Pricing of Catastrophe reinsurance and derivatives with a Cox process using shot noise intensity

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(work jointly done with Ji-Wook Jang)

Papers:

Dassios, Angelos; Jang, J-W. 'Pricing of catastrophe reinsurance and derivatives using the Cox process with shot noise intensity.' *Finance and Stochastics* **7**, no. 1 (2003), pp. 73-95.

Dassios, Angelos; Jang, J-W. 'Kalman-Bucy filtering for linear systems driven by the Cox process with shot noise intensity and its application to the pricing of reinsurance contracts.' *Journal of Applied Probability* **42**, no. 1 (2005), pp. 93-107.

References for this talk can be found in the two papers.

We use the Cox process (or a doubly stochastic Poisson process) to model the claim arrival process for catastrophic events. The shot noise process is used for the claim intensity function within the Cox process. The Cox process with shot noise intensity is examined by piecewise deterministic Markov process theory. We apply the model to price stop-loss catastrophe reinsurance contract & catastrophe insurance derivatives. The asymptotic distribution of the claim intensity is used to derive pricing formulae for stop-loss reinsurance contract for catastrophic events & catastrophe insurance derivatives. We assume that there is an absence of arbitrage opportunities in the market to obtain the gross premium for stop-loss reinsurance contract and arbitrage-free prices for insurance derivatives. This can be achieved by using an equivalent martingale probability measure in the pricing models. The Esscher transform is used for this purpose.

In practical situations, we observe the number of claims to an insurance portfolio but not the claim intensity. It is therefore of interest to try to solve the'filtering problem', that is to obtain the best estimate of the claim intensity on the basis of reported claims. In order to use the Kalman-Bucy filter, based on the Cox process incorporating a shot noise process as claim intensity, we need to approximate it by a Gaussian process. We demonstrate that if the primary event arrival rate of the shot noise process is reasonably large, we can then approximate the intensity, claim arrival and aggregate loss processes by a three-dimensional Gaussian process. We establish weak convergence results. We then use the Kalman-Bucy filter and we obtain the price of reinsurance contracts involving high frequency events.

The CBOT devised a loss ratio index as the underlying instrument for catastrophe insurance futures and options contracts. The Insurance Service Office calculates the index from loss data reported by at least 25 selected companies (CBOT, 1994, 1995a and 1995b). The loss ratio index is the reported losses incurred in a given quarter and reported by the end of the following quarter, L_t, divided by one fourth of the premiums received in the previous year, Π , i.e. L_t/Π.

The value of an insurance future, F_t, at maturity t is the nominal contract value, US$25,000, times the loss ratio index capped at 2, i.e.

$$F_t = 25,000 \times Min\left(\frac{L_t}{\Pi}, 2\right). \tag{1}$$

The CBOT capped the maximum loss ratio at 200% in order to limit the credit risk from unexpected huge losses and to make the contract look like a non-proportional reinsurance policy. However, to date there has not been an incident where the maximum loss ratio has been reached; the highest estimated loss ratio being 179% for Hurricane Andrew. Therefore ignoring the maximum loss ratio, the value of a catastrophe insurance call option on the future of the option, P_t, at maturity t is given by

$$P_t = \text{Max}(F_t - K, 0) = (F_t - K)^+ = \left(25,000 \times \frac{L_t}{\Pi} - K\right)^+ = \frac{25,000}{\Pi}(L_t - B)^+ \quad (2)$$

where K is the exercise price and $B = \frac{\Pi K}{25,000}$.

Let $Z_i,\ i = 1, 2, \cdots,$ be the claim amounts, which are assumed to be independent and identically distributed with distribution function $H(z)$ $(z > 0)$. The total loss excess over b, which is a retention limit, up to time t is

$$(C_t - b)^+ \quad (3)$$

where $C_t = \sum_{i=1}^{N_t} Z_i,\ N_t$ is the number of claims up to time t and $(C_t - b)^+ = Max(C_t - b, 0)$. Therefore the stop-loss reinsurance premium at present time 0 is

$$E\left\{(C_t - b)^+\right\} \quad (4)$$

where the expectation is calculated under an appropriate probability measure. Throughout the paper, for simplicity, we assume interest rates to be constant.

If we assume that $L_t = C_t$, the price of the insurance future at time 0 is

$$E\left[25,000 \times Min\left(\frac{C_t}{\Pi}, 2\right)\right] \tag{5}$$

and ignoring the maximum loss ratio, the price at time 0 of the call option on the insurance future is

$$\frac{25,000}{\Pi} E\left[(C_t - B)^+\right] \tag{6}$$

where the expectations are calculated under an appropriate probability measure. If we substitute 'b' with 'B' in the formula of the stop-loss reinsurance premium at time 0 excluding $\frac{25,000}{\Pi}$, the two formulae (1.4) and (1.6) are equivalent.
There has been discussion and research into the possibility of using catastrophe insurance futures and options contracts rather than conventional reinsurance contracts (Lomax & Lowe, 1994, Smith, 1994, Ryan, 1994, Sutherland, 1995, Kielholz & Durrer, 1997 and Smith, Canelo & Di Dio, 1997). The competitiveness of the reinsurance market emphasises the need for an appropriate pricing model for reinsurance contracts and catastrophe insurance derivatives. This also causes reinsurance companies to assess their strategies for the type of products offered to the market.

For catastrophic events, the assumption that resulting claims occur in terms of the Poisson process is inadequate as it has deterministic intensity. Therefore an alternative point process needs to be used to generate the claim arrival process. We will employ a doubly stochastic Poisson process, or the Cox process (Cox, 1955, Bartlett, 1963, Serfozo, 1972, Grandell, 1976, 1991, Bremaud, 1981 and Lando, 1994). Under a doubly stochastic Poisson process, or the Cox process, the claim intensity function is assumed to be stochastic.

The doubly stochastic Poisson process provides flexibility by letting the intensity not only depend on time but also allowing it to be a stochastic process. Therefore the doubly stochastic Poisson process can be viewed as a two step randomisation procedure. A process λ_t is used to generate another process N_t by acting as its intensity. That is, N_t is a Poisson process conditional on λ_t which itself is a stochastic process (if λ_t is deterministic then N_t is a Poisson process).

Definition 2.1 *Let* (Ω, F, P) *be a probability space with information structure given by* $F = \{\Im_t, t \in [0, T]\}$*. Let* N_t *be a point process adapted to* F*. Let* λ_t *be a non-negative process adapted to* F *such that*

$$\int_0^t \lambda_s d\, s < \infty \text{ almost surely } \quad (\text{no explosions}).$$

If for all $0 \le t_1 \le t_2$ *and* $u \in \Re$

$$E\left\{e^{iu(N_{t_2} - N_{t_1})} | \Im_{t_2}^{\lambda}\right\} = \exp\left\{\left(e^{iu} - 1\right) \int_{t_1}^{t_2} \lambda_s d\, s\right\} \tag{7}$$

then N_t *is call a* $\Im_t$*-doubly stochastic Poisson process with intensity* λ_t *where* $\Im_t^{\lambda} = \sigma\{\lambda_s; s \le t\}$.
Equation (2.1) gives us

$$\Pr\{N_{t_2} - N_{t_1} = k | \lambda_s; t_1 \le s \le t_2\} = \frac{\exp\left(-\int_{t_1}^{t_2} \lambda_s d\, s\right)\left(\int_{t_1}^{t_2} \lambda_s d\, s\right)^k}{k!}. \tag{8}$$

Now consider the process $X_t = \int_0^t \lambda_s d\, s$ (*the aggregated process*), then from (2.2) we can easily find that

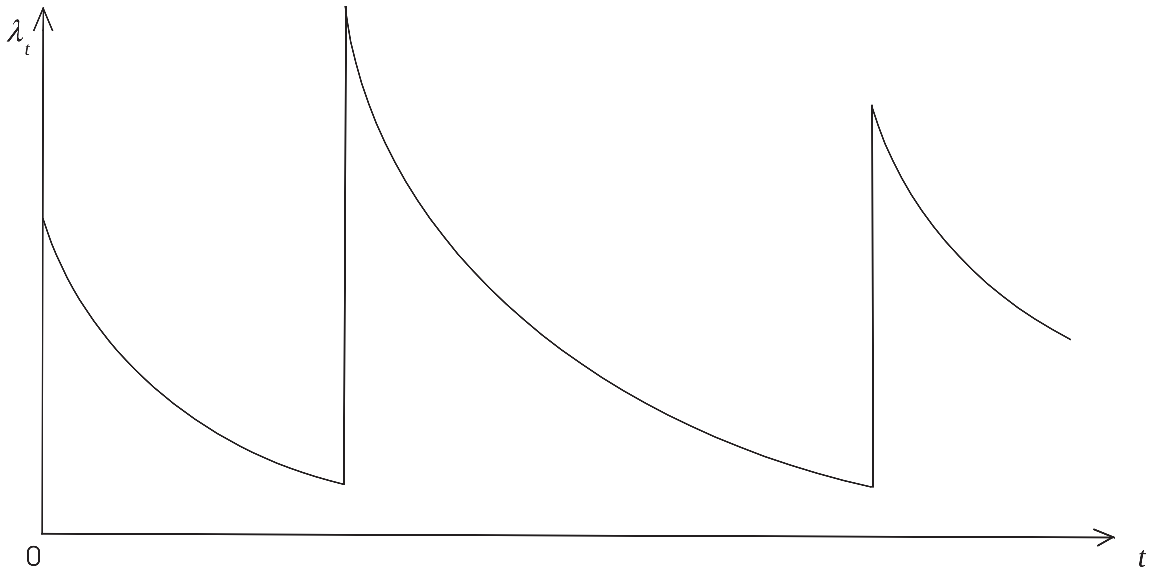
$$E\left(\theta^{N_{t_2} - N_{t_1}}\right) = E\left\{e^{-(1-\theta)(X_{t_2} - X_{t_1})}\right\}. \tag{9}$$

The shot-noise process

$$\lambda_t = \lambda_0 e^{-\delta t} + \sum_{i:s_i \leqslant t} y_i e^{-\delta(t-s_i)}$$

where:
λ_0 is the initial value
y_i is the size of catastrophe i.
s_i is the time that it happens and they are the event times of a Poisson process with parameter ρ
δ is the rate of exponential decay

Here is a sample path



The three parameters of the shot noise process described are homogeneous in time. We are now going to generalise the shot noise process by allowing the parameters to depend on time. The rate of jump arrivals, $\rho(t)$, is bounded on all intervals $[0,\ t)$ (no explosions). $\delta(t)$ is the rate of decay and the distribution function of jump sizes at any time t is $G(y;\ t)$ $(y > 0)$ with $E(y;\ t) = \mu_1(t) = \int_0^\infty y\ d\ G(y;\ t)$. We assume that $\delta(t),\ \ \rho(t)$ and $G(y;\ t)$ are all Riemann integrable functions of t and are all positive.

The generator of the process $(X_t,\ N_t,\ \lambda_t,\ t)$ acting on a function $f(x,\ n,\ \lambda,\ t)$ belonging to its domain is given by

$$\begin{aligned} Af(x,n,\lambda,t) &= \frac{\partial f}{\partial t} + \lambda\frac{\partial f}{\partial x} + \lambda[f(x,n+1,\lambda,t) - f(x,n,\lambda,t)] - \delta(t)\lambda\frac{\partial f}{\partial \lambda} \\ &+ \rho(t)\Big[\int_0^\infty f(x,n,\lambda+y,t)d\,G(y;t) - f(x,n,\lambda,t)\Big]. \end{aligned}$$

For $f(x,\ n,\ \lambda,\ t)$ to belong to the domain of the generator A, it is sufficient that $f(x,n,\lambda,t)$ is differentiable w.r.t. $x,\ \lambda,\ t$ for all $x,\ n,\ \lambda,\ t$ and that $\Big|\int_0^\infty f(\cdot,\lambda+y,\cdot)d\,G(y;t) - f(\cdot,\lambda,\cdot)\Big| < \infty.$

Result:

Let $v_1 \geq 0,\ v_2 \geq 0,\ v \geq 0,\ 0 \leq \theta \leq 1$. *Then*

$$
\begin{aligned}
&E\Big\{e^{-v_1(X_{t_2}-X_{t_1})}e^{-v_2\lambda_{t_2}}|X_{t_1},\lambda_{t_1}\Big\}\\
&=\exp\Big[-\Big\{\frac{v_1}{\delta}+\Big(v_2-\frac{v_1}{\delta}\Big)e^{-\delta(t_2-t_1)}\Big\}\lambda_{t_1}\Big]\\
&\cdot\exp\Big[-\int_{t_1}^{t_2}\rho(s)\Big[1-\hat{g}\Big\{\frac{v_1}{\delta}+\Big(v_2-\frac{v_1}{\delta}\Big)e^{-\delta(t_2-s)};s\Big\}\Big]d\,s\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
&E\Big\{\theta^{(N_{t_2}-N_{t_1})}e^{-v\lambda_{t_2}}|N_{t_1},\lambda_{t_1}\Big\}\\
&=\exp\Big[-\Big\{\frac{1-\theta}{\delta}+\Big(v-\frac{1-\theta}{\delta}\Big)e^{-\delta(t_2-t_1)}\Big\}\lambda_{t_1}\Big]\\
&\cdot\exp\Big[-\int_{t_1}^{t_2}\rho(s)\Big[1-\hat{g}\Big\{\frac{1-\theta}{\delta}+\Big(v-\frac{1-\theta}{\delta}\Big)e^{-\delta(t_2-s)};\\
&s\Big\}\Big]d\,s\Big].
\end{aligned}
$$

The Laplace transforms of the distribution of λ_t *and* X_t *are given by*

$$E\Big\{e^{-v\lambda_{t_2}}|\lambda_{t_1}\Big\} = \exp\Big[-v\,e^{-\delta(t_2-t_1)}\lambda_{t_1}\Big] \cdot \exp\Big[-\int_{t_1}^{t_2}\rho(s)\Big[1-\hat{g}\Big\{v\;e^{-\delta(t_2-s)};\,s\Big\}\Big]d\,s\Big],$$

$$E\Big\{e^{-v(X_{t_2}-X_{t_1})}|\lambda_{t_1}\Big\} = \exp\Big[-\frac{v}{\delta}\Big\{1-e^{-\delta(t_2-t_1)}\Big\}\lambda_{t_1}\Big] \cdot \exp\Big[-\int_{t_1}^{t_2}\rho(s)\Big[1-\hat{g}\Big\{\frac{v}{\delta}\Big(1-e^{-\delta(t_2-s)}\Big);\,s\Big\}\Big]d\,s\Big]$$

and the probability generating function of N_t *is given by*

$$E\Big\{\theta^{(N_{t_2}-N_{t1})}|\lambda_{t_1}\Big\}=\exp\Big[-\frac{1-\theta}{\delta}\Big\{1-e^{-\delta(t_2-t_1)}\Big\}\lambda_{t_1}\Big]$$

$$\cdot\ \exp\Big[\ -\ \int_{t_1}^{t_2}\ \rho(s)\Big[1\ -\ \hat{g}\Big\{\frac{1-\theta}{\delta}\Big(1\ -$$

$$e^{-\delta(t_2-s)}\Big);s\Big\}\Big]d\,s\Big].$$

It will be interesting to find the Laplace transforms of the distribution of λ_t, X_t and the p.g.f. (probability generating function) N_t at time t, using a specific jump size distribution of $G(y;t)$ $(y>0)$. We use an exponential jump size distribution, i.e. $g(y;t) = \left(\alpha + \gamma e^{\delta t}\right) e^{-(\alpha+\gamma e^{\delta t})y}$, $y > 0$, $-\alpha e^{-\delta t} < \gamma \leq 0$. In practice, other thick-tail distributions such as log-normal, gamma and Pareto, etc. can also be applied for jump size distribution of $G(y;t)$ $(y>0)$. Examining the effect on stop-loss reinsurance premiums and prices for catastrophe insurance derivatives caused by changes in the jump size distribution will be also of interest.

Let us assume that $\rho(t) = \rho \dfrac{\alpha}{\alpha + \gamma e^{\delta t}}$. The reason for this particular assumption will become apparent later when we change the probability measure.

Let the jump size distribution be exponential, i.e. $g(y;t) = \big(\alpha + \gamma e^{\delta t}\big)\exp\big\{-\big(\alpha + \gamma e^{\delta t}\big)y\big\}$, $y > 0$, $\alpha e^{-\delta t} < \gamma \leq 0$, *and assume that* $\rho(t) = \rho\frac{\alpha}{\alpha + \gamma e^{\delta t}}$. *Then*

$$E\Big\{e^{-v\lambda_{t_1}}|\lambda_{t_0}\Big\} = \exp\Big\{-v\lambda_{t_0}e^{-\delta(t_1-t_0)}\Big\} \cdot \left(\frac{\gamma e^{\delta t_0} + \alpha e^{-\delta(t_1-t_0)}}{\gamma e^{\delta t_0} + \alpha}\right)^{\frac{\rho}{\delta}} \cdot \left(\frac{\gamma e^{\delta t_0} + v\, e^{-\delta(t_1-t_0)} + \alpha}{\gamma e^{\delta t_0} + (v+\alpha)e^{-\delta(t_1-t_0)}}\right)^{\frac{\rho}{\delta}},$$

$$E\Big\{e^{-v(X_{t_2}-X_{t_1})}|\lambda_{t_1}\Big\} = \exp\Big[-\frac{v}{\delta}\Big\{1 - e^{-\delta(t_2-t_1)}\Big\}\lambda_{t_1}\Big]\left(\frac{\gamma e^{\delta t_1} + \alpha e^{-\delta(t_2-t_1)}}{\gamma e^{\delta t_1} + \alpha}\right)^{\frac{\rho}{\delta}} \cdot \left(\frac{\gamma e^{\delta t_1} + \alpha + \frac{v}{\delta}\Big(1 - e^{-\delta(t_2-t_1)}\Big)}{\gamma e^{\delta t_1} + \alpha e^{-\delta(t_2-t_1)}}\right)^{\frac{\alpha\rho}{\delta\alpha + v}} \tag{10}$$

and

$$E\Big\{\theta^{(N_{t_2}-N_{t_1})}|\lambda_{t_1}\Big\} = \exp\Big[-\frac{1-\theta}{\delta}\Big\{1 - e^{-\delta(t_2-t_1)}\Big\}\lambda_{t_1}\Big]\left(\frac{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}{\gamma e^{\delta t_1}+\alpha}\right)^{\frac{\rho}{\delta}} \cdot \left(\frac{\gamma e^{\delta t_1}+\alpha+\frac{1-\theta}{\delta}\Big(1-e^{-\delta(t_2-t_1)}\Big)}{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}\right)^{\frac{\alpha\rho}{\delta\alpha+(1-\theta)}}. \tag{11}$$

If λ_t *is* $'-\infty'$ *asymptotic,*

$$E\Big(e^{-v\lambda_{t_1}}\Big) = \left(\frac{\gamma+\alpha e^{-\delta t_1}}{\gamma+(v+\alpha)e^{-\delta t_1}}\right)^{\frac{\rho}{\delta}}, \tag{12}$$

$$E\Big\{e^{-v(X_{t_2}-X_{t_1})}\Big\} = \left(\frac{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}{\gamma e^{\delta t_1}+\alpha+\frac{v}{\delta}\Big(1-e^{-\delta(t_2-t_1)}\Big)}\right)^{\frac{\rho}{\delta}}$$

$$\cdot\left(\frac{\gamma e^{\delta t_1}+\alpha+\frac{v}{\delta}\Big(1-e^{-\delta(t_2-t_1)}\Big)}{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}\right)^{\frac{\alpha\rho}{\delta\alpha+v}}$$

and

$$E\Big\{\theta^{(N_{t_2}-N_{t_1})}\Big\}=\left(\frac{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}{\gamma e^{\delta t_1}+\alpha+\frac{1-\theta}{\delta}\Big(1-e^{-\delta(t_2-t_1)}\Big)}\right)^{\frac{\rho}{\delta}}$$
$$\cdot\left(\frac{\gamma e^{\delta t_1}+\alpha+\frac{1-\theta}{\delta}\Big(1-e^{-\delta(t_2-t_1)}\Big)}{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}\right)^{\frac{\alpha\rho}{\delta\alpha+(1-\theta)}}. \tag{13}$$

Now let us derive the expected value of claim number process, N_t.

The expectation of claim number process, N_t *is given by*

$$E(N_{t_2}-N_{t_1})=\int_{t_1}^{t_2}E(\lambda_s)d\,s$$

$$= \left(\frac{1 - e^{-\delta(t_2 - t_1)}}{\delta}\right) E(\lambda_{t_1}) + \frac{1}{\delta} \int_{t_1}^{t_2} \Big(1 - e^{-\delta(t_2 - s)}\Big) \rho(s) \mu_1(s) d\, s. \tag{14}$$

If the jump size distribution is exponential, i.e. $g(y;t) = \big(\alpha + \gamma e^{\delta t}\big)\exp\big\{-\big(\alpha + \gamma e^{\delta t}\big)y\big\}$, $y > 0$, $-\alpha e^{-\delta t} < \gamma \leq 0$ *with* $\rho(t) = \rho\frac{\alpha}{\alpha + \gamma e^{\delta t}}$ *and* λ_t *is* $'-\infty'$ *asymptotic, then*

$$E(N_{t_2} - N_{t_1}) = \frac{\rho}{\delta\alpha}(t_2 - t_1) - \frac{\rho}{\delta^2\alpha}\ln\left(\frac{\gamma e^{\delta t_2} + \alpha}{\gamma e^{\delta t_1} + \alpha}\right). \tag{15}$$

Let us assume that there exist a liquid reinsurance market, i.e. at any time $t \leq T$, the insurer can decide to sell any part of the risk of C_u, $t \leq u \leq T$, based on the information available at time t where C_u follows doubly stochastic compound Poisson process with shot noise intensity defined on the probability space (Ω, F, P). Let PR_u denote the total value of premiums received up to time u defined on (Ω, F, P) and define a reinsurance strategy.

Let $s \in [0, T]$, a reinsurance strategy $\{\xi_u; t \leq u \leq T\}$ is a predictable stochastic process on (Ω, F, P) with $0 \leq \xi_u \leq 1$ for all $u \in [t, T]$.

Assuming that interest rates is constant, let us define the specified process R_t, $0 \leq t \leq T$, given by

$$R_t = PR_t - C_t \ (0 \leq t \leq T)$$

denoting the net surplus from insurance business up to time t. If the insurer choose at time t some reinsurance strategy $\{\xi_u; t \leq u \leq T\} \in H_t$ where H_t denotes the set of all reinsurance strategies starting at time t, then the company's final gain at time T is given by

$$G_T(\xi) = \int_t^T \xi_u d R_u$$

where it is assumed that the reinsurer receives direct insurer's premiums for his engagement. A strategy $\{\xi_u; t \leq u \leq T\}$ allowing for a possible profit without the possibility of a loss is called an arbitrage strategy, i.e. a strategy $\{\xi_u; t \leq u \leq T\}$ satisfying

(i). $G_T(\xi) \geq 0$, $P-$ almost surely

(ii). $E_P[G_T(\xi)] > 0$

is called an arbitrage strategy. Therefore, for the reinsurance market (Ω, F, P), R_t does not allow for arbitrage strategies if there is an equivalent probability measure P^* such that the process R_t is a martingale. A probability measure P^* is called an equivalent martingale probability measure if:

(i). $P^*(A) = 0$ iff $P(A) = 0$, for any $A \in \Im_t$;

(ii). The Radon-Nikodym derivative $\frac{dP^*}{dP}$ belongs to $L^2(\Omega, \Im_t, P)$;

(iii). R_t is a martingale under P^*, i.e.

$$E^*[R_t | \Im_s] = R_s, \;\; P^* - \text{a.s.}$$

for any $0 \leq s \leq t \leq T$, where E^* denotes the expectation with respect to P^* (Harrison & Kreps, 1979 and Sondermann, 1991).

If the market is complete, the fair price of a contingent claim is the expectation with respect to exactly one equivalent martingale probability measure (i.e. by assuming that there is an absence of arbitrage opportunities in the market). For example, when the underlying stochastic process follows geometric Brownian motion or homogeneous Poisson process, we can obtain the fair price with respect to a unique equivalent martingale probability measure. However, as the underlying stochastic process for the claim arrival process is the Cox process, we will have infinitely many equivalent martingale probability measures. In other words, we will have several choices of equivalent martingale probability measures to price a stop-loss reinsurance contract & insurance derivatives as the market is incomplete.

It is not our purpose to decide which is the appropriate one to use. The insurance companies' attitude towards risk determines which equivalent martingale probability measure should be used. The attractive thing about the Esscher transform is that it provides us with at least one equivalent martingale probability measure in incomplete market situations.

We need to obtain a martingale that can be used to define a change of probability measure, i.e. it can be used to define the Radon-Nikodym derivative $\frac{dP^*}{dP}$ where P is the original probability measure and P^* is the equivalent martingale probability measure with parameters involved.

Let M_t be the total number of catastrophe jumps up to time t and C_t the sum of all catastrophe sizes up to that time. We will assume that claim points and catastrophe jumps do not occur at the same time.

Considering constants θ^*, v^*, ψ^* and γ^* such that $\theta^* \geq 1$, $v^* \leq 0$, $\psi^* \geq 1$ and $\gamma^* \leq 0$,

$$\theta^{*N_t} e^{-v^* C_t} e^{-\left\{\theta^* \hat{h}(v^*) - 1\right\} \int_0^t \lambda_s ds} \psi^{*M_t} e^{-\gamma^* \lambda_t e^{\delta t}} \exp[\rho \int_0^t \left\{1 - \psi^* \hat{g}\left(\gamma^* e^{\delta s}\right)\right\} ds] \quad (16)$$

is a martingale where $\hat{h}(v^*) = \int_0^\infty e^{-v^* z} dH(z)$.

Using the martingale above to change the measure we see the following:

(i). The claim intensity function λ_t has changed to $\lambda_t \theta^* \hat{h}(v^*)$;

(ii). The rate of jump arrival ρ has changed to $\rho^*(t) = \rho\psi^* \hat{g}\big(\gamma^* e^{\delta t}\big)$ (it now depends on time);

(iii). The jump size measure $d\,G(y)$ has changed to $d\,G^*(y;\ \ t) = \frac{\exp(-\gamma^* e^{\delta t} y) d\,G(y)}{\hat{g}\big(\gamma^* e^{\delta t}\big)}$ (it now depends on time);

(iv). The claim size measure $d\,H(z)$ has changed to $d\,H^*(z) = \frac{e^{-v^* z} d\,H(z)}{\hat{h}(v^*)}$.

In other words, the risk-neutral Esscher measure is the measure with respect to which N_t becomes the Cox process with parameter where three parameters $\lambda_t \theta^* \hat{h}(v^*)$ where three parameters of the shot noise process λ_t are δ, $\rho^*(t) = \rho\psi^* \hat{g}\big(\gamma^* e^{\delta t}\big)$, $d\,G^*(y;t) = \frac{\exp(-\gamma^* e^{\delta t} y) d\,G(y)}{\hat{g}\big(\gamma^* e^{\delta t}\big)}$ and claim size distribution becomes $d\,H^*(z) = \frac{e^{-v^* z} d\,H(z)}{\hat{h}(v^*)}$.

In practice, the reinsurer will calculate the values of a stop-loss contract & insurance derivatives using $\theta^* > 1$, $\psi^* > 1$, $\gamma^* < 0$ and $v^* < 0$. This results in the reinsurer assuming that there will be a higher value of claim intensity itself, a higher value of the damage caused by the catastrophe, more catastrophes occurring in a given period of time and a higher value of claim size.

Let the jump size distribution be exponential. Consider constants θ, θ^*, v^*, ψ^* and γ^* such that $0 \leq \theta \leq 1$, $\theta^* \geq 1$, $v^* = 0$, $\psi^* = 1$ and $\gamma^* \leq 0$. Furthermore if λ_t is $'-\infty'$ asymptotic, then

$$E^*\left(\theta^{N_{t_2}-N_{t_1}}\right) = \left\{\frac{\gamma^* e^{\delta t_1} + \alpha e^{-\delta(t_2-t_1)}}{\gamma^* e^{\delta t_1} + \alpha + \frac{\theta^*(1-\theta)}{\delta}\left(1 - e^{-\delta(t_2-t_1)}\right)}\right\}^{\frac{\rho}{\delta}}$$

$$\cdot$$

$$\left\{\frac{\gamma^* e^{\delta t_1} + \alpha + \frac{\theta^*(1-\theta)}{\delta}\left(1 - e^{-\delta(t_2-t_1)}\right)}{\gamma^* e^{\delta t_1} + \alpha e^{-\delta(t_2-t_1)}}\right\}^{\frac{\alpha\rho}{\delta\alpha+\theta^*(1-\theta)}}$$

It will be interesting to derive the premium and pricing formulae, using a specific claim size distribution of $H(z)$ $(z > 0)$. We assume that the claim size distribution is gamma, i.e. $h(z) = \frac{\beta^{\varphi} z^{\varphi-1} e^{-\beta z}}{(\varphi-1)!}$, $z > 0$, $\beta > 0$, $\varphi \geq 1$. Then

$$E^{*}\Big[(C_t - b)^{+}\Big] = \sum_{n=1}^{\infty} a_n^{*}\left\{\frac{n\varphi}{\beta}\int_b^{\infty}\frac{\beta^{n\varphi+1}c^{n\varphi}e^{-\beta c}}{(n\varphi)!}d\,c - b\int_b^{\infty}\frac{\beta^{n\varphi}c^{n\varphi-1}e^{-\beta c}}{(n\varphi-1)!}d\,c\right\}, \tag{17}$$

Now let us illustrate the calculation of stop-loss reinsurance gross premium for catastrophic events & the arbitrage-free prices of the catastrophe insurance derivatives using the models derived previously. From (3.13), the p.g.f. of N_t is

$$E^{*}\big(\theta^{N_t}\big) = \sum_{n=1}^{\infty}\theta^n P^{*}(N_t = n) = \sum_{n=0}^{\infty}\theta^n a_n^{*}$$
$$= \left\{\frac{\gamma^{*} + \alpha e^{-\delta t}}{\gamma^{*} + \alpha + \frac{\theta^{*}(1-\theta)}{\delta}\big(1 - e^{-\delta t}\big)}\right\}^{\frac{\rho}{\delta}}\left\{\frac{\gamma^{*} + \alpha + \frac{\theta^{*}(1-\theta)}{\delta}\big(1 - e^{-\delta t}\big)}{\gamma^{*} + \alpha e^{-\delta t}}\right\}^{\frac{\alpha\rho}{\delta\alpha + \theta^{*}(1-\theta)}}$$

Let the parameter values be

$$\theta^{*} = 1.1, \gamma^{*} = -0.1, \alpha = 1, \delta = 0.3, \rho = 4, t = 1.$$

By expanding using the *MAPLE* algebraic manipulations package we can obtain $a_n^* = P^*(N_t = n)$ which is as follows:

$$E^*\left(\theta^{N_t}\right) = \sum_{n=1}^{\infty} \theta^n P^*(N_t = n)$$

$$= \sum_{n=0}^{\infty} \theta^n a_n^* = \left\{\frac{0.64082}{0.9} + 0.95033(1 - \theta)\right\}^{\frac{4.4(1-\theta)}{0.09+0.33(1-\theta)}}$$

$$= 0.000014982 + 0.000116628\theta + 0.00048266\theta^2 + 0.00142250\theta^3 + 0.00333550\theta^4 + 0.006615\theta^5 + 0.011523\theta^6 + 0.018086\theta^7 + 0.02\backslash 6045\theta^8 + 0.034881\theta^9 + 0.0439\theta^{10} + 0.052349\theta^{11} + 0.059537\theta^{12} + 0.064932\theta^{13} + 0.068214\theta^{14} + 0.06929\theta^{15} + 0.068273\theta^{16} + 0.065434\theta^{17} + 0.061148\theta^{18} + 0.055831\theta^{19} + 0.049898\theta^{20}$$

$$
\begin{aligned}
&+\ 0.043723\theta^{21}\ +\ 0.037616\theta^{22}\ +\ 0.031815\theta^{23}\ + \\
&0.0264484\theta^{24} + 0.02172\theta^{25} \\
&+\ 0.017567\theta^{26}\ +\ 0.014023\theta^{27}\ +\ 0.011056\theta^{28}\ + \\
&0.0086166\theta^{29} + 0.0066419\theta^{30} \\
&+ 0.0050667\theta^{31} + 0.0038272\theta^{32} + 0.0028639\theta^{33} + \\
&0.00212410\theta^{34} + 0.00156210\theta^{35} \\
&+ 0.0011396\theta^{36} + 0.00082497\theta^{37} + 0.0005282\theta^{38} + \\
&0.000423010\theta^{39} \\
&+ 0.00029981\theta^{40} + 0.00021112\theta^{41} + 0.000775\theta^{42} + \\
&0.00010279\theta^{43} + \\
&0.0000711010\theta^{44}\ \ +\ \ 0.000048911^{45}\ \ +\ \ 0.00003\backslash \\
&3469\theta^{46} + 0.0000227850\theta^{47} + \\
&0.0000154360\theta^{48}\ \ +\ \ 0.0000104070\theta^{49}\ \ +\ \ 0.00000\backslash \\
&6985\theta^{50} + 0.0000046672\theta^{51} + \\
&0.00000310510\theta^{52}\ +\ 0.0000020573\theta^{53}\ +\ 0.000001\backslash \\
&3575\theta^{54} + O\big(\theta^{55}\big).
\end{aligned}
$$

We can now claculate the reinsurance premium for various retention levels:

Retention level	Premium
0	16.58403
5	11.61916
10	7.06779
16.61	2.83349
20	1.58701
25	0.59582
30	0.19512

The shot noise process λ_t has been taken to be unobservable. This implies that catastrophes can only be observed on the basis of an observed process N_t of reported claims. However in practical situation, as we observe catastrophes, we can trace back which and how many claims are caused by them. Therefore "the filtering problem" can be applied to obtain the best estimate λ_t on the basis of the observed process N_t of reported claims or observed catastrophes.

We start by introducing the following linear transformations of the processes λ_t, N_t and C_t:

$$Z_t^{(\rho)} = \frac{\lambda_t - \frac{\mu_1\rho}{\delta}}{\sqrt{\frac{\mu_2\rho}{2\delta}}} \quad \text{i.e.} \quad \lambda_t = \frac{\mu_1\rho}{\delta} + Z_t^{(\rho)}\sqrt{\frac{\mu_2\rho}{2\delta}} \tag{18}$$

$$W_t^{(\rho)} = \frac{N_t - \frac{\mu_1\rho}{\delta}t}{\sqrt{\frac{\mu_2\rho}{2\delta}}} \quad \text{i.e.} \quad N_t = \frac{\mu_1\rho}{\delta}t + W_t^{(\rho)}\sqrt{\frac{\mu_2\rho}{2\delta}} \tag{19}$$

$$U_t^{(\rho)} = \frac{C_t - m_1\frac{\mu_1\rho}{\delta}t}{\sqrt{\frac{\mu_2\rho}{2\delta}}} \quad \text{i.e.} \quad C_t = m_1\frac{\mu_1\rho}{\delta}t + U_t^{(\rho)}\sqrt{\frac{\mu_2\rho}{2\delta}}. \tag{20}$$

Assuming that $\rho \to \infty$ and that λ_0 is a random variable that is independent of everything else such that $\frac{\lambda_0 - (\mu_1\rho/\delta)}{\mu_2\rho/2\delta}$ converges in distribution to Z_0, $Z_t^{(\rho)}$, $W_t^{(\rho)}$ *and* $U_t^{(\rho)}$ *converge in law to* Z_t, W_t *and* U_t *where*

$$d\,Z_t = -\,\delta Z_t d\,t + \sqrt{2\delta}\,d\,B_t^{(1)} \tag{21}$$

$$d\,W_t = Z_t d\,t + \sqrt{\frac{2\mu_1}{\mu_2}}\,d\,B_t^{(2)} \tag{22}$$

$$d\,U_t = m_1 d\,W_t + \sqrt{k_2\frac{2\mu_1}{\mu_2}}\,d\,B_t^{(3)} = m_1 Z_t d\,t + \sqrt{m_2\frac{2\mu_1}{\mu_2}}\,d\,B_t^{(4)} \tag{23}$$

where $B_t^{(1)}$, $B_t^{(2)}$, $B_t^{(3)}$ *are three independent standard Brownian motions and* $B_t^{(4)} = \frac{m_1\sqrt{\frac{2\mu_1}{\mu_2}}B_t^{(2)} + \sqrt{k_2\frac{2\mu_1}{\mu_2}}B_t^{(3)}}{\sqrt{(m_1^2+k_2)\frac{2\mu_1}{\mu_2}}}$ (*also a standard Brownian motion*).

This implies that Z_t, W_t and U_t are normally distributed. Therefore we can define $\tilde{\lambda}_t$, $\tilde{N}_t$ and $\tilde{C}_t$ as Gaussian approximations of λ_t, N_t and C_t;

$$\tilde{\lambda}_t = \frac{\mu_1\rho}{\delta} + Z_t\sqrt{\frac{\mu_2\rho}{2\delta}} \text{ i.e. } \quad Z_t = \frac{\tilde{\lambda}_t - \frac{\mu_1\rho}{\delta}}{\sqrt{\frac{\mu_2\rho}{2\delta}}} \tag{24}$$

$$\tilde{N}_t = \frac{\mu_1\rho}{\delta} + W_t\sqrt{\frac{\mu_2\rho}{2\delta}} \text{ i.e. } \quad W_t = \frac{\tilde{N}_1 - \frac{\mu_1\rho}{\delta}t}{\sqrt{\frac{\mu_2\rho}{2\delta}}} \tag{25}$$

and

$$\tilde{C}_t = m_1\frac{\mu_1\rho}{\delta} + U_t\sqrt{\frac{\mu_2\rho}{2\delta}} \text{ i.e. } \quad U_t = \frac{\tilde{C}_t - m_1\frac{\mu_1\rho}{\delta}t}{\sqrt{\frac{\mu_2\rho}{2\delta}}}. \tag{26}$$

Theorem 1. *Let $(Z_t,\, W_t)$ be a two-dimensional normal process satisfying the system of equations above. Then the estimate of Z_t based on the observed $\{W_s;\, 0 \le s \le t\}$ is*

$$\hat{Z}_t = E(Z_t \mid W_s;\ 0 \le s \le t) = \exp\left\{\int_0^t \Psi(s)\,d\,s\right\}\hat{Z}_0 + \frac{\mu_2}{2\mu_1}\int_0^t \exp\left\{\int_s^t \Psi(u)\,d\,u\right\} S(s)\,d\,W_s \tag{27}$$

where

$$S(s) = \frac{\xi(1+\eta)}{\eta-1} - 2\delta\frac{\mu_1}{\mu_2} \tag{28}$$

and

$$\Psi(s) = -\frac{\xi(1+\eta)}{\frac{2\mu_1}{\mu_2}(\eta-1)} \tag{29}$$

where

$$\xi = \sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}, \qquad \eta =$$

$$\frac{a^2+\frac{2\delta\mu_1}{\mu_2}+\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\Big(\frac{2\delta\mu_1}{\mu_2}+2\Big)}}{a^2+\frac{2\delta\mu_1}{\mu_2}-\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\Big(\frac{2\delta\mu_1}{\mu_2}+2\Big)}}\exp\left(\frac{\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\Big(\frac{2\delta\mu_1}{\mu_2}+2\Big)}}{\frac{\mu_1}{\mu_2}}s\right)$$

$and\ S(0)=a^2.$

Let Z_t, W_t, $\hat{Z}_t$ and $S(t)$ be as defined. Then the conditional distribution of Z_t, given $\{W_s; 0 \le s \le t\}$ is given by

$$E\big(e^{-\gamma Z_t}\,|\,W_s; 0\le s\le t\big)=\exp\left\{-\gamma\hat{Z}_t+\frac{1}{2}\gamma^2 S(t)\right\}. \tag{30}$$

We have transformed and approximated λ_t and N_t as normal variables Z_t and W_t from which we have obtained the conditional distribution of Z_t, given $\{W_s; 0 \leq s \leq t\}$. Now let us derive the pricing model for stop-loss reinsurance contract using normal variables Z_t and W_t. As mentioned earlier, as we have assumed that $\rho \to \infty$, this approach can be used for the pricing of common events with high frequency such as car accidents or accidents from a large collective insurance portfolio.

Let $\aleph_i$, $i = 1, 2, \cdots$, be the claim amounts, which are assumed to be independent and identically distributed with distribution function. The actuarial stop-loss reinsurance premium at time t is

$$E\left[\left(\sum_{i=1}^{N_T - N_t} \aleph_i - b\right)^+ \mid N_s; 0 \leq s \leq t\right] \tag{31}$$

where b is a suitably large retention limit. In particular we set

$$b = \sqrt{\frac{\mu_2 \rho}{2\delta}}\beta + m_1 \frac{\mu_1 \rho}{\delta}(T - t). \tag{32}$$

Let $C_T - C_t$ be the total amount of claims between time T and t. Then from (32), the stop-loss reinsurance premium at time t becomes

$$E\big[\{(C_T - C_t) - b\}^+ \mid N_s; 0 \le s \le t\big]. \tag{33}$$

Since we have obtained $\tilde{C}_t$ and $\tilde{N}_t$ which are Gaussian approximations of C_t and N_t, we will use these approximations. Therefore set $\tilde{C}_t = m_1 \frac{\mu_1 \rho}{\delta} t + U_t \sqrt{\frac{\mu_2 \rho}{2\delta}}$; then

$$E\Big[\Big\{\Big(\tilde{C}_T - \tilde{C}_t\Big) - b\Big\}^+ \mid \tilde{N}_s;\, 0 \le s \le t\Big] = \sqrt{\frac{\mu_2 \rho}{2\delta}} E\Big[\{U_T - U_t - \beta\}^+ \mid W_s;\, 0 \le s \le t\Big] \tag{34}$$

The stop-loss reinsurance premium at time t based on the observations $\{W_s; 0 \le s \le t\}$ is given by

$$E\Big[\Big\{\Big(\tilde{C}_T - \tilde{C}_t\Big) - b\Big\}^+ \mid W_s;\, 0 \le s \le t\Big] = \sqrt{\frac{\mu_2 \rho \Sigma}{4\delta\pi}} e^{-\frac{1}{2}L^2} + \sqrt{\frac{\mu_2 \rho}{2\delta}} (\Omega - \beta)\Phi(-L) \tag{35}$$

where

$$\Omega = E(U_T - U_t \mid W_s; 0 \leq s \leq t) = m_1 \frac{1 - e^{-\delta(T-t)}}{\delta} \hat{Z}_t, \tag{36}$$

$$\Sigma = Var(U_T - U_t \mid W_s; 0 \leq s \leq t) = \tag{37}$$

$$\left(\frac{m_1}{\delta}\right)^2 \left[\left\{1 - e^{-\delta(T-t)}\right\}^2 S(t) - e^{-2\delta(T-t)} + 4e^{-\delta(T-t)} - 3 \right] + 2\left(\frac{m_1^2}{\delta} + \frac{m_2 \mu_1}{\mu_2}\right)(T - t), \tag{38}$$

$L = \frac{\beta - \Omega}{\sqrt{\Sigma}}$ and $\Phi(\,\cdot\,)$ is the cumulative normal distribution function.

Example The numerical values used to simulate the claim arrival process are $\delta = 0.5$, $\lambda_0 = 200$. We will assume that $\rho = 100$, i.e. the interarrival time between jumps is exponential with mean 0.01 and that the jump size follows an exponential with mean 1. We generate random values and to simulate the claim arrival process. The numerical values used are

$$\begin{aligned}
\hat{Z}_0 &= 0, S(0) = 0, \theta = 0.1, \mu_1 = 1, \mu_2 = 2, m_1 = 1, m_2 = 3, \\
&\quad t = 1, T = 2, \\
b &= 0, 180, 190, 200, 210, 220
\end{aligned}$$

where we have

$$E = (C_T - C_t) = E(N_T - N_t)E(\aleph) = \frac{\mu_1 \rho}{\delta} m_1 = 200.$$

By computing the quantities above, where $\hat{Z}_1 = 0.5579152$, the calculation of stop-loss reinsurance premiums for high frequency events at each retention level b, with/without a relative security loading factor θ, are shown in the table.

Retention level b	Net reinsurance premium ($\theta = 0$)	Risk reinsurance premium ($\theta = 0.1$)
0	206.21	226.83
180	26.58	29.24
190	18.06	19.87
200	11.00	12.10
210	5.77	6.35
220	2.41	2.66