Implications of Return Predictability across Horizons for Asset Pricing Models

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Abstract

We analyze predictors-based variance bounds, i.e bounds on the variance of the stochastic discount factors (SDFs) that price a given set of returns conditional on the information contained in a vector of return predictors. For an asset pricing model identified by its state variables, information structure and model SDF, we supply a sufficient condition under which our predictors-based bounds constitute bona fide lower bounds on the variance of the SDF of the model. Using our predictors-based bounds we document that consumption-based asset pricing models such as long-run risk and habit models do not produce SDFs volatile enough at the one-year horizon. When we look at long-horizons our evidence shows that it is the habit model, not the long-run risk model, that satisfies our bounds. As a consequence, the investment horizon and the use of conditioning information emerge as fundamental ingredients that permit either to set models apart, or to select the common behavior among apparently different models.
1 Introduction

“There is no way to predict the price of stocks and bonds over the next few days or weeks. But it is quite possible to foresee the broad course of these prices over longer periods, such as the next three to five years.”

Press release of the The Royal Swedish Academy of Sciences for the 2013 Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel

If there is valuable information for predicting stock and bond prices over time, and the more so the longer the horizon, when and how can we use this information to discriminate among competing asset pricing models? The answer we give in this paper is both methodological and empirical. From a methodological point of view, given a set of stock and bond predictors we offer a simple condition under which a variance bound that incorporates conditioning information from the predictors constitutes a bona fide lower bound on the variance of the Stochastic Discount Factor (SDF) of a given asset pricing model. From an empirical point of view, we examine two leading asset pricing models, the External Habit model of Campbell and Cochrane (1999), and the model of Long Run Risk (LRR) of Bansal and Yaron (2004) and its variant Bansal, Kiku, and Yaron (2012a), we show that both models satisfy the condition for the predictors-based bounds to be bona fide lower bounds on the variance of their SDFs, and then we use these bounds to explore the role that short and long horizon predictability plays in the econometric evaluation of these models.

The empirical evidence on return predictability has shown that the statistical significance of some predictors increase with the horizon and becomes quite strong at long horizons. If predictability is not a symptom of market malfunction but rather the consequence of a fair compensation for risk taking, then it should reflect attitudes toward risk and variation in market risk over time. Different theories on the relationship between risk and asset prices should then be assessed on the basis of their ability of explaining the predictability that emerges from the data. The predictability of returns on financial assets can only be consistent with arbitrage-free markets if the discount factor is highly variable over time. The question then is whether
a theoretical model is able to generate sufficient variability in the discount factor and, as a consequence, the predictability patterns found in the data. This is why we want to understand when and how we can employ the information contained in a set of predictors, and synthesized in the predictors-based bounds, to evaluate different asset pricing models. In fact, the predictability of returns at different horizons implied by asset pricing models with time-varying discount factor is conditional on some model specific state variables that are in general different from those used in the predictive regressions. The possible discrepancy between the informational content of the predictors versus that of the state variables may render the empirical evidence on predictability not informative to reject some models, on one side, and decisive to draw inference on other models on the other.

Our simple condition that implies that the variance of the model stochastic discount factor must satisfy the variance bounds obtained by conditioning on the predictors is illustrated in Proposition 1. The condition is on the returns discounted by the SDF of a given asset pricing model, and it requires the predictability of these discounted returns not to increase when the information in the predictors is added to information in the state variables of the model. To enhance the intuition, the implication of our condition can also be summarized as follows: since discounted returns are unpredictable when a model’s SDF satisfies the Euler equation, if that model’s SDF fails to achieve the variance threshold dictated by our predictors-based bound, then the discounted returns on some assets must become predictable when the information in the state variables is augmented with the information in the predictors.

In the empirical part of the paper, we first provide robust evidence that long horizon predictability translates into tight lower bounds on the variance of the SDFs. We then test our condition for both the habit-formation model of [Campbell and Cochrane (1999)] and the long run risk model of [Bansal and Yaron (2004)], together with its variant due to [Bansal et al. (2012a)], and show that the condition cannot be rejected for these two models: our predictors-based bounds are bona fide bounds on the variance of the SDF of both the habit and the LRR models. Recall now that these two models match closely both the historical unconditional annual real return on the risk-free bond and the equity market. Moreover, they both incorporate a low frequency component that should make asset pricing puzzles less pronounced at longer horizons. Consis-
tent with these statements, the conclusion drawn from the standard unconditional Hansen and Jagannathan (1991) bounds are not surprising: both models satisfy these unconditional cups at medium and long horizons. The conclusions are different when we use our predictors-based bounds. Our bounds show that all these models share a common feature at the one-year horizon: the variance of their SDFs is too small when compared with the lower bound computed from the data and summarized in the predictors-based bounds. Over the long horizons is where we are able to tell models apart: the LRR model has problem in fitting the 5-year predictors-based bound, whereas the habit model is well within the admissible region. Therefore if one were to look only at the long horizon bounds with no conditioning information, the conclusion would be that the long run equity premium puzzle can be resolved as long as sufficient time-nonseparability is incorporated in the preferences. However, our predictor-based bounds highlight that time-nonseparable preferences are not the full story: we show that the long run risk model does not generate returns that are predictable enough compared to the predictability that emerges from the data, while the habit model does. Hence time-nonseparability and the ability of a model to produce empirically plausible patterns of predictability in stock returns are both features necessary to satisfy our bounds.

These results show the importance of understanding when and how we can employ the information contained in a set of predictors. The dynamic asset pricing models under consideration are constructed from a mixture of assumptions about preferences (such as recursive utility or habit persistence, etc) and exposure to fundamental shocks. Our predictors-based bounds allow us to abstract from these model ingredients since they highlight the transitory and long run implications of these models. Our bounds are simultaneously able to detect the common feature across these models, i.e. the low variance of their SDFs at the 1-year horizon, and to tell one model apart from the other upon looking at long horizons.

Our work is related to Kirby (1998), who provides an explicit link between linear predictability and the Hansen and Jagannathan (1991) bounds. Whereas Kirby (1998) investigates whether the ability of predictors to forecast a given set of return is correctly priced by some rational asset pricing model, in the sense that there exist SDFs that price correctly those dynamic strategies which condition on the predictors, our interest here is different: we want to exploit the informa-
tional content of a given set of predictors to investigate the potential of a given asset pricing model to price a given set of returns. Our work is also related to the recent literature which, using a decomposition of the model’s dynamics into transient and permanent components, investigates the implications of these components for valuation (see Hansen and Scheinkman (2009) and Borovicka, Hansen, Hendricks, and Scheinkman (2011)). In particular we view our predictors-based bounds as a useful tool for understanding the high- and low-frequency components of such models. Finally, our work is related to the recent information-theoretic literature that uses entropy bounds to restrict the admissible regions for the SDF and its components (see Bakshi and Chabi-Yo (2012) and Ghosh, Julliard, and Taylor (2011)). In particular our conclusions are in line with Backus, Chernov, and Zin (2011b) who show that the entropy of a model should be sufficiently large to account for observed excess returns.

The rest of this paper is organized as follows. Sections 2 introduces our predictors-based variance bounds and provides the condition under which these bounds are indeed bona fide lower bounds on the variance of the SDF of a given asset pricing model. Section 3 documents the existence of significant predictable variation in stock and bond returns and shows how conditioning information plays an important role in the construction of our bounds at different horizons. We then assess whether various SDF specifications are consistent with our predictors-based bounds. Section 4 addresses two questions: which among the asset classes considered in the paper, stocks and bonds, is key to our results; and how the comparison of model-implied return predictability versus the historical one is connected to our predictors-based variance bounds. Section 5 concludes.

2 Variance Bounds, Predictability and Asset Pricing

In this section we first define our predictors-based variance bounds, which are bounds on the variance of the SDFs that price a given set of returns conditional on the information contained in a vector $Z_t$ of return’s predictors. Given then any asset pricing model with SDF $m_{t+h}^{X_t}$, we ask: under what conditions does a predictors-based bound constitute a bona fide lower bound on the variance of $m_{t+h}^{X_t}$? We answer this question by identifying in Proposition 1 a simple condition,
under which the variance of \(m_{t+h}^X\) must indeed satisfy the bound obtained by conditioning of the predictors \(Z_t\). In Proposition 2, moreover, we rephrase our sufficient condition in terms of an upper bound on the \(R^2\) from predictive regressions of future returns on the current values of the predictors.

2.1 Variance bounds when returns are predictable

We consider an environment with \(N\) random returns on a set of assets traded at a given time \(t\). We denote the return on each asset by \(R_{j,t+h}\), with \(h = 1, 2, \ldots\) the investment horizon, and we let \(R_{t+h}\) denote the vector collecting these \(N\) returns. Alongside the returns we consider a vector \(Z_t\) of return’s predictors, and we denote with \(\mathcal{F}_t^Z\) the informational content of these predictors.

By saying that \(Z_t\) predicts the return \(R_{j,t+h}\) on some asset \(j\) we mean \(\text{Var}\left[\mathbb{E}\left(R_{j,t+h} | F_Z^t\right)\right] > 0\) over some holding period \(h\).

We denote with \(\mathcal{M}^Z\) the set of SDFs that price returns conditionally on the realizations of the predictors \(Z_t\), that is

\[
\mathcal{M}^Z = \{m_{t+h} \mid E(m_{t+h}^2) < \infty, E(m_{t+h}R_{t+h} | \mathcal{F}_t^Z) = e\} \tag{1}
\]

where \(e\) denotes the unit vector. We assume \(\mathcal{M}^Z\) non-empty, which from the standpoint of interpretation means that the Law of One Price holds in the linear space of payoffs obtained by managed portfolios that condition on the predictors’ realization. In other words, we assume that the returns incorporate efficiently the information contained in the predictors \(Z_t\).

Given the SDFs in \(\mathcal{M}^Z\), we call predictors-based variance bound, denoted with \(\sigma^2_Z(v)\), the lower envelope of the set of all variances of SDFs in \(\mathcal{M}^Z\), that is the map

\[
\sigma^2_Z(v) = \inf \left\{\text{Var}\left(m_{t+h}\right) \mid m_{t+h} \in \mathcal{M}^Z, E(m) = v, v \in \mathbb{R}\right\} \tag{2}
\]

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1We assume that all the random variables are defined over a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), so that \(\mathcal{F}_t^Z\) is formally the \(\sigma\)-algebra generated by the vector of predictors \(Z_t\), and hence \(\mathcal{F}_t^Z \subset \mathcal{F}\). We also assume that returns have finite unconditional first and second moments, \(\mu\) and \(E^2 \triangleq E[R_{t+h}R_{t+h}']\), with unconditional variance-covariance matrix \(\Sigma = E^2 - \mu\mu'\). Moreover, we assume that the matrix \(E^2_t \triangleq E[R_{t+h}R_{t+h}' | \mathcal{F}_t^Z]\) of second moments conditional on the predictors is (almost surely) positive-definite and hence invertible, i.e. returns are linearly independent conditionally on information in the predictors. Therefore, denoting with \(\mu_t \triangleq E[R_{t+h} | \mathcal{F}_t^Z]\) the vector of conditional expected returns, both the conditional variance-covariance matrix \(\Sigma_t = E^2_t - \mu_t\mu_t'\) and its unconditional counterpart \(\Sigma\) are positive-definite (and hence invertible) as well.
where the variable \( v \) assumes the interpretation of shadow price of a unit risk-free zero-coupon bond with maturity \( t + h \). The parabolic function \( \sigma_Z^2(v) \) represents an unconditional frontier for SDFs in the sense of Gallant, Hansen, and Tauchen (1990): since it considers all SDFs that price returns conditionally on \( Z_t \) it takes full advantage of the predictive power of the vector \( Z_t \) while maintaining the simplicity of concentrating on the unconditional moments of such SDFs.

As observed by Bekaert and Liu (2004), when the conditional moments of returns are not correctly specified the predictors-based bound \( \sigma_Z^2(v) \) may fail to be a valid lower bound for the volatility of SDFs in \( \mathcal{M}_Z \). To obviate to this problem, we extend to this conditional setting the duality between mean-variance frontiers for SDFs and maximum Sharpe ratios first illustrated for the unconditional case by Hansen and Jagannathan (1991) in their seminal work. More specifically, define the following set of returns from managed portfolios

\[
\mathcal{R}_Z = \{ R_{t+h}^w \mid R_{t+h}^w = w_t R_{t+h}, \ w_t \mathcal{F}_t^Z - \text{measurable s.t. } E(w_t) = 1 \} \tag{3}
\]

This set collects all the payoffs that are generated by trading strategies that exploit the information contained in the predictors at time \( t \). As long as \( \nu \neq 0 \) one can show (see also Abhyankar, Basu, and Stremme (2007), Peñaaranda and Sentana (2013)) that

\[
\sigma_Z^2(v) = \nu^2 \sup_{R_{t+h}^w \in \mathcal{R}_Z} \left( \frac{E(R_{t+h}^w) - \nu^{-1}}{\text{Var}(R_{t+h}^w)} \right)^2 \tag{4}
\]

In words, for any given level of the risk-free rate the predictors-based bound is proportional to the square of the maximum Sharpe ratio that can be generated by managed portfolios that exploit the information contained in the predictors \( Z_t \). Observing that a mis-specification of the conditional expected returns and variances introduces a duality gap in (4), Bekaert and Liu (2004) suggest to always use the right-hand side to actually compute a variance bound that incorporates conditioning information since, by the very own definition of sup, this right hand side will always constitute a valid lower bound on the variance of the SDFs in \( \mathcal{M}_Z \) (albeit, not\footnote{When \( \nu = 0 \) the sup coincides with the reciprocal of the global minimum portfolio variance over the set \( \mathcal{R}_Z \). Moreover, the sup is always attained with the only exception of the case in which \( \nu \) is set equal to the expected return on the global minimum variance portfolio, case in which the sup is attained by a return whose expected price is zero.}}
necessarily the highest lower bound if mis-specification of the first two conditional moments of returns actually occurs). In the empirical part of this paper we follow the lead of Bekaert and Liu (2004), and estimate our predictors-based bounds using the solution to the left-hand side of (4) supplied by Bekaert and Liu (2004).

2.2 Predictors-based bounds and asset pricing modelling

Let’s consider now the asset pricing modelling side of our argument. Our main interest is to understand when and how we can employ information contained in the set of predictors $Z_t$, and synthesized in the predictors-based bound $\sigma^2_Z(v)$, to evaluate a given asset pricing model. To formalize our discussion, we identify any given asset pricing model with the triple $(X_t, F^X_t, m^X_{t+h})$ where $X_t$ denotes the set of state variables of the model, $F^X_t$ denotes the informational content in state variables $X_t$, and $m^X_{t+h}$ denotes the SDF of the given asset pricing model. Since from the standpoint of a given asset pricing model agents maximize their utility based on the information contained in the state variables $X_t$, the SDF $m^X_{t+h}$ together with the returns $R_{t+h}$ must satisfy the first order condition

$$E(m^X_{t+h}R_{t+h} | F^X_t) = e$$

(5)

More generally, the SDF $m^X_{t+h}$ must price all the managed portfolio that condition on the state variables $X_t$ of the given asset pricing model.

To help the intuition it is useful to exemplify this general framework with the two asset pricing models that we analyze in the empirical part. The first example is the Bansal et al. (2012a) model of long run risk, where the state variables are the first two conditional moments of log consumption growth $g_t$, that is $X_t = (x_t, \sigma^2_t)$, the information $F^X_t$ is generated by the innovations in these first two conditional moments, and the SDF takes the form

$$\ln(m^X_{t+h}) = A + Bg_{t+h} + Cr_{a,t+h}$$

where $r_{a,t+h}$ denotes the (continuously compounded) return on an asset that delivers a dividend.

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3Formally, $F^X_t$ is the $\sigma-$algebra generated by the vector of state variables $X_t$, and hence $F^X_t \subset F$ where $F$ is the $\sigma-$algebra of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ over which all the random variables are defined.
equal to aggregate consumption, and $A, B, C$ are functions of the discount factor, risk-aversion coefficient and intertemporal elasticity of substitution of the representative investor. A second example is the External Habit model of [Campbell and Cochrane 1999], where the state variables are log consumption growth $g_t$ and log consumption surplus $s_t$, so that in this case $X_t = (g_t, s_t)$, the information $\mathcal{F}_t^X$ is generated by the innovations in consumption growth, and the SDFs takes the form

$$\ln (m_{t+1}^X) = A' + B' (g_{t+1} + s_{t+1})$$

The question we want to address is: under what conditions does the predictors-based bound $\sigma_Z^2(v)$ constitutes a *bona fide* lower bound on the variance of the SDF of a given asset pricing model? To address this question, given an asset pricing model $(X_t, \mathcal{F}_t^X, m_{t+h}^X)$ we denote with $\mathcal{F}_t^{X,Z}$ the information set obtained by adjoining to the vector of state variables $X_t$ the vector $Z_t$ of predictors.\footnote{Formally, $\mathcal{F}_t^{X,Z} = \sigma (\mathcal{F}_t^X \cup \mathcal{F}_t^Z)$, i.e. it is the smallest $\sigma-$algebra that contains all the information in $X_t$ and $Z_t$, therefore $\mathcal{F}_t^{X,Z} \subset \mathcal{F}$.} Moreover, we let $\nu^X \equiv E (m_{t+h}^X)$ denote the price assigned by the SDF $m_{t+h}^X$ to a unit zero-coupon bond with maturity $t + h$. With this notation at hand we are now able to state the following sufficient condition for the predictors-based bound $\sigma_Z^2$ to constitute a bona fide volatility bound for $m_{t+h}^X$.

**Proposition 1.** Suppose that the asset pricing model $(X_t, \mathcal{F}_t^X, m_{t+h}^X)$ satisfies

$$E \left( m_{t+h}^X R_{t+h} \big| \mathcal{F}_t^X \right) = E \left( m_{t+h}^X R_{t+h} \big| \mathcal{F}_t^{X,Z} \right) \quad (*)$$

Then the predictors-based frontier for SDFs $\sigma_Z^2(v)$ constitutes a *bona fide* lower bound for the volatility of $m_{t+h}^X$, in the sense that

$$\text{Var} (m_{t+h}^X) \geq \sigma_Z^2 (\nu^X)$$

is a necessary condition for $(*)$ to hold.
Proof. The iterative property of conditional expectation implies that

\[
E \left( m_{t+h}^X R_{t+h} \bigg| \mathcal{F}_t^Z \right) = E \left[ E \left( m_{t+h}^X R_{t+h} \bigg| \mathcal{F}_t^{X,Z} \right) \bigg| \mathcal{F}_t^Z \right]
\]

This, together with the orthogonality condition \( E \left[ m_{t+h}^X R_{t+h} \bigg| \mathcal{F}_t^X \right] = \epsilon \) and \((*)\) implies that

\[
E \left( m_{t+h}^X R_{t+h} \bigg| \mathcal{F}_t^Z \right) = \epsilon
\]

that is \( m_{t+h}^X \in \mathcal{M}^Z \), from which

\[
\text{Var} \left( m_{t+h}^X \right) \geq \sigma_Z^2 \left( \nu^X \right)
\]

follows readily.  

To better place our result in the vast literature on predictability and asset pricing observe that, from Kirby (1998) on, it is standard in that literature to assume that the predictors belong to a general information set \( \mathcal{F}_I^t \) which investors condition on when pricing assets. More formally, in the literature it is customary to concentrate on those SDFs \( m_{t+h} \) which satisfy

\[
E \left( m_{t+h} R_{t+h} \bigg| \mathcal{F}_t^I \right) = \epsilon \quad (6)
\]

for some information set \( \mathcal{F}_I^t \) such that \( \mathcal{F}_I^t \supset \mathcal{F}_t^Z \). This perspective is clearly useful to investigate if the ability of \( Z_t \) to predict a given set of return is correctly priced by some rational asset pricing model, since whenever \( \mathcal{F}_I^t \supset \mathcal{F}_t^Z \) then any SDF that satisfies \((6)\) must also price those dynamic strategies that condition on the predictors \( Z_t \). If, however, one wants to exploit the informational content of a given set of predictors to investigate the potential of a given asset pricing model to price a given set of returns, then the information sets \( \mathcal{F}_t^Z \) and \( \mathcal{F}_t^X \) must be taken as given, there is no guarantee that \( \mathcal{F}_t^Z \subset \mathcal{F}_t^X \), and this is where our condition \((*)\) finds its bite.

Whenever condition \((*)\) holds, therefore, \( \sigma_Z^2 \left( \nu \right) \) is a legitimate lower bound on the volatility of the SDF of the given asset asset pricing model. If \( \text{Var} \left( m_{t+h}^X \right) < \sigma_Z^2 \left( \nu^X \right) \) but condition \((*)\)
fails, however, we cannot reject out of hand the asset pricing model \((X_t, F_t^X, m_{t+h}^X)\), since in that case the orthogonality condition \(E(m_{t+h}^X R_{t+h} | F_t^X) = e\) is in principle compatible with a volatility level lower than the one dictated by conditioning on the predictors \(Z_t\). An alternative, but logically equivalent, way to express the implication in Proposition 1 is contained in the next result.

**Corollary 1.** If, given the predictors-based bound \(\sigma_Z^2 (\nu)\), an asset pricing model \((X_t, F_t^X, m_{t+h}^X)\) satisfies \(E(m_{t+h}^X R_{t+h} | F_t^X) = e\) and \(\text{Var}(m_{t+h}^X) < \sigma_Z^2 (\nu^X)\), then for some return \(R_{j,t+h}\)

\[
\text{Var} \left[ E \left( m_{t+h}^X R_{j,t+h} | F_t^X, Z_t \right) \right] > 0
\]

**Proof.** By Proposition 1, if \(E(m_{t+h}^X R_{t+h} | F_t^X) = e\) and \(\text{Var}(m_{t+h}^X) < \sigma_Z^2 (\nu^X)\) then \(E \left[ m_{t+h}^X R_{j,t+h} | F_t^X, Z_t \right] \neq 1\) for some return \(R_{j,t+h}\), hence

\[
\text{Var} \left( E \left[ m_{t+h}^X R_{j,t+h} | F_t^X, Z_t \right] \right) = \text{Var} \left( E \left[ m_{t+h}^X R_{t+h} | F_t^X, Z_t \right] - 1 \right) > 0
\]

Corollary 1 supplies a dual interpretation of Proposition 1 in terms of predictability of discounted returns. Given an asset pricing model \((X_t, F_t^X, m_{t+h}^X)\), the discounted returns \(m_{t+h}^X R_{t+h}\) cannot be predicted by the state variables \(X_t\) alone if the model satisfies the Euler equation. If the SDF \(m_{t+h}^X\) satisfies the conditional Euler equation and yet it fails to achieve the variance threshold dictated by the predictors-based bound \(\sigma_Z^2 (\nu^X)\), however, then the discounted return of some asset becomes predictable upon augmenting the state variables \(X_t\) with the predictors \(Z_t\).

### 2.3 Predictors-based bounds and predictive \(\mathcal{R}^2_s\)

We show now that the predictors-based bound \(\sigma_Z^2 (\nu)\) generates also an upper bound for the \(\mathcal{R}^2_s\) from regressions of the returns \(R_{t+h}\) on the predictors \(Z_t\). When taken together with Proposition 1, this implies that the variance of the SDF of any asset pricing model \((X_t, F_t^X, m_{t+h}^X)\) that satisfies condition \(\text{[\text{[}}\) bounds from above these predictive \(\mathcal{R}^2_s\) as well. We establish these facts
in the next Proposition, under the assumption that a risk-free return $R_{f,t+h}$ is available to the investors.

**Proposition 2.** Given an asset pricing model $(X_t, F_t^X, m_{t+h})$ and the return $R_{j,t+h}$ on a traded asset, suppose that $\text{Var} \left( R_{j,t+h} | F_t^Z \right)$ is constant and condition (4) holds. Then

$$R_j^2 \equiv \frac{\text{Var} \left[ E \left( R_{j,t+h} | F_t^Z \right) \right]}{\text{Var} \left( R_{j,t+h} \right)} \leq R_{f,t+h}^2 \sigma^2_Z (\nu_{\text{min}}) \leq R_{f,t+h}^2 \text{Var} \left( m_{t+h} \right)$$

where $\sigma^2_Z (\nu)$ is the global minimum variance over all SDFs in $\mathcal{M}^Z$.

**Proof.** Since under condition (4) the second inequality follows readily from Proposition 1, we only need to establish the first inequality. To this end, denoting with $R_{e,j,t+h} = R_{j,t+h} - R_{f,t+h}$ the excess return on asset $j$, for any $m_{t+h} \in \mathcal{M}^Z$ we have $E \left( m_{t+h} R_{e,j,t+h} | F_t^Z \right) = 0$, that is

$$E \left( R_{e,j,t+h} | F_t^Z \right) = -R_{f,t+h} \text{cov} \left( m_{t+h}, R_{e,j,t+h} | F_t^Z \right)$$

Squaring both sides up, exploiting the conditional Cauchy-Schwarz inequality, taking expectations and exploiting the fact that the variance cannot exceed the second moment, we have

$$\text{Var} \left[ E \left( R_{j,t+h} | F_t^Z \right) \right] = \text{Var} \left[ E \left( R_{e,j,t+h} | F_t^Z \right) \right] \leq R_{f,t+h}^2 E \left[ \text{Var} \left( R_{j,t+h} | F_t^Z \right) \right] \text{Var} \left( m_{t+h} | F_t^Z \right)$$

where the second inequality follows from the assumption of constant conditional variance and the last inequality follows from decomposing the total variance of $m_{t+h}$ into the sum of average conditional variance plus variance of the conditional expectation. Therefore

$$R_j^2 \equiv \frac{\text{Var} \left( E \left[ R_{j,t+h} | F_t^Z \right] \right)}{\text{Var} \left( R_{j,t+h} \right)} \leq R_{f,t+h}^2 \text{Var} \left( m_{t+h} \right) , \quad \forall m_{t+h} \in \mathcal{M}^Z$$

from which the first inequality in (7) follows from the definition of $\sigma^2_Z (\nu)$ in (2). ■

It is useful to compare this result with the literature, in particular with Proposition 5 in
Ross (2005) (see also Zhou (2010)). In line with our general approach of allowing the information in the predictors to be not necessarily included in the information in the state variables, a first contribution of our Proposition 2 is to show that, if condition (*) is violated, the $R^2$ from a predictive regression is not constrained to be below the volatility of the SDF of a given pricing model, that is, $R^2_{j,t+h} \text{Var} (m^X_{t+h}) < R^2_j$ is potentially compatible with the model Euler equation $E \left( m^X_{t+h} R_{t+h} \mid F^X_t \right) = e$. This fact can not emerge from Proposition 5 in Ross (2005), since there agents are assumed to price conditionally on a generic information set $F^I_t$ which is implicitly assumed to satisfy $F^I_t \supset F^Z_t$, i.e. to contain all the information in the predictors. Our proof, moreover, highlights the importance of assuming returns to have constant conditional variance, which implies

$$E \left[ \text{Var} \left( R_{j,t+h} \mid F^Z_t \right) \text{Var} \left( m_{t+h} \mid F^Z_t \right) \right] \leq \text{Var} \left( R_{j,t+h} \right) \text{Var} \left( m_{t+h} \right)$$

from which the bound on $R^2_j$ obtained in Proposition 2 follows. Without constant conditional second moments, that is, in the case of stochastic volatility, it is not at all obvious that a bound as sharp as the one in (7) can be established at all.

### 3 Empirical Results

In this section we put the theoretical framework introduced in the previous section to work. We first introduce the linear predictive model for returns that we use throughout the empirical part of the paper and use it to actually compute the predictors-based bound $\sigma^2_Z (\nu)$ for different sets of assets and different horizons. We then test condition (*) for both the long run risk model and the External Habit model. Since the condition is not rejected in either case, we go ahead and compare the volatilities of the models SDFs with our predictors-based bounds. We complement our analysis with a parameters’ sensitivity analysis, we reinterpret our findings in terms of upper bounds on the $R^2$s from predictive regressions, and finally we analyze the robustness of our results to the possibility of misspecification of the model for the conditional moments of returns.
3.1 The predictive model

The empirical analysis is based on the (gross) return on the constant maturity bonds, and on the
(gross) return on the value weighted portfolio of all stocks traded in the NYSE, the AMEX, and
NASDAQ. Table I presents full-sample statistics of the the 5-year constant maturity bond and
stock returns for the common sample period (1952Q2 to 2012Q4). Over this sample period, the
mean nominal return on stocks was 11.45% per annum, the mean nominal return on bonds was
6.31% per annum, and the mean short-term interest rate - not shown in the table - was about
5.65% per annum. The standard deviation of stock nominal returns was 16.68% per annum, and
the standard deviation of bond log returns was 5.77% per annum.

We consider two different set of assets:

1. SET A: risk-free bond, 5-year Treasury constant maturity and equity market returns
2. SET B: risk-free bond, 5-, 7-, 10- and 30-year Treasury constant maturity government
   bonds and equity market returns.

These two sets correspond to a universe of equity and bond portfolios whose return properties
are the subject of much scrutiny in the empirical asset pricing research.

In contrast to the simple random walk view, stock and bond returns do seem predictable.
We use a typical specification that regresses rates of return on lagged predictors to review this
claim. In particular we consider the following linear predictive system:

\[ R_{i+h}^t = \beta_{0,i}^t + \beta_{1,i}^t z_i^t + u_{i+h} \]  

(8)

where \( i = S, B \) stands for stocks and bonds, respectively, \( Z_t = (z_t^S, z_t^P) \) denotes the vector of
returns’ predictors, potentially different for stock and bonds. We let the holding period range
from one quarter to five years, i.e. \( h = 1, 4, 20 \) quarters. Table 2: Panel A presents regressions
of the real stock returns \( R_{t+h}^i \) on the dividend-price ratio \( pd_t \) and the consumption-wealth ratio

\footnote{For a detailed description of data construction see the Appendix.}
The choice of these two stock market predictors is motivated by the present value logic, see Campbell and Shiller (1988), and a linearization of the accumulation equation for aggregate wealth in a representative agent economy, see Campbell and Mankiw (1989) and Lettau and Ludvigson (2001). They are both “noisy” predictors of future asset returns. Although the $R^2 = 4\%$ at quarterly horizon does not look that impressive, the $R^2$ rises with horizon, reaching a value of about 50%, at the 5 years horizon. Each variable has an important impact on forecasting long horizon returns: using the dividend yield as the sole forecasting variable, for instance, would lead to an $R^2$ of “only” 22% at the 5-years horizons. Table 2 presents regressions of the 5-year maturity bond log return onto the lagged short-term nominal interest rate $y_t$, the lagged yield spread $spr_t$ and the Cochrane and Piazzesi (2005) CP$_t$ factor. The results show that our predictive system is able to capture fluctuations in bond excess returns at all horizons. These results are consistent with much of the recent empirical research on the predictability of stock and bond returns (see Fama and French (1989), Campbell (1987) and Cochrane (2001; 2008) among others).

[Insert Table 2]

### 3.2 Predictors-based bounds across horizons

It seems apparent from Table 2 that expected returns vary over time. To examine the ability of the predictors to improve the variance bounds as a diagnostic tool, in this section we compare the predictors-based bounds to the classical, unconditional HJ bounds. Along with the predictive versus unconditional dimension, moreover, we also analyze the effect of altering the investment horizon. Whereas Cochrane and Hansen (1992) were the first to carry out this exercise on the unconditional variance bounds, our analysis extends their results and highlights the interaction of conditioning information with the horizon dimension.

To compute the predictors-based bounds we use the solution to the left-hand side of Eq. (4). Bekaert and Liu (2004) show that the optimal trading strategy $w_t$ that incorporates information

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6 Using excess returns yields similar results.
7 Our results are not driven by the inclusion of the CP factor. In a system where the CP factor is not included the Newey-West corrected t-statistic on the yield spread is above 2.2 at all horizons and the predictive system achieves a high $R^2$ value of 44.5% at the five year horizon.
takes the following expression:

\[ w_t = (\mu_t' \mu_t + \Sigma_t)^{-1} (1 - w\mu_t) \]

where \( \mu_t \) and \( \Sigma_t \) are the vector of conditional expected returns and the conditional variance-covariance matrix, respectively, and 

\[ w = (\nu - b)/(1 - d), \quad \nu = E[M_{t+1}], \quad b = E\left[\epsilon' (\mu_t' \mu_t + \Sigma_t)^{-1} \mu_t \right] \]

and 

\[ d = E \left[ \mu_t' (\mu_t' \mu_t + \Sigma_t)^{-1} \mu_t \right]. \]

Since to compute the predictors-based bounds we need a model of the first and second conditional moments of asset returns, \( \mu_t \) and \( \Sigma_t \), we use to this end the linear predictive model in equation (9) discussed in the previous subsection. For simplicity, we assume the conditional covariance matrix for returns to be constant, and estimate it as the residual in the forecasting regressions.

Figure 3 presents our results for SET A. We consider investment horizons of 1 quarter, 1 year, and 5 years. The shortest investment horizon coincides with the sampling interval of returns. In each panel we report the efficient bounds generated with conditioning information (solid lines) along with the unconditional HJ bounds (dashed lines) that make no use of conditioning information. Similar to Cochrane and Hansen (1992), Figure 3 shows that the bottom of the mean standard deviation frontier shifts up and to the left as we increase the investment horizon. At the same time, although the lower bound for volatility increases, it does so slowly. Importantly, the picture shows that the strong predictability at long horizons documented in Table 2 translates into a tight lower bound on the variance of the SDF. In particular Figure 3 imparts two conclusions. First, the estimates of our predictor-based bounds are sharper relative to the unconditional ones. In Figure 3, for instance, the minimum point of the frontier at the 5-year horizon obtained using conditioning information is about 1.6 times sharper than the unconditional lower bound, thereby substantiating the incremental value of conditioning information in asset pricing applications. The difference between the bounds with and without conditioning information at the 5-year horizon reflects the considerable long run predictability documented in Table 2. Second, the implications of using conditioning information strongly depend on the holding period: while conditioning information adds little at short horizons, the volatility impli-

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8 We investigate the effect of potential model misspecification in our regressions on the construction of the bounds in Section 3.3.5.
cations of the long horizon returns are more dramatic and reveal the fundamental role played by conditioning information over time. Consistently with the results on the predictive regressions, therefore, the role of the information contained in the predictors becomes more apparent as we lengthen the investment horizon.

Figure 3 presents the same analysis for set B. Comparing Figures 3 with 5, we observe that expanding the number of assets, i.e. moving from SET A to SET B, leads to a bound that is intrinsically tighter.

Taken together figure 3 and 5 highlight the two effects that are at work simultaneously: the conditioning information embedded in the conditional moments of returns and the horizon at which this information becomes relevant. The tightening of the volatility bounds is the combined result of these two forces at work simultaneously.

3.3 Predictors-based bounds and asset pricing models

We compare now our predictors-based bounds to the external habit-formation model of Campbell and Cochrane (1999), and the long run risk models of Bansal and Yaron (2004) and Bansal et al. (2012a) [9] There are two main reasons for concentrating our attention on these two models. First, although they are far from exhaustive, they still embed different utility specifications and specify the long run and short run risk in distinct ways (see also Hansen (2009)). Second, to assess the ability of a model to produce realistic SDF dynamics we need to compute its first and second unconditional moments. While the conditional variances are amenable to closed-form characterization, the unconditional variances are tractable only via simulations [10] Therefore we rely on a statistic generated from a simulation procedure and we look for models for which the solution methods are well established and from which it is easy to simulate. To simulate the series of $m^X_{t+h}$ we use either calibrated or estimated values. Tables 8 and 9 report the complete specification of the parameter values for preferences and exogenous dynamics, along with their standard errors in case the parameters are estimated. The calibrated parameters for the long run risk and habit models are taken from Bansal et al. (2012a) and Campbell and Cochrane (1999).

9In a previous version of this paper we also consider the rare event of Backus, Chernov, and Martin (2011a) and an affine 3-factors model suggested by Koijen, Lustig, and Nieuwerburgh (2012) to explain the cross section of bond and stock returns.

10There are some exception, see for instance the rare disaster model given i.i.d. uncertainties.
respectively. The estimated parameter values are taken from Bansal, Kiku, and Yaron (2012b) and Aldrich and Gallant (2011)\textsuperscript{11}

\textbf{[Insert Tables 8 and 9 about here]}

### 3.3.1 Testing for condition (*)

We have seen that a violation of condition (*) would prevent us from using the predictors-based bounds to test the validity of a stochastic discount factor. Therefore before exploiting the conditioning information in our predictors to sharpen the unconditional bounds, we first check condition (*) for each asset class, models, and horizons.

To test condition (*) we run the following two regressions:

\[
\begin{align*}
    m_{t+h}^X R_{t+h}^i &\equiv \alpha_{1,h} + \beta_{1,h} X_t + \varepsilon_{1,t+h} \\
    m_{t+h}^X R_{t+h} &\equiv \alpha_{2,h} + \beta_{2,h} X_t + \gamma_h Z_t + \varepsilon_{2,t+h}
\end{align*}
\]

where \(i = S, B\) denotes the asset class, and we compute the differences of the fitted values

\[
m_{t+h}^X R_{t+h}^{X,Z} - m_{t+h}^X R_{t+h}^X
\]

If 90\% of the distribution of these differences includes zero, then we do not reject condition (*).

With the test assets, the set of candidate predictors and the model-implied stochastic discount factors at hand, we are now ready to test condition (*). Figure \textsuperscript{11} displays the results for the LRR (Panel A) and the habit models (Panel B) when the test asset is the Equity Market. Empirically, there is no horizon at which we reject condition (*), and this is true for both models. We obtain analogous results when we consider the yield on the Treasury Bill and the constant maturity coupon bond returns as test assets\textsuperscript{12}

In summary, we can conclude that the predictors that we employ in our linear prediction model are indeed useful in sharpening the unconditional variance bounds, since the predictors-based bounds that we derive are indeed bona fide lower

\textsuperscript{11}Aldrich and Gallant (2011) present for each parameter the posterior mean and posterior standard deviation. We refer to the posterior mean of each parameter as our point estimate for that parameter.

\textsuperscript{12}Results are available upon requests from the authors.
bounds for the volatility of the SDF of both models.

[Insert Figure 1 about here]

To conclude this section we remark that, although we are in no position to discuss the power properties of our procedure to test condition (*), we are at least able to show that the condition is in fact rejected in cases in which a rejection would be the expected outcome. To see this, consider the simplest possible consumption-based asset pricing model, i.e. the model with a representative consumer with CCRA utility and whose endowment/consumption process exhibits i.i.d. growth. Figure 2 shows that in this case our procedure does reject condition (*) soundly, as one would definitely expect.

[Insert Figure 2 about here]

### 3.3.2 Model-implied SDFs and predictors-based bounds

We compare now the SDFs implied by two asset pricing models under scrutiny at the light of the information contained in the returns’ predictors. To do so, since condition (*) is satisfied we can use our predictors-based bounds as a bona fide differentiating diagnostic, that is, it makes sense to check whether the variance of the $h$-period SDFs implied by each of the two models satisfies the predictors-based variance bounds at the corresponding horizon.

Figures 3, 4, 5, and 6 compare the volatility of the SDFs generated by the two competing models with the predictors-based bounds at different horizons and for different sets of test assets. The (blue) stars and (red) triangles represent population values using calibrated and estimated parameters, respectively. In particular we use the dynamics of consumption growth and of the state variables to simulate 600,000 monthly observations (50,000 years) of the model-implied SDF. From this long time series we then calculate the SDF unconditional moments.\footnote{Using a single simulation run to infer the population values for the entities of interest is consistent with, among others, the approach of Campbell and Cochrane (1999) and Beeler and Campbell (2009).} Since the volatility bound itself is estimated from the data, it is random. Moreover, the computation of the mean and standard deviation of the SDF using a specific utility function relies on estimates of the moments of the consumption process, and so it is random as well. To account for the...
first source of uncertainty, we construct confidence intervals for the bounds with conditioning information based on 50,000 random samples of size 234 from the data and a block bootstrap. The bounds along with their lowest 90% confidence interval are reported in Figure 4 and 6.

To account for the second source of uncertainty, we follow an approach similar to Cecchetti, Lam, and Mark (1994) and Burnside (1994).

Figures 3 to 6 provide a visual representation of the importance of jointly considering conditioning information and horizons. Start with Figure 3 which shows that at the yearly horizon the SDF obtained using calibrated values satisfies for both models the unconditional HJ bounds. This is not surprising: both models are calibrated closely to offer conformity with the historical unconditional annual real return on risk-free bond and the equity market. The picture offers different conclusions when we incorporate conditioning information. In this case the LRR model falls largely below the bounds and hence cannot be considered a valid SDF. Going on to Figure 4 we see that even accounting for the uncertainty that arises in the comparison of the mean and standard deviation of the SDF implied by a particular model of preferences with the bound that is computed from asset returns data, the conclusion does not change: the LRR model has problem in fitting the 1-year HJ conditional bound. On the other hand, the habit model falls within the lowest 90% confidence interval.

Recent theoretical and empirical research in macro-finance has highlighted the importance of capturing low frequencies components for an asset pricing model to be successful. One would then expect this low frequency component to make asset pricing puzzles less pronounced at longer horizons. With the visual aid of Figure 3 we can see that this statement holds true when no conditioning information is incorporated: the 5-year unconditional HJ bounds are satisfied by both models. However, the conclusion changes significantly when we look at the 5-year predictors-based bound: in this case while the habit-model satisfies the predictors-based bound with a good margin, the LRR models clearly struggles. As expected, using SET B and hence expanding the set of assets, exacerbates this results: Figure 6 shows that, after accounting for uncertainty, while the habit model still satisfies the bound at long horizons, the long run risk models generates instead an SDF not volatile enough at long horizons. Hence as the investment

\[14\] We select the optimal block length for the bootstrap according to Politis and White (2001). We thank Andrew Patton for making the code available on his website.
horizon increases, it is quite possible that the equity premium puzzle does not vanish, but becomes instead more pronounced. This shows why an asset pricing model may reproduce some (unconditional) asset market phenomena at short horizon, while finding it onerous to satisfy bounds at longer horizons.

It is important to stress that if we looked instead at the long-horizon bounds with no conditioning information, we would have concluded that the long-run equity premium puzzle can be resolved as long as sufficient time-nonseparability is incorporated in preferences. However, our predictors-based bounds highlights that time-nonseparable preferences are not the full story: we show in Section 4.2 that time-nonseparability and the ability of a model to lead to empirically plausible patterns of predictability in stock returns are jointly important to satisfy variance bounds, as ours, which are based on well-established predictors.

The results discussed so far are summarized in Table 3. The last two columns report the minimum value of the predictors-based bound for SET A and SET B, respectively. The remaining columns report the unconditional variance of the model-implied SDF, using estimated parameters. We compare the minimum point estimate of the predictors-based cup with the population value of the volatility of the SDF. Although the unconditional variance of the SDF should be compared with the point on the parabola corresponding to the unconditional mean of the model-implied SDF, we can safely conclude that an SDF is not a valid one if its variance is below the minimum point. The Table reveals that at the yearly horizon the standard deviation implied by the Bansal et al. (2012a) and the Campbell and Cochrane (1999) models are 0.47, and 0.74, respectively. The LRR value is lower than 0.73, the minimum volatility suggested by the predictors-based bound constructed from SET A. For the habit-model the value is close to the minimum point, although as we have seen in Figure 4 the bounds are still not satisfied. Hence when we consider SET A we observe that, although both models satisfy the unconditional bounds, none of them is able to satisfy the 1-year conditional bounds. When we look at long horizons, the SDF from the long run risk model has a volatility of 1.53. Although this value is pronounced it is still lower than the minimum volatility obtained from the SET B, namely 2.44. On the other hand the habit-model, with a sound 4.29, satisfies the bound easily.

In sum, this section shows that our predictors-based bounds incorporating conditioning in-
formation from a well-established set of stock and bond predictors are a useful tool to assess the performance of candidate asset pricing models at multiple horizons. It is noteworthy that each asset pricing model parametrization reasonably mimics the (annual) unconditional equity premium and the real risk-free return, while simultaneously calibrating closely to the first two moments of consumption growth. However our evidence reveals that the variance of the SDF from each model fails to meet the restrictions imposed by the predictor-based HJ bounds at the 1– year horizon. Moreover at a long 5-years investment horizon, and using SET B, the standard deviation of the SDF implied by the LRR never approaches the bounds; it is instead the habit-model that turns out to be able to generate enough volatility at long horizons to satisfy the long run equity premium puzzle.

3.3.3 Parameters sensitivity analysis

Table 4 reports the population variance of the simulated SDFs for the long run risk and external habit formation models, obtained using both calibrated and estimated parameters. This exercise shows that the population values for the volatility of the SDF implied by the long run risk model and external habit model would be slightly greater when using calibrated values. For the LRR model this is mainly driven by the lower persistence in the consumption growth volatility: compared to the benchmark calibration, where the half-life is essentially infinite (58-year), the estimated value implies a half-life slightly over 33-year. For the habit model the results are due to the lower value for the \( \gamma \) parameter which enters the coefficient of relative risk aversion. At the same time if we look once again at Figures 4 and 6 we see that our conclusions are largely unaffected by either using calibrated or estimated parameters: the evidence reveals that the variance of the SDF from each model fails to meet the lower bound restriction at the 1-year horizon for both SET A and SET B, and only the \textit{Campbell and Cochrane (1999)} model is able to resolve the long horizon equity premium.

[Insert Table 4 about here]
3.3.4 Bounds, models and $R^2$

We now evaluate the asset pricing models scrutinized so far through the lenses of the $R^2$s of the predictive model (8) that underlies our predictive-based bounds. In Table 2 we compare the variance, scaled by the squared gross risk-free rate, of the SDFs of the LRR and habit models with the $R^2$s of predictive regressions for stock and 5-year constant maturity government bond real gross returns. The table, moreover, reports the minimum variance of both the unconditional HJ bounds and of our predictors-based bounds, both scaled by the squared gross risk-free rate as well.

[Insert Table 5 about here]

The implication of Proposition 2 in Section 2.3 is that, as long as Condition (*) holds, for any asset class the minimum scaled variance of the predictors-based bounds $R^2_{f,t+h} \sigma_Z^2(\nu_{min})$ should be intermediate between the predictive $R^2$ and the scaled variance of the models SDFs. After having analyzed in the previous sections the failure of the models to meet the predictors based bounds, the fact that this implication is challenged by the data at different horizons and for different sets of assets should not come at a surprise. What is more interesting here is to observe that at shorter horizons the scaled variance of the models SDFs are very close to, and in certain cases outright below, the predictive $R^2$s. This happens, in particular, for the case of the LRR model and the $R^2$ of the 5-year constant maturity government bond real gross returns, where the variance of the scaled SDF is well below the predictive $R^2$ at both the 1-quarter and the 1-year horizons (although, to be fair, these $R^2$ still belong to the confidence interval around the scaled variance of the SDF). When compared to the scaled minimum variance of the predictors-based bounds, however, the predictive $R^2$s line up nicely below the minimum values at all horizons.

The findings in this section are interesting from two points of view. First, the fact that the scaled variance of the LRR model falls below the $R^2$ of the 5-year constant maturity government bond real gross returns at the 1-quarter and 1-year horizon implies a short horizon challenge to the model that complements the long horizon challenge discussed above. Second, this inversion between the scaled variance of a model SDFs and a predictive $R^2$, while on the other side the $R^2$ falls nicely below the scaled minimum variance of the predictors-based bound, reinforces the
point made in Proposition 2 that the difference between the information sets associated with
the predictors and that posited by a given model must be dutifully accounted for when employing
predictive $R^2$ as model diagnostics.

3.3.5 Predictability, model mis-specification and variance bounds

We conclude this section by investigating the performance of our bounds along two further
dimensions: robustness and efficiency. Recall that the results presented so far are obtained
under the assumptions of a time-invariant variance-covariance matrix for returns and a linear
model for their conditional mean. To investigate possible mis-specification of the conditional
moments and the efficiency of our bound we plot in Figure 9 alternative implementations of
the variance bounds: specifically, the optimal bounds of Ferson and Siegel (2003, 2009) (FS),

[Bekaert and Liu (2004) show that their bound, obtained by maximizing the Sharpe ratio over
all returns obtained from portfolios that condition on $Z_t$ and that cost 1 on average (see (4) in
Section 2.1), must be a parabola under the null of correct moments specification. Figure 9 shows
that in our case we obtain a smooth parabola indeed. The figure, moreover, shows that the GHT
bound, obtained via the inf in (4), and the BL bound are virtually on top of each another, i.e.
there is no duality gap. This suggests that the BL bound closely approximate the efficient use of
conditioning information. Overall the three alternative implementations of the variance bounds
that incorporate information from the predictors $Z_t$ generate similar bounds with no visible
misspecification. The FS is the lowest bound, see also Table 6; this is readily understood by
observing that the FS bound collects all those payoffs that are generated by trading strategies
that reflect the information available at time $t$, and that have unit price almost surely equal
to one, and not just on average as for the BL case.\textsuperscript{15} Although the FS approach yields the
most conservative bounds, the variance bounds do not shift enough to change our conclusions.

\textsuperscript{15}More formally, the FS bound (see Ferson and Siegel (2003)) is defined as

$$
\sigma^2_{FS}(\nu) = \nu^2 \sup_{R_t^w \in R_{FS}} \left( \frac{E(R_t^w) - \nu^{-1}}{\text{Var}(R_t^w)} \right)^2
$$

25
The evidence suggests that misspecification of the conditional moments does not seem to play a major role to change the results.

4 Extensions

In Section 3.1 we have shown that incorporating predictability of asset returns does make the variance bounds tighter. In this Section we answer the following two questions: first, the predictability of which asset class, stocks or bonds, contributes the most to the sharper variance bounds exhibited in the previous section? Second, does the failure of an asset pricing models relates to the comparison of model-implied returns predictability versus historical data predictability and, if so, how?

4.1 Stock-based versus bonds-based variance bounds

To check the relative importance of different asset classes for predictability, and hence for sharpening the bounds, we consider the following experiment. We build the variance bounds according to different scenarios. Each scenario imposes some restrictions on the predictive system in $\mathcal{S}$.

In the first case (restriction I) stock returns are unpredictable. In the second case (restriction II), treasury government bonds returns are unpredictable. In both cases, we maintain the assumption that the risk-free rate is predictable.

Figure 7 displays the results for both SET A (see Panel A) and SET B (see Panel B): the variance bounds without information (dashed black line), with conditioning information (solid red line), with conditioning information together with restriction I (solid blue line) and with conditioning information together with restriction II (solid green line).

\[ R_{FS} = \{ R_{t+h} \in \mathcal{R}^2 \mid w'e = 1 \text{ almost surely} \} \]

where

\[ \mathcal{R}_{FS} = \{ R_{t+h} \in \mathcal{R}^2 \mid w'e = 1 \text{ almost surely} \} \]

i.e. the FS variance bound follows from maximizing the Sharpe ratio over the set of returns from portfolios that, while conditioning on $Z_t$, are required to have unit price almost surely, and not just on average. Therefore, it is evident that $\sigma_{FS}^2(v) \leq \sigma_Z^2(v)$.

For SET A, this implies that the 5-year maturity treasury government bond return is assumed to be unpredictable. For SET B, we shut down the predictability of 5-, 7-, 10-, 20- and 30-year maturity treasury government bonds returns simultaneously.
From Figure 7 we can draw two main conclusions. First, it is the predictability in stock returns that really tightens the variance bounds, particularly so at long horizon. In particular, under Restriction I the minimum point of the frontier for the volatility of SDF at the 5-year horizon based on the return properties of SET A (SET B) is about 1.54 (1.16) times less than the one with conditioning information and otherwise unrestricted. Second, the additional tightening due to the predictability of treasury government bond is instead marginal. Figure 7(b) shows that even when we shut down the predictability of all the treasury government bonds simultaneously, the minimum value of the variance bound at the 5-year horizon is still 0.93 times that of the benchmark case (i.e. the variance bound generated with conditioning information, red solid line).

Summing up, the shape of variance bounds on SDFs essentially depends on the model we choose for predicting stock returns, whereas the predictability of bond returns plays a somehow more marginal role.

4.2 Historical versus model-implied predictability

In this section we investigate whether the asset pricing models analyzed in this paper can capture the predictability in the data. Our main conclusion is that, to produce SDFs with enough volatility to satisfy our predictors-based bounds, an asset pricing model must generate a significant degree of predictability for excess return.

Table 7 displays results for the regressions of future excess returns over the horizons of 1, 3, 5 and 8 years, on the log of the price-dividend ratio, both in the data and in the models. Model-implied predictive regression results are listed in Column 2 (Bansal and Yaron (2004) model), Column 3 (Bansal et al. (2012a)) and Column 4 (Campbell and Cochrane (1999)); in these cases the regressions are run on simulated data, and the results are the average of 1000 simulations. Once again we observe that in the data excess returns are predictable at long horizons, the $R^2$ rising with maturity, from 7% at the 1-year horizon to about 25% at the 8-year horizon. When we look at the model-implied results, we observe that the long run risk model by Bansal and Yaron (2004) features modest predictability, with an $R^2$ in the range of 0.0%-3%, and the slope coefficients varying from $-0.05$ at the one-year horizon to $-0.34$ at the eight-year
horizon. The model-implied regression slope is instead, on average, close to the corresponding estimates across horizons. However this model yields a quite low $R^2$ in the range of 0.0%-9% which is much lower than the one based on real data. On the other hand, the 
$R^2$s implied by Campbell and Cochrane (1999) start low, 8%, but then rise to impressive values, 33.7% which is even higher than the one generated from real data, 25%. As shown in Table 7, the outstanding $R^2$s are obtained at a cost of a very high magnitude of slope coefficients. For instance, the model implied slope coefficients are $-0.32$ at one-year horizon and $-1.32$ at eight-year horizon while the corresponding point estimates for the historical data are $-0.14$ and $-0.69$, respectively. The predictive power for the price-dividend ratio is amplified by a factor of almost two.

[Insert Tables 7 about here]

Figure 8 provides a visual representation of these results.

[Insert Figure 8 about here]

In all, the long run risk models produce low predictability for excess returns at long horizons: the implied $R^2$ is too small compared to the data. On the other hand, the external habit model of Campbell and Cochrane (1999) generates overly predictable excess returns compared with the results from historical data. Counterpart to failing or satisfying the bounds seems to be the ability of a model to generate predictable enough returns.

5 Conclusion

We analyze predictors-based variance bounds, i.e bounds on the variance of those SDFs that price a given set of returns conditional on the information contained in a vector of returns’predictors. We identify a simple sufficient condition under which the predictors-based bounds constitute bona fide lower bounds on the variance of the SDF of a given asset pricing model. We use our predictors-based bounds to assess the performance of two leading consumption-based asset pricing models: the long run risk model of Bansal and Yaron (2004) with its variant due to Bansal et al. (2012a), and the habit-formation model of Campbell and Cochrane (1999).
Our results point to the importance of jointly considering conditioning information and horizons. The asset pricing models we consider reasonably mimic the annual unconditional equity premium and the real risk-free return. However, our evidence reveals that the variance of all of the model-implied SDFs considered fails to meet the lower bound restriction at 1-year horizon once conditioning information is accounted for. Consistent with the idea that asset pricing puzzles are less pronounced at longer horizons, the 5-year unconditional HJ bounds are always satisfied. However, this conclusion does not hold when conditioning information is brought into the picture: the habit model solves the equity premium at long horizons whereas the long run risk model produces an SDF that is not volatile enough. As a consequence, it is quite possible that the equity premium puzzle does not vanish, but rather gets more pronounced as the investment horizon increases.

Our predictors-based bounds represent a convenient tool for researchers. First, they are informative of the dynamics that an admissible SDF should have at short-, medium- and long-term horizons. Second, our bounds yield a graphical and intuitive comparison of the performance of asset pricing models by isolating the common feature behind them at a specific horizon. In fact, the dynamic asset pricing models under consideration are constructed from a mixture of assumptions about preferences (such as recursive utility or habit persistence, etc) and exposure to fundamental shocks. The predictors-based bounds allow one to highlight the transitory and long run implications of these economic models. Consistent with the idea that all models are approximations of reality and as such likely to be misspecified along some dimensions, our predictors-based bounds use the investment horizon and conditioning information as the fundamental ingredients that allow researchers to set models apart models (as, for us, at long horizons), or to identify the common behavior among apparently different models (as, in our case, at the 1-year horizon).
Data

We consider a set of quarterly equity and bond returns over the period 1952Q2 to 2012Q4. Our choice of the start date is dictated by the availability of data for our predictors. Real returns are computed by deflating nominal returns by the Consumer Price Index inflation. We obtain the time series of bond and stock returns using monthly daily returns on stocks and bonds. Quarterly returns are constructed by compounding their monthly counterparts. The $h$-horizon continuously compounded excess return is calculated as $r_{t,t+h} = r_{t+1}^e + \ldots + r_{t+h}^e$ where $r_{t+j}^e = \ln(R_{t+j}) - \ln(R_{f,t+j})$ is the 1-year excess log stock return between dates $t + j - 1$ and $t + j$; $R_{t+j}$ is the simple gross return; and $R_{f,t+j}$ is the gross risk-free rate (3-month Treasury bill) at the beginning of period $t + j$.

1. Stock returns: Return data on the value-market index are obtained from the Center for Research in Security Prices (CRSP) at the University of Chicago. We use the NYSE/Amex value-weighted index with dividends as our market proxy, $R_{t+1}$.

2. Bond returns: Returns on bonds are extracted from the US Treasuries and Inflation Indices File and the Stock Indices File of the Center of Research in Security Prices (CRSP) at the University of Chicago. The CRSP US Treasuries and Inflation Indices File provides returns on constant maturity coupon bonds, with maturities ranging from 1 year to 30 years, starting on January, 1942. The nominal short-term rate ($R_{f,t+1}$) is the annualized yield on the 3-month Treasury bill taken from the CRSP treasury files.


4. Inflation: we use the seasonally unadjusted CPI from the Bureau of Labor Statistics. Quarterly inflation is the log growth rate in the CPI.
References


Figure 1: Empirical verification of Condition(*) - Market returns. Dashed blue lines give the 90% confidence interval of the differences between the estimated values of discounted returns with and without using predictors, at horizon 1-Quarter, 1-Year, 5-Year. The discounted returns are the product of model generated SDFs and real market index returns. Dotted red lines locate the benchmark of zero value. We do 1000 times simulations over 241 quarters for both long run risk model by Bansal et al. (2012a) (Panel A) and external habit model by Campbell and Cochrane (1999) (Panel B). We use estimated parameters as given in Tables 8 and 9. We regress the discounted returns on model generated state variables and model generated state variables plus predictors, respectively. Finally we plot the 90% confidence interval of the difference between the fitted discount returns from the two regressions. Sample: 1952Q2 - 2012Q3.
**Figure 2** An analytical example in which Condition(*) fails - Market returns. Dashed blue lines give the 90% confidence interval of the differences between the estimated values of discounted returns with and without using predictors, at horizon 1-Quarter, 1-Year, 5-Year. The discounted returns are the product of model generated SDFs and real market index returns. Dotted red lines locate the benchmark of zero value. We do 1000 times simulations over 241 quarters of a consumption-based model with i.i.d consumption growth and CRRA utility. In this example, the risk aversion coefficient equals 7.42 and the subjective discounted factor equals 0.9989. Then we regress the discounted returns on model generated state variables and model generated state variables plus predictors, respectively. Finally we plot the 90% confidence interval of the difference between the fitted discount returns from the two regressions. Sample: 1952Q2 - 2012Q3.
Figure 3 Volatility bounds on stochastic discount factors for different investment horizons, SET A. Dashed line gives the volatility bound when no conditional information is used. Solid line gives the volatility bound using conditional information based on Bekaert and Liu’s (2004) specification. The red triangle reports average mean and standard deviation values from 10 simulations run of 600,000 months with estimated parameters. The blue star reports the same objects computed with calibrated parameters. The ellipses (dashed dotted area) show the uncertainty (one standard deviation) in the calculation of the mean and standard deviation of the model implied SDFs. We only account for uncertainty in the parameter values for the state dynamics (i.e. we take as given parameters that characterize the preferences). Long horizon returns are computed by compounding quarterly returns. Sample: 1952Q2 - 2012Q3.
Figure 4: Volatility bounds on stochastic discount factors for different investment horizons, SET A. Solid line gives the volatility bound using conditional information based on Bekaert and Liu’s (2004) specification. Dashed line gives the bootstrapped 90% confidence interval for corresponding variance bound. The red triangle reports average mean and standard deviation values from 10 simulations run of 600,000 months with estimated parameters. While the five angle star reports the same objects computed with calibrated parameters. The dashed dotted area are the one standard deviation confidence ellipses for simulated point with estimated parameters. The ellipses show the sensitivity of simulated SDFs to the estimated value of dynamics parameters. Long horizon returns are computed by compounding quarterly returns. Sample: 1952Q2 - 2012Q3.
Figure 5 Volatility bounds on stochastic discount factors for different investment horizons, SET B. Dashed line gives the volatility bound when no conditional information is used. Solid line gives the volatility bound using conditional information based on Bekaert and Liu’s (2004) specification. The red triangle reports average mean and standard deviation values from 10 simulations run of 600,000 months with estimated parameters. The blue star reports the same objects computed with calibrated parameters. The ellipses (dashed dotted area) show the uncertainty (one standard deviation) in the calculation of the mean and standard deviation of the model implied SDFs. We only account for uncertainty in the parameter values for the state dynamics (i.e. we take as given parameters that characterize the preferences). Long horizon returns are computed by compounding quarterly returns. Sample: 1952Q2 - 2012Q3.
Figure 6 Volatility bounds on stochastic discount factors for different investment horizons, SET B. Solid line gives the volatility bound using conditional information based on Bekaert and Liu’s(2004) specification. Dashed line gives the bootstrapped 90% confidence interval for corresponding variance bound. The red triangle reports average mean and standard deviation values from 10 simulations run of 600,000 months with estimated parameters. While the five angle star reports the same objects computed with calibrated parameters. The dashed dotted area are the one standard deviation confidence ellipses for simulated point with estimated parameters. The ellipses show the sensitivity of simulated SDFs to the estimated value of dynamics parameters. Long horizon returns are computed by compounding quarterly returns. Sample: 1952Q2 - 2012Q3.
Figure 7 Volatility bounds on stochastic discount factors for different investment horizons. Black Dashed line gives the volatility bound when no conditional information is used. Solid red line gives the volatility bound using conditional information by Bekaert and Liu (2004). Solid blue line gives the volatility bound with restriction which implies stock returns are unpredictable. Solid green line gives the volatility bound with restriction that Treasury bonds returns are unpredictable (for SET A is 5-year treasury government bond and for SET B are 5-, 7-, 10-, 20- and 30-year maturity treasury government bonds). The bounds are generated using SET A (see Panel A) and SET B (see Panel B). Long horizon returns are computed by compounding quarterly returns. Sample: 1952Q2 - 2012Q3.
Figure 8 Predictability of excess return by PD-ratio: this figure presents $R^2$s from projecting one-, three-, five- and eight-year excess return of the aggregate stock market portfolio onto lagged price-dividend ratio. Red triangle stands for the results by using annulized real data from 1952Q2 to 2012Q3. Blue, magenta and green triangles presents the results by using model generated simulated data.

<table>
<thead>
<tr>
<th>Asset</th>
<th>Stocks</th>
<th>Bonds</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean return (% p.a.)</td>
<td>11.45</td>
<td>6.31</td>
</tr>
<tr>
<td>Standard deviation (% p.a.)</td>
<td>16.68</td>
<td>5.77</td>
</tr>
</tbody>
</table>

Table 1 This table reports sample statistics of quarterly nominal stock and bond total returns. Stock returns are nominal returns on the stock total returns on the value weighted portfolio of all stocks traded in the NYSE, the AMEX, and NASDAQ from CRSP. Bond returns are nominal returns on the 5-year constant maturity bond from the CRSP Fixed Term Indices File. Sample: 1947Q2: 2012Q3.

Panel A: Predictive regressions for stock returns

<table>
<thead>
<tr>
<th>Horizon h (in quarters)</th>
<th>pd_t [t-stat]</th>
<th>cay_t [t-stat]</th>
<th>R^2(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.03 (1.99)</td>
<td>0.85 (2.93)</td>
<td>4.9</td>
</tr>
<tr>
<td>4</td>
<td>-0.13 (2.37)</td>
<td>3.24 (2.75)</td>
<td>17.4</td>
</tr>
<tr>
<td>20</td>
<td>-0.60 (4.70)</td>
<td>16.23 (4.17)</td>
<td>50.0</td>
</tr>
</tbody>
</table>

Panel B: Predictive regressions for bond returns

<table>
<thead>
<tr>
<th>Horizon h (in quarters)</th>
<th>spr_t [t-stat]</th>
<th>y_t [t-stat]</th>
<th>CP_t [t-stat]</th>
<th>R^2(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7.92 (2.97)</td>
<td>0.76 (1.60)</td>
<td>-0.01 (0.06)</td>
<td>7.1</td>
</tr>
<tr>
<td>4</td>
<td>7.68 (1.66)</td>
<td>2.32 (2.24)</td>
<td>1.41 (4.48)</td>
<td>29.1</td>
</tr>
<tr>
<td>20</td>
<td>12.40 (0.68)</td>
<td>17.97 (4.20)</td>
<td>2.16 (2.42)</td>
<td>44.9</td>
</tr>
</tbody>
</table>

Table 2 Panel A reports quarterly overlapping regressions of multiple horizon real gross stock returns onto a constant, the log price-dividend ratio and cay_t. Panel B reports monthly overlapping regressions of multiple horizon real gross return on a 5-year constant maturity coupon bond from CRSP onto a constant, the log short rate y(t) and the yield spread spr(t). The short rate is the log yield on the 30-day Treasury Bill from CRSP, and the spread is the difference between the log yield on a 5-year artificial zero-coupon bond from the CRSP Fama-Bliss Discount Bond File, and the log yield on the Treasury Bill (T-bill). The table reports coefficient estimates, the R^2 of the regression, and, in brackets, the Newey-West corrected t-statistics. Sample: 1947Q2: 2012Q3.
Table 3 BL Bounds with conditioning information. We compute the variance of the SDFs at different holding horizons, via simulations, respectively, for the models that incorporate long-run risk, external habit persistence, and rare disasters. All estimation are based on model parameters tabulated Table Appendix I - III, and the reported values are the average standard deviation from 10 simulations run of 600,000 month. Then we construct the related 1-quarter, 1-year and 5-year SDFs. We construct Hansen-Jagannathan Bounds at different horizons, using the optimally scaled bounds of Bekaert and Liu (2004), based on the return predictive system (see equation(8)). The quarterly data used in the construction of σ is from 1952Q2 to 2012Q3, with the 90% confidence intervals. To compute the confidence intervals, we create 50,000 random samples of sample size from the data, where the sampling in the block bootstrap is based on the optimal block length we calculated for each asset return regression residuals. Real returns are computed by deflating the nominal returns by the Consumer Price Index inflation. 1-year, 5-year holding returns are computed by compounding related quarterly returns of each asset.
Table 4 BL Bounds with conditioning information. We compute the variance of the SDFs at different holding horizons, via simulations, respectively, for the models that incorporate long-run risk and external habit persistence with both calibrated and estimated parameters which are based on model parameters tabulated in Table Appendix I - III. The reported values are the average values of standard deviation from 10 single simulations run of 600,000 month. Then we construct the related 1-quarter, 1-year and 5-year SDFs. We construct Hansen-Jagannathan Bounds at different horizons, using the optimally scaled bounds of Bekaert and Liu (2004), based on the return predictive system (see equation (8)). The quarterly data used in the construction of $\sigma$ is from 1952Q2 to 2012Q3, with the 90% confidence intervals. To compute the confidence intervals, we create 50,000 random samples of sample size from the data, where the sampling in the block bootstrap is based on the optimal block length we calculated for each asset return regression residuals. Real returns are computed by deflating the nominal returns by the Consumer Price Index inflation. 1-year, 5-year holding returns are computed by compounding related quarterly returns of each asset.
Table 5 Upper bound on the $R^2$ of return predictive regressions. This table presents the upper bound on $R^2$ of returns’ predictive regressions. We compute the upper bound $R^2_{f,t+h}\sigma^2_Z(v_{min})$ of $R^2$, together with the corresponding 90% confidence intervals for SETA and SETB. For the unconditional HJ bounds, in the first two columns, we reported their minimum value at each horizon, 1-quarter, 1-year and 5-year. For the predictor-based bounds, we report $\sigma^2_Z(v_{min})$, i.e. the minimum value of the predictor-based bounds, computed using Bekaert and Liu (2004) specification, again for each horizon. The brackets below the columns of the predictirs-based bounds report the values of the 90% bootstrapped confidence intervals. For the scaled variance of the models $R^2_{f,t+h} \text{Var}(m_{t+h})$, we report our the values for the Bansal, Kiku and Yaron (2012) model and the Campbell and Cochrane(1999) model. The 90% confidence interval of these scaled variances are reported in the brackets. The last two columns report, from Table 2, the predictive regression $R^2$ of stock returns and 5-year constant maturity government bond returns. For the risk free rate, we adopt the mean of the 3-month Tbill returns. 1-year, 5-year holding returns computed compounding related quarterly returns of each asset. Sample: 1952Q2-2012Q3.

<table>
<thead>
<tr>
<th>Horizon</th>
<th>HJ bounds SET A</th>
<th>HJ bounds SET B</th>
<th>predictor-based bounds SET A</th>
<th>predictor-based bounds SET B</th>
<th>Bansal-Kiku-Yaron</th>
<th>Campbell-Cochrane</th>
<th>regression $R^2$ Stock Returns</th>
<th>Bond Returns</th>
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<tbody>
<tr>
<td>1</td>
<td>0.065</td>
<td>0.088</td>
<td>0.166</td>
<td>0.243</td>
<td>0.051</td>
<td>0.083</td>
<td>0.049</td>
<td>0.071</td>
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<tr>
<td></td>
<td>[ 0.145 0.454 ]</td>
<td>[ 0.231 0.626 ]</td>
<td></td>
<td></td>
<td>[ 0.018 0.100 ]</td>
<td>[ 0.079 0.087 ]</td>
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<td>4</td>
<td>0.238</td>
<td>0.494</td>
<td>0.536</td>
<td>0.855</td>
<td>0.227</td>
<td>0.562</td>
<td>0.174</td>
<td>0.291</td>
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<td>[ 0.505 1.304 ]</td>
<td>[ 0.807 2.221 ]</td>
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<td>[ 0.067 0.481 ]</td>
<td>[ 0.502 0.604 ]</td>
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<tr>
<td>20</td>
<td>0.993</td>
<td>4.342</td>
<td>2.129</td>
<td>6.383</td>
<td>2.503</td>
<td>19.703</td>
<td>0.500</td>
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<tr>
<td></td>
<td>[ 1.628 6.549 ]</td>
<td>[ 5.158 13.563 ]</td>
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<td></td>
<td>[ 0.154 7.677 ]</td>
<td>[ 10.364 32.016 ]</td>
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</table>
Table 6 FS Bounds with conditioning information. We construct Hansen-Jagannathan Bounds at different horizon, using Ferson and Siegel (2003) approach, based on the return predictive system (see equation(8)). The quarterly data used in the construction of $\sigma$ is from 1952Q2 to 2012Q3, with the 90% confidence intervals. To compute the confidence intervals, we create 50,000 random samples of sample size from the data, where the sampling in the block bootstrap is based on the optimal block length we calculated for each asset return regression residuals. Real returns are computed by deflating the nominal returns by the Consumer Price Index inflation. 1-year, 5-year holding returns are computed by compounding related quarterly returns of each asset.
Table 7 Predictability of Returns, and Price–Dividend Ratios. This table presents $R^2$s and slope coefficients from projecting one-, three-, five- and eight-year excess return of the aggregate stock market portfolio onto lagged price-dividend ratio. The results correspond to regressing $r_{t+1} + r_{t+2} + \cdots + r_{t+h} = \alpha(h) + \beta(h) \log \left( \frac{P_t}{D_t} \right) + \varepsilon_{t+h}$, where $r_{t+1}$ is the excess return, and $h$ denotes the forecast horizon in year. Data statistics are reported in the “Data” column, sample 1952Q2-2012Q3. The "Bansal-Yaron" column presents predictability evidence implied by Bansal and Yaron’s (2004) model. "Bansal-Kiku-Yaron" column presents predictability evidence implied by the Bansal Kiku and Yaron’s (2012) model. And "Campbell-Cochrane" column presents predictability evidence implied by Campbell and Cochrane’s (1999) model with estimated parameters reported in [9]. The entries for the models are based on 1,000 simulations each with 724 monthly observations that are time-aggregated to an annual frequency. We report the median of simulated slope and mean of simulated $R^2$ for each model. Standard errors which reported in parentheses are Newey-West corrected.

<table>
<thead>
<tr>
<th>Horizon year</th>
<th>Data $\beta$</th>
<th>$R^2$</th>
<th>Bansal-Yaron $\beta$</th>
<th>$R^2$</th>
<th>Bansal-Kiku-Yaron $\beta$</th>
<th>$R^2$</th>
<th>Campbell-Cochrane $\beta$</th>
<th>$R^2$</th>
</tr>
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<tr>
<td>1</td>
<td>-0.115 (0.053)</td>
<td>0.067</td>
<td>-0.047 (0.109)</td>
<td>0.000</td>
<td>-0.080 (0.083)</td>
<td>0.000</td>
<td>-0.210 (0.005)</td>
<td>0.084</td>
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<td>3</td>
<td>-0.266 (0.084)</td>
<td>0.158</td>
<td>-0.150 (0.213)</td>
<td>0.004</td>
<td>-0.229 (0.173)</td>
<td>0.028</td>
<td>-0.554 (0.011)</td>
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<td>5</td>
<td>-0.339 (0.076)</td>
<td>0.193</td>
<td>-0.244 (0.279)</td>
<td>0.018</td>
<td>-0.375 (0.227)</td>
<td>0.059</td>
<td>-0.801 (0.015)</td>
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<td>8</td>
<td>-0.460 (0.107)</td>
<td>0.247</td>
<td>-0.390 (0.318)</td>
<td>0.030</td>
<td>-0.547 (0.268)</td>
<td>0.091</td>
<td>-1.045 (0.020)</td>
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<td>BKY2012b estimation</td>
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<td>Time preference</td>
<td>$\delta$ 0.9989</td>
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<td>EIS</td>
<td>$\psi$ 1.5</td>
<td>2.05 (0.84)</td>
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<tr>
<td><strong>Consumption growth dynamics</strong>, $g_t$</td>
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<td></td>
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<tr>
<td>Mean</td>
<td>$\mu$ 0.0015</td>
<td>0.0012 (0.0007)</td>
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<td><strong>Dividends growth dynamics</strong>, $g_{d,t}$</td>
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<tr>
<td>Mean</td>
<td>$\mu_d$ 0.0015</td>
<td>0.0020 (0.0017)</td>
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<td>4.45 (1.63)</td>
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<td>$\varphi_d$ 5.96</td>
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<tr>
<td>Correlation between innovations</td>
<td>$\rho_{dc}$ 0.49</td>
<td>0.49 (0.33)</td>
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<tr>
<td>Persistence</td>
<td>$\rho$ 0.975</td>
<td>0.9812 (0.0086)</td>
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<tr>
<td>Volatility parameter</td>
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<td>0.0306 (0.0160)</td>
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<td><strong>Consumption growth volatility</strong>, $\sigma_t$</td>
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<tr>
<td>Mean</td>
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<tr>
<td>Persistence</td>
<td>$v$ 0.999</td>
<td>0.9983 (0.0021)</td>
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<tr>
<td>Volatility parameter</td>
<td>$\sigma_w$ $2.8 \times 10^{-5}$</td>
<td>$2.62 \times 10^{-5}$ (3.10$\times 10^{-6}$)</td>
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<td></td>
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</tbody>
</table>

Table 8 Appendix-I: Parametrization of asset pricing models incorporating long-run risk. Our parameterizations of calibrated values are taken from Bansal et al. (2012a) Table 1. The estimated parameters are from Bansal et al. (2012b) Table II and standard errors of estimation are reported in parentheses. The models are simulated at the monthly frequency.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Campbell-Cochrane calibration</th>
<th>Campbell-Cochrane estimation</th>
</tr>
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<tbody>
<tr>
<td>Preference</td>
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<tr>
<td>Time preference</td>
<td>$\delta$</td>
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<tr>
<td>Risk aversion</td>
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<tr>
<td>Consumption growth dynamics, $g_t$</td>
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<tr>
<td>Mean</td>
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<tr>
<td>Volatility parameter</td>
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<tr>
<td>Dividend growth dynamics, $\Delta d_t$</td>
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<tr>
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<td>Corr between innovations</td>
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<tr>
<td>Steady state surplus consumption ratio</td>
<td>$\bar{S}$</td>
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<tr>
<td>Persistence in consumption surplus ratio</td>
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<tr>
<td>Log of risk-free rate</td>
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Table 9 Appendix-II: Parametrization of asset pricing models incorporating external habit persistence. Our parameterizations of calibrated values are taken from Campbell and Cochrane (1999). The estimated values for Campbell-Cochrane model are from (Aldrich and Gallant 2011) and standard errors of estimation are reported in parentheses. The models are simulated at the monthly frequency.