Optimal partial hedging of options with small transaction costs

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Abstract

We use asymptotic analysis to derive the optimal hedging strategy for an option portfolio hedged using an imperfectly correlated hedging asset with small transaction costs, both fixed per trade and proportional to the value traded. In special cases we opbtain explicit formulae. The hedging strategy involves holding a position in the hedging asset whose *value* lies between two bounds, which are independent of the value of the hedging asset itself, as is the option value. The bounds depend on the leading order Gamma and Delta of the option position. Decreasing absolute correlation decreases the absolute value of the centre of the no-transaction band, the no-transaction bandwidth and the certainty equivalent value of transaction costs. However generally the increase in unhedgeable risk has a greater effect on option values, which decrease with $|\rho|$.

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1 Introduction

Dynamic hedging of derivative securities written on a portfolio of assets, basket options, often proves impractical because of the probibitive associated cost of transacting in each asset. One alternative is to hedge the derivatives using an imperfectly correlated hedging asset with lower associated transaction costs.

In this paper we use asymptotic analysis[1] to derive formulae for the hedging strategy in such a situation, taking account of both the transaction costs and the imperfect correlation. We also derive the equations satisfied by the resulting certainty equivalent value of the derivatives. We find that both hedging strategy and option values are independent of the current value of the hedging asset, so the equations have the same dimension as in the case of perfect correlation. However with imperfect correlation the formulae describing the hedging strategy depend on the Delta of the derivative position as well as its Gamma, and the leading order option value in the absence of transaction costs is no longer the Black-Scholes value, but the nonlinear certainty equivalent value for costless imperfect hedging derived by Henderson (2005) [13].

This work extends two lines of literature in the valuation of financial options: hedging under transaction costs and hedging with an imperfectly correlated hedging asset. The incorporation of transaction costs into the valuation of financial options and the associated hedging strategy has been extensively studied. Leland (1985) [18] first recognised that hedging options using the Black-Scholes delta-hedging strategy in the presence of non-zero transaction costs led to potentially unbounded costs. He proposed a discrete hedging strategy of transacting at fixed points in time and derived the value of plain-vanilla European options under this hedging strategy in closed form. This was extended to value portfolios of options by Hoggard, Whalley & Wilmott (1992) [17]; other models incorporating exogenous hedging

[1] We exploit the generally small magnitude of transaction costs

strategies include Henrotte (1993) [14], Whalley & Wilmott (1993) [24], Grannan & Swindle (1996) [12] and Avellaneda & Paras (1993) [2] and, using a binomial framework, Boyle & Vorst (1992) [5], Bensaid, Lesne, Pages & Schienkman (1992) [3] and Edirisinghe, Naik & Uppal (1993) [11].

Hodges & Neuberger (1989) [16] were the first to derive an *optimal* hedging strategy and associated reservation value (certainty equivalent value) for options incorporating transaction costs in a utility-based framework. They found that the optimal hedging strategy involves maintaining the number of the underlying asset held within an endogenously determined no-transaction band, transacting only when the edge of the band is reached. Davis, Panas & Zariphopoulou (1993) [9], Hodges & Clewlow (1997) [6], Andersen & Damgaard (1999) [1], Damgaard (2003, 2000) [7], [8], Ho (2003) [15], Monoyios (2004) [19] Zakamouline (2003, 2004, 2004b) [27], [28], [29], and Subramanian (2005) [22] considered similar models and Whalley & Wilmott (1997, 1999) [25], [26] found formulae for the hedging strategy and less computationally demanding approximations to the option values for small cost levels using asymptotic analysis.

All the above papers consider the case where the asset used for hedging purposes is identical to (perfectly correlated with) the asset underlying the option portfolio. Thus in theory with zero transaction costs and continuous transacting, perfect hedging (complete elimination of the risk associated with the hedging strategy) is possible. We extend this to allow the asset used for hedging purposes to differ from (be imperfectly correlated with) the asset underlying the derivatives contract. In this case the minimum possible hedging error, even with zsero transaction costs, is strictly positive, and this reduces certainty equivalent option values.

The hedging strategy for and valuation of options hedged costlessly using an imperfectly correlated hedging asset were derived by Henderson (2005) [13], who applied the model to executive stock option valuation. She derived a non-linear partial differential equation for the certainty equivalent option value and found that option values always lie strictly below the Black-Scholes value. Other analyses of costless hedging with an imperfectly correlated asset include Musiela & Zariphopoulou (2004) and Monoyios (2004b) [20], who considered optimal and 'naive' suboptimal[2] hedging of basket options using imperfectly correlated index futures as the hedging asset and used asymptotic analysis (assuming the effect of the imperfect hedging on the option value was small) to derive closed form solutions for the optimal hedging strategy. However, neither Henderson nor Monoyios incorporated transaction costs into their analysis.

We recover earlier results as limiting cases of our analysis: in the limit as transaction costs tend to zero we obtain the partial differential equation in Henderson (2005) [13]. and for perfectly correlated hedging assets we recover results given in Whalley & Wilmott (1997, 1999) ([25], [26]). However, for the case of proportional transaction costs we extend the asymptotic analysis of the effects of transaction costs to higher order so the extended asymptotic scheme now also has the characteristic features of transaction cost model hedging strategies: the centre of the hedging band differs from the costlessly hedged option delta systematically. For perfectly correlated hedging assets the difference depends on the sign of the option's Gamma; more generally it depends on both the Delta and Gamma of the option position.

We find that both increasing transaction costs and decreasing absolute correlation decrease option values. Intuitively, decreasing absolute correlation increases the minimum possible hedging error (with costless hedging); increasing transaction cost levels increases the optimal hedging error further by widening the no-transaction band, as well as increasing expected total transaction costs. Increased undiversifiedrisk or hedging error in the investor's portfolio reduces their utility and hence the certainty equivalent option value. The relative magnitude of the effect of each depends on the level of risk aversion, the size of the option position, the correlation between returns on the

[2] *i.e.* using a time-based strategy of hedging to the Black-Scholes delta

hedged and hedging assets and the level of transaction costs.

In the presence of an imperfectly correlated hedging asset, the optimal hedging strategy changes from holding a *number* of the underlying asset which lies within a no transaction band centred (to leading order) on the option's delta to having a position in the hedging asset, the *value* of which lies between two bounds. The hedging band is thus defined in terms of the value held in the hedging asset and is centred to leading order on the value which would be held in the hedging asset in the absence of transaction costs. Decreasing absolute correlation decreases both the absolute value of the centre of the band and also its width; this leads to a reduction in the certainty equivalent value of transaction costs. However for the parameter values we consider the decrease in option values due to increasing undiversifiable risk dominates any increase in value due to a reduction in the certainty equivalent transaction cost value, so option values decrease with absolute correlation. Finally, we find that the nonlinearlity of the problem means portfolio effects are significant, so in particular the certainty equivalent value per option (and also the marginal value of an option) depends on the size and sign of the option portfolio as a whole. The magnitude of the effect of the nonlinearity depends on the absolute value of the Delta, and thus also differs between calls and puts.

In section 2 we set out the optimisation problem, derive the partial differential equations and associated boundary conditions for the certainty equivalent option values, and perform asymptotic analysis to simplify their solution. Section 3 considers special cases where explicit formulae are obtainable for the hedging strategy, hence simplifying the partial differential equations for the leading order terms in the asymptotic expansion of the option value. Section 4 investigates the relative importance and sensitivity of option prices to the correlation of hedging and hedged assets and size of the option position. Section 5 concludes and considers further work.

2 The model

We consider a risk-averse investor who can trade in a risky asset, M, and a riskless asset, B, and derive his optimal trading strategy. Transfers between the risky (hedging) asset, M, and the riskless asset, B, incur a cost with two components: a fixed amount per trade and a cost proportional to the value traded

$$k(M, dy) = k_f + k_p M|dy|$$

where dy is the number of hedging assets traded. Thus M and B evolve respectively as:

$$dM = (r + \lambda\sigma)Mdt + \sigma MdZ$$

$$dB = rB - Mdy - k(M, dy)$$

with $\lambda = \frac{\mu_M - r}{\sigma}$ the Sharpe ratio of the hedging asset, M.

The investor will value any option position by its certainty equivalent value assuming optimal hedging of his portfolio including the option position. The hedging strategy for the option position can be represented by the change in the trading strategy resulting from the option position.

We assume the investor holds an option position written on an asset, S, whose returns are imperfectly correlated with M.

$$dS = (r + \xi\eta)Sdt + \eta SdZ_S$$

with $\xi = \frac{\mu_S - r}{\eta}$ similarly the Sharpe ratio of the hedged asset, S, and where $dZdZ_S = \rho dt$ with $|\rho| \leq 1$.

We further assume the investor does not[3] trade in S, but may partially hedge the risk associated with the option position by trading in the hedging asset, M, and consider only European-style options.

[3]Reasons for not hedging the option position with the underlying asset include impracticality (due to the number and cost of transactions, particularly if S is composed of a number of thinly-traded or high transaction cost assets) or regulation (as in the case of executive stock options, whose holders are prohibited from trading in the underlying shares of the firms they manage). See Whalley (2006) [23] for an application to executive stock options.

The investor thus maximises his utility of terminal wealth, solving

$$J(S,M,B,t,y) = \sup_{dy_s, t\le s\le T} E_t\left[U(W_T+\Lambda(S_T))\right] \tag{1}$$

where $\Lambda(S)$ represents the cash payoff to the option position[4], *e.g.* for n call options

$$\Lambda(S) = n\max(S-K,0)$$

and terminal wealth, W_T, consists of holdings of B_T in the riskless asset and a number y_T of the risky asset and is assumed to be net of transaction costs incurred in liquidating the risky asset position

$$W_T = B_T + M_T y_T - k(M_T, y_T).$$

Expanding (1) we have

$$\begin{aligned}
&J(S,M,B,t,y) = \\
&\quad = \sup_{dy_s, t\le s\le T} E_t\left[E_{t+dt}\left[U(W_T+\Lambda(S_T))\right]\right] \\
&\quad = \sup_{dy_t} E_t\left[J(S+dS, M+dM, B+dB, t+dt, y+dy)\right]
\end{aligned}$$

Applying It$\hat{o}$'s lemma gives the differential equation satisfied by J:

$$\begin{aligned}
0 \;=\; & +J_t + (r+\xi\eta)SJ_S + (r+\lambda\sigma)MJ_M + rBJ_B \\
& + \frac{\eta^2}{2}S^2 J_{SS} + \rho\eta\sigma SMJ_{SM} + \frac{\sigma^2}{2}M^2 J_{MM} \\
& + \sup_{dy}\left\{J(S,M,B-Mdy-k(M,dy),t,y+dy) - J(S,M,B,t,y)\right\}
\end{aligned} \tag{2}$$

As in [16], [9], [25], we find that there are three regions:

- a **no transaction** region, within which y, the number of units of the market held, remains constant. Thus $dy^* = 0$ and J satisfies:

$$\begin{aligned}
0 \;=\; & J_t + (r+\xi\eta)SJ_S + (r+\lambda\sigma)MJ_M + rBJ_B \\
& + \frac{\eta^2}{2}S^2 J_{SS} + \rho\eta\sigma SMJ_{SM} + \frac{\sigma^2}{2}M^2 J_{MM}
\end{aligned} \tag{3}$$

  In this region the decrease in utility from holding a suboptimal number of units of the market is lower than the marginal utility

[4] We assume cash settlement of the option position

loss arising from the transaction costs of adjusting the position

$$
\begin{aligned}
& J(S, M, B - M dy - (k_f + k_p M|dy|), t, y + dy) \\
< \quad & J(S, M, B, t, y) \qquad\qquad\qquad\qquad \forall\ dy \neq 0
\end{aligned} \tag{4}
$$

Thus y is close to its "optimal" value, $y^*(S, M, t)$, *i.e.* the number of units of the market which would be held in the absence of transaction costs. Writing the edges of the no-transaction region as y^+ and y^- respectively, we thus have

$$
y^-(S, M, B, t) \leq y \leq y^+(S, M, B, t)
$$

- a **buy** region in which $dy^* > 0$ and

  $$J(S, M, B - M(1 + k_p)dy^* - k_f, t, y + dy^*) > J(S, M, B, t, y) \tag{5}$$

  holds. Utility is maximised by choosing dy as large as possible (as long as (S_t, M_t, t) remains in this region) Hence if (S_t, M_t, t) moves into this region where $y < y^-$, a transaction is made (y is increased) until y lies within the no transaction region again.

- a **sell** region in which $dy^* < 0$ and

  $$J(S, M, B - M(1 - k_p)dy^* - k_f, t, y + dy^*) > J(S, M, B, t, y) \tag{6}$$

  holds. Utility is maximised by choosing dy as negative as possible (as long as (S_t, M_t, t) remains in this region) Again, if (S_t, M_t, t) moves into this region where $y > y^+$, a transaction is made (y is decreased) until y lies within the no transaction region again.

For the case of negative exponential utility, $U(x) = \frac{1}{\gamma}e^{-\gamma x}$ the solution is independent of the investor's initial wealth, which has the effect of reducing the dimension of the problem by one. We thus look for a solution of the form

$$
J(S, M, B, t, y) = -\frac{1}{\gamma} e^{-\hat{\gamma}(t)(B + h^w(S,M,t,y))}
$$

where $\hat{\gamma}(t) = \gamma e^{r(T-t)}$, for the problem including the option position and similarly $J^0(M, B, t, y) = -\frac{1}{\gamma}e^{-\hat{\gamma}(t)(B+h^0(M,t,y))}$ for the portfolio without. h^w and h^0 thus represent certainty equivalent values of the trading strategy net of transaction costs for the portfolios including and excluding the option position respectively.

h^w satisfies:

$$\begin{aligned} 0 &= h^w_t - rh^w + (r+\xi\eta)Sh^w_S + (r+\lambda\sigma)Mh^w_M \\ &\quad + \frac{\eta^2}{2}S^2(h^w_{SS} - \hat{\gamma}(t)h^{w2}_S) + \rho\eta\sigma SM(h^w_{SM} - \hat{\gamma}(t)h^w_S h^w_M) \\ &\quad + \frac{\sigma^2}{2}M^2(h^w_{MM} - \hat{\gamma}(t){h^w_M}^2) \end{aligned} \tag{7}$$

where $\hat{\gamma}(t) = \gamma e^{r(T-t)}$ and subject to:

$$h^w(S,M,t,y+dy) \leq h^w(S,M,t,y) + Mdy + k(M,dy) \quad \forall\ dy \neq 0$$

$$h^w(S,M,T,y) = \Lambda(S,M) + h^0(M,T,y)$$

Similarly the certainty equivalent value for the non-option problem, h^0, satisfies:

$$0 = h^0_t - rh^0 + (r+\lambda\sigma)Mh^0_M + \frac{\sigma^2}{2}M^2(h^0_{MM} - \hat{\gamma}(t){h^0_M}^2) \tag{8}$$

subject to:

$$h^0(M,t,y+dy) \leq h^0(M,t,y) + Mdy + k(M,dy) \quad \forall\ dy \neq 0$$

$$h^0(M,T,y) = yM - kM|y|$$

These non-linear three or four dimensional problems must be solved numerically. We can, however, simplify them by exploiting the fact that transaction costs, $k \ll 1$ are generally very small. We thus expand in powers of k using an asymptotic expansion in order to obtain the leading order behaviour.

2.1 Asymptotic expansion

We follow [25] and [26] in expanding the value function, h, and the width of the no-transaction band, $y^+ - y^-$ as asymptotic expansions in fractional powers of a small parameter, $\epsilon \ll 1$, which we take to represent the transaction cost magnitude.

$$k(S, M, u) = \epsilon\kappa(S, M, U)$$

with $\kappa = O(1)$, in changing variables from (S, M, t, y) to (S, M, t, Y) where Y represents the 'unhedgedness' of the portfolio, *i.e.* the difference between the actual number of units of the market held, y and the "optimal" number in the absence of transaction costs, $y^*(S, M, t)$, and in finding that the width of the no transaction band is $O(\epsilon^{\frac{1}{4}})$ so

$$y = y^*(S, M, t) + \epsilon^{\frac{1}{4}} Y.$$

We denote by Y^+ and $-Y^-$ the sell and buy edges of the no-transaction band respectively in terms of the new unhedgedness variable, Y, and by $\hat{Y}^+ (< Y^+)$ and $-\hat{Y}^- (> -Y^-)$ the optimal rebalance points (levels of unhedgedness after transaction) on sale and purchase respectively.

We thus expand the value functions for the part of the investor's portfolio held in risky assets, h^w and h^0, in powers of $\epsilon^{\frac{1}{4}}$:

$$\begin{aligned} h^w &= M(y^*(S, M, t) + \epsilon^{\frac{1}{4}} Y) + h_0^w(S, M, t, Y) \\ &\quad + \epsilon^{\frac{1}{4}} h_1^w(S, M, t, Y) + \epsilon^{\frac{1}{2}} h_2^w(S, M, t, Y) + \epsilon^{\frac{3}{4}} h_3^w(S, M, t, Y) + \ldots \\ h^0 &= M(y_0^*(S, M, t) + \epsilon^{\frac{1}{4}} Y) + h_0^0(S, M, t, Y) \\ &\quad + \epsilon^{\frac{1}{4}} h_1^0(S, M, t, Y) + \epsilon^{\frac{1}{2}} h_2^0(S, M, t, Y) + \epsilon^{\frac{3}{4}} h_3^0(S, M, t, Y) + \ldots \end{aligned}$$

Here the first two terms in each equation represent the value of the holding in the hedging asset, $x = My$, so the certainty equivalent value of the option is given by the difference between the remaining terms

$$H(S, M, t, y) = \sum_i (h_i^w(S, M, t, y) - h_i^0(M, t, y)).$$

We consider successive orders of magnitude of the terms obtained on substitution into the differential equations [5] and boundary condi-

[5]Note since the derivatives in (7) and (8) are keeping y fixed, this change of variables

alters the differential equation, since

$$\frac{\partial}{\partial y} \to \epsilon^{-\frac{1}{4}} \frac{\partial}{\partial Y},$$

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - \epsilon^{-\frac{1}{4}} y_t^* \frac{\partial}{\partial Y}, \qquad \frac{\partial}{\partial M} \to \frac{\partial}{\partial M} - \epsilon^{-\frac{1}{4}} y_M^* \frac{\partial}{\partial Y}, \qquad \frac{\partial}{\partial S} \to \epsilon^{-\frac{1}{4}} y_S^* \frac{\partial}{\partial Y}.$$

$$\frac{\partial^2}{\partial S^2} \to \frac{\partial^2}{\partial S^2} - \epsilon^{-\frac{1}{4}} \left(y_{SS}^* \frac{\partial}{\partial Y} + 2y_S^* \frac{\partial^2}{\partial Y \partial S} \right) + \epsilon^{-\frac{1}{2}} {y^*}_S^2 \frac{\partial^2}{\partial Y^2},$$

$$\frac{\partial^2}{\partial S \partial M} \to \frac{\partial^2}{\partial S \partial M} - \epsilon^{-\frac{1}{4}} \left(y_{SM}^* \frac{\partial}{\partial Y} + y_S^* \frac{\partial^2}{\partial Y \partial M} + y_M^* \frac{\partial^2}{\partial Y \partial S} \right) + \epsilon^{-\frac{1}{2}} y_S^* y_M^* \frac{\partial^2}{\partial Y^2},$$

$$\frac{\partial^2}{\partial M^2} \to \frac{\partial^2}{\partial M^2} - \epsilon^{-\frac{1}{4}} \left(y_{MM}^* \frac{\partial}{\partial Y} + 2y_M^* \frac{\partial^2}{\partial Y \partial M} \right) + \epsilon^{-\frac{1}{2}} {y^*}_M^2 \frac{\partial^2}{\partial Y^2},$$

These are reflected in the equations we obtain at successive orders of $\epsilon^{\frac{1}{4}}$.

tions[6]. The problems satisfied by h^w and h^0 differ only in the final boundary conditions. We will thus work with h; differences between the solutions for h^w and h^0 will be discussed later.

[6]The free boundary conditions between the transaction (buy and sell) and no-transaction regions become

$$\begin{aligned} & h_0^w(S,M,t,\pm\hat{Y}^\pm) + \epsilon^{\frac{1}{4}} h_1^w(S,M,t,\pm\hat{Y}^\pm) + \epsilon^{\frac{1}{2}} h_2^w(S,M,t,\pm\hat{Y}^\pm) + \epsilon^{\frac{3}{4}} h_3^w(S,M,t,\pm\hat{Y}^\pm) \\ & \quad +\epsilon h_4^w(S,M,t,\pm\hat{Y}^\pm) + \ldots \qquad (9) \\ = & \quad h_0^w(S,M,t,\pm Y^\pm) + \epsilon^{\frac{1}{4}} h_1^w(S,M,t,\pm Y^\pm) + \epsilon^{\frac{1}{2}} h_2^w(S,M,t,\pm Y^\pm) \\ & \quad +\epsilon^{\frac{3}{4}} h_3^w(S,M,t,\pm Y^\pm) + \epsilon h_4^w(S,M,t,\pm Y^\pm) + \ldots + \epsilon(\kappa_f + \kappa_p M| \pm Y^\pm - (\pm\hat{Y}^\pm)| \end{aligned}$$

$$\begin{aligned} & h_{0_Y}^w(S,M,t,\pm\hat{Y}^\pm) + \epsilon^{\frac{1}{4}} h_{1_Y}^w(S,M,t,\pm\hat{Y}^\pm) + \epsilon^{\frac{1}{2}} h_{2_Y}^w(S,M,t,\pm\hat{Y}^\pm) \qquad (10) \\ & \quad +\epsilon^{\frac{3}{4}} h_{3_Y}^w(S,M,t,\pm\hat{Y}^\pm) + \epsilon h_{4_Y}^w(S,M,t,\pm\hat{Y}^\pm) + \ldots = \epsilon\kappa_p M sgn(\pm Y^\pm - (\pm\hat{Y}^\pm)) \\ & h_{0_Y}^w(S,M,t,\pm Y^\pm) + \epsilon^{\frac{1}{4}} h_{1_Y}^w(S,M,t,\pm Y^\pm) + \epsilon^{\frac{1}{2}} h_{2_Y}^w(S,M,t,\pm Y^\pm) \qquad (11) \\ & \quad +\epsilon^{\frac{3}{4}} h_{3_Y}^w(S,M,t,\pm Y^\pm) + \epsilon h_{4_Y}^w(S,M,t,\pm Y^\pm) + \ldots = \epsilon\kappa_p M sgn(\pm Y^\pm - (\pm\hat{Y}^\pm)) \end{aligned}$$

where

$$sgn(x) = \begin{cases} +1 & \text{if} x > 0 \\ -1 & \text{if} x < 0 \end{cases}$$

and the smooth pasting conditions (11) and (10) represent the optimality of the choice of the edge of the no-transaction region and optimal rebalance point respectively. Note since the hedging bandwidth is $O(\epsilon^{\frac{1}{4}})$, transaction costs are

$$k(M,dy) = k_f + k_p M|dy| = \epsilon\kappa(M,dY) = \epsilon\kappa_f + (\epsilon^{\frac{3}{4}}\kappa_p)M|\epsilon^{\frac{1}{4}}dY|$$

so $k_f = \epsilon\kappa_f$ and $k_p = \epsilon^{\frac{3}{4}}\kappa_p$. For the case of purely proportional transaction costs, the edge of the no-transaction region and the optimal rebalance point coincide and the value matching and smooth pasting conditions become:

$$\begin{aligned} & h_{0_Y}^w(S,M,t,\pm Y^\pm) + \epsilon^{\frac{1}{4}} h_{1_Y}^w(S,M,t,\pm Y^\pm) + \epsilon^{\frac{1}{2}} h_{2_Y}^w(S,M,t,\pm Y^\pm) \\ & \qquad +\epsilon^{\frac{3}{4}} h_{3_Y}^w(S,M,t,\pm Y^\pm) + \epsilon h_{4_Y}^w(S,M,t,\pm Y^\pm) = -\epsilon\kappa_p M \qquad (12) \\ & h_{0_{YY}}^w(S,M,t,\pm Y^\pm) + \epsilon^{\frac{1}{4}} h_{1_{YY}}^w(S,M,t,\pm Y^\pm) + \epsilon^{\frac{1}{2}} h_{2_{YY}}^w(S,M,t,\pm Y^\pm) \\ & \qquad +\epsilon^{\frac{3}{4}} h_{3_{YY}}^w(S,M,t,\pm Y^\pm) + \epsilon h_{4_{YY}}^w(S,M,t,\pm Y^\pm) = 0 \qquad (13) \end{aligned}$$

Considering each successive power in $\epsilon^{i/4}$ separately gives the respective boundary conditions for h_i.

2.1.1 The $O(\epsilon^{-\frac{1}{2}})$ equation

The leading order terms in the equation are at $O(\epsilon^{-\frac{1}{2}})$ and involve derivatives with respect to the level of unhedgedness, Y:

$$\mathcal{A}(y^*)\left(h_{0_{YY}} - \hat{\gamma}(t)h_{0_Y}^2\right) = 0 \tag{14}$$

subject to

$$h_{0_Y}(Y^+) = h_{0_Y}(-Y^-) = 0$$

$$h_{0_{YY}}(Y^+) = h_{0_{YY}}(-Y^-) = 0$$

$$h_0^0(M, Y, T) = 0; \qquad h_0^w(S, M, Y, T) = \Lambda(S, M)$$

where

$$\mathcal{A}(y^*) \equiv \frac{1}{2}\eta^2 S^2 y_S^{*^2} + \rho\sigma\eta SMy_S^* y_M^* + \frac{1}{2}\sigma^2 M^2 y_M^{*^2}. \tag{15}$$

This has solution $h_{0_Y} = 0$, so to leading order the option value is independent of the current level of unhedgedness and $h_0 \equiv h_0(S, M, t)$.

2.1.2 The $O(\epsilon^{-\frac{1}{4}})$ equation

Considering the terms at $O(\epsilon^{-\frac{1}{4}})$ and using $h_{0_Y} = 0$, we find an equation for the dependence of h_1 on Y:

$$\mathcal{A}(y^*)h_{1_{YY}} = 0 \tag{16}$$

which can be solved subject to

$$h_{1_Y}(Y^+) = h_{1_Y}(-Y^-) = 0$$

$$h_{1_{YY}}(Y^+) = h_{1_{YY}}(-Y^-) = 0$$

$$h_1(S, M, Y, T) = 0$$

to give $h_{1_Y} = 0$, so $h_1 \equiv h_1(S, M, t)$.

2.1.3 The $O(1)$ equation

Now considering the $O(1)$ terms and using $h_{0_Y} = h_{1_Y} = 0$ we obtain

$$\mathcal{A}(y^*)h_{2_{YY}} + \sigma M y^* \left(\lambda - \hat{\gamma}(t)\left(\sigma M\left(h_{0_M} + \frac{y^*}{2}\right) + \rho\eta S h_{0_S}\right)\right)$$
$$+\mathcal{L}(h_0) - \hat{\gamma(t)}\left(\frac{\eta^2}{2}S^2 h_{0_S}^2 + \rho\eta\sigma S M h_{0_S} h_{0_M} + \frac{\sigma^2}{2}M^2 h_{0_M}^2\right) = 0$$

where

$$\begin{aligned}\mathcal{L}(h) \quad \equiv \quad & h_t - rh + (r+\xi\eta)Sh_S + (r+\lambda\sigma)Mh_M \\ & + \frac{1}{2}\eta^2 S^2 h_{SS} + \rho\sigma\eta S M h_{SM} + \frac{1}{2}\sigma^2 M^2 h_{MM}. \end{aligned} \tag{17}$$

Regarding this as an ordinary differential equation in Y for h_2, which can be solved subject to

$$h_{2_Y}(Y^+) = h_{2_Y}(-Y^-) = 0$$

$$h_{2_{YY}}(Y^+) = h_{2_{YY}}(-Y^-) = 0$$

we similarly find $h_{2_Y} = 0$, so $h_2 \equiv h_2(S, M, t)$.

2.1.4 The $O(\epsilon^{\frac{1}{4}})$ equation

We can similarly view the $O(\epsilon^{\frac{1}{4}})$ terms as an ordinary differential equation in Y for h_3:

$$\begin{aligned}&\mathcal{A}(y^*)h_{3_{YY}} + \mathcal{L}(h_1) + \sigma M Y\left(\lambda - \hat{\gamma}(t)\left(\sigma M\left(h_{0_M} + y^*\right) + \rho\eta S h_{0_S}\right)\right) \\ &- \hat{\gamma}(t)\left(\eta^2 S^2 h_{0_S} h_{1_S} + \rho\eta\sigma S M(h_{0_S} h_{1_M} + h_{1_S} h_{0_M}) + \sigma^2 M^2 h_{0_M} h_{1_M}\right) \\ &\quad - \hat{\gamma}(t) y^*\left(\sigma^2 M^2 h_{1_M} + \rho\eta\sigma S M h_{1_S}\right) = 0\end{aligned} \tag{18}$$

subject to

$$h_{3_Y}(Y^+) = h_{3_Y}(-Y^-) = 0$$

$$h_{3_{YY}}(Y^+) = h_{3_{YY}}(-Y^-) = 0$$

and obtain $h_{3_Y} = 0$, so $h_3 \equiv h_3(S, M, t)$.

Hence both the coefficient of Y and the terms independent of Y must seperately equal zero. From the coefficient of Y we obtain an equation for $My^*(S,M,t)$, the amount invested in the market to leading order:

$$My^*(S,M,t) = \frac{\lambda}{\sigma\hat{\gamma}(t)} - \frac{\rho\eta S}{\sigma} h_{0_S} - M h_{0_M} \tag{19}$$

and from the terms independent of Y we obtain the differential equation satisfied by $h_1(S,M,t)$:

$$\begin{aligned} 0 &= \mathcal{L}(h_1) - \hat{\gamma}(t)\left(\left(\frac{\lambda}{\hat{\gamma}(t)} - \sigma M h_{1_M} - \rho\eta S h_{0_S}\right)(\sigma M h_{1_M} + \rho\eta S h_{1_S})\right) \\ &- \hat{\gamma}(t)\left(\eta^2 S^2 h_{0_S} h_{1_S} + \rho\eta\sigma SM(h_{0_S} h_{1_M} + h_{1_S} h_{0_M}) + \sigma^2 M^2 h_{0_M} h_{1_M}\right) \end{aligned}$$

Solving this subject to $h_1(S,M,T) = 0$ we find $h_1 = 0$.

Returning to the $O(1)$ equation and substituting for y^* from (19) we obtain the equation satisfied by the leading order option value, h_0:

$$\begin{aligned} &\mathcal{L}(h_0) + \frac{\hat{\gamma}(t)}{2}\left((\frac{\lambda}{\hat{\gamma}(t)} - (\sigma M h_{0_M} + \rho\eta S h_{0_S})\right)^2 \\ &\quad - \hat{\gamma(t)}\left(\frac{\eta^2}{2} S^2 h_{0_S}^2 + \rho\eta\sigma SM h_{0_S} h_{0_M} + \frac{\sigma^2}{2} M^2 h_{0_M}^2\right) = 0 \end{aligned}$$

This reduces to

$$\begin{aligned} &h_{0_t} + rMh_{0_M} + rSh_{0_S} - rh_0 + \frac{\sigma^2}{2} M^2 h_{0_{MM}} + \rho\eta\sigma SM h_{0_{SM}} \\ &\quad + \frac{\eta^2}{2} S^2 h_{0_{SS}} - \hat{\gamma}(t)\frac{\eta^2}{2}(1-\rho^2) S^2 h_{0_S}^2 = -\frac{\lambda^2}{2\hat{\gamma}(t)} \end{aligned} \tag{20}$$

which must be solved subject to $h_0^w(S,M,T) = \Lambda(S,M)$ or $h_0^0(S,M,T) = 0$ as appropriate.

2.1.5 The $O(\epsilon^{\frac{1}{2}})$ equation

The $O(\epsilon^{\frac{1}{2}})$ terms are

$$\begin{aligned} &\mathcal{A}(y^*) h_{4_{YY}} - \frac{\hat{\gamma}(t)\sigma^2}{2} M^2 Y^2 + \mathcal{L}(h_2) \\ &- \hat{\gamma}(t)\left(\eta^2 S^2 h_{0_S} h_{2_S} + \rho\eta\sigma SM(h_{0_S} h_{2_M} + h_{2_S} h_{0_M}) + \sigma^2 M^2 h_{0_M} h_{2_M}\right) \\ &\quad - \hat{\gamma}(t) y^*\left(\sigma^2 M^2 h_{2_M} + \rho\eta\sigma SM h_{2_S})\right) = 0 \end{aligned} \tag{21}$$

These give us an ordinary differential equation in Y for h_4

$$h_{4_{YY}} = \alpha_4 Y^2 - \beta_4 \tag{22}$$

where

$$\alpha_4 = \frac{\hat{\gamma}(t)\sigma^2 M^2}{2\mathcal{A}(y^*)}$$

and

$$\beta_4 = \frac{1}{\mathcal{A}(y^*)}\Big(\mathcal{L}(h_2) - \hat{\gamma}(t)y^*\left(\sigma^2 M^2 h_{2_M} + \rho\eta\sigma S M h_{2_S}\right)$$
$$-\hat{\gamma}(t)\left(\eta^2 S^2 h_{0_S} h_{2_S} + \rho\eta\sigma S M(h_{0_S} h_{2_M} + h_{2_S} h_{0_M}) + \sigma^2 M^2 h_{0_M} h_{2_M}\right)\Big)$$

If $k_f \neq 0$, this must be solved subject to[7]

$$\begin{aligned} h_4(Y^+) &= h_4(\hat{Y}^+) - (\kappa_f + \kappa_p M|Y^+ - \hat{Y}^+|) && (25)\\ h_4(-Y^-) &= h_4(-\hat{Y}^-) - (\kappa_f + \kappa_p M|-Y^- + \hat{Y}^-|) && (26)\\ h_{4_Y}(Y^+) &= h_{4_Y}(\hat{Y}^+) = -\kappa_p M && (27)\\ -h_{4_Y}(-Y^-) &= -h_{4_Y}(-\hat{Y}^-) = -\kappa_p M && (28) \end{aligned}$$

where Y^+, $-Y^-$ represent the edges of the no-transaction region and $\hat{Y}^+$, $-\hat{Y}^-$ the optimal rebalance points, which lie strictly within the no-transaction region. Thus if the Y reaches $-Y^-$, more of the hedging asset is bought so the new value of Y (number of the hedging asset held in excess of y^*) is $-\hat{Y}^-$; similarly if Y reaches Y^+, the hedging asset is sold until $Y = \hat{Y}^+$. Note the free boundaries Y^+, $-Y^-$, $\hat{Y}^+$ and $-\hat{Y}^-$ are as yet unknown.

The solution to (22) is

$$h_4 = \frac{\alpha_4}{12}Y^4 - \frac{\beta_4}{2}Y^2 + a_4 Y + b_4$$

with a_4 and b_4 unknown functions of S, M, t.

Applying the boundary conditions (25)-(28) gives $Y^- = Y^+$, $\hat{Y}^- = \hat{Y}^+$, $a_4 = 0$ and

$$\beta_4 = \frac{\alpha_4}{3}\left((Y^+)^2 + Y^+\hat{Y}^+ + (\hat{Y}^+)^2\right)$$

[7]If $\kappa_f = 0$, $Y^+ = \hat{Y}^+$ and $-Y^- = -\hat{Y}^-$ (each edge of the no transaction region and the appropriate optimal rebalance point coincide) and the equations are instead:

$$\begin{aligned} h_{4_Y}(Y^+) = -h_{4_Y}(-Y^-) &= -\kappa_p M && (23)\\ h_{4_{YY}}(Y^+) = h_{4_{YY}}(-Y^-) &= 0. && (24) \end{aligned}$$

We will consider this case separately in section 3.1.

or

$$
\begin{aligned}
&\mathcal{L}(h_2) - \hat{\gamma}(t) y^* \left(\sigma^2 M^2 h_{2_M} + \rho\eta\sigma S M h_{2_S})\right) \\
&- \hat{\gamma}(t) \left(\eta^2 S^2 h_{0_S} h_{2_S} + \rho\eta\sigma S M (h_{0_S} h_{2_M} + h_{2_S} h_{0_M}) + \sigma^2 M^2 h_{0_M} h_{2_M}\right) \\
&= \quad \hat{\gamma}(t) \frac{\sigma^2 M^2}{2} \frac{\left((Y^+)^2 + Y^+ \hat{Y}^+ + (\hat{Y}^+)^2\right)}{3}
\end{aligned}
\tag{29}
$$

which must be solved subject to $h_2(S, M, T) = 0$. This gives the leading order correction to the option value due to transaction costs.

The semibandwidth, Y^+, and optimal rebalance point, $\hat{Y}^+$, are given by the solution to

$$
(Y^+ + \hat{Y}^+)(Y^+ - \hat{Y}^+)^3 \quad = \quad \mathcal{Q} = \frac{24\kappa_f \mathcal{A}(y^*)}{\hat{\gamma}(t)\sigma^2 M^2} \tag{30}
$$

$$
Y^+ \hat{Y}^+ (Y^+ + \hat{Y}^+) \quad = \quad \mathcal{P} = \frac{6\kappa_p \mathcal{A}(y^*)}{\hat{\gamma}(t)\sigma^2 M} \tag{31}
$$

In general (30) and (31) must be solved numerically[8]; explicit solutions are however available in some special cases which will be considered in section 3.

2.1.6 The $O(\epsilon^{\frac{3}{4}})$ equation

As above, we first regard the $O(\epsilon^{\frac{3}{4}})$ terms as an ordinary differential equation in Y for h_5:

$$
h_{5_{YY}} \quad = \quad \alpha_5 Y^3 - \beta_5 Y + \omega_5 \tag{32}
$$

with

$$
\alpha_5 \quad = \quad \frac{1}{3\mathcal{A}(y^*)} \left(\alpha_4 \frac{\mathcal{B}_M(y^*)}{M} - \mathcal{B}_M(y^*)\alpha_{4_M} - \mathcal{B}_S(y^*)\alpha_{4_S}\right) \tag{33}
$$

[8] They can be transformed into a ninth-order polynomial satisfied by Y^+,

$$
8\mathcal{P}{Y^+}^9 + \mathcal{Q}{Y^+}^* - 12\mathcal{P}^2{Y^+}^6 - 10\mathcal{P}\mathcal{Q}{Y^+}^5 - \mathcal{Q}^2{Y^+}^4 + 6\mathcal{P}^3{Y^+}^3 - 2\mathcal{P}^2\mathcal{Q}{Y^+}^2 - \mathcal{P}^4 = 0
$$

together with

$$
\hat{Y}^+ = \frac{{Y^+}^2}{2}\left(-1 + \sqrt{\left(1 + \frac{4\mathcal{P}}{{Y^+}^3}\right)}\right),
$$

$$\beta_5 = \frac{1}{\mathcal{A}(y^*)}\Big(-\beta_4\frac{\mathcal{B}_M(y^*)}{M} + \mathcal{B}_M(y^*)\beta_{4_M} + \mathcal{B}_S(y^*)\beta_{4_S} - \hat{\gamma}(t)\rho\sigma\eta SMh_{2_S}\Big) \tag{34}$$

and

$$\omega_5 = \frac{1}{\mathcal{A}(y^*)}\Big(\mathcal{L}(h_3) - \hat{\gamma}(t)y^*\left(\sigma^2M^2h_{3_M} + \rho\eta\sigma SMh_{3_S}\right) - \hat{\gamma}(t)\left(\eta^2S^2h_{0_S}h_{3_S} + \rho\eta\sigma SM(h_{0_S}h_{3_M} + h_{3_S}h_{0_M}) + \sigma^2M^2h_{0_M}h_{3_M}\right)\Big)$$

This has general solution

$$h_5 = \frac{\alpha_5}{20}Y^5 - \frac{\beta_5}{6}Y^3 + \frac{\omega_5}{2}Y^2 + a_5Y + b_5$$

with a_5 and b_5 unknown functions of S, M, t.

The free boundary conditions (25), (26), (27) and (28) and the free boundaries must now be expanded to higher order: we write the top and bottom free boundaries as $\bar{Y}^+ + \epsilon^{\frac{1}{4}}z^+$ and $-\bar{Y}^- + \epsilon^{\frac{1}{4}}z^-$ respectively, the optimal rebalance points as $\hat{\bar{Y}}^+ + \epsilon^{\frac{1}{4}}\hat{z}^+$ and $-\hat{\bar{Y}}^- + \epsilon^{\frac{1}{4}}\hat{z}^-$, and $\bar{h}_n^+$ and $\bar{h}_n^-$ for evaluations at the leading order boundaries $\bar{Y}^+$ and $-\bar{Y}^-$ and and $\hat{\bar{h}}_n^+$ and $\hat{\bar{h}}_n^-$ for evaluations at the leading order boundaries $\hat{\bar{Y}}^+$ and $-\hat{\bar{Y}}^-$ respectively. We will deal with the case when $k_f \neq 0$; the analysis for $k_f = 0$ differs and will be covered in section 3.1.

Expanding the value matching and smooth pasting conditions to higher order we obtain

$$(h_4(\pm Y^\pm) - h_4(\pm\hat{Y}^\pm)) + \epsilon^{\frac{1}{4}}(h_5(\pm Y^\pm) - h_5(\pm\hat{Y}^\pm)) + O(\epsilon^{\frac{1}{2}}) = -\kappa_f - \kappa_p M| \pm Y^\pm - (\pm\hat{Y}^\pm)| \tag{35}$$

$$-h_{4_Y}(\pm Y^\pm) - \epsilon^{\frac{1}{4}}h_{5_Y}(\pm Y^\pm) + O(\epsilon^{\frac{1}{2}}) = \pm\kappa_p M \tag{36}$$

$$-h_{4_Y}(\pm\hat{Y}^\pm) - \epsilon^{\frac{1}{4}}h_{5_Y}(\pm\hat{Y}^\pm) + O(\epsilon^{\frac{1}{2}}) = \pm\kappa_p M \tag{37}$$

or, transferring the boundary conditions to the *leading order* boundaries $\pm\bar{Y}^\pm$,

$$\left(\bar{h}_4^\pm - \hat{\bar{h}}_4^\pm\right) + \epsilon^{\frac{1}{4}}\left(z^\pm\bar{h}_{4_Y}^\pm + \bar{h}_5^\pm - (\hat{z}^\pm\hat{\bar{h}}_{4_Y}^\pm + \hat{\bar{h}}_5^\pm)\right) + O(\epsilon^{\frac{1}{2}}) = -\kappa_f - \kappa_p M| \pm \bar{Y}^\pm - (\pm\hat{\bar{Y}}^\pm) \pm \epsilon^{\frac{1}{4}}(z^\pm - \hat{z}^\pm)| \tag{38}$$

$$-\bar{h}^{\pm}_{4_Y} - \epsilon^{\frac{1}{4}}(z^{\pm}\bar{h}^{\pm}_{4_{YY}} - \bar{h}^{\pm}_{5_Y}) + O(\epsilon^{\frac{1}{2}}) \quad = \quad \pm\kappa_p M \qquad (39)$$

$$-\hat{\bar{h}}^{\pm}_{4_Y} - \epsilon^{\frac{1}{4}}(\hat{z}^{\pm}\bar{h}^{\pm}_{4_{YY}} - \bar{h}^{\pm}_{5_Y}) + O(\epsilon^{\frac{1}{2}}) \quad = \quad \pm\kappa_p M \qquad (40)$$

From (38) we obtain $\bar{h}^{+}_{5} - \hat{\bar{h}}^{+}_{5} = 0$ (since $-\bar{h}^{\pm}_{4_Y} = -\hat{\bar{h}}^{\pm}_{4_Y} = \pm\kappa_p M$) and hence $\omega_5 = 0$. After some simplification, this gives the differential equation satisfied by h_3:

$$h_{3_t} + rMh_{3_M} + rSh_{3_S} - rh_3 + \frac{\sigma^2}{2}M^2 h_{3_{MM}} + \rho\eta\sigma SMh_{0_{SM}} + \frac{\eta^2}{2}S^2 h_{3_{SS}} - \hat{\gamma}(t)\eta^2(1-\rho^2)S^2 h_{0_S} h_{3_S} = 0 \quad (41)$$

which must be solved subject to $h_3^w(S,M,T) = -\kappa_p|My^*(T)|$ or $h_3^w(S,M,T) = -\kappa_p|My_0^*(T)|$ as appropriate.

We also obtain the corrections to the leading order boundaries:

$$z^+ = -\frac{\bar{h}^{+}_{5_Y}}{\bar{h}^{+}_{4_{YY}}}, \qquad z^- = -\frac{\bar{h}^{-}_{5_Y}}{\bar{h}^{-}_{4_{YY}}} \qquad (42)$$

$$\hat{z}^+ = -\frac{\hat{\bar{h}}^{+}_{5_Y}}{\hat{\bar{h}}^{+}_{4_{YY}}}, \qquad \hat{z}^- = -\frac{\hat{\bar{h}}^{-}_{5_Y}}{\hat{\bar{h}}^{-}_{4_{YY}}} \qquad (43)$$

Since both h_{5_Y} and $h_{4_{YY}}$ contain only even powers of Y, we find

$$z^- = z^+, \qquad \hat{z}^- = \hat{z}^+$$

so the difference between the unhedgedness of the two edges of the band $Y^+ - (-Y^-) = \bar{Y}^+ + \epsilon^{\frac{1}{4}}z^+ - (-\bar{Y}^- + \epsilon^{\frac{1}{4}}z^-) = 2\bar{Y}^+$ does not change; its location (*i.e.* the centre of the band) merely shifts.

2.2 Option value

In principle the value of the option and the location of the no-transaction band could be functions of S, M, y and t; however we find that, at least to the orders of magnitude we consider, successive terms in the expansion of the option value, and also the locations of the centre and edges of the no-transaction band in the amount of the hedging asset held are all *independent* of the actual value of the hedging asset, M. This simplifies their calculation significantly.

We reflect this independence of M and thus write the hedging strategy in terms of the value held in the hedging asset, so $x = My$, $X = MY$ *etc.* and also $H(S,x,t) = \sum_i (h_i^w - h_i^0)$ for the value of the option once part of the investor's portfolio. Note the initial price the investor should be willing to pay to take on the option position will also take account of the initial costs of setting up the hedge and so will be

$$H(S,0,t) = H(S,x,t) - k_1 - k_3 \frac{\rho\eta}{\sigma} S H_{0_S}(S,t)$$

where

$$H(S,x,t) = H_0(S,t) + \epsilon^{\frac{1}{2}} H_2(S,t) + \epsilon^{\frac{3}{4}} H_3(S,t) + \epsilon H_4(S,X,t) + \ldots$$

with $H_i = (h_i^w - h_i^0)$. We thus have that H_0 satisfies

$$H_{0_t} + rSH_{0_S} - rH_0 + \frac{\eta^2}{2} S^2 H_{0_{SS}} - \hat{\gamma}(t)\frac{\eta^2}{2}(1-\rho^2) S^2 H_{0_S}^2 = 0$$

subject to final condition $H_0(S,T) = \Lambda(S)$. This is exactly the equation satisfied by a costlessly imperfectly hedged option value as in Henderson (2005) [13].

The leading order correction to the option value, H_2, satisfies

$$\begin{aligned} H_{2_t} + (r + \eta(\xi - \lambda\rho)) S H_{2_S} - rH_2 + \frac{\eta^2}{2} S^2 H_{2_{SS}} & \\ - \hat{\gamma}(t)\eta^2(1-\rho^2) S^2 H_{0_S} H_{2_S} & \qquad (44) \\ = \quad \hat{\gamma}(t)\frac{\sigma^2}{2}\left(\frac{(X^{+2} + X^+\hat{X}^+ + \hat{X}^{+2})}{3} - \frac{(X_0^{+2} + X_0^+\hat{X}_0^+ + \hat{X}_0^{+2})}{3}\right) & \end{aligned}$$

subject to $H_2(S,T) = 0$. Here $X^+(S,t) = MY^+$ and $\hat{X}^+(S,t) = M\hat{Y}^+$ are the semi-bandwidth and optimal rebalance point in terms of the values held in the hedging asset and are given by the solution to (47) and (48) below. H_2 represents to leading order the certainty equivalent value of transaction costs incurred during the life of the option as a result of hedging due to moving outside the transaction band. Note the right-hand side of this equation involves only derivatives of H_0, so the equation is linear.

H_3, which represents the leading order effects of certainty equivalent cost of unwinding the hedge at expiry, satisfies

$$H_{3_t} + (r + \eta(\xi - \lambda\rho))SH_{3_S} - rH_3 + \frac{\eta^2}{2}S^2 H_{3_{SS}} - \hat{\gamma}(t)\eta^2(1-\rho^2)S^2 H_{0_S} H_{3_S} = 0 \qquad (45)$$

with final condition

$$H_3(S,T) \quad = \quad -\kappa_3(|x^*(S,T)| - |x_0^*(T)|) \qquad (46)$$

where $x^* = My^*$ and $x_0^* = My_0^*$.

Equations (44), (44) and (45) and associated final conditions are straightforward to solve numerically using finite difference methods.

2.3 Hedging strategy

The optimal trading strategy consists of a no transaction region, within which the *number*, y, of the hedging aset hold remains constant. If the prices of the hedged (underlying share price) or hedging asset move so the *value* held in the hedging asset, $My \equiv x$, moves outside this band, transactions are made to bring x back within the band. The no transaction band can be written as

$$x^* - \epsilon^{\frac{1}{4}}X^+ = x^-(S,t) \leq x \leq x^+(S,t) = x^* + \epsilon^{\frac{1}{4}}X^+$$

with $My^* = x^*(S,t)$, $MY^+ = X^+(S,t)$ and $MY^- = X^-(S,t)$ to emphasise the lack of dependence of the location of the band on the current value of the hedging asset, M. The value of x thus changes with changes in M, whereas the location and width of the band change with S and t.

The centre of the no-transaction band for the problem with the option grant is related to the Delta of the leading order option value

$$x^*(S,t) = x_0^*(t) - \frac{\rho\eta}{\sigma}Sh_{0_S}(S,t)$$

where $x_0^* = \frac{\lambda e^{-r(T-t)}}{\gamma\sigma}$ is the amount of the hedging asset held without the option grant in the absence of transaction costs.

Thus the leading order difference between the amounts held in the hedging asset, *i.e.* the *extra* amount held due to the option grant,

$$x^* - x_0^* \approx -\frac{\rho\eta S}{\sigma} h_{0_S},$$

is proportional to the firm's stock price multiplied by the option's delta. Note however that, since h_0 does not satisfy the Black-Scholes equation, h_{0_S} is not the Black-Scholes delta and must be calculated numerically.

We further find that the no-transaction band is symmetrical to leading order.The leading order values for the semi bandwidth, X^+, and optimal rebalance point, $\hat{X}^+$, are given by the solutions to

$$(X^+ + \hat{X}^+)(X^+ - \hat{X}^+)^3 = Q = \frac{24\kappa_f A(x^*)}{\hat{\gamma}(t)\sigma^2} \tag{47}$$

$$X^+\hat{X}^+(X^+ + \hat{X}^+) = P = \frac{6\kappa_p A(x^*)}{\hat{\gamma}(t)\sigma^2} \tag{48}$$

where $A(x^*) = M^2\mathcal{A}(y^*)$ is independent of M. X^+ and $\hat{X}^+$ thus depend on the level and form of the transaction costs, the investor's risk aversion and the Delta and Gamma (first and second derivatives) of the option position. In the special cases in section 3 we can obtain explicit formulae for the semibandwidth; however from (47) and (48) we can see in all cases both X^+ and $\hat{X}^+$ depend positively on $A(x^*)$ given by

$$\begin{aligned} A(x^*) &= \frac{\eta^2}{2}\left[\rho^2\left(\frac{\lambda}{\hat{\gamma}(t)\eta} + \left(\frac{\eta}{\sigma} - \rho\right) S h_{0_S} + \frac{\eta}{\sigma} S^2 h_{0_{SS}}\right)^2 \right. \\ &\quad \left. + (1-\rho^2)\left(\frac{\lambda}{\hat{\gamma}(t)\eta} - \rho S h_{0_S}\right)^2\right] \end{aligned} \tag{49}$$

Note if $\rho = 1$ and $\sigma = \eta$, the equations for the semibandwidth and optimal rebalancing point reduce to those of Whalley & Wilmott (1999) [26], for the case of perfect correlation. These depend only on the option's Gamma, h_{SS} since

$$A(x^*) = \frac{1}{2}\left(\frac{\lambda}{\hat{\gamma}(t)} + \eta S^2 h_{0_{SS}}\right)^2 \tag{50}$$

Imperfect correlation between the underlying and hedging asset thus introduces dependence also on the option's Delta, h_S into the bandwidth and hence the certainty equivalent value of transaction costs.

3 Special cases

3.1 Proportional transaction costs only

When costs are purely proportional to the value of the hedging asset traded (so $k_f = 0$)[9] the optimal rebalance point co-incides with the edge of the hedging band, so $\hat{X}^+ = X^+$, $\hat{X}^- = X^-$ and we obtain explicit formulae for the leading order hedging strategy in terms of the leading order option Greeks.

$$X^+_{(p)} \equiv \bar{X}^+ = \bar{X}^- \quad = \quad \left(\frac{3\kappa_p}{\sigma^2\hat{\gamma}(t)}\right)^{\frac{1}{3}} |A(x^*)|^{\frac{1}{3}} \tag{51}$$

where $A(x^*)$ is as given in (49) above.

Note there is also a hedging bandwidth for the problem without the option given by

$$\bar{X}^+_0 \quad = \quad \frac{1}{\hat{\gamma}(t)}\left(\frac{3\kappa_p\lambda^2}{\sigma^2}\right)^{\frac{1}{3}} \tag{52}$$

In this case[10] we also find that the leading order correction to the hedging band is given by

$$w^- = w^+ = -\frac{\rho\eta}{\sigma} S h_{2_S},$$

[9]From the value matching and smooth pasting conditions (23) - (24) for this case we can deduce $Y^+ = Y^- = \hat{Y}^+ = \hat{Y}^-$ and $\beta_4 = \frac{\alpha_4}{Y}^{+2}$. Substituting into (23) we obtain an explicit formula for Y^+ equivalent to (51).

[10]Expanding the value matching and smooth pasting conditions (23) - (24) to higher order we obtain:

$$h_{4_Y}(\pm Y^\pm) + \epsilon^{\frac{1}{4}} h_{5_Y}(\pm Y^\pm) + O(\epsilon^{\frac{1}{2}}) \quad = \quad \pm\kappa_3 M \tag{53}$$

$$h_{4_{YY}}(\pm Y^\pm) + \epsilon^{\frac{1}{4}} h_{5_{YY}}(\pm Y^\pm) + O(\epsilon^{\frac{1}{2}}) \quad = \quad 0 \tag{54}$$

or, transferring the boundary conditions to the *leading order* boundaries $\pm\bar{Y}^\pm$,

$$\bar{h}^\pm_{4_Y} + \epsilon^{\frac{1}{4}}(z^\pm \bar{h}^\pm_{4_{YY}} + \bar{h}^\pm_{5_Y}) + O(\epsilon^{\frac{1}{2}}) \quad = \quad \pm\kappa_3 M \tag{55}$$

$$\bar{h}^\pm_{4_{YY}} + \epsilon^{\frac{1}{4}}(z^\pm \bar{h}^\pm_{4_{YYY}} + \bar{h}^\pm_{5_{YY}}) + O(\epsilon^{\frac{1}{2}}) \quad = \quad 0 \tag{56}$$

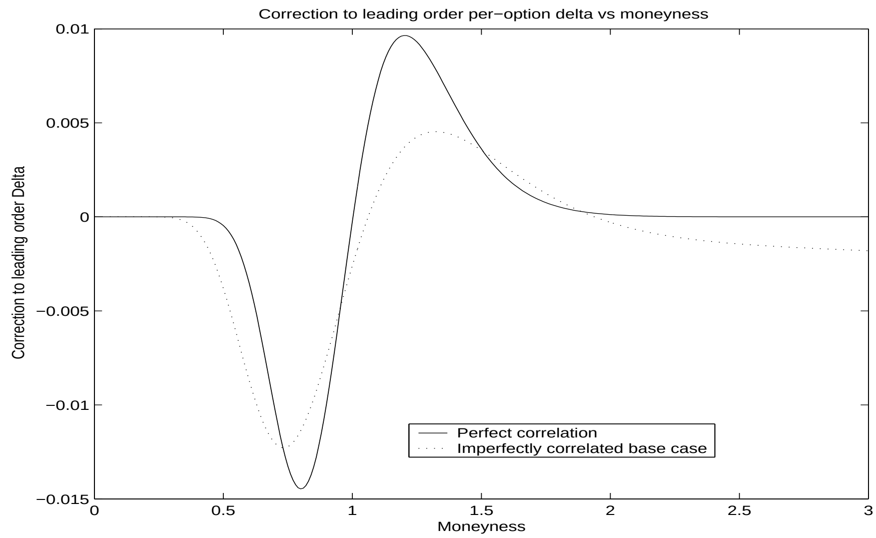

Figure 1: $O(\epsilon^{\frac{1}{2}})$ correction to per option Delta values as a function of moneyness, S/K. Parameter values for perfect correlation: $T = 1$, $\sigma = \eta = 0.3$, $\rho = 1$, $r = 0.04$, $\gamma = 1 \times 10^{-6}$, $nK = 1 \times 10^{6}$ and $k_p = 0.01$. Parameter values for imperfect correlation base case: $T = 1$, $\sigma = 0.3$, $\sigma = 0.2$, $\rho = 0.8$, $r = 0.04$, $\gamma = 1 \times 10^{-6}$, $nK = 1 \times 10^{6}$ and $k_p = 0.01$.

where $w^+ = Mz^+$, $w^- = Mz^-$ and so $X^+_{(p)} + \epsilon^{\frac{1}{4}} w^+$ and $-X^+_{(p)} + \epsilon^{\frac{1}{4}} w^-$ represent the edges of the no transaction band in terms of values held in the hedging asset. We see that this correction does not change the width of the band; its location (*i.e.* the centre of the band) merely shifts to reflect the leading order correction to the delta.

Figure 1 shows the effect of the corrections to the centre of the hedging band for the special case of long call options with $\rho = 1$ and for the base case partially correlated value. In the perfect correlation case, since the hedging and underlying assets are identical, we

The leading order corrections to the boundaries are given by

$$z^+ = z^- = -\frac{\bar{h}^+_{5_{YY}}}{\bar{h}^+_{4_{YYY}}} = -\frac{\alpha_5 {Y^+}^2 + \beta_5}{2\alpha_4}$$

After some tedious calculation we find all the terms involving h_0 and y^* cancel and we are left with $z^\pm = -\frac{\rho\eta}{\sigma} S h_{2_S}$.

can express the band in terms of the option's delta. Here the leading order delta is given by Black Scholes; note the higher order effect on the centre of the band gives rise the characteristic effect that option values incorporating transaction costs change the effective volatility of the option depending on the sign of the option's Gamma. Imperfect correlation introduces an additional negative effect on per-option Delta values due to the unhedgeable risk, particularly for asset values where $|h_{0_S}|$ is large (large S for long call options).

Thus optimal hedging occurs only when the *value* of the amount held of the hedging asset, M, moves outside the hedging band given by

$$\begin{aligned} x_0^*(t) - \frac{\rho\eta}{\sigma} S(h_{0_S} + \epsilon^{\frac{1}{2}} h_{2_S}) - \epsilon^{\frac{1}{4}} \bar{X}^+ \\ \leq x \leq x_0^*(t) - \frac{\rho\eta}{\sigma} S(h_{0_S} + \epsilon^{\frac{1}{2}} h_{2_S}) + \epsilon^{\frac{1}{4}} \bar{X}^+ \end{aligned}$$

with $\bar{X}^+$ given by (51).

If the investor initially holds x_0^* in the hedging asset[11], then the option value is given by

$$\begin{aligned} H(S, x_0^*, t) &= H_0 + \epsilon^{\frac{1}{2}} H_2 + \epsilon^{\frac{3}{4}} \left(H_3 - \kappa_p \left| -\frac{\rho\eta}{\sigma} S h_{0_S} \right| \right) \\ &\quad + \epsilon \left(H_4(S, \pm\bar{X}^{\pm}, t) + \kappa_p \bar{X}^+ \right) + \ldots \\ &= H_0 + \epsilon^{\frac{1}{2}} H_2 + \epsilon^{\frac{3}{4}} \left(H_3 - \kappa_p \frac{\rho\eta}{\sigma} S |h_{0_S}| \right) + \epsilon \frac{3}{8} \kappa_p \bar{X}^+ + \ldots \end{aligned}$$

since $H_4(S, \pm\bar{X}^{\pm}.t) = -\frac{5}{8}\kappa_p \bar{X}^+$. The leading order value for portfolios of European options is the zero-transaction cost value, which is

[11] This assumes x_0^* is outside the no-transaction region for the problem including the option position. If the investor holds $x_0 = x_0^* + \epsilon^{\frac{1}{4}} X_0$ (with $-\bar{X}_0^+ \leq X_0 \leq \bar{X}_0^+$, the option value would decrease by ϵX_0 if $x_0 > x^* + \epsilon^{\frac{1}{4}} X^+$, so the initial transaction is to sell the hedging asset. If x_0 is initially in the buy region for the problem including the option position, $x_0 < x^* - \epsilon^{\frac{1}{4}} X^+$, then the option value would increase by ϵX_0, and if x_0 is within the no-transaction region for the problem including the option position, then no initial transaction would be required and

$$H(S, x_0, t) = H_0 + \epsilon^{\frac{1}{2}} H_2 + \epsilon^{\frac{3}{4}} H_3 + \epsilon H_4(S, \epsilon^{-\frac{1}{4}}(x_0 - x^*), t) + \ldots$$

the solution to (44) with appropriate final condition $H_0(S,T) = \Lambda(S)$. The leading order component of the option value resulting from transaction costs, H_2, now has an explicit partial differential equation to be solved subject to $H_2(S,T) = 0$:

$$\begin{aligned} H_{2_t} + (r + \eta(\xi - \lambda\rho))SH_{2_S} - rH_2 + \frac{\eta^2}{2}S^2 H_{2_{SS}} \\ - \hat{\gamma}(t)\eta^2(1-\rho^2)S^2 H_{0_S} H_{2_S} \quad (57) \\ = \quad \hat{\gamma}(t)\frac{\sigma^2}{2}\left(\bar{X}^{+2} - \bar{X}_0^{+2}\right) \\ = \quad \hat{\gamma}(t)\frac{\sigma^2}{2}\left(\frac{3\kappa_p}{\sigma^2\hat{\gamma}(t)}\right)^{\frac{2}{3}}\left(|A(x^*)|^{\frac{2}{3}} - \left(\frac{\lambda^2}{2\hat{\gamma}^2(t)}\right)^{\frac{2}{3}}\right) \end{aligned}$$

where $\bar{X}^+$ and $\bar{X}_0^+$ are given by (51) and (52) respectively, and $A(x^*)$ is given by (49).

The component of the option value reflecting the leading order certainty equivalent value of final transaction costs, H_3, is given by the solution to (45) with final condition (46).

3.2 Fixed costs only

When the only component of transaction costs is the fixed cost per trade (so $k_p = 0$) we can solve (47) and (48) to obtain explicit formulae for the leading order hedging strategy in terms of the leading order option Greeks. In this case the optimal rebalance point differs from the edge of the no-transaction band:

$$\bar{\hat{X}}^+ = \bar{\hat{X}}^- \quad = \quad 0 \tag{58}$$

$$X_{(f)}^+ \equiv \bar{X}^+ = \bar{X}^- \quad = \quad \left(\frac{24\kappa_f}{\sigma^2\hat{\gamma}(t)}\right)^{\frac{1}{4}} |A(x^*)|^{\frac{1}{4}} \tag{59}$$

where $A(x^*)$ is as given in (49) above.

Similarly the leading order optimal rebalance point and edge of band for the problem without the option are given by

$$\bar{X}_0^+ \quad = \quad \left(\frac{12\kappa_f\lambda^2}{\hat{\gamma}^3(t)\sigma^2}\right)^{\frac{1}{4}}, \qquad \bar{\hat{X}}_0^+ = 0 \tag{60}$$

If the investor initially holds x_0 in the hedging asset[12], then the option value is given by

$$\begin{aligned} H(S, x_0^*, t) &= H_0 + \epsilon^{\frac{1}{2}} H_2 + \epsilon^{\frac{3}{4}} H_3 + \epsilon \left(H_4(S, \pm\bar{X}^{\pm}, t) - \kappa_f \right) + \dots \\ &= H_0 + \epsilon^{\frac{1}{2}} H_2 - 2\epsilon\kappa_f + \dots \end{aligned}$$

since the $O(\epsilon^{\frac{3}{4}})$ term in the option value, H_3, is given by the solution to the inhomogeneous linear equation (45) with final condition $H_3(S,T) = 0$ and is thus identically zero $H_3(S,t) = 0$.

The leading order value for portfolios of European options is again the zero-transaction cost value, which is the solution to (44) with appropriate final condition $H_0(S,T) = \Lambda(S)$. The leading order component of the option value resulting from transaction costs, H_2, has an explicit partial differential equation to be solved subject to $H_2(S,T) = 0$:

$$\begin{aligned} & H_{2_t} + (r + \eta(\xi - \lambda\rho)) S H_{2_S} - r H_2 + \frac{\eta^2}{2} S^2 H_{2_{SS}} \\ & \qquad\qquad\qquad\qquad - \hat{\gamma}(t)\eta^2(1-\rho^2) S^2 H_{0_S} H_{2_S} \qquad (61) \\ &= \hat{\gamma}(t)\frac{\sigma^2}{2}\left(\frac{\bar{X}^{+2}}{3} - \frac{\bar{X}_0^{+2}}{3}\right) \\ &= \hat{\gamma}(t)\frac{\sigma^2}{6}\left(\frac{24\kappa_f}{\sigma^2\hat{\gamma}^3(t)}\right)^{\frac{1}{2}}\left(|A(x^*)|^{\frac{1}{2}} - \left(\frac{\lambda^2}{2\hat{\gamma}^2(t)}\right)^{\frac{1}{2}}\right) \end{aligned}$$

where $A(x^*)$ given by (49).

4 Results

4.1 Effect of the inability to hedge perfectly

In the case where hedging with the asset underlying the derivatives contract is possible, so $\rho = 1$ and $\sigma = \eta$, we recover the results of

[12] If x_0 lies within the no-transaction region for the problem including the option position, then no initial transaction would be required and

$$H(S, x, t) = H_0 + \epsilon^{\frac{1}{2}} H_2 + \epsilon H_4(S, \epsilon^{-\frac{1}{4}}(x_0 - x^*), t) + \dots$$

Whalley & Wilmott (1997, 1999) [25, 26] for the optimal hedging strategy and resulting option valuation in the presence of small but arbitraty transaction costs. In particular, since in theory with costless hedging the hedging error is zero, the leading order option value satisfies the Black-Scholes equation. The hedging strategy is given by a no transaction band in the number of assets held, $y^- \leq y \leq y^+$ which corresponds to our hedging band where the *value* held in the hedging asset lies between two bounds, $x^- = x^* - \epsilon^{\frac{1}{4}} X^+$ and $x^+ = x^* + \epsilon^{\frac{1}{4}} X^+$. They show the leading order difference between the centres of the hedging bands for the problems with and without the option is, in our notation, $x^* = -SH_{0_S}$. For proportional costs we have shown there is an $O(\epsilon^{\frac{1}{2}})$ correction to this so the centre of the band in this case is

$$-S(H_{0_S} + H_{2_S})$$

The width of the hedging band, X^+, and the optimal rebalance point, $\hat{X}^+$, are functions of $A(x^*)|_{\rho=1,\sigma=\eta}$, which from (50) depends only on the Gamma of the option position.

For imperfectly correlated hedging assets, any trading strategy must involve a level of unhedgeable risk. In our certainty equivalent framework, this reduces option values relative to the case of perfect correlation, even in the absence of transaction costs. This 'cost of unhedgeable risk' is reflected by additional terms in each differential equation for successive terms h_i. These extra terms all have the form

$$-c_i \hat{\gamma}(t) \eta^2 (1 - \rho^2) S^2 H_{0_S} H_{i_S}$$

for constants c_i. In particular, the term in the leading order equation,

$$-\frac{1}{2} \hat{\gamma}(t) \eta^2 (1 - \rho^2) S^2 H_{0_S}^2$$

is unambiguously negative if $|\rho| < 1$ and therefore has the effect of reducing option values[13]. The magnitude of this term increases, though at an decreasing rate, as the (absolute) correlation between the hedging and hedged assets decreases from 1, causing the option value in the absence of costs to decrease similarly.

[13] This can be shown using a comparison argument.

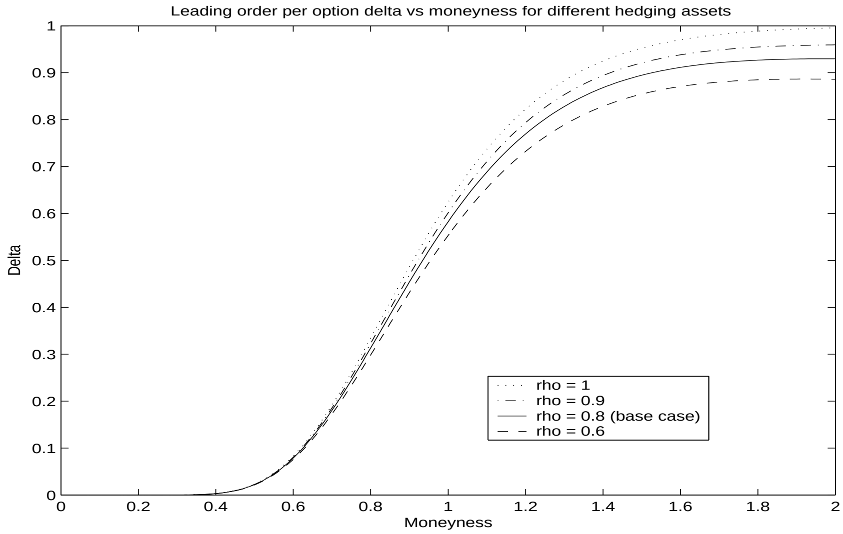

Figure 2: Leading order per-option deltas for different hedging assets (ρ) as a function of moneyness, S/K. Parameter values where not stated: $T = 1$, $\eta = 0.3$, $r = 0.05$, $\gamma = 1 \times 10^{-6}$, $nK = 1 \times 10^{6}$.

There is also an impact on the leading order option delta, H_{0_S}. As shown in Figure 2, the delta for a long call option decreases as the potential hedging effectiveness of the hedging asset (absolute correlation) decreases from 1. The leading order Gamma, $H_{0_{SS}}$ is also affected.

The location of the difference in the centre of the no transaction band due to the option holding,

$$x^* - x_0^* = -\frac{\rho\eta}{\sigma} S H_{0_S}$$

is thus affected by differences in ρ both directly (due to the multiplier $\frac{\rho\eta}{\sigma}$) and indirectly because of the effect on H_{0_S}. For long call options, as $|\rho|$ decreases both effects reduce the absolute value of the centre of the band, reducing the difference between the trading portfolios with and without the option position. The width of the no-transaction band is also affected through its dependence on $A(x^*)$, which itself depends on the correlation both explicitly and implicitly, through the Delta and Gamma of the option position.

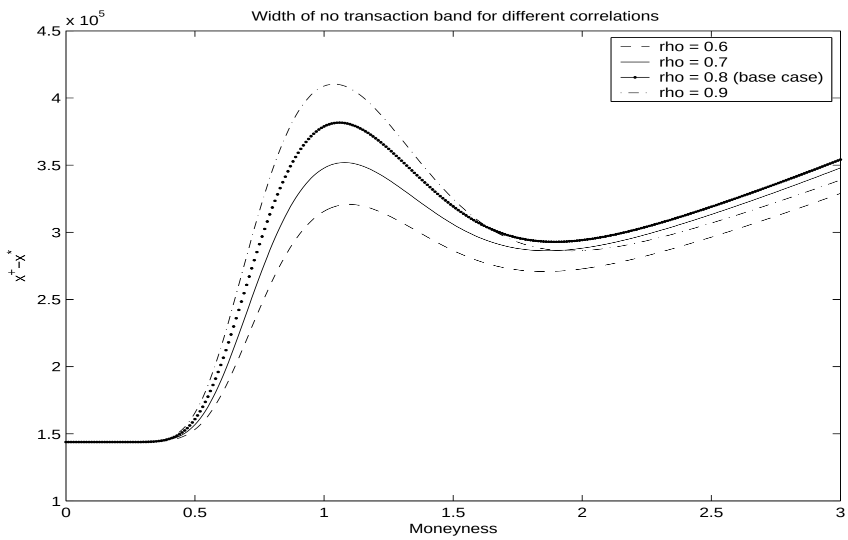

Figure 3: No-transaction band semi-bandwidth per option for different hedging assets (ρ) as a function of moneyness, S/K. Parameter values where not stated: $T = 1$, $\eta = 0.3$, $r = 0.05$, $\gamma = 1 \times 10^{-6}$, $nK = 1 \times 10^6$, $\sigma = 0.25$ and $k = 0.01$.

Figure 3 shows the semibandwidth for a long call option for a range of values of ρ. We can see the effct of the Gamma term in $A(x^*)$ in the widening of the band close to the money, which decreases in magnitude as $|\rho|$ decreases, and, for in-the-money asset values, the effect of the term including the Delta, which for these parameter values first increases as $|\rho|$ decreases from 1 and then decreases for lower $|\rho|$.

The location of the no-transaction band is shown for two values of ρ in Figure 4. Note that the magnitude of the changes in the centre of the band with differences in ρ is much greater than the effects of changes in the width of the band. The square of the bandwidth affects the leading order reduction in the certainty equivalent option value due to transaction costs. This, together with the initial and final costs, both of which increase with increases in $|\rho|$, is shown in Figure 5. Thus overall the certainty equivalent value of transaction costs decreases as $|\rho|$ decreases. All else equal, the lower the proportion of risk which is

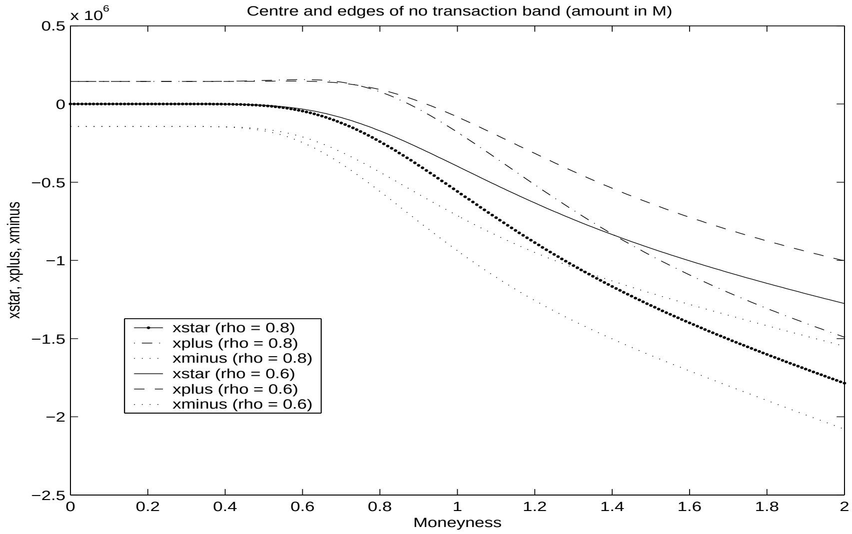

Figure 4: Centre and edges of no-transaction band per option for different hedging assets (ρ) as a function of moneyness, S/K. Parameter values where not stated: $T = 1$, $\eta = 0.3$, $r = 0.05$, $\gamma = 1 \times 10^{-6}$, $nK = 1 \times 10^{6}$, $\sigma = 0.25$ and $k = 0.01$.

hedgeable, the lower the CE transaction costs incurred in hedging.

However, the general decrease in the certainty equivalent value of transaction csots for lower $|\rho|$, which has a positive effect on long option values, is offset by the decrease in the certainty equivalent option value because of the unhedgeable risk. This effect occurs in the leading order equation and thus, for the parameter values we consider, dominates. Hence, as shown in Figure 6, overall long certainty equivalent option values inclusive of transaction costs *decrease* as $|\rho|$ decreases from 1. The rate of decrease is greatest for $|\rho|$ close to 1 and for asset values for which $|h_{0_S}|$ is greatest (large S for calls).

4.2 Nonlinearities and large option portfolios

It has been recognised in earlier work, (*e.g.* Whalley & Wilmott (1999) [26]), Damgaard (2003) [7], Zakamouline (2003) [27]) that utility based or certainty equivalent valuation of options results in nonlinear valua-

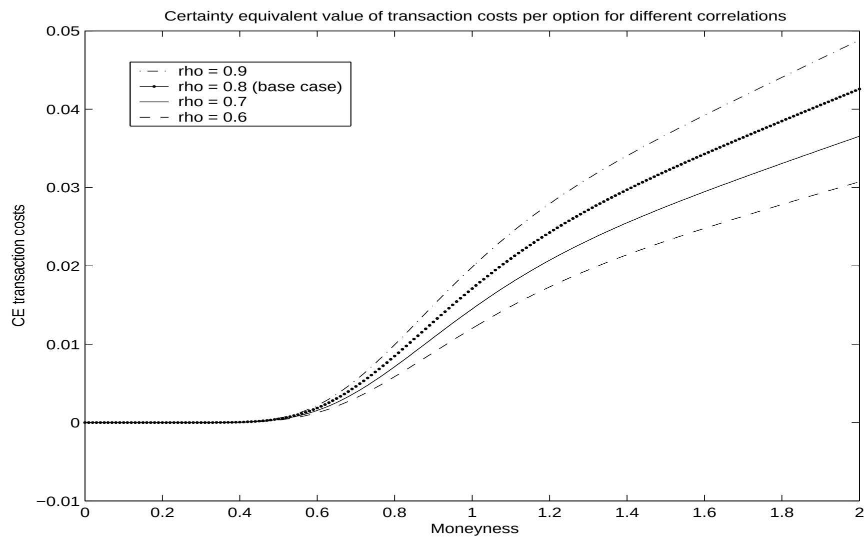

Figure 5: Total certainty equivalent value of transaction costs under optimal hedging for n long call options as a function of moneyness, S/K, for different choices of hedging asset (ρ). Parameter values where not stated: $T = 1$, $\eta = 0.3$, $r = 0.05$, $\gamma = 1 \times 10^{-6}$, $nK = 1 \times 10^{6}$, $\sigma = 0.25$, $k_f = 0$ and $k_p = 0.01$.

tion equations. This means that option values are no longer additive, that portfolios of options need to be valued as a whole and that the value of an option to an investor equals its marginal value and thus depends on the investor's existing portfolio.

In particular, the size of an option portfolio will affect its value. We will consider the simplest case of a portfolio of n identical options and investigate how the hedging strategy and option value vary with n.

Increasing the size of the option portfolio has three effects. Firstly it increases the amount of unhedgeable risk (due to the imperfect correlation). This decreases certainty equivalent values in the absence of transaction costs at an increasing rate. For our portfolio of n identical options, we write $H = nh^{(n)}$ so $h^{(n)}$ represents the certainty equivalent value *per option* when there are n options in the portfolio in total. We

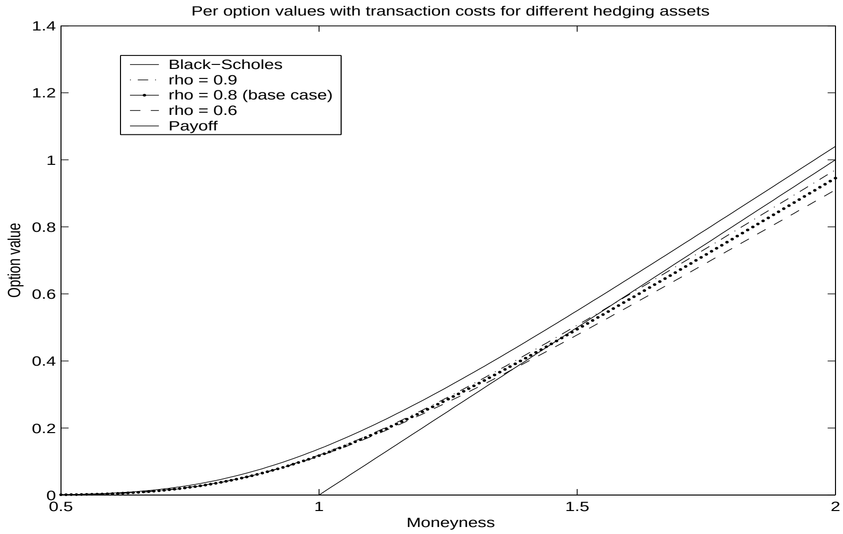

Figure 6: Per option value under optimal hedging with transaction costs for n long call options as a function of moneyness, S/K, for different choices of hedging asset (ρ). Parameter values where not stated: $T = 1$, $\eta = 0.3$, $r = 0.05$, $\gamma = 1 \times 10^{-6}$, $nK = 1 \times 10^{6}$, $\sigma = 0.25$, $k_f = 0$ and $k_p = 0.01$.

find that increasing the number of options in the portfolio increases the reduction in value per option from the Black-Scholes value even in the costless case[14]: $h_0^{(n)}$ satisfies

$$h_{0_t}^{(n)} + rSh_{0_S}^{(n)} - rh_0^{(n)} + \frac{\eta^2}{2}S^2h_{0_{SS}}^{(n)} - n\hat{\gamma}(t)\frac{\eta^2}{2}(1-\rho^2)S^2h_{0_S}^{(n)2} = 0 \quad (62)$$

subject to *e.g.* $h^{(n)}(S,T) = \max(S-K, 0)$ for a portfolio of n call options.

There are terms in each of the higher order equations reflecting this 'cost of unhedgeable risk' of the form

$$-n\hat{\gamma}(t)\eta^2(1-\rho^2)S^2h_{0_S}^{(n)}h_{i_S}^{(n)}$$

the effect of which also increases with n.

[14]The extra negative term in the differential equation for the per option value increases with n. Thus by a comparison argument, the difference in value between the partially and fully hedged option values is negative and the magnitude increases with n.

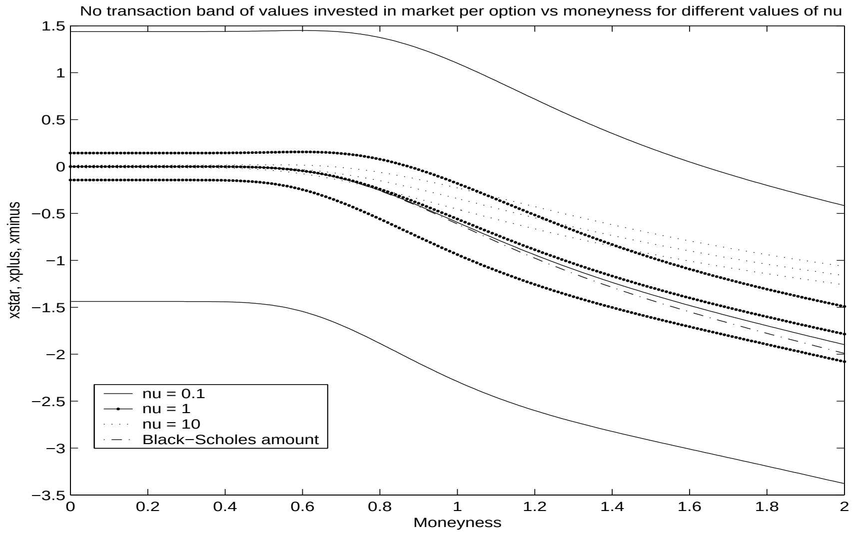

Figure 7: No-transaction band per option for different values of $\nu = nK\gamma$, as a function of moneyness, S/K. Parameter values where not stated: $T = 1$, $\eta = 0.3$, $r = 0.05$, $\sigma = 0.25$, $\rho = 0.8$, $k_f = 0$ and $k_p = 0.01$.

The second and third effects relate to the certainty equivalent value of transaction costs and arise for hedges with both partially and perfectly correlated hedging assets. Considering initially proportional costs only, the hedging bandwidth per option, $X^{(n)^+} = \bar{X}^+/n$ is given by

$$X_{(p)}^{(n)^+} \equiv \frac{X_{(p)}^+}{n} \approx \frac{1}{n}\left(\frac{3\kappa_p}{\sigma^2\hat{\gamma}(t)}\right)^{\frac{1}{3}}\left|n^2 A^{(n)}(x^*) + O(n)\right|^{\frac{1}{3}} \quad (63)$$

where

$$A^{(n)}(x^*) \equiv \frac{A(x^*)}{n^2} = \frac{\eta^2}{2}\left[\rho^2\left(\frac{\lambda}{n\hat{\gamma}(t)\eta} + \left(\frac{\eta}{\sigma} - \rho\right)Sh_{0_S}^{(n)} + \frac{\eta}{\sigma}S^2 h_{0_{SS}}^{(n)}\right)^2 + (1-\rho^2)\left(\frac{\lambda}{n\hat{\gamma}(t)\eta} - \rho S h_{0_S}^{(n)}\right)^2\right] \quad (64)$$

Note $A^{(n)}(x^*)$ depends on n through $h_{0_S}^{(n)}$ and $h_{0_{SS}}^{(n)}$ as well as x_0^*/n (all of which decrease as n increases).

The per-option semi-bandwidth thus decreases as n increases. This is shown in Figure 7 and reflects the trade-off in the utility maximi-

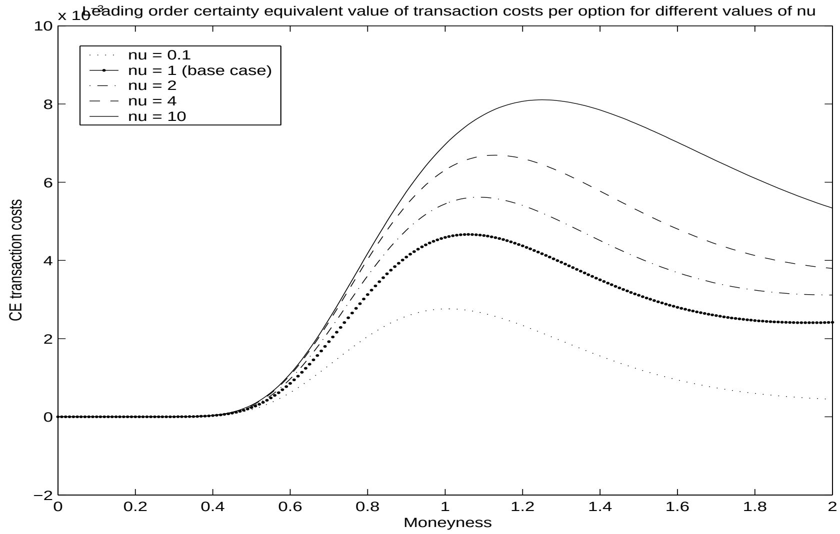

Figure 8: Leading order certainty equivalent value of transaction costs per option under optimal hedging as a function of moneyness, S/K, for different values of $\nu = nK\gamma$. Parameter values where not stated: $T = 1$, $\eta = 0.3$, $r = 0.05$, $\sigma = 0.25$, $\rho = 0.8$, $k_f = 0$ and $k_p = 0.01$.

sation between additional cost and additional risk[15] for proportional transaction costs and the negative exponential utility function. Note the amount equivalent to holding Sh_S^{BS} where h_S^{BS} is the Black-Scholes delta lies inside the no-transaction band only for low values of ν (*i.e.* smaller option positions).

The effect of the change in the bandwidth on the per-option certainty equivalent of transaction costs during the life of the option, $h_2^{(n)}$, for proportional costs only is shown in Figure 8. The transaction cost related terms in (44) are proportional to the square of the total semi-bandwidth, and hence the terms in the per-option equa-

[15] Increasing the number of options whilst keeping the width of the hedging band per option unchanged would increase the risk in the investor's portfolio, decreasing its certainty equivalent value by an amount which increases greater than proportionally with the risk, due to the assumed convexity of the utility function. The investor is thus willing to trade more frequently to offset part of the additional risk and so the width of the band per option decreases.

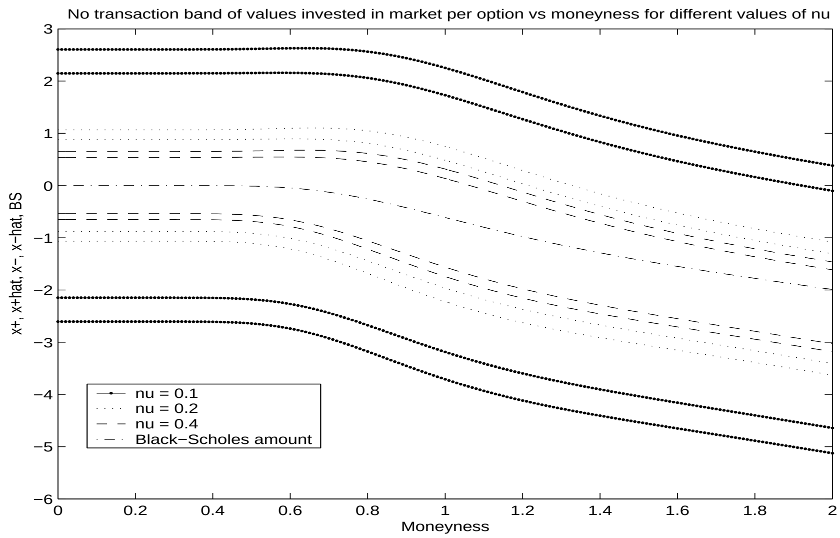

Figure 9: Optimal rebalance points and and edges of no-transaction band per option for different values of $\nu = nK\gamma$, as a function of moneyness, S/K. Parameter values where not stated: $T = 1$, $\eta = 0.3$, $r = 0.04$, $\gamma = 1 \times 10^{-6}$, $K = 1$, $\sigma = 0.2$, $\rho = 0.8$, $k_f = 100$ and $k_p = 0.01$.

tion for $h_2^{(n)}$ are proportional to n times the square of the per-option semi-bandwidth, which for our parameter values decrease with n.

The final effect concerns the relative magnitude of the fixed and proportional transaction cost terms. As the number of options in the portfolio increases, the relative effect of the fixed costs, k_f, decreases whilst that of the proportional costs, $k_p|dx|$ increases. For a very large portfolio of options, the proportional terms will have the dominant effect, so the bandwidth becomes proportional to $k_p^{\frac{1}{3}}|A(x^*)|^{\frac{1}{3}}$ and rehedging is of the minimum possible amount in order to stay within the no transaction band. For intermediate sizes of option portfolios, where the relative sizes of the transaction cost parameters become comparable, the optimal rebalance points will gradually move away from the edges of the band towards the centre as the size of the portfolio decreases.

These effects can be seen in Figure 9, which shows how the location

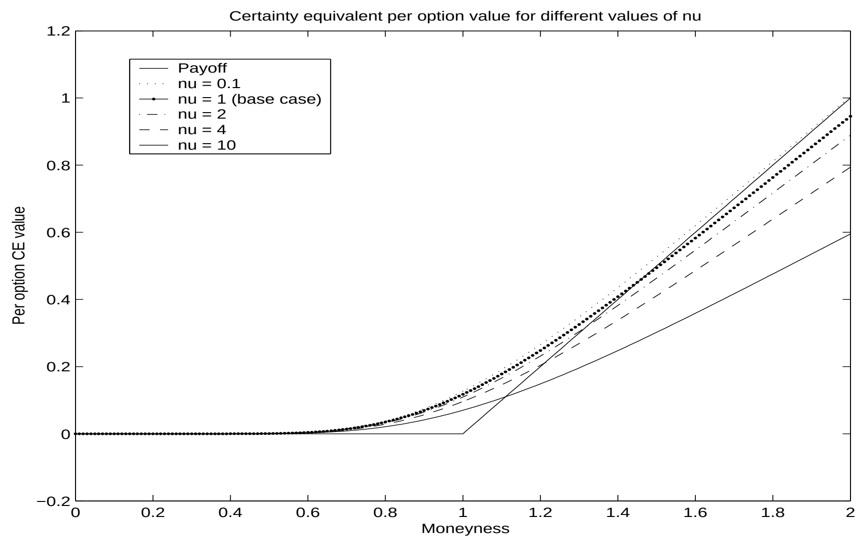

Figure 10: Leading order per-option certainty equivalent value as a function of moneyness, S/K, for different values of n. Parameter values where not stated: $T = 1$, $\eta = 0.3$, $r = 0.05$, $\gamma = 1 \times 10^{-6}$, $K = 1$, $\sigma = 0.25$, $\rho = 0.8$, $k_f = 0$ and $k_p = 0.01$.

of the edges of the hedging band and the optimal rebalance points vary with n. Note that for the smaller values of n in this figure (compared to Figure 7) the centre of the band does not change as significantly with n.

However overall for large n, the dominant effect again arises from the cost of unhedgeable risk, which increases with n, and this outweighs any savings in per-option certainty equivalent transaction costs due to changes in the hedging strategy. This is illustrated in Figure 10. Note in particular how the per-option value for large European option grants lies below the option payoff.

5 Conclusions and further work

In this paper we have used asymptotic analysis to derive explicit formulae in terms of the leading order Greeks for the optimal trading strategy for hedging an option position using a potentially imperfecly correlated hedging asset when the associated transaction costs are either fixed per trade or proportional to the value traded. For more general forms of transaction costs characteristics of the trading strategy are given in terms of the roots of a polynomial equation. This extends not only the range of scenarios for which relatively simple formulae for such hedging strategies are available, but, by considering higher orders than in previous work, also increases their accuracy.

The optimal hedging strategy depends on the Delta and the Gamma of the option position. Imperfect correlation introduces a level of unhedgeable risk which cannot be eliminated at any cost into the trade-off between the level of residual risk, or hedging error, resulting from the trading strategy and the transaction costs incurred. This has the effect of reducing the additional risk incurred as a result of the trading strategy incorporating transaction costs (the hedging bandwidth dcreases as $|\rho|$ decreases) and also reducing the certainty equivalent value of transaction costs relative to the case of perfect correlation. However for the parameter values we consider, overall the unhedgeable risk has a greater effect on the certainty equivalent option value, which decreases as $|\rho|$ decreases from 1.

The nonlinearity of the equations, both resulting from the transaction cost terms, but also because of the inherent non-linearity of imperfectly hedged option values in the absence of costs, has significant effects on the evaluation of portfolios of options which cannot be hedged perfectly. The effects are particularly important for large option portfolios and when the absolute correlation between the hedging and hedged asset is low. Thus whilst both (high) absolute correlations and (low) transaction cost levels add value when choosing a hedging asset, the former is generally more important, especially for large option positions and longer times to maturity.

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