Macroeconomic Dynamics, **6**, 2002, 167–185. Printed in the United States of America.
DOI: 10.1017.S1365100501010240

ROBUST DECISION THEORY AND THE LUCAS CRITIQUE

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In this paper, we reconsider the *theoretical* basis for the Lucas critique from the point of view of robust decision theory. We first emphasize that the Lucas critique rests on a weak theoretical paradigm in that it fails to consider the motivation for the policy change by the government and hence inconsistently assumes limited rationality by the government. When placed in a proper dynamic general equilibrium framework of a dynamic game between the government and the private sector, much of the force of the critique simply vanishes. We also reconsider the critique by adopting an alternative theoretical paradigm and notion of rationality based on robust decision theory. This view of rationality might be regarded as more relevant than the nonrobust rationality employed by Lucas and, critically, it is one in which the Lucas critique can be shown simply not to apply, provided the private sector has adopted suitably robust decision rules.

Keywords: Lucas Critique, H Infinity, Robust Decision Theory

1. INTRODUCTION

The Lucas critique [Lucas (1976)] is approximately 25 years old and it may be difficult for some to appreciate the fundamental impact that it had on econometric model building, macroeconomic theory, and policy analysis. Given that it denied the possibility of using macroeconometric models for policy simulation, and since this was a basic objective of the research program that started with Tinbergen and Klein, it represented a devastating criticism of existing econometric practice.

Over the past 25 years, the economics profession as a whole seems to have adopted a disturbingly unscientific, somewhat schizophrenic pragmatism with regard to the relevance of the critique, driven more by an ideology of convenience than by scholarship or empirical evidence. Despite its obvious importance, the critique has received relatively limited theoretical investigation while gaining an enormous citation record [see Ericsson and Irons (1995)]. A number of economists seem to

First Version 1996. This paper incorporates arguments made originally in previously unpublished papers by the two authors: Salmon (1981, 1984) and Marcellino (1995) and represents an abridged version of Marcellino and Salmon (2002). We would like to thank Tom Sargent, Ken Kasa, and Ian Petersen for comments on earlier versions and helpful discussions. Address correspondence to: Mark Salmon, City University Business School, London EC2Y 8HB, England; e-mail: m.salmon@city.ac.uk.

 1365-1005/02 $9.50

regard the critique as valid (almost without question) and, indeed, it has, at least in part, been responsible for stimulating entirely new methodological paradigms such as equilibrium macromodeling (calibration) and the rational-expectations econometrics program [see Hansen and Sargent (1980), Sargent (1981)]. On the other hand, many economists simply ignore the critique, apparently viewing it as irrelevant.[1]

Let us quote Lucas at the outset so that we are clear in what follows as to what constitutes the critique[2]:

> Given that the structure of an econometric model consists of optimal decision rules for economic agents, and that optimal decision rules vary systematically with changes in the structure of series relevant to the decision maker, it follows that any change in policy will systematically alter the structure of econometric models. Lucas (1976, p. 41).

While it is clear, therefore, that the critique is an observation on the use of empirical macroeconometric models for policy analysis, it is based in *theory* and on an assumed form of behavior and the corresponding decision rules that Lucas believes are adopted by economic agents. Hence, the critique can be addressed both on its practical relevance and on its basis in theory.

The critique has, of course, been attacked over the years on both theoretical [see, in particular, Sims (1980, 1982a,b, 1986, 1987, 1988), Cooley et al. (1984), and LeRoy (1995)] and empirical grounds [see Favero and Hendry (1992) and Ericsson and Irons (1995), for instance] but the impression still remains that there is an open question as to whether the critique is valid and hence how macroeconometric models might be used for policy analysis, if at all.

In this paper, we reconsider the *theoretical* justification for the critique, and hence its relevance, by reviewing the notion of rational behavior assumed by Lucas. We have been led to take two separate positions: *First*, we adopt the rationality paradigm that Lucas himself uses but extend it to include rational decisionmaking by the policymaker. Since Lucas did not offer any model as to why policy should change and did not attempt to explain the policymaker's behavior, yet regarded the private sector as reactive and rational, it seems only natural to impose rationality on the policymaker as well.[3] *Second*, we suggest an alternative "rational" paradigm for the development of decision rules that acknowledge the need for a degree of robustness that we believe to be potentially more relevant than the nonrobust decision theory employed by Lucas. However, it is a notion of rationality in which the Lucas critique may simply not apply if the private sector has adopted robust rules of behavior that account for potential variations in government policy.

To state the obvious, it is important to recognize from the outset that the Lucas critique represents a classic exercise in economic *theory* and, as with all such theoretical propositions in all disciplines, it may or may not have any relevance in the real world. Lucas made his critique by *imposing* on the policymaking process a particular theoretical view of rational behavior, for both the policymaker and economic agents, and it is this that we believe requires more rigorous theoretical and empirical investigation.

We also need to maintain a clear separation between potentially plausible but purely theoretical constructions and empirical models based on observed behavior that may bear no relationship whatsoever to the theory. The direct transfer of the implications of *any* theory to an empirical model by supposition, including the robust paradigm put forward below, is surely poor scientific method and needs to be supported by empirical evidence as to the relevance of the theory. However, this is precisely the form of argument used in the Lucas critique.

2. THE LUCAS CRITIQUE: A STATEMENT AND SOME OBSERVATIONS

As the earlier quote from Lucas indicates, the critique rests on the following syllogism: Econometric models represent the behavior of rational (optimizing) agents, who change their behavior when there are changes in economic policy. Hence, when there are changes in policy, there are also implied changes in econometric models that seek to capture their behavior, and we cannot use the "old" model to analyze the effects of the "new" policy.

The following characterization in terms of rational expectations provides a standard description of the critique:

$$y_{t+1} = \gamma E(x_{t+1} \mid I_t) + \varepsilon_t, \tag{1}$$

$$x_t = \lambda x_{t-1} + v_t, \tag{2}$$

$$y_{t+1} = \vartheta x_t + \eta_t \quad \text{with} \quad \vartheta = \lambda\gamma. \tag{3}$$

Equation (1) is assumed to represent the private sector's *optimal* decision rule for some variable y_t in terms of rational expectations of policy in the next period. This equation, together with the policy function of the government, equation (2), is often described as a structural form of the model. Equation (3) then is the derived "reduced form" that is assumed to represent the estimated econometric model on which policy simulations would be based. Lucas's argument is that a change in policy, as represented by a new value for instance of λ, *implies* a changed value of θ. Hence, an econometric model that uses a given estimated value of θ would not be relevant for exploring the value of a new policy that implied a different value of λ and, hence, a different value of θ.[4]

The relevance of the Lucas critique rests on an understanding of what constitutes *a change in policy*, *why* it comes about, and whether it necessarily *implies* a change in behavior of the private sector *that is not already captured by the model*. If the private sector's behavior has anticipated the potential policy changes and this has been properly captured in the econometric model, then the Lucas critique and the potential self-conflictory nature of policy simulation stressed by Ericsson et al. (1998) will not apply.

3. RATIONALITY AND OPTIMALITY

Since Lucas assumes a rational, optimizing private sector, we need, at the outset, to make a relatively simple point that is easily overlooked and that is to be absolutely clear about *what is meant by rational behavior*, whether it be by the private sector or the government. Some reflection based on standard decision theory leads to the conclusion that rational behavior, as assumed by Lucas, implies the use of an optimal decision rule that is derived from an *explicit specification* of all the information needed to solve the initial statement of the intertemporal optimization problem facing the agent. For instance, in a stochastic decision problem, this implies that the agents have assumed a knowledge of the stochastic processes that drive the uncertainty they face when they derive their optimal decision rule. Their ignorance of the real world is thus summarized in this *stochastic* specification, and the resulting optimal decision rule is only rational *given* this specification. Similarly, the constraints they face in this optimization problem define the rationality of the optimal rule at the time when the rule is derived. A rational agent will not knowingly ignore information about his economic environment when forming optimal decision rules.

It is now possible to see a basic theoretical ambiguity and contradiction within the Lucas critique. Once agents have determined their optimal decision rules, there is no need to reconsider them unless the original policy formation failed in some way.[5] However, Lucas asks us to consider changes in the *optimal* rule of the private sector. An optimal rule, as we have just stressed, reflects, by assumption, all potential environments that the private agents could imagine that they would face, including that of government policy. It would be a reflection of irrationality, which Lucas rules out *ab initio* by explicitly assuming the use of *optimal* decision rules, if the need arose for the private sector to reconsider its original decision rules. As it stands from Lucas's formulation of the policy problem, either the *optimal* decision rule is chosen by the rational private sector in the first place, and hence there would be no need to reconsider any change in their decision rule since it would already incorporate an optimal response to changing government policy actions, or the original decision rule was not fully optimal, which contradicts Lucas's basic initial assumption.[6]

Furthermore, in a "strict" rational-expectations paradigm, following Lucas, all agents including the policymaker are supposed to be able to formulate their desires in terms of objective functions, to know the constraints they face, and to be capable of solving the resulting optimization problem. Notice then that the use of rational expectations also implies that agents employ the *true* probability distribution as their subjective distributions when forming their expectations. Is this the "true" distribution before the assumed policy change or after? Full rationality, in a dynamic general equilibrium context, must surely imply that this distribution includes a specification of how and why government policy changes and hence would be applicable throughout any change in policy if it is indeed the *true* distribution.

If these rationality propositions are not upheld, then the private sector is not, in fact, forming optimal rules in the first place, as assumed by Lucas, and his theoretical framework collapses. The Lucas critique rests paradoxically on a restricted specification of the economic environment that serves as the constraint for the private sector's optimization problem and, in fact, irrational expectations,[7] and the implied behavior is thus, at best, only a *partial* equilibrium in nature and *not* descriptive of a general equilibrium. This becomes apparent with the assumed policy change, when Lucas assumes that agents are forced to reoptimize and derive new decision rules. His initial optimal rules for the private sector are, in contradiction, then seen to be suboptimal. To properly assess the implications of imposing rational, optimizing behavior on the policy invariance of macroeconometric models, we need to *fully* specify the dynamic general equilibrium model, including the behavior of the government and its interaction with the private sector.

These points are fundamental and question the nature of the economic theory on which Lucas calls for support. Perhaps more importantly, this discussion also asks us to consider the nature of the decision rules that are used every day by economic agents and hence captured in the observed data and empirical econometric models.

3.1. What Constitutes a Policy Change?

We now also need to understand the nature of the policy change that Lucas considers and whether it necessarily induces a change in private-sector behavior. For the time being, we overlook the argument made in the immediately preceding section regarding the invariance of a truly optimal rule and temporarily accept Lucas's partial equilibrium framework since it is, in fact, possible to demonstrate the failure of the critique even within this theoretically inappropriate setting.

The macroeconometric models and econometric practice of the time that Lucas was also explicitly criticizing did not, in general, assume endogenous policy reaction functions but made a modeling assumption that, at the estimation stage, treated policy variables as "exogenous." Hence, simulation exercises that involved a selection of new values for these variables were used to evaluate the effects of different policies on the model. We can think of three forms of policy change relative to a given econometric model: a change in the values taken by a policy variable from those in the estimation period, changes in the parameters of an equation that describes a policy rule or reaction function captured in the model, or introduction of a completely new policy regime by either a new rule or an instrument that is not present in the model.

Notice that the Lucas critique trivially does not apply in the last and most dramatic case of a policy change simply because the situation would, in fact, never arise. After all, it is, impossible to simulate the effects of changing a policy variable that does not appear in the model. Everyone would recognize that econometric policy evaluation would be worthless in that case.

Lucas's view of what constitutes a policy change seems to be that economic agents will *necessarily* treat changed values for the policy variables as arising from a changed government policy rule (although invariably unseen and often unannounced) that they had recognized as an explicit constraint when forming their original decisions.[8] The claim, then, is that changing a constraint's form through a change in policy will change the structure of the agents' optimal decision rules, invalidating the econometric model for policy analysis. Both parts of this supposition as to the nature and effect of a policy change can be questioned on theoretical grounds.

The first part turns on the question of whether the private sector treats the government's *actions* as given, as in an open-loop dynamic Nash game (i.e., conditions on a sequence $\{x_t\}_1^\infty$), or whether they take the government's *policy rule* as given, as in a feedback Nash game [i.e., conditions on equation (2)]. Clearly, if the private sector forms open-loop decision rules (or sequential open-loop or adaptive decision rules), then *by definition* they treat the stochastic process generating the values of the government's policy variables as superexogenous for the parameters of their own decision rules, and the Lucas critique will not apply.[9] Taking the values of the policy variables as given assumes in effect that the private sector cannot or does not need to learn the form of the implicit policy rule.[10] At a deeper theoretical level, it may simply be that an atomistic private sector does not play a strategic game with the policymaker at all and each individual agent simply adopts a passive role, treating the government's policy actions as an exogenous stochastic process and an input into their optimal feedback decision rules. The standard use of the fictional representative agent to represent the private sector and to support models of strategic interaction is theoretically flawed, as demonstrated by Kirman (1992), and, until this issue of establishing rigorous theoretical (not necessarily micro) foundations for aggregate behavior[11] can be convincingly resolved, it would seem to be impossible to argue one way or the other between the use of (sequential) open-loop or feedback decision rules by the private sector theoretically. Nevertheless, it is clear that Lucas's presumption that policy variables will not be superexogeneous is not the only theoretical position that could be taken, even in a strategic game.[12]

Turning to the second part of the supposition (and ignoring the ambiguity over the first part), we need to address the question of how the private sector would act if it were faced with uncertainty about the relevant constraints, including the government's policy rules (or actions), when forming its *rational* decision rules. One approach would be to formalize the constraints as inexact [through v_t in equation (2)] and employ the methods of *stochastic* decision theory—the route that is standard throughout economics.[13] Sims has frequently made this point when discussing the Lucas critique and has emphasized that it would only be rational for the private sector to incorporate a stochastic specification for government policy whose range space was sufficiently broad to incorporate any reasonable policy change. Given this specification, the appropriate reaction to potential future policies would then be considered within the design of the optimal decision rule at the outset and thus there would be no need to derive a new decision rule simply because a new drawing of the random policy variable had been obtained.[14]

4. ROBUST DECISION RULES

This argument to treat the private sector's views as to the potential range of policy changes stochastically returns us to the earlier discussion of what constitutes rational behavior and hence what is an *optimal* decision rule. It is now critically important to recognize that simply adopting a stochastic approach to uncertainty and using a stochastic decision theory does *not* guarantee robustness in the sense of performance invariance in the face of misspecifications in the assumed constraints or disturbances. A much more powerful approach to dealing with an uncertain environment, which has not yet generally been appreciated as a good basis for rational economic behavior, but has been used by control engineers for a number of years, is to explicitly employ *robust* decision theory.[15] Critically, this robust decision theory largely treats uncertainty as the existence of deterministic shocks and *non*, stochastic. Hence, using robust decision theory, we can still maintain Lucas's position of viewing a policy change as a parametric or structural change in a rule [i.e., a change in λ in (2)], be it either a deterministic or a stochastic rule, and yet show that the private sector's optimal decision rule will *not* be required to change with a change in policy. This runs in direct contradiction to Lucas's basic argument. The distinction between the two approaches to forming decision rules under ignorance as to the constraints (and other aspects of the decision problem), either stochastic or robust decision theory can be seen to be an aspect of the difference between risk and uncertainty as identified by Knight (1921). Robust decision theory has recently emphasized a deterministic approach to modeling the unstructured shocks hitting the decision problem (uncertainty), whereas standard stochastic decision theory has employed probability models (risk). Essentially, the robust approach does not presume that economic agents are able to employ probability distributions but argues that they might treat shocks on an individual basis and employ decision rules that would be able to achieve their ultimate objective for a given class of such deterministic shocks.

What we now question is the notion of optimality and rationality that Lucas employs to characterize economic behavior under uncertainty. Recognizing that rationality is only defined relative to a given loss function, we can see that, instead of using the standard theoretical approach in economics through utility maximization (which describes how economists believe economic agents *should* act), we might wish to consider robust behavioral motivations that might better describe how people actually *do* behave. Rosenbrock and McMorran (1971), for instance, distinguish between good, bad, and optimal decision rules: Their point is simple. *Optimal* rules are often not *good* rules in the sense that optimality follows from an exact specification of the optimization problem, including the constraints faced. Any deviation from the assumed form of the constraints may yield extremely poor performance from the *ex ante* "optimal rule." This poor performance in the optimal rule is exactly what leads Lucas to claim that agents' decision rules will change in the face of deviation from the assumed form for the government's policy rules, stating "everything we know about dynamic economic theory indicates that this presumption (that the decision rule will not change) is unjustified." We beg to differ. Under certain conditions, assuming either a different formulation for the loss function employed by economic agents or different state information would lead to optimal but robust decision rules. Perhaps, as we shall see later, this would involve integral action, which would remain invariant in the face of disturbances and shifts in the constraints (such as those induced by policy changes) while still achieving desired performance objectives such as a zero-steady-state error in tracking some desired equilibrium position.

4.1. Robustness and Integral Action

It has been standard practice for control engineers to design controllers that achieve their objective, given an inexact model of the system to be controlled, and hence to allow deviations from the assumed constraints. One simple step in this direction is to suggest that agents formally use proportional, integral, and derivative (PID) decision rules, which, by incorporating integral action, recognize the error between current behavior and some desired position and adjust actions accordingly to ensure that the error is asymptotically zero; see Phillips (1954). Salmon (1982), following standard control theory, showed how such PID rules could be derived from within classic intertemporal optimization problems and led to econometric specifications that have become well known as error correction mechanisms (ECM's) [see, for instance, Hendry and Anderson (1977), Hendry and von Ungern-Sternberg (1981), Nickell (1985), and Hendry (1995) for a recent discussion]. In fact, Salmon and Young (1978) had, earlier demonstrated the ability of such PID rules to achieve a zero-steady-state error precisely in the case of a parameter change in an equation of an econometric model within a policy optimization exercise. The point is then that, contrary to Lucas's claim, it is possible to consider that the private sector uses optimal (PID) decision rules that have an invariant structure that is unaffected by perturbations in the assumed constraints but robustly delivers an optimal response in the face of changing government policy of the form considered by Lucas.[16]

Following Athans (1971) we can demonstrate this invariance argument and the need for integral action in robust decision rules with a simple example. Consider a decisionmaker faced with a constraint described by the following first-order system:

$$\dot{y}(t) = -\alpha_0 y(t) + \beta_0 v(t). \tag{4}$$

This constraint could represent the reduced form of the economic system he faces, including the government's policy reaction function; so, a change in the parameters of (4) could represent a change in the government's policy rule. His decision variable is $v(t)$ and his objective is to drive $y(t)$ toward some desired position, y^*, [a requirement that we can express as $\lim_{t\to\infty} y(t) = y^*$], which may, for instance, be the equilibrium described by some classical micro theory. The problem that our representative agent faces is to find a practical decision rule recognizing his uncertainty about the parameters α_0 and β_0. Let us define the error $e(t) = y^* - y(t)$; then, the error dynamics implied by the system are given by

$$\dot{e}(t) = -\alpha_0 e(t) - \beta_0 v(t) + \alpha_0 y^*. \tag{5}$$

A standard approach would then be to set up the optimization problem:

$$\min_{v(t)} \int_0^\infty [g e^2(t) + v^2(t)]\, dt. \tag{6}$$

Without discounting, there is no solution to this problem because the convergence of the cost integral requires $\lim_{t\to\infty} e(t) = 0$ and $\lim_{t\to\infty} v(t) = 0$, but the optimal solution implies that $\lim_{t\to\infty} v(t) = (\alpha_0/\beta_0) y^* \neq 0$. More importantly for the present discussion, the dependence of the optimal decision variable on the assumed values of α_0 and β_0 is also clear and, if the true constraint were to be given by different values for these parameters, say α and β, then the desired target would not be achieved. Given in this case that the target is a constant, there would be a constant steady-state error given by

$$e_{ss}(t) = \left(1 - \frac{\beta\alpha_0}{\alpha\beta_0}\right) y^*.$$

One way to introduce integral action into economic agents' decision rules, which then ensures that the specified target behavior will be achieved *independently* of the specific values taken by the constraint parameters, is to allow the rate of change of the decision variable to enter into the cost function. Let us define $u(t)$ as $\dot{v}(t)$. Then, we can set up the state-space model and optimization problem as follows: Let the state variables be defined as $x_1(t) = e(t)$ and $x_2(t) = \dot{e}(t)$, so that

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= -\alpha_0 x_2(t) - \beta_0 u(t). \end{aligned} \tag{7}$$

Then, by optimizing

$$\int_0^\infty \left[q x_1^2(t) + u^2(t)\right] dt$$

subject to the state-space model (7), we obtain, by standard linear quadratic optimization theory,

$$u(t) = g_1 x_1(t) + g_2 x_2(t), \tag{8}$$

where

$$\begin{aligned} g_1 &= \sqrt{q}, \\ g_2 &= \frac{1}{\beta_0}\left[-\alpha_0 + \sqrt{\left(\alpha_0^2 + 2\beta_0\sqrt{q}\right)}\right]. \end{aligned} \tag{9}$$

Hence, the optimal decision rule is given by

$$v(t) = g_2 e(t) + g_1 \int e(\tau)\, d\tau,$$

which has proportional and integral terms acting on the "disequilibrium error," $e(t)$. Notice, however, that this decision rule ensures a zero-steady-state error for any simple parameter changes, as can be seen since the closed-loop gain is unity in the limit. Alternatively, by applying the final value theorem of Laplace transforms to the error dynamics, which are given by

$$\ddot{e}(t) = -\alpha_0\dot{e}(t) - \beta_0\dot{v}(t) = -(\alpha + \beta g_2)\dot{e}(t) - \beta g_1 e(t), \tag{10}$$

we can see that $\lim_{t\to\infty} e(t) = 0$ for *any* value of α, $\beta > 0$ and y^*, which ensure stability of the error dynamics. The critical insight is that, by introducing the derivative of control action into the cost function, we ensure that $\lim_{t\to\infty}\dot{v}(t) = 0$, which is equivalent to ensuring that $\lim_{t\to\infty} v(t) = c$, where c is *any* unknown constant.[17]

There are several different ways of introducing integral action into agents' decision rules (see virtually any control text). Indeed, many naturally occurring economic variables may induce integral action and, through a proper specification of stock and flow equilibrium, economic behavior may naturally ensure a zero-steady-state error without any particular need to respecify the agents' loss function, as suggested by the earlier example. Notice also, as discussed by Salmon (1982), that simple integral action may not be sufficient in the face of more general changes in the constraint structure than simple parameter shifts. The most general theory developed in the control literature following this approach for this disturbance rejection and tracking problem implies, as described by Salmon (1988), that the economic agent should form an "internal model" [see Francis and Wonham (1976)] of the government's behavior, which is then formally exploited in the design of its *invariant* decision rule, which would then be able to achieve the desired optimal economic objective regardless of any changes in government actions. *However, this is no more and no less than the assumption that Lucas himself uses when he assumes that economic agents employ rational expectations or in the dynamic general equilibrium implied by a dynamic game*. The subjective model of the true environment that economic agents use to form their rational expectations and their decision rules must include this internal model. Then, the internal-model principle implies that they will therefore be able to construct invariant robust decision rules in the face of changing government behavior. Dynamic general equilibrium, rational macro theory would then in fact seem to deny the Lucas critique.[18]

4.2. Robust Decision Rules and H^∞

Robust control theory has recently developed along a number of different lines; see Zhou et al. (1996) and Petersen et al. (2000). The most significant advance in robust decision theory has been the development of H^∞ theory [see Başar and Bernhard (1995) or Green and Limebeer (1995)], which has considerably relaxed the nature of the lack of knowledge regarding the environment facing the decision-maker, enabling robust rules to be used in the face of norm-bounded deviations from a nominal model. It is important to recognize, however, that the standard economist-theorist's route of adopting a statistical framework for decisionmaking under uncertainty is not normally employed today by practical control engineers seeking robust control rules. H^{∞} theory was developed within a deterministic frequency-domain environment and can be seen as an extension of decisionmaking under Knightian uncertainty in which a worst-case scenario is conceived and the best decision in the face of this potential worst case is then computed. The disturbances may, in fact, have no stochastic interpretation at all and may be purely deterministic once off shocks. The result is a maxmin strategy that is common in other areas of robust analysis, such as statistics, where the objective is to ensure that disturbances have as little effect as possible on the output.

The environment that we envisage the rational economic agent as inhabiting is dynamic and hence there is a dynamic mapping from the shocks to the output variables that is captured by a suitably defined transfer function or state-space form for the nominal model. Rationality now implies the need to minimize the impact of mispecifications, or shocks, so that the agents' decision rules satisfy given performance bounds for all disturbances within some norm-bounded set. Structured or unstructured disturbances to this nominal model can take a wide variety of forms, either as additive or multiplicative mispecifications in terms of parameters or quite general frequency-dependent disturbances bounded within some region of the nominal model. H^{∞} theory has established several alternative routes to obtaining linear-feedback decision rules that provide the desired performance in the face of arbitrary misspecifications satisfying the norm-bounded constraints. Bąsar and Bernhard (1995) provide a dynamic game interpretation of H^{∞} decision theory in which nature, as the other player, attempts to construct the worst-case strategy that one could face and then one constructs the best possible decision rule, given that Nash assumption. The saddle-point structure that results from this game provides a minmax solution to the robust policy design problem. Whittle (1990, 1996) has discussed the relationship between risk-sensitive decision theory and H^{∞} theory and shows how the stochastic formulation of risk sensitivity can lead to a decision rule equivalent to that from H^{∞}, which is, in principle, deterministic. So, a demand for robustness to deterministic shocks to utility can be equivalent to risk-sensitive behavior. Hansen et al. (1999) also discuss how this demand for robustness ties in with Epstein and Zin's (1989) recursive utility theory and Gilboa and Shmeidler's (1989) version of Knightian uncertainty.

In the context of the Lucas critique, we consider the case of economic agents who face uncertainty in the policy rule of the government and hence, in response, employ H^{∞} decision rules, which incorporate in their potential range of uncertainty all rules that the government may adopt. The argument here is simply that they form the best-response decision rule, given a prespecified range of alternative rules that could describe the government's actions. Given a reference model M, they seek, in the terminology of Onatski and Stock (2002), to minimize the risk $R(K, M + \Delta)$ associated with employing decision rule K, given a potential range of perturbed models, $M + \Delta$, where $\Delta \in D$, corresponding to different government policy rules. The robust decision problem facing the private-sector agent then corresponds to solving

$$\min_{\{K\}} \sup_{\Delta \in D} R(K, M + \Delta),$$

where the range of alternative policies that could be delivered by the government could extend as far as the worst possible policy that the agent could face. The result is a fixed coefficient feedback rule that is designed to provide good behavior over a range of alternative constraints rather than be optimal for just one. The latter is, of course, the nonrobust case considered by Lucas in the critique.

In a general dynamic setting, we could consider the private sector constructing decision rules in the face of both additive (different government policy rules) and multiplicative (e.g., parameter or nonstochastic) uncertainty. As a simple example, we assume a state-space form for the constraints that the private sector faces, which includes the government's decision rules as additive adjustments to a nominal model, as in Onatski and Stock (2002):

$$x(t) = Ax(t) + B_u u(t) + B_w w(t),$$
$$y(t) = Cx(t) + Du(t),$$

where the private sector wants to determine its optimal rule for the decision variable u, w represents disturbances or the arbitrarily specified alternative forms of government policy expressed as deviations from the nominal model; and y represents the observable output variables. A number of different formulations of the form of the uncertainty could be made. Notice that the state variable $x(t)$ may include both backward-looking and forward-looking variables such as asset prices, as in Miller and Salmon (1985a), and hence may also include forward-looking expectational terms.

The H^∞ decision problem then seeks to find the feedback rule that minimizes the closed-loop H^∞ norm. Alternatively, a quadratic objective function can be constructed by considering the bound on the closed-loop ∞ norm, representing the linear operator or transfer function of the disturbances to the output variables,

$$\|G_{yw}\|_\infty = \sup_{\|w(t)\|_2 \neq 0} \frac{\|y(t)\|_2}{\|w(t)\|_2} < \gamma,$$

where γ is called the performance bound. Notice that γ is inversely related to the norm bound on the uncertain inputs $\|w(t)\|_2$. A decision rule satisfying this will also satisfy the squared bound, and so,

$$\|G_{yw}\|_\infty^2 = \sup_{\|w(t)\|_2 \neq 0} \left\{ \frac{\|y(t)\|_2^2}{\|w(t)\|_2^2} \right\} < \gamma^2.$$

For the supremum to satisfy the strict inequality, the term within the brackets must be bounded away from γ^2 so that, for some ε,

$$\frac{\|y(t)\|_2^2}{\|w(t)\|_2^2} \leq \gamma^2 - \varepsilon^2.$$

Then, multiplying through by the denominator yields

$$\|y(t)\|_2^2 - \gamma^2\|w(t)\|_2^2 \leq -\varepsilon^2\|w(t)\|_2^2.$$

The satisfaction of this inequality for all disturbances and some ε is equivalent to the bound on the closed-loop ∞ norm, and the left-hand side of this expression can be used as an objective function:

$$J_\gamma(x, u, w) = \|y(t)\|_2^2 - \gamma^2\|w(t)\|_2^2.$$

Standard tools of differential game theory can now be applied to find the decision rule that minimizes this objective function in the presence of the worst possible disturbance. This leads to a guaranteed performance given by the upper value of the dynamic game

$$\inf_u \sup_w J_\gamma(x, u, w). \qquad \mathbf{(11)}$$

If we let $\gamma^* \equiv \inf\{\gamma$: the uppervalue in (11) is finite$\}$, then γ^* is the minimum value of $\|G_{yw}\|_\infty$ that can be obtained. So,

$$\gamma^* = \inf_u \|G_{yw}\|_\infty$$

and the decision rule that obtains this infinum is known as the H^∞ *optimal* rule. However, in many cases, it is sufficient to design a decision rule that corresponds to a *suboptimal* case that guarantees $\|G_{yw}\|_\infty < \gamma$ with $\gamma > \gamma^*$. This can be achieved by approaching as closely as desired on the optimal value, $\inf_u \|G_{yw}\|_\infty$, iterating on γ. This suboptimal controller attains the upper value for the game corresponding to $\gamma < \gamma^*$. Whittle (2002) and Bernhard (2002) have discussed, in their contributions to this special issue, the relationship between the parameterization of the stochastic risk-sensitivity formulation (LEQG) of the decision problem and its relationship to γ^* in this deterministic H^∞ problem; see also Dupuis et al. (2000).

Since the dynamic game above is nonstochastic, standard open-loop Pontryagin methods can be used to determine the solution, with the first-order conditions or Hamiltonian system obtained from an unconstrained minmax problem with objective function defined using the costate variables p:

$$J_\gamma^*(u, w, p) = \int_0^\infty y'y - \gamma^2 w'w + 2p'(Ax + B_u u + B_w w - \dot{x})\, dt.$$

The resulting Hamiltonian system is given by

$$\begin{bmatrix} \dot{x}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} A & -B_u B_u' + \gamma^{-2} B_w B_w' \\ -C'C & -A' \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix},$$

with a resulting decision rule given by

$$u(t) = -B_u' p(t).$$

The uniqueness of the robust decision rule for the range of different models defined by the uncertainty bound arises from the uniqueness of the costate process $p(t)$, which can be obtained either through an eigenvector decomposition and the solution to the two-point boundary value problem as in Miller and Salmon (1985a) or through the solution to the implied Riccati equation. Notice the generality provided by this Hamiltonian formulation of the H^∞ problem. The state vector can be set up so as to include a variety of strategic dynamic games, as in Miller and Salmon (1985b), including expectational or forward-looking variables (such as costate variables in the solution of strategically asymmetric or Stackelberg games). The solution is then obtained as a single consistent mapping of the unstable variables onto the stable variables under control, or onto the stable manifold. This single mapping through the solution to the Riccati matrix ensures that policy is consistent with expectations and expectations are consistent with policy.

Formally, a specification of the norm-bounded uncertainty is equivalent via the small-gain theorem to solving a (state feedback) H^∞ problem for the nominal system. This H^∞ problem is, in turn, equivalent to a particular dynamic game problem with a corresponding Riccati solution. The initial specification of the norm-bounded uncertainty set describing the private sector's uncertainty as the government's decision rule translates into a unique costate process for *this set* of deviations from the nominal model. Thus a single robust decision rule can be used by the private sector that satisfies the desired robust performance criteria, regardless of changes in the government's policy function within this set. Clearly, if the government were to choose a policy that was not incorporated within the norm-bounded uncertainty set considered by the private sector initially, then the Lucas critique would continue to apply. The question of whether or not the Lucas critique then applies in practice becomes a question of whether or not we believe rational economic agents adopt sufficient robustness within their decision rules.

This very rapid description of the H^∞ approach has highlighted the main issue that robust rules may remain invariant within a set of predefined norm-bounded uncertainty. However, we should also note that H^∞ rules in common with H^2 do not naturally incorporate integral action, although, as with H^2 decision theory, integral effects can be included by construction; see Zhou et al. (1996, Sect. 17.4). Integral action along with the internal model and H^∞ decision rules may provide a better *behavioral* basis for economic theorists to characterize rational economic behavior than the "knife edge," nonrobust utility maximers envisaged in the Lucas critique. Fundamentally, in this case the force of the critique is critically weakened and may vanish completely. This analysis can be extended to consider economic agents using H^∞ decision rules in strategic settings and also to form robust expectations using H^∞ filtering methods.

4.3. Conclusions

The Lucas critique raised a range of fundamental issues, of both economic theory and econometric practice at that time. It emphasized the endogeneity of policy and expectations. At a very basic level, it asked the question of what do we mean by rational economic behavior and hence what is the nature of the empirical relations captured in macroeconometric models. We feel that the critique failed to recognize the implications of the dynamic general equilibrium that exists between the government and the private sector and hence misinterpreted the nature of agents' decision rules and the information within the data sets on which the empirical models may be built. If the deep strategic relationship between the government and the private sector were to change, then the critique would have more force, but that would require a rational explanation within a dynamic game that we find hard to accept. By failing to recognize the endogeneity of the policy change, the critique is logically internally inconsistent in its assumption of rational behavior and hence in the adoption of optimal decision rules both before and after the assumed policy change.

Moreover, the form of rational behavior under uncertainty embodied within the Lucas critique presumes that neither party in the dynamic game adopts robust decision rules. We have stressed that, if the notion of rationality is extended so that economic agents are seen to employ robust decision rules, then these can remain invariant in the face of a wide class of potential changes in government policy, and the critique fails.

NOTES

1. We suspect that this view is often by default since it is clearly "inconvenient" to recognize the critique without a suitable theoretical response or the tools to provide a valid practical response.

2. Ericsson et al. (1998) have expressed a somewhat more general statement as follows: "*A model cannot be used for policy if implementing the policy would change the implications from the model, since then the policy outcome would differ from that predicted by the model. A policy which leads to a change in the model parameters which are assumed constant contradicts the basis of itself.*"

3. See Sims (1982a), Sargent (1984), and Marcellino and Salmon (2002) for further discussion of this view.

4. When viewed as a criticism of estimating reduced-form models, the critique is really making the same argument made several years earlier by Goldfeld and Blinder (1972) when discussing the endogeneity of stabilization policy and the "St. Louis Approach."

5. Under standard conditions, whether in a single-player context or the strategic context, if an infinite horizon is maintained, then both players will choose an optimal decision rule that is invariant, *given* the initial formulation of the policy problem. If a finite policy horizon is considered, then a rule that implies continuously changing actions is implied for both players, but this systematically changing policy rule is not what is considered in the Lucas critique since both players perfectly anticipate and account for it in formulating their optimization problems.

6. This fundamental argument against the Lucas critique has been made at various times by Sims (1982a, 1982b, 1987, 1988) and Blinder (1984). Sargent (1984) also seems to accept this point, but argues from an empirical point of view regarding the optimality of government policy within the sample period consistent with that in the simulation period.

7. LeRoy (1995) refers to this distinction as between *stationary* and *rational* expectations.

8. Notice, however, that a very wide range of values for policy instruments can follow from a single fixed-policy rule with changing input data. Similarly, although policy action is obviously intended to alter the dynamic response or properties of the economy, this *does not* imply that the structure of the economy changes.

9. See Hendry and Mizon (1996) for a discussion of superexogeneity and Başar and Olsder (1995) for a discussion of information patterns in dynamic games.

10. In fact, Sims (1980, p. 12, and 1982a, p. 110) questions whether, in reality, government policy is ever sufficiently systematic that a coherent rule could ever be identified.

11. See also Hahn and Solow (1995).

12. In fact, we believe that the only sensible way to resolve this issue in practice is to formally test the superexogeneity of policy variables using the tools described by Engle and Hendry (1993).

13. Lucas (1976, Sect. 6) effectively makes this suggestion but continues to regard a policy change as a parameter shift rather than as a new realization of the stochastic term within the fixed structure of the policy rule.

14. Of course, this does not mean that policy is random, but that uncertainty about policy implies that a stochastic description is feasible. The stochastic specification also captures what Sims means by a regime change, which is effectively a policy change that is not captured in the range space of the stochastic policy variable, and this is likely to be a very rare event. This, in turn, may make the whole issue of econometric policy evaluation irrelevant if, as mentioned earlier, such a policy is not captured in the model. Sims's position, then, is that there may be poor evidence empirically in an existing model if the "input topology" regarding potential policy variation is not rich enough at the time of estimation, but this does not necessarily imply that the model is *theoretically* inadequate, as suggested by the Lucas critique—just poorly estimated.

15. Similar arguments for robust decision rules can be found in the earlier work of behavioral economists in the 1950's, such as Alchian (1950), Simon (1959), and Baumol and Quandt (1964). It turns out that effectively equivalent arguments can also be found in the adjustment cost literature to which Lucas (1967) also contributed.

16. Aside from recognizing the power of simple integral action, adaptive control theory attempts to estimate the unknown parameters of the system while constructing an optimal decision rule—a method that economists have accepted with the development of theories of learning. However, notice that the incorporation of integral action into a decision rule denies the need to learn the unknown parameters perfectly because the decision rule robustly takes up the slack created by imperfect knowledge of the constraints. For further discussion, see Salmon (1993).

17. For the present, the target is taken as a constant but the theory generalizes to more general dynamic forms for y^* with different costs of adjustment.

18. Notice also that the wide use of error or equilibrium correction mechanisms in the specification of large macroeconometric models would seem to have been empirically justified and is one clear difference between present-day macroeconometric models and their predecessors.

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