1.000	0.366	0.354	490.986	1.000	429.748	1722.300
Avg		0.000	0.091	*0.091*	4.050	0.266	0.020	0.060	2.282	0.270	1.418	7.578
StdDev		0.000	0.003	*0.000*	55.255	0.411	0.161	0.080	36.481	0.387	32.147	120.795
Utils												
Min		0.000	0.080	*0.091*	-409.394	0.000	-0.076	0.000	-272.647	0.000	-242.338	-932.225
Max		0.000	0.108	*0.091*	570.034	0.000	1.153	0.788	374.190	0.031	327.540	1286.574
Avg		0.000	0.091	*0.091*	3.285	0.000	0.460	0.406	2.101	0.000	1.577	6.309
StdDev		0.000	0.003	*0.000*	39.644	0.000	0.312	0.258	26.126	0.002	22.984	95.378
Other												
Min		0.000	0.085	*0.091*	-822.766	0.000	-1.460	0.000	-548.358	0.000	-487.538	-2224.590
Max		0.000	0.098	*0.091*	1052.178	1.000	0.447	0.000	690.304	0.401	604.106	2456.929
Avg		0.000	0.091	*0.091*	9.104	0.011	-0.604	0.000	4.748	0.008	2.473	15.258
StdDev		0.000	0.002	*0.000*	75.934	0.082	0.402	0.000	50.014	0.043	44.027	214.367
Mkt-RF												
Min		1.000	0.087	*0.091*	-9971.132	0.000	-3.812	0.000	-6544.101	0.000	-5727.790	-22815.791
Max		1.000	0.095	*0.091*	7731.958	0.000	4.835	0.266	5150.971	0.000	4578.913	17310.647
Avg		1.000	0.091	*0.091*	-58.478	0.000	0.486	0.006	-32.222	0.000	-18.902	-113.622
StdDev		0.000	0.001	*0.000*	715.310	0.000	2.036	0.029	471.645	0.000	415.155	1729.156

Table 5: Nine international equity indexes

The table reports a variety of statistics for eleven different static asset allocation strategies. Panel A gives the in-sample mean, variance and Sharpe ratio of excess returns; Panel B gives the out-of-sample mean, variance, Sharpe ratio and Certainty Equivalent (CEQ) of excess returns, along with the P-values that the Sharpe ratio and CEQ are different from those for the $1/N$ strategy (without and with rebalancing); Panel C gives the minimum, maximum, average and standard deviation over time of the portfolio weight allocated to a particular asset. The investor has available for investment a US T-bill, 8 international equity indices, and the world equity index. There are a total of 379 monthly observations over the period January 1970 to July 2001. The nine equity indices are for Canada, Japan, France, Germany, Italy, Switzerland, United Kingdom, United States and the World. Data are from MSCI (Morgan Stanley Capital International). The risk-free rate is the 90-day T-bill, obtained from CRSP.

Statistic / Strategy	Single asset	$1/N$ (no rebal)	$1/N$ (*with rebal*)	Mean-var	Mean-var constr.	Min-var	Min-var constr.	Bayes-Stein	Bayes-Stein constr.	3-fund	Data&model $\omega = 0.50$
Panel A: Statistics about in-sample performance											
Mean	0.00525	0.00555	*0.00555*	0.01105	0.00679	0.00654	0.00624	0.00693	0.00661	0.00574	0.00651
Variance	0.00179	0.00189	*0.00189*	0.00279	0.00181	0.00165	0.00167	0.00176	0.00181	0.00207	0.00190
Sharpe Ratio	0.12392	0.12767	*0.12767*	0.20902	0.15952	0.16083	0.15270	0.16529	0.15549	0.12635	0.14936
Panel B: Statistics about out-of-sample performance											
Mean	0.00525	0.00498	*0.00555*	-0.03519	0.00496	0.00633	0.00623	-0.01171	0.00453	0.00486	0.01058
Variance	0.00179	0.00195	*0.00189*	0.23951	0.00342	0.00181	0.00172	0.04907	0.00285	0.00722	0.01702
Sharpe Ratio	0.12392	0.11286	*0.12767*	-0.07191	0.08480	0.14896	0.15014	-0.05284	0.08476	0.05724	0.08111
pVal.-($1/N$-rebal.)	0.43233	0.05442	*0.50000*	0.00973	0.17411	0.20912	0.15833	0.01256	0.14731	0.12904	0.25897
pVal.-($1/N$-no rebal.)	0.29735	0.50000	*0.05442*	0.01520	0.24587	0.09942	0.07061	0.01993	0.21755	0.18834	0.32892
CEQ	0.00435	0.00401	*0.00461*	-0.15495	0.00325	0.00543	0.00537	-0.03624	0.00310	0.00125	0.00207
pVal.-($1/N$-rebal.)	0.39318	0.06924	*0.50000*	0.00000	0.29186	0.23238	0.21172	0.00149	0.23210	0.23195	0.00000
pVal.-($1/N$-no rebal.)	0.35172	0.50000	*0.06924*	0.00000	0.36785	0.12078	0.10550	0.00172	0.31013	0.27519	0.00000
Turnover	0.00000	0.00000	*0.02931*	124.14658	0.21196	0.21400	0.06647	51.59943	0.17886	24.01118	40.89281
Panel C: Statistics about portfolio weights											
Canada											
Min	0.000	0.099	*0.111*	-100.443	0.000	-0.152	0.000	-61.238	0.000	-50.784	-67.241
Max	0.000	0.125	*0.111*	123.586	0.415	0.289	0.302	57.876	0.444	19.670	29.587
Avg	0.000	0.111	*0.111*	-0.068	0.006	0.059	0.088	0.057	0.005	0.255	-0.015
StdDev	0.000	0.004	*0.000*	15.567	0.043	0.137	0.110	7.662	0.039	4.037	5.720
France											
Min	0.000	0.090	*0.111*	-14.640	0.000	-0.146	0.000	-8.954	0.000	-7.438	-9.801
Max	0.000	0.124	*0.111*	18.686	1.000	0.126	0.052	8.719	0.842	2.924	7.157
Avg	0.000	0.111	*0.111*	0.239	0.139	-0.062	0.002	0.069	0.099	-0.056	0.135
StdDev	0.000	0.004	*0.000*	2.320	0.293	0.057	0.008	1.147	0.208	0.602	0.871
Germany											
Min	0.000	0.095	*0.111*	-72.513	0.000	0.024	0.000	-33.643	0.000	-22.326	-29.718
Max	0.000	0.125	*0.111*	83.761	0.244	0.349	0.277	39.391	0.272	13.592	15.709
Avg	0.000	0.111	*0.111*	0.246	0.003	0.149	0.120	0.216	0.011	0.248	0.179
StdDev	0.000	0.004	*0.000*	9.767	0.023	0.086	0.077	4.636	0.044	2.012	2.683

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Table 5 (cont.): Nine international equity indexes

Statistic / Strategy	Single asset	$1/N$ (no rebal)	$1/N$ (*with rebal*)	Mean-var	Mean-var constr.	Min-var	Min-var constr.	Bayes-Stein	Bayes-Stein constr.	3-fund	Data&model $\omega = 0.50$
Panel C (cont.): Statistics about portfolio weights											
Italy											
Min	0.000	0.091	*0.111*	-17.089	0.000	0.015	0.008	-8.010	0.000	-2.800	-4.316
Max	0.000	0.133	*0.111*	35.657	0.574	0.136	0.111	21.800	0.369	18.106	23.871
Avg	0.000	0.111	*0.111*	0.172	0.034	0.075	0.062	0.121	0.044	0.086	0.089
StdDev	0.000	0.006	*0.000*	2.818	0.098	0.028	0.023	1.576	0.085	1.169	1.561
Japan											
Min	0.000	0.090	*0.111*	-247.675	0.000	-0.194	0.009	-115.370	0.000	-72.607	-96.187
Max	0.000	0.128	*0.111*	321.523	1.000	0.364	0.244	150.699	1.000	51.376	80.426
Avg	0.000	0.111	*0.111*	2.215	0.467	0.078	0.129	1.144	0.402	0.598	1.229
StdDev	0.000	0.006	*0.000*	37.378	0.426	0.126	0.071	17.726	0.376	7.491	10.508
Switzerland											
Min	0.000	0.102	*0.111*	-52.351	0.000	-0.146	0.000	-31.958	0.000	-26.521	-35.046
Max	0.000	0.123	*0.111*	39.380	1.000	0.220	0.166	18.380	1.000	7.115	12.417
Avg	0.000	0.111	*0.111*	0.197	0.205	0.047	0.051	0.137	0.165	0.131	0.141
StdDev	0.000	0.004	*0.000*	5.861	0.350	0.089	0.045	3.062	0.259	1.963	2.779
UK											
Min	0.000	0.098	*0.111*	-106.734	0.000	-0.276	0.000	-49.807	0.000	-22.854	-30.190
Max	0.000	0.122	*0.111*	136.233	0.806	0.435	0.252	63.764	1.000	21.628	35.068
Avg	0.000	0.111	*0.111*	0.959	0.061	-0.042	0.025	0.449	0.076	0.167	0.580
StdDev	0.000	0.004	*0.000*	15.652	0.150	0.159	0.064	7.329	0.132	2.799	3.872
US											
Min	0.000	0.102	*0.111*	-599.852	0.000	-0.148	0.378	-279.532	0.000	-136.736	-181.236
Max	0.000	0.122	*0.111*	765.660	1.000	1.052	0.622	358.756	1.000	122.167	189.040
Avg	0.000	0.111	*0.111*	4.063	0.084	0.437	0.505	2.320	0.159	1.575	2.629
StdDev	0.000	0.004	*0.000*	85.998	0.232	0.250	0.058	40.381	0.263	15.794	22.141
World											
Min	1.000	0.105	*0.111*	-1481.952	0.000	-1.407	0.000	-693.868	0.000	-235.645	-364.088
Max	1.000	0.118	*0.111*	1168.282	0.000	1.244	0.244	544.931	0.000	322.161	426.549
Avg	1.000	0.111	*0.111*	-7.023	0.000	0.259	0.016	-3.514	0.000	-2.004	-3.968
StdDev	0.000	0.002	*0.000*	169.285	0.000	0.576	0.048	80.064	0.000	33.228	46.685

Table 6: Market, HML and SMB portfolios

The table reports a variety of statistics for eleven different static asset allocation strategies. Panel A gives the in-sample mean, variance and Sharpe ratio of excess returns; Panel B gives the out-of-sample mean, variance, Sharpe ratio and Certainty Equivalent (CEQ) of excess returns, along with the P-values that the Sharpe ratio and CEQ are different from those for the $1/N$ strategy (without and with rebalancing); Panel C gives the minimum, maximum, average and standard deviation over time of the portfolio weight allocated to a particular asset. The investor has available for investment a US T-bill, the market portfolio, defined as the value-weighted return on all NYSE, AMEX and NASDAQ stocks (from CRSP) minus the one-month Treasury bill rate (from Ibbotson Associates), and the Fama-French portfolios, HML (a zero-cost portfolio that is long in high book-to-market stocks and short in low book-to-market stocks), and SMB (a zero-cost portfolio that is long in small stocks and short in big stocks). There are a total of 497 monthly observations from July 1963 to November 2004. The data are taken from Kenneth French's website. The risk-free rate is the 90-day T-bill, obtained from CRSP.

Statistic \ Strategy	Single asset	$1/N$ (no rebal)	$1/N$ (*with rebal*)	Mean-var	Mean-var constr.	Min-var	Min-var constr.	Bayes-Stein	Bayes-Stein constr.	3-fund	Data&model $\omega = 0.50$
Panel A: Statistics about in-sample performance											
Mean	0.00532	0.00411	*0.00411*	0.00480	0.00534	0.00428	0.00428	0.00436	0.00470	0.00434	0.00311
Variance	0.00219	0.00034	*0.00034*	0.00028	0.00038	0.00025	0.00025	0.00026	0.00038	0.00026	0.00032
Sharpe Ratio	0.11377	0.22403	*0.22403*	0.28507	0.27391	0.26910	0.26910	0.27136	0.24111	0.27123	0.17276
Panel B: Statistics about out-of-sample performance											
Mean	0.00532	0.00397	*0.00411*	0.00471	0.00358	0.00400	0.00400	0.00445	0.00419	0.00452	0.00465
Variance	0.00219	0.00031	*0.00034*	0.00047	0.00109	0.00026	0.00026	0.00031	0.00076	0.00032	0.00054
Sharpe Ratio	0.11377	0.22711	*0.22403*	0.21859	0.10837	0.24925	0.24925	0.25363	0.15136	0.25462	0.19984
pVal.-($1/N$-rebal.)	0.00238	0.43899	*0.50000*	0.46061	0.01869	0.22931	0.22931	0.24665	0.08745	0.22476	0.32281
pVal.-($1/N$-no rebal.)	0.00492	0.50000	*0.43899*	0.42085	0.00631	0.24698	0.24698	0.20548	0.03930	0.17741	0.24912
CEQ	0.00423	0.00381	*0.00394*	0.00448	0.00303	0.00387	0.00387	0.00430	0.00380	0.00437	0.00438
pVal.-($1/N$-rebal.)	0.43879	0.35968	*0.50000*	0.30941	0.27869	0.45002	0.45002	0.31878	0.45710	0.27610	0.39738
pVal.-($1/N$-no rebal.)	0.41642	0.50000	*0.35968*	0.21282	0.28607	0.45951	0.45951	0.18921	0.49592	0.13986	0.33944
Turnover	0.00000	0.00000	*0.02370*	0.06720	0.09761	0.02630	0.02630	0.04378	0.08641	0.06175	0.07148
Panel C: Statistics about portfolio weights											
SMB											
Min	0.000	0.268	*0.333*	-0.686	0.000	0.180	0.180	-0.412	0.000	-0.473	-0.397
Max	0.000	0.366	*0.333*	0.562	1.000	0.370	0.370	0.342	1.000	0.455	0.538
Avg	0.000	0.332	*0.333*	0.075	0.129	0.281	0.281	0.170	0.109	0.167	0.082
StdDev	0.000	0.009	*0.000*	0.230	0.321	0.053	0.053	0.125	0.201	0.140	0.197
HML											
Min	0.000	0.295	*0.333*	0.353	0.000	0.425	0.425	0.399	0.000	0.393	0.217
Max	0.000	0.382	*0.333*	0.984	1.000	0.630	0.630	0.735	1.000	0.734	1.072
Avg	0.000	0.334	*0.333*	0.636	0.426	0.520	0.520	0.567	0.477	0.548	0.566
StdDev	0.000	0.012	*0.000*	0.147	0.429	0.060	0.060	0.079	0.335	0.070	0.177
Mkt-RF											
Min	1.000	0.282	*0.333*	-0.234	0.000	0.082	0.082	-0.006	0.000	0.015	-0.478
Max	1.000	0.383	*0.333*	1.023	1.000	0.303	0.303	0.797	1.000	0.847	1.013
Avg	1.000	0.334	*0.333*	0.290	0.445	0.199	0.199	0.262	0.414	0.285	0.352
StdDev	0.000	0.012	*0.000*	0.215	0.441	0.042	0.042	0.122	0.382	0.120	0.256

Table 7: Market, HML, SMB, and twenty size- and B/M-sorted portfolios

The table reports a variety of statistics for eleven different static asset allocation strategies. Panel A gives the in-sample mean, variance and Sharpe ratio of excess returns; Panel B gives the out-of-sample mean, variance, Sharpe ratio and Certainty Equivalent (CEQ) of excess returns, along with the P-values that the Sharpe ratio and CEQ are different from those for the $1/N$ strategy (without and with rebalancing); Panel C gives the minimum, maximum, average and standard deviation over time of the portfolio weight allocated to a particular asset. The investor has available for investment a US T-bill, the market portfolio, defined as the value-weighted return on all NYSE, AMEX and NASDAQ stocks (from CRSP) minus the one-month Treasury bill rate (from Ibbotson Associates), the Fama-French factors HML and SMB, and 20 Fama and French portfolios of firms sorted according to size and book to market. There are a total of 497 monthly observations from July 1963 to November 2004. The data are taken from Kenneth French's website. We use the methodology of Pástor (2000) to construct Bayesian portfolios, and here the market portfolio serves as a factor portfolio. The risk-free rate is the 90-day T-bill, obtained from CRSP.

Statistic / Strategy	Single asset	$1/N$ (no rebal)	$1/N$ (*with rebal*)	Mean-var	Mean-var constr.	Min-var	Min-var constr.	Bayes-Stein	Bayes-Stein constr.	3-fund	Data&model $\omega = 0.50$
Panel A: Statistics about in-sample performance											
Mean	0.00532	0.00827	*0.00827*	0.03583	0.01284	0.00015	0.00428	0.02273	0.01018	0.02190	0.01571
Variance	0.00219	0.00244	*0.00244*	0.00496	0.00371	0.00002	0.00027	0.00315	0.00359	0.00293	0.00333
Sharpe Ratio	0.11377	0.16744	*0.16744*	0.50882	0.21090	0.03317	0.26176	0.40478	0.17001	0.40478	0.27233
Panel B: Statistics about out-of-sample performance											
Mean	0.00532	0.00922	*0.00827*	-0.00488	0.00845	-0.00019	0.00411	-0.00541	0.00814	-0.00598	0.02754
Variance	0.00219	0.00244	*0.00244*	1.09248	0.00200	0.00008	0.00026	0.71386	0.00173	0.67103	0.46431
Sharpe Ratio	0.11377	0.18656	*0.16744*	-0.00467	0.18906	-0.02181	0.25574	-0.00641	0.19577	-0.00730	0.04042
pVal.-($1/N$-rebal.)	0.00974	0.00003	*0.50000*	0.00945	0.22637	0.00595	0.04149	0.00886	0.19244	0.00857	0.03755
pVal.-($1/N$-no rebal.)	0.00176	0.50000	*0.00003*	0.00451	0.46384	0.00286	0.07760	0.00420	0.38504	0.00405	0.02005
CEQ	0.00423	0.00800	*0.00705*	-0.55112	0.00745	-0.00023	0.00398	-0.36234	0.00727	-0.34149	-0.20462
pVal.-($1/N$-rebal.)	0.00511	0.00002	*0.50000*	0.00000	0.38434	0.00266	0.08351	0.00000	0.44096	0.00000	0.00000
pVal.-($1/N$-no rebal.)	0.00076	0.50000	*0.00002*	0.00000	0.33691	0.00082	0.03283	0.00000	0.31081	0.00000	0.00000
Turnover	0.00000	0.00000	*0.01850*	63.35506	0.30349	0.13283	0.03235	49.28308	0.25722	47.41400	151.83048
Panel C: Statistics about portfolio weights											
SMB											
Min	0.000	0.031	*0.043*	-109.275	0.000	0.370	0.154	-78.897	0.000	-71.283	-234.890
Max	0.000	0.053	*0.043*	53.013	0.000	0.752	0.332	43.449	0.000	42.366	53.429
Avg	0.000	0.043	*0.043*	0.564	0.000	0.642	0.249	0.583	0.000	0.578	-0.928
StdDev	0.000	0.002	*0.000*	7.215	0.000	0.115	0.040	5.321	0.000	4.873	17.096
HML											
Min	0.000	0.036	*0.043*	-436.447	0.000	0.252	0.425	-356.637	0.000	-347.597	-339.724
Max	0.000	0.061	*0.043*	55.953	1.000	0.609	0.687	40.518	0.960	36.605	110.536
Avg	0.000	0.043	*0.043*	-1.203	0.147	0.352	0.554	-0.882	0.212	-0.822	-0.768
StdDev	0.000	0.003	*0.000*	23.842	0.280	0.107	0.063	19.269	0.321	18.675	19.644
Mkt-RF											
Min	1.000	0.040	*0.043*	-13.866	0.000	0.507	0.000	-10.771	0.000	-10.305	-410.803
Max	1.000	0.046	*0.043*	185.254	1.000	0.852	0.303	151.547	1.000	147.729	69.373
Avg	1.000	0.043	*0.043*	1.338	0.023	0.723	0.114	1.188	0.025	1.135	-0.703
StdDev	0.000	0.001	*0.000*	9.943	0.138	0.099	0.108	8.057	0.140	7.814	26.085

Table 8: Market, HML, SMB, MOM, and twenty size- and B/M-sorted portfolios

The table reports a variety of statistics for eleven different static asset allocation strategies. Panel A gives the in-sample mean, variance and Sharpe ratio of excess returns; Panel B gives the out-of-sample mean, variance, Sharpe ratio and Certainty Equivalent (CEQ) of excess returns, along with the P-values that the Sharpe ratio and CEQ are different from those for the $1/N$ strategy (without and with rebalancing); Panel C gives the minimum, maximum, average and standard deviation over time of the portfolio weight allocated to a particular asset. The investor has available for investment a US T-bill, the market portfolio, defined as the value-weighted return on all NYSE, AMEX and NASDAQ stocks (from CRSP) minus the one-month Treasury bill rate (from Ibbotson Associates), the Fama-French factors HML and SMB, the momentum (MOM) portfolio, and 20 Fama and French portfolios of firms sorted according to size and book to market. There are a total of 497 monthly observations from July 1963 to November 2004. The data are taken from Kenneth French's website. We use the methodology of Pástor (2000) to construct Bayesian portfolios, and here the market, HML, SMB, and MOM portfolios serve as a factor portfolio. The risk-free rate is the 90-day T-bill, obtained from CRSP.

Strategy Statistic	Single asset	$1/N$ (no rebal)	$1/N$ (*with rebal*)	Mean-var	Mean-var constr.	Min-var	Min-var constr.	Bayes-Stein	Bayes-Stein constr.	3-fund	Data&model $\omega = 0.50$
				Panel A: Statistics about in-sample performance							
Mean	0.00532	0.00793	*0.00793*	0.00099	0.01284	0.00006	0.00007	0.00064	0.01007	0.00062	0.00049
Variance	0.00219	0.00224	*0.00224*	0.00000	0.00372	0.00000	0.00000	0.00000	0.00359	0.00000	0.00000
Sharpe Ratio	0.11377	0.16749	*0.16749*	0.52372	0.21060	0.12970	0.14517	0.42024	0.16821	0.42021	0.37934
				Panel B: Statistics about out-of-sample performance							
Mean	0.00532	0.00906	*0.00793*	0.00890	0.00844	0.00006	0.00007	0.00583	0.00817	0.00418	-0.00075
Variance	0.00219	0.00235	*0.00224*	0.00964	0.00199	0.00000	0.00000	0.00325	0.00172	0.00124	0.00062
Sharpe Ratio	0.11377	0.18701	*0.16749*	0.09064	0.18921	0.08622	0.14768	0.10231	0.19664	0.11905	-0.03008
pVal.-($1/N$-rebal.)	0.00968	0.00004	*0.50000*	0.14547	0.22561	0.13804	0.38940	0.18435	0.18572	0.25043	0.00162
pVal.-($1/N$-no rebal.)	0.00165	0.50000	*0.00004*	0.09257	0.46775	0.08886	0.28850	0.12108	0.37832	0.17178	0.00063
CEQ	0.00423	0.00789	*0.00681*	0.00408	0.00745	0.00006	0.00007	0.00421	0.00730	0.00357	-0.00106
pVal.-($1/N$-rebal.)	0.00784	0.00000	*0.50000*	0.31311	0.31323	0.00284	0.00285	0.24589	0.36678	0.13895	0.13718
pVal.-($1/N$-no rebal.)	0.00089	0.50000	*0.00000*	0.24895	0.36343	0.00086	0.00086	0.16704	0.34001	0.07637	0.08132
Turnover	0.00000	0.00000	*0.01938*	12.39385	0.30307	0.01561	0.00066	23.99381	0.25821	4.55204	6.71060
				Panel C: Statistics about portfolio weights							
SMB											
Min	-0.649	0.030	*0.042*	-12.031	0.000	-0.001	0.000	-6.504	0.000	-2.919	-67.719
Max	5.033	0.050	*0.042*	122.355	0.000	0.054	0.003	65.637	0.000	28.101	16.456
Avg	0.022	0.041	*0.042*	0.609	0.000	0.023	0.001	0.350	0.000	0.191	-0.236
StdDev	0.281	0.002	*0.000*	6.982	0.000	0.016	0.001	3.780	0.000	1.721	4.251
HML											
Min	-9.289	0.035	*0.042*	-5.734	0.000	0.001	0.000	-3.097	0.000	-1.386	-26.377
Max	14.010	0.057	*0.042*	42.009	1.000	0.020	0.004	22.537	0.962	11.940	6.862
Avg	0.072	0.042	*0.042*	0.277	0.148	0.010	0.002	0.150	0.212	0.069	-0.080
StdDev	0.901	0.003	*0.000*	3.038	0.281	0.004	0.001	1.668	0.320	0.817	1.687
MOM											
Min	-16.885	0.034	*0.042*	-164.957	0.000	0.929	0.994	-88.029	0.000	-37.117	-21.498
Max	11.469	0.055	*0.042*	18.467	0.000	1.002	1.000	10.439	0.000	5.232	90.631
Avg	0.895	0.041	*0.042*	0.050	0.000	0.966	0.997	0.448	0.000	0.688	1.282
StdDev	1.112	0.002	*0.000*	9.861	0.000	0.020	0.001	5.363	0.000	2.503	5.715
Mkt-RF											
Min	-1.159	0.039	*0.042*	-14.749	0.000	0.001	0.000	-7.709	0.000	-3.456	-77.468
Max	0.259	0.044	*0.042*	152.945	1.000	0.064	0.001	82.048	1.000	35.128	19.727
Avg	0.012	0.042	*0.042*	0.848	0.023	0.028	0.000	0.493	0.025	0.279	-0.294
StdDev	0.090	0.001	*0.000*	9.125	0.138	0.018	0.000	4.968	0.140	2.335	5.208

Table 9: Market portfolio and 10-year bond with stochastic interest rates: Risk aversion = 3

The table reports a variety of statistics for eleven different static asset allocation strategies. Panel A gives the in-sample mean, variance and Sharpe ratio of excess returns; Panel B gives the out-of-sample mean, variance, Sharpe ratio and Certainty Equivalent (CEQ) of excess returns, along with the P-values that the Sharpe ratio and CEQ are different from those for the $1/N$ strategy (without and with rebalancing); Panel C gives the minimum, maximum, average and standard deviation over time of the portfolio weight allocated to a particular asset. There are a total of 171 quarterly observations from 1954:1 to 1996:3. The riskless rate is represented by the real Treasury bill rate, constructed as the difference between the yield on a 90-day T-bill and log inflation. The two risky asset are, respectively, the value-weighted excess return (including dividends) on the NYSE, NASDAQ and AMEX market and the excess return on the 10-year bond. The data and model are the same as in Campbell and Viceira (2001) and Chapter 3 of Campbell and Viceira (2002). The investor is assumed to have a risk aversion of $\gamma = 3$.

Statistic / Strategy	Single asset	$1/N$ (no rebal)	$1/N$ (*with rebal*)	Mean-var	Mean-var constr.	Min-var	Min-var constr.	Bayes-Stein	Bayes-Stein constr.	3-fund	Dynamic
Panel A: Statistics about in-sample performance											
Mean	0.02472	0.01728	*0.01728*	0.01421	0.02421	0.01102	0.01102	0.01138	0.01249	0.01143	0.02436
Variance	0.00530	0.00177	*0.00177*	0.00115	0.00529	0.00089	0.00089	0.00094	0.00140	0.00095	0.00539
Sharpe Ratio	0.33968	0.41105	*0.41105*	0.41952	0.33296	0.36936	0.36936	0.37180	0.33404	0.37171	0.33189
Panel B: Statistics about out-of-sample performance											
Mean	0.02472	0.01768	*0.01728*	0.01779	0.01920	0.01027	0.01027	0.01249	0.01321	0.01288	0.02214
Variance	0.00530	0.00195	*0.00177*	0.00803	0.00410	0.00085	0.00085	0.00293	0.00179	0.00385	0.00394
Sharpe Ratio	0.33968	0.40084	*0.41105*	0.19857	0.29979	0.35229	0.35229	0.23067	0.31188	0.20751	0.35284
pVal.-($1/N$-rebal.)	0.08695	0.22321	*0.50000*	0.01356	0.00674	0.32837	0.32837	0.02106	0.01281	0.01449	0.10455
pVal.-($1/N$-no rebal.)	0.08607	0.50000	*0.22321*	0.01768	0.00517	0.36411	0.36411	0.03203	0.03724	0.02124	0.09573
CEQ	0.00229	0.00217	*0.00219*	0.00192	0.00217	0.00201	0.00201	0.00194	0.00204	0.00191	0.00223
Turnover	0.00000	0.00000	*0.02464*	18.46392	0.08653	0.00565	0.00565	1.22629	0.09242	1.92817	0.04225
Panel C: Statistics about portfolio weights											
LongBond											
Min	0.000	0.451	*0.500*	-14.953	0.000	0.954	0.954	-8.422	0.000	-10.255	0.029
Max	0.000	0.570	*0.500*	0.534	0.412	0.980	0.980	0.811	0.771	0.787	0.316
Avg	0.000	0.497	*0.500*	-0.270	0.222	0.968	0.968	0.329	0.570	0.205	0.128
StdDev	0.000	0.018	*0.000*	2.255	0.145	0.007	0.007	1.354	0.232	1.618	0.069
Equity											
Min	1.000	0.430	*0.500*	0.466	0.588	0.020	0.020	0.189	0.229	0.213	0.684
Max	1.000	0.549	*0.500*	15.953	1.000	0.046	0.046	9.422	1.000	11.255	0.971
Avg	1.000	0.503	*0.500*	1.270	0.778	0.032	0.032	0.671	0.430	0.795	0.872
StdDev	0.000	0.018	*0.000*	2.255	0.145	0.007	0.007	1.354	0.232	1.618	0.069

Table 10: Market portfolio and 5-year bond with time-varying expected returns: Risk aversion = 3

The table reports a variety of statistics for eleven different static asset allocation strategies. Panel A gives the in-sample mean, variance and Sharpe ratio of excess returns; Panel B gives the out-of-sample mean, variance, Sharpe ratio and Certainty Equivalent (CEQ) of excess returns, along with the P-values that the Sharpe ratio and CEQ are different from those for the $1/N$ strategy (without and with rebalancing); Panel C gives the minimum, maximum, average and standard deviation over time of the portfolio weight allocated to a particular asset. The investor has available for investment a US T-bill, a 5-year bond, and the US market portfolio. There are a total of 185 quarterly observations from 1952:2 to 1999:3. The riskless rate is represented by the real Treasury bill rate, constructed as the difference between the yield on a 90-day T-bill and log inflation. The two risky asset are, respectively, the value-weighted excess return (including dividends) on the NYSE, NASDAQ and AMEX market and the excess bond return on the 5-year bond. The state variables used to predict returns are the nominal log yield on a 90-day Treasury bill, the dividend-price ratio and the yield spread between the 5-year zero coupon bond yield. This is the dataset used in Campbell, Chan, and Viceira (2003). The investor is assumed to have a risk aversion of $\gamma = 3$.

Statistic / Strategy	Single asset	$1/N$ (no rebal)	$1/N$ (*rebal*)	Mean var	Mean-var constr.	Min-var	Min-var constr.	Bayes Stein	Bayes-Stein constr.	3-fund	Dynamic re-est	Dynamic (one-est)	Dynamic constr.
Panel A: Statistics about in-sample performance													
Mean	0.02329	0.01469	*0.01469*	0.00873	0.00873	0.01129	0.01954	0.00909	0.00968	0.00943	0.01094	0.01094	0.01094
Variance	0.00582	0.00172	*0.00172*	0.00066	0.00066	0.00088	0.00375	0.00075	0.00097	0.00067	0.00077	0.00077	0.00077
Sharpe Ratio	0.30517	0.35383	*0.35383*	0.33910	0.33910	0.38078	0.31903	0.33235	0.31060	0.36425	0.39357	0.39357	0.39357
Panel B: Statistics about out-of-sample performance													
Mean	0.02329	0.01620	*0.01469*	0.14163	0.01186	0.00801	0.00801	0.12526	0.00656	0.00901	0.02784	0.32795	0.01717
Variance	0.00582	0.00165	*0.00172*	0.16218	0.00344	0.00068	0.00068	0.13869	0.00108	0.00397	0.05248	0.81453	0.00446
Sharpe Ratio	0.30517	0.39866	*0.35383*	0.35169	0.20207	0.30686	0.30686	0.33634	0.19936	0.14312	0.12151	0.36337	0.25696
pVal.-($1/N$-rebal.)	0.05869	0.00006	*0.50000*	0.13240	0.00051	0.38137	0.38137	0.14002	0.03676	0.07369	0.00391	0.01285	0.01744
pVal.-($1/N$-no rebal.)	0.00883	0.50000	*0.00006*	0.08348	0.00017	0.25574	0.25574	0.09009	0.01132	0.04221	0.00391	0.01285	0.01744
CEQ	0.00349	0.00324	*0.00311*	0.00276	0.00274	0.00318	0.00318	0.00272	0.00254	0.00237	0.00191	0.00201	0.00277
Turnover	0.00000	0.00000	*0.02677*	5.40181	0.12014	0.00500	0.00500	4.28032	0.07281	1.21319	1.27328	34.84525	0.05542
Panel C: Statistics about portfolio weights													
5yrBond													
Min	0.000	0.452	*0.500*	-6.956	0.000	0.959	0.959	-3.141	0.000	-7.694	-18.776	-37.199	0.000
Max	0.000	0.577	*0.500*	48.577	0.720	0.980	0.980	45.709	0.950	9.865	5.900	278.641	0.737
Avg	0.000	0.497	*0.500*	1.438	0.395	0.970	0.970	1.706	0.755	1.197	0.071	8.804	0.188
StdDev	0.000	0.019	*0.000*	7.579	0.251	0.006	0.006	6.596	0.302	1.869	2.912	36.199	0.243
Market													
Min	1.000	0.423	*0.500*	-47.577	0.280	0.020	0.020	-44.709	0.050	-8.865	-4.900	-277.641	0.263
Max	1.000	0.548	*0.500*	7.956	1.000	0.041	0.041	4.141	1.000	8.694	19.776	38.199	1.000
Avg	1.000	0.503	*0.500*	-0.438	0.605	0.030	0.030	-0.706	0.245	-0.197	0.929	-7.804	0.812
StdDev	0.000	0.019	*0.000*	7.579	0.251	0.006	0.006	6.596	0.302	1.869	2.912	36.199	0.243

Table 11: Simulated data: Estimation window of 10 years and 4 risky assets

The table reports a variety of statistics for twelve different static asset allocation strategies. Panel A gives the in-sample mean, variance and Sharpe ratio of excess returns; Panel B gives the out-of-sample mean, variance, Sharpe ratio and Certainty Equivalent (CEQ) of excess returns, along with the P-values that the Sharpe ratio and CEQ are different from those for the $1/N$ strategy (without and with and without rebalancing); Panel C gives the minimum, maximum, average and standard deviation over time of the portfolio weight allocated to a particular asset. Details about how the data was simulated are given in Section 5.

Statistic / Strategy	Single asset	$1/N$ (no rebal)	$1/N$ (*with rebal*)	Mean-var	Mean-var constr.	Min-var	Min-var constr.	Bayes-Stein	Bayes-Stein constr.	3-fund	Data&model $\omega = 0.50$	Mean-var true
Panel A: Statistics about in-sample performance												
Mean	0.00684	0.00674	0.00674	0.00682	0.00867	0.00509	0.00567	0.00668	0.00841	0.00670	0.00683	0.00684
Variance	0.00215	0.00267	0.00267	0.00213	0.00455	0.00159	0.00177	0.00208	0.00433	0.00209	0.00213	0.00215
Sharpe Ratio	0.14773	0.13045	0.13045	0.14794	0.12857	0.12785	0.13492	0.14635	0.12785	0.14635	0.14777	0.14771
Panel B: Statistics about out-of-sample performance												
Mean	0.00684	0.01002	0.00674	0.01662	0.00784	0.00503	0.00563	0.00887	0.00689	0.00319	-0.00756	0.00684
Variance	0.00215	0.00823	0.00267	1.58105	0.00501	0.00163	0.00179	0.38661	0.00336	0.53413	7.34277	0.00215
Sharpe Ratio	0.14773	0.11038	0.13046	0.01322	0.11075	0.12450	0.13299	0.01426	0.11896	0.00436	-0.00279	0.14773
pVal.-($1/N$-rebal.)	0.00000	0.00000	0.50000	0.00000	0.00000	0.09168	0.17361	0.00000	0.00094	0.00000	0.00000	0.00000
pVal.-($1/N$-no rebal.)	0.00000	0.50000	0.00000	0.00000	0.46485	0.02039	0.00001	0.00000	0.03310	0.00000	0.00000	0.00000
CEQ	0.00577	0.00590	0.00540	-0.77390	0.00533	0.00421	0.00474	-0.18444	0.00522	-0.26388	-3.67894	0.00577
pVal.-($1/N$-rebal.)	0.00741	0.07415	0.50000	0.00000	0.39815	0.00000	0.00000	0.00000	0.17993	0.00000	0.00000	0.00741
pVal.-($1/N$-no rebal.)	0.37711	0.50000	0.07415	0.00000	0.05986	0.00073	0.00478	0.00000	0.04333	0.00000	0.00000	0.37711
Turnover	0.00000	0.00000	0.03349	12.71952	0.13146	0.05560	0.02972	6.02862	0.12818	5.02943	5.57196	0.00000
Panel C: Statistics about portfolio weights												
Asset1												
Min	0.000	0.206	0.250	-859.654	0.000	0.026	0.082	-292.042	0.000	-466.699	-331.275	0.000
Max	0.000	0.306	0.250	1972.865	1.000	0.567	0.591	534.758	1.000	443.069	3132.383	0.000
Avg	0.000	0.249	0.250	0.189	0.140	0.318	0.354	0.251	0.180	0.261	0.191	0.000
StdDev	0.000	0.012	0.000	20.355	0.306	0.071	0.074	7.724	0.254	7.795	22.905	0.000
Asset2												
Min	0.000	0.206	0.250	-1183.001	0.000	-0.213	0.000	-646.162	0.000	-607.779	-356.988	0.000
Max	0.000	0.291	0.250	815.185	1.000	0.273	0.260	430.765	1.000	480.897	2434.671	0.000
Avg	0.000	0.250	0.250	0.018	0.215	0.012	0.039	0.020	0.154	0.034	0.216	0.000
StdDev	0.000	0.011	0.000	14.791	0.358	0.075	0.049	7.326	0.266	8.241	22.724	0.000
Asset3												
Min	0.000	0.188	0.250	-1572.134	0.000	-0.400	0.000	-336.931	0.000	-144.123	-725.693	0.000
Max	0.000	0.301	0.250	1021.694	1.000	0.012	0.012	557.999	1.000	809.264	565.264	0.000
Avg	0.000	0.251	0.250	0.109	0.419	-0.209	0.000	-0.040	0.224	-0.055	0.023	0.000
StdDev	0.000	0.014	0.000	15.238	0.436	0.058	0.000	5.931	0.344	8.028	9.670	0.000
Asset4												
Min	1.000	0.228	0.250	-2161.562	0.000	0.454	0.297	-905.144	0.000	-1355.893	-3719.930	1.000
Max	1.000	0.278	0.250	2635.739	1.000	1.306	0.889	565.326	1.000	512.712	485.951	1.000
Avg	1.000	0.250	0.250	0.684	0.216	0.879	0.607	0.769	0.409	0.760	0.570	1.000
StdDev	0.000	0.006	0.000	31.679	0.369	0.125	0.088	12.464	0.369	14.818	34.952	0.000

Table 12: Simulated data: Estimation window of 100 years and 4 risky assets

The table reports a variety of statistics for eleven different static asset allocation strategies. Panel A gives the in-sample mean, variance and Sharpe ratio of excess returns; Panel B gives the out-of-sample mean, variance, Sharpe ratio and Certainty Equivalent (CEQ) of excess returns, along with the P-values that the Sharpe ratio and CEQ are different from those for the $1/N$ strategy (without and with rebalancing); Panel C gives the minimum, maximum, average and standard deviation over time of the portfolio weight allocated to a particular asset. Details about how the data was simulated are given in Section 5.

Strategy Statistic	Single asset	$1/N$ (no rebal)	$1/N$ (*with rebal*)	Mean-var	Mean-var constr.	Min-var	Min-var constr.	Bayes-Stein	Bayes-Stein constr.	3-fund	Data&model $\omega = 0.50$	Mean-var true
Panel A: Statistics about in-sample performance												
Mean	0.00684	0.00674	0.00674	0.00682	0.00867	0.00509	0.00567	0.00668	0.00841	0.00670	0.00683	0.00684
Variance	0.00215	0.00267	0.00267	0.00213	0.00455	0.00159	0.00177	0.00208	0.00433	0.00209	0.00213	0.00215
Sharpe Ratio	0.14773	0.13045	0.13045	0.14794	0.12857	0.12785	0.13492	0.14635	0.12785	0.14635	0.14777	0.14771
Panel B: Statistics about out-of-sample performance												
Mean	0.00684	0.01002	0.00674	0.00708	0.00872	0.00510	0.00569	0.00638	0.00789	0.00635	0.00694	0.00684
Variance	0.00215	0.00823	0.00267	0.00261	0.00510	0.00159	0.00177	0.00206	0.00368	0.00209	0.00226	0.00215
Sharpe Ratio	0.14773	0.11038	0.13046	0.13843	0.12211	0.12771	0.13523	0.14047	0.12995	0.13896	0.14597	0.14773
pVal.-($1/N$-rebal.)	0.00000	0.00000	0.50000	0.02148	0.00524	0.26578	0.03838	0.00395	0.43277	0.01405	0.00000	0.00000
pVal.-($1/N$-no rebal.)	0.00000	0.50000	0.00000	0.00000	0.00009	0.00585	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
CEQ	0.00577	0.00590	0.00540	0.00577	0.00617	0.00430	0.00481	0.00535	0.00604	0.00531	0.00581	0.00577
pVal.-($1/N$-rebal.)	0.00741	0.07415	0.50000	0.03344	0.00055	0.00000	0.00001	0.38190	0.00019	0.30374	0.00689	0.00741
pVal.-($1/N$-no rebal.)	0.37711	0.50000	0.07415	0.38256	0.16694	0.00122	0.00730	0.11298	0.32961	0.09753	0.41292	0.37711
Turnover	0.00000	0.00000	0.03349	0.04901	0.03147	0.04127	0.02069	0.03827	0.03181	0.04135	0.02384	0.00000
Panel C: Statistics about portfolio weights												
Asset1												
Min	0.000	0.206	0.250	-0.714	0.000	0.244	0.295	-0.480	0.000	-0.522	-0.273	0.000
Max	0.000	0.306	0.250	0.480	0.425	0.390	0.413	0.415	0.489	0.489	0.256	0.000
Avg	0.000	0.249	0.250	0.005	0.016	0.317	0.355	0.115	0.052	0.119	0.007	0.000
StdDev	0.000	0.012	0.000	0.197	0.058	0.023	0.020	0.152	0.105	0.170	0.098	0.000
Asset2												
Min	0.000	0.206	0.250	-0.654	0.000	-0.041	0.000	-0.498	0.000	-0.524	-0.288	0.000
Max	0.000	0.291	0.250	1.145	1.000	0.076	0.089	0.798	1.000	0.830	0.588	0.000
Avg	0.000	0.250	0.250	-0.028	0.164	0.010	0.017	-0.026	0.111	-0.033	-0.012	0.000
StdDev	0.000	0.011	0.000	0.266	0.292	0.021	0.018	0.169	0.203	0.168	0.135	0.000
Asset3												
Min	0.000	0.188	0.250	-0.385	0.000	-0.263	0.000	-0.322	0.000	-0.328	-0.213	0.000
Max	0.000	0.301	0.250	0.534	1.000	-0.159	0.000	0.347	1.000	0.377	0.240	0.000
Avg	0.000	0.251	0.250	0.017	0.500	-0.209	0.000	-0.063	0.301	-0.065	0.005	0.000
StdDev	0.000	0.014	0.000	0.170	0.360	0.018	0.000	0.124	0.319	0.135	0.083	0.000
Asset4												
Min	1.000	0.228	0.250	-0.339	0.000	0.769	0.554	0.054	0.000	0.021	0.326	1.000
Max	1.000	0.278	0.250	1.943	1.000	1.015	0.705	1.715	1.000	1.758	1.445	1.000
Avg	1.000	0.250	0.250	1.007	0.321	0.882	0.628	0.974	0.536	0.980	0.999	1.000
StdDev	0.000	0.006	0.000	0.418	0.330	0.046	0.028	0.273	0.329	0.277	0.212	0.000

Table 13: Simulated data: Estimation window of 100 years and 100 risky assets

The table reports a variety of statistics for eleven different static asset allocation strategies. Panel A gives the in-sample mean, variance and Sharpe ratio of excess returns; Panel B gives the out-of-sample mean, variance, Sharpe ratio and Certainty Equivalent (CEQ) of excess returns, along with the P-values that the Sharpe ratio and CEQ are different from those for the $1/N$ strategy (without and with rebalancing); Panel C gives the minimum, maximum, average and standard deviation over time of the portfolio weight allocated to four of the 100 assets. Details about how the data was simulated are given in Section 5.

Strategy Statistic	Single asset	$1/N$ (no rebal)	$1/N$ (*with rebal*)	Mean-var	Mean-var constr.	Min-var	Min-var constr.	Bayes-Stein	Bayes-Stein constr.	3-fund	Data&model $\omega = 0.50$	Mean-var true
Panel A: Statistics about in-sample performance												
Mean	0.00684	0.00683	0.00683	0.00701	0.01006	0.00113	0.00398	0.00520	0.00850	0.00506	0.00668	0.00684
Variance	0.00215	0.00219	0.00219	0.00195	0.00480	0.00031	0.00089	0.00145	0.00467	0.00137	0.00197	0.00215
Sharpe Ratio	0.14773	0.14609	0.14609	0.15868	0.14508	0.06374	0.13354	0.13674	0.12439	0.13672	0.15061	0.14742
Panel B: Statistics about out-of-sample performance												
Mean	0.00684	0.00950	0.00683	0.00815	0.00912	0.00115	0.00402	0.00505	0.00801	0.00267	0.00685	0.00684
Variance	0.00215	0.00572	0.00218	0.02864	0.00488	0.00034	0.00090	0.00880	0.00365	0.00164	0.00454	0.00215
Sharpe Ratio	0.14773	0.12554	0.14640	0.04814	0.13064	0.06189	0.13366	0.05380	0.13250	0.06597	0.10158	0.14773
pVal.-($1/N$-rebal.)	0.05302	0.00000	0.50000	0.00000	0.00000	0.00000	0.00001	0.00000	0.00000	0.00000	0.00000	0.05302
pVal.-($1/N$-no rebal.)	0.00000	0.50000	0.00000	0.00000	0.10700	0.00000	0.03383	0.00000	0.04049	0.00000	0.00001	0.05302
CEQ	0.00577	0.00664	0.00574	-0.00617	0.00669	0.00098	0.00357	0.00065	0.00618	0.00185	0.00458	0.00577
pVal.-($1/N$-rebal.)	0.23688	0.00049	0.50000	0.00000	0.00003	0.00000	0.00000	0.00000	0.00561	0.00000	0.00015	0.23688
pVal.-($1/N$-no rebal.)	0.00074	0.50000	0.00049	0.00000	0.43421	0.00000	0.00000	0.00000	0.05638	0.00000	0.00000	0.00074
Turnover	0.00000	0.00000	0.04666	5.21064	0.08417	0.19197	0.04517	1.73650	0.08648	0.43471	0.59255	0.00000
Panel C: Statistics about portfolio weights												
Asset1												
Min	0.000	0.008	0.010	-2.001	0.000	0.044	0.053	-1.008	0.000	-0.477	-0.236	0.000
Max	0.000	0.013	0.010	5.450	0.000	0.089	0.160	2.839	0.121	0.527	0.308	0.000
Avg	0.000	0.010	0.010	0.006	0.000	0.066	0.111	0.032	0.001	0.050	0.010	0.000
StdDev	0.000	0.001	0.000	0.234	0.000	0.009	0.018	0.128	0.006	0.050	0.095	0.000
Asset2												
Min	0.000	0.008	0.010	-6.285	0.000	0.024	0.076	-3.220	0.000	-1.247	-0.346	0.000
Max	0.000	0.012	0.010	0.798	0.367	0.097	0.167	0.448	0.419	0.172	0.422	0.000
Avg	0.000	0.010	0.010	-0.099	0.002	0.063	0.113	-0.030	0.014	0.016	-0.019	0.000
StdDev	0.000	0.001	0.000	0.378	0.019	0.012	0.016	0.211	0.057	0.089	0.131	0.000
Asset3												
Min	-0.000	0.007	0.010	-1.570	0.000	0.017	0.021	-0.959	0.000	-0.608	-0.207	-0.000
Max	-0.000	0.013	0.010	10.483	0.063	0.061	0.095	5.413	0.167	0.917	0.269	-0.000
Avg	-0.000	0.010	0.010	0.009	0.000	0.034	0.058	0.020	0.003	0.029	-0.002	-0.000
StdDev	0.000	0.001	0.000	0.288	0.001	0.007	0.014	0.156	0.015	0.057	0.077	0.000
Asset4												
Min	-0.000	0.008	0.010	-2.387	0.000	0.017	0.017	-1.226	0.000	-0.266	-0.202	-0.000
Max	-0.000	0.013	0.010	2.031	0.000	0.054	0.096	1.154	0.109	0.731	0.253	-0.000
Avg	-0.000	0.010	0.010	0.019	0.000	0.036	0.060	0.029	0.001	0.040	0.006	-0.000
StdDev	0.000	0.001	0.000	0.205	0.000	0.007	0.014	0.115	0.005	0.050	0.071	0.000

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