Using Copulas to Construct Bivariate Foreign Exchange Distributions

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York Financial Econometrics Meeting, June 2006

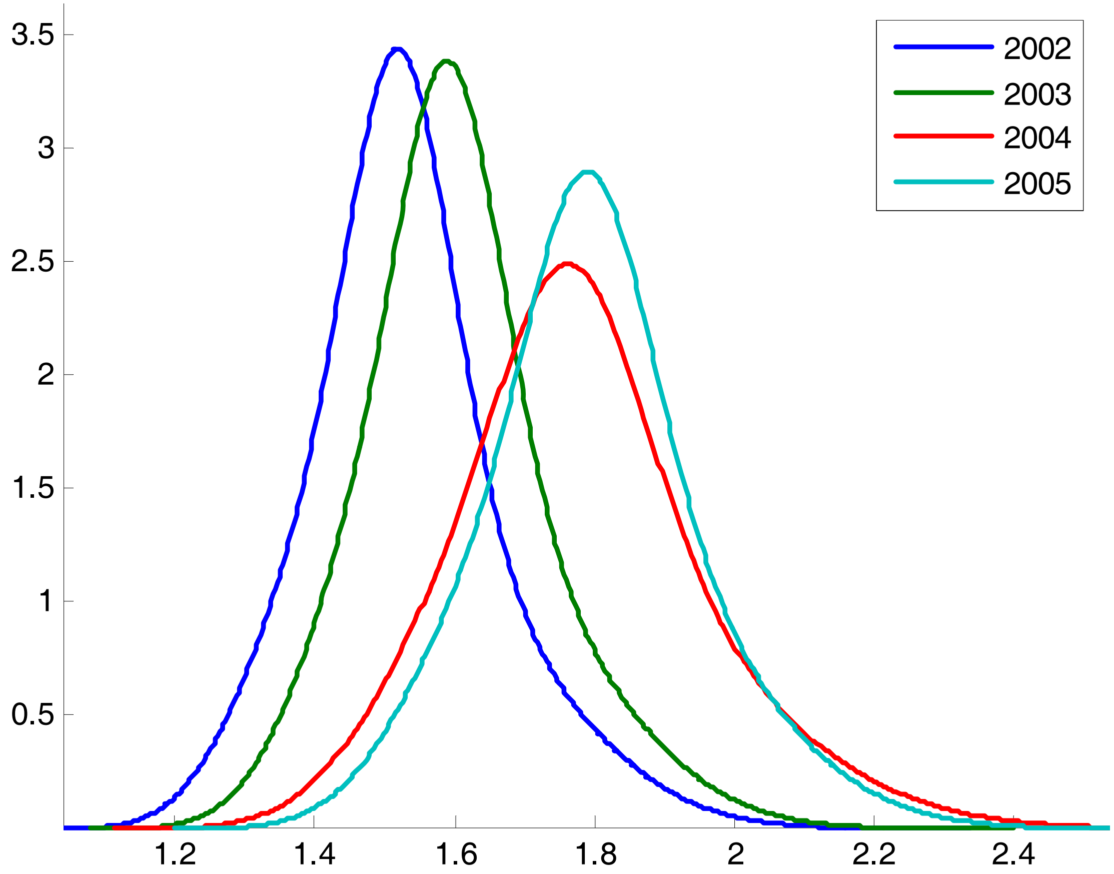
Overview

1. Risk neutral densities and copulas
2. The Sterling ERI
3. Pricing index options and multivariate contingent claims in general

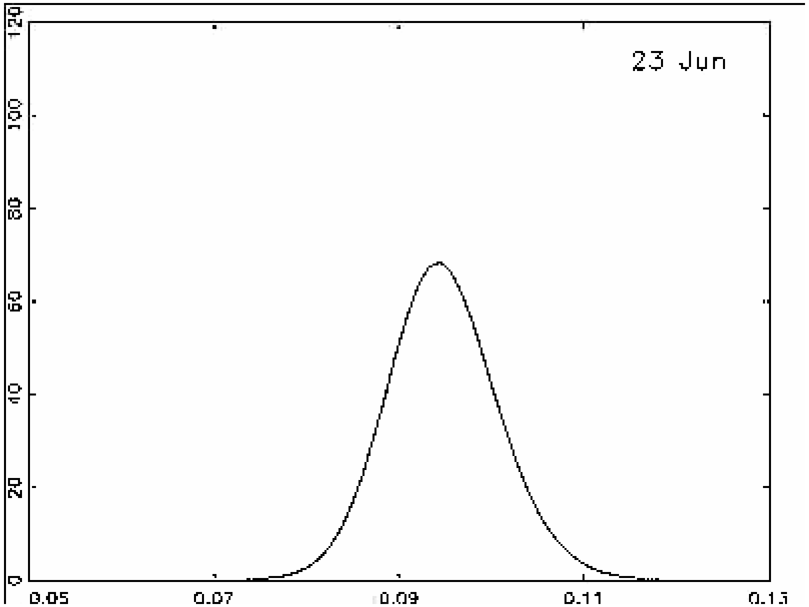
Derivatives and perceptions of risk

- *Central projection*: Futures and Forwards (+ risk premium)
- *Variance*: Implied Volatility
- *Balance of risk*: Risk Reversals
- *Likelihood of extreme events*: Strangles/Butterflies

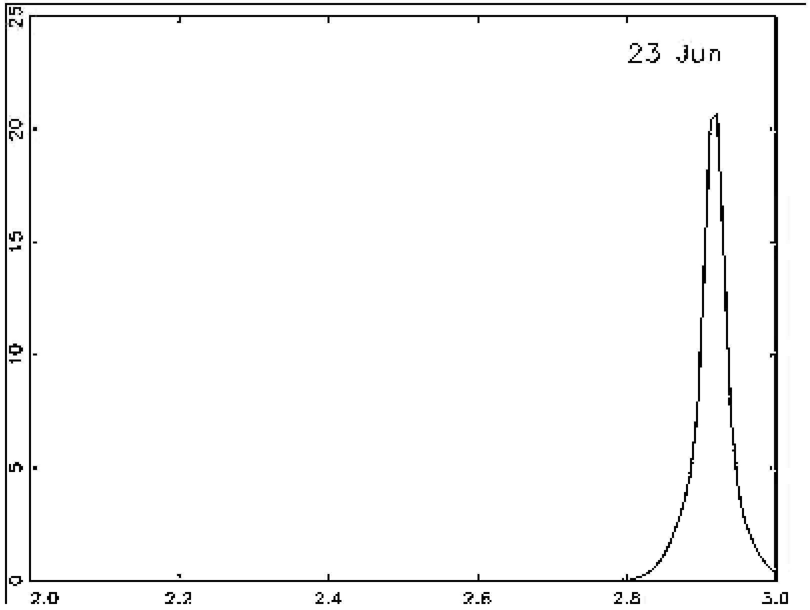
Option Implied PDFs: Eg. 12 month cable



Sterling ERM Crisis 16th Sept 1992
GBP Short rate



Sterling ERM Crisis 16th Sept 1992
DmGBP 1 month forward

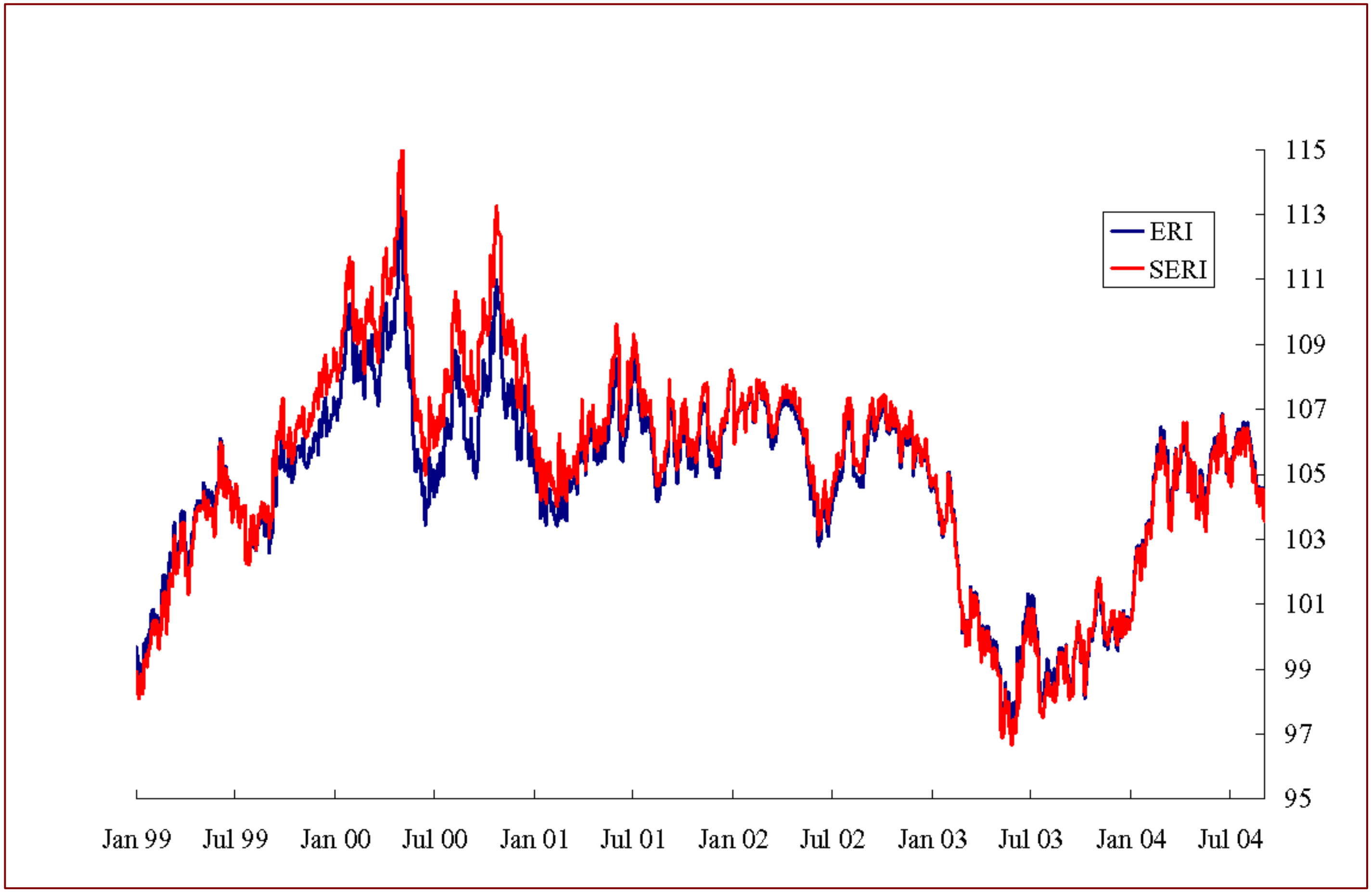


What can we do for the Sterling Exchange Rate Index (ERI)?

- Geometric weighted average of 21 currencies
- Would need to model a 21-dimensional multivariate distributions

Simplified ERI (SERI)

- Geometric mean of only 2 currencies (Dollar and Euro)



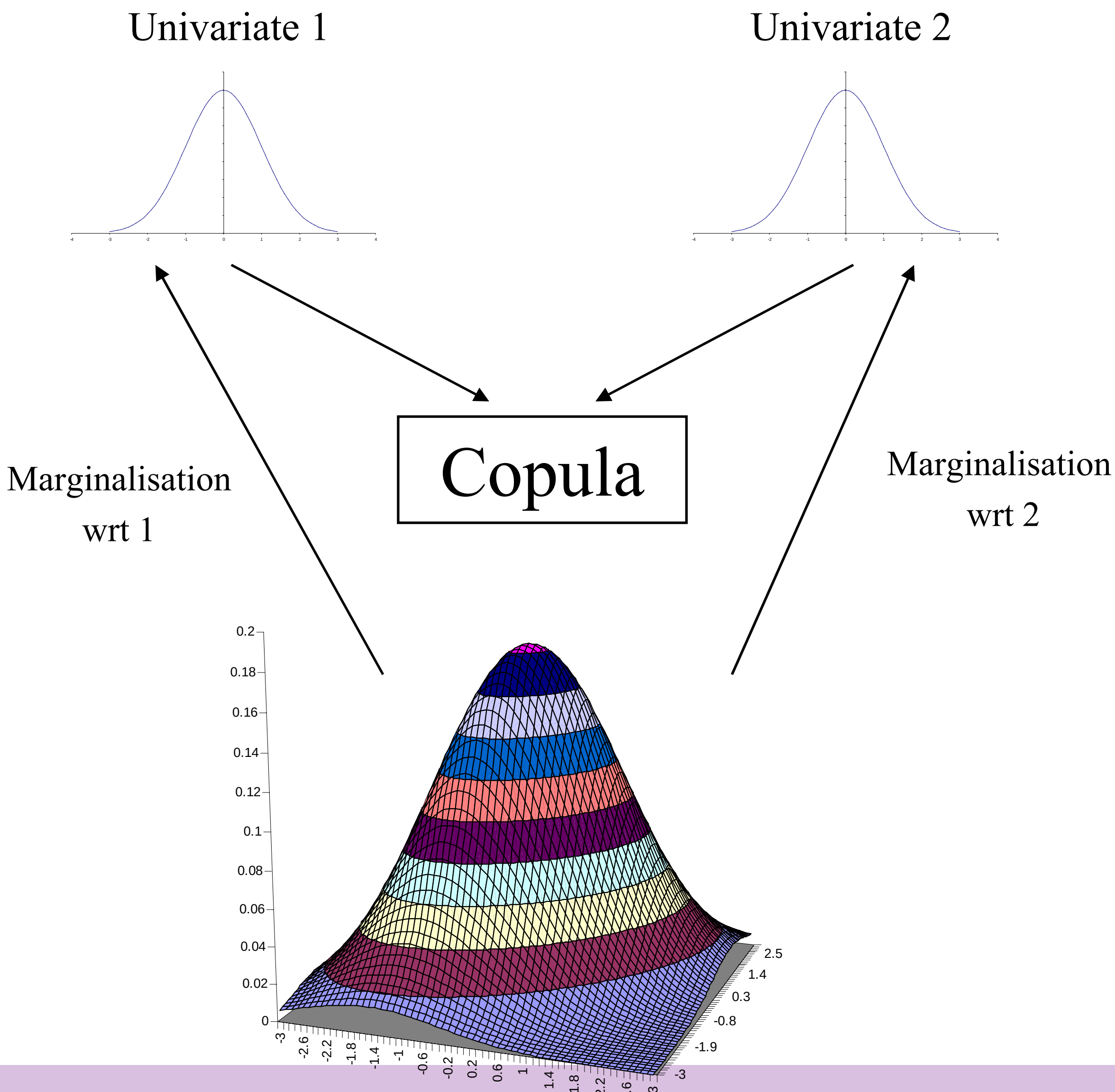
Need to model the joint distribution of euro-sterling and dollar-sterling

2 ad-hoc possibilities:

1. Use forwards and implied vols and implied correlation => *assume* joint normality
2. Use implied PDFs and assume independence

Ideally we would like to:

- Make *full* use of the information of the univariate distributions
- *And* model their joint dependence


Univariate 1
Univariate 2
Copula
Marginalisation
wrt 1
Marginalisation
wrt 2
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
-3
-2.6
-2.2
-1.8
-1.4
-1
-0.6
-0.2
0.2
0.6
1
1.4
1.8
2.2
2.6
3
2.5
1.4
0.3
-0.8
-1.9
-3

Copula C:

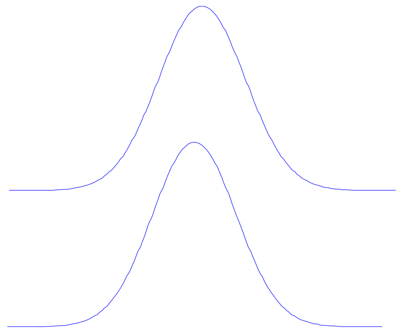
$$F_{xy}(x,y) = C(F_x(x),\ F_y(y))$$

Density:

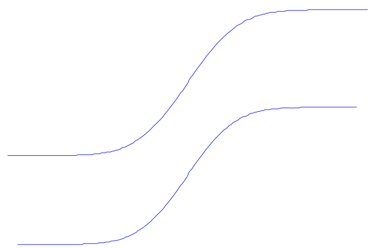
$$f_{xy}(x,y) = c(F_x(x),\ F_y(y))\ f_x(x)\ f_y(y)$$

Example 1: Bivariate Normal Distribution (r = 0.5)

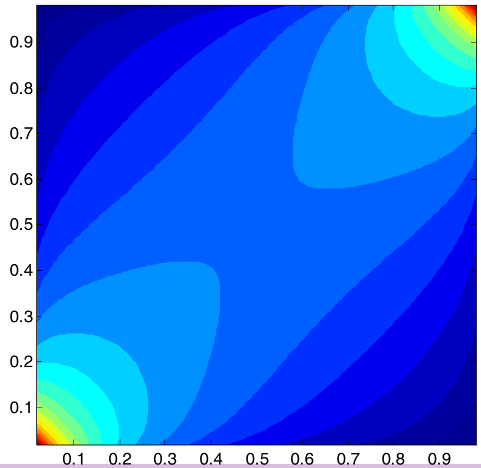
Normal
Marginal
PDFs

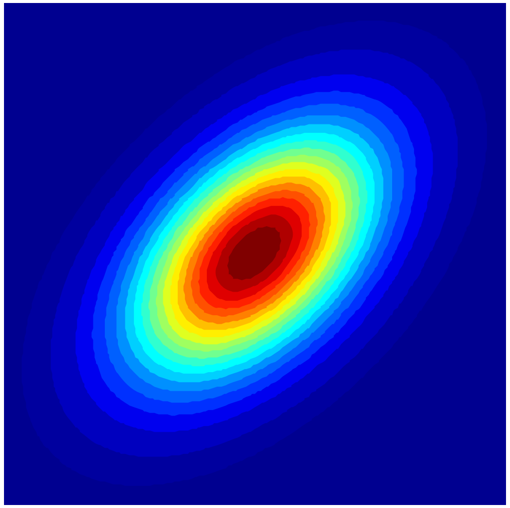


Normal
Marginal
CDFs



Copula

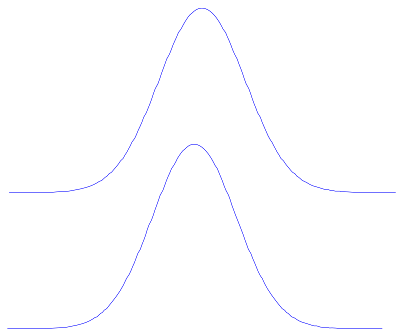




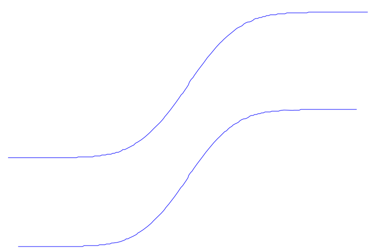
Bivariate Distribution

Example 2: Bivariate Normal Distribution (r = -0.5)

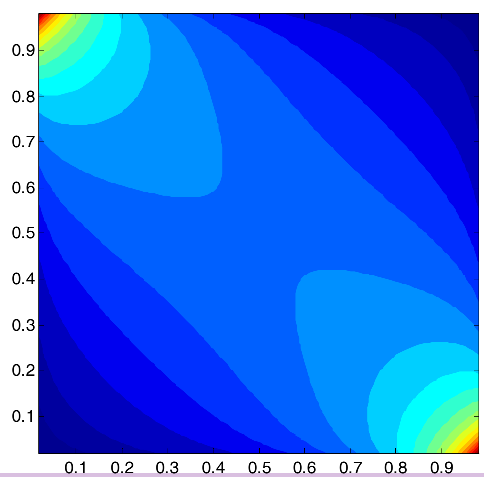
Normal
Marginal
PDFs

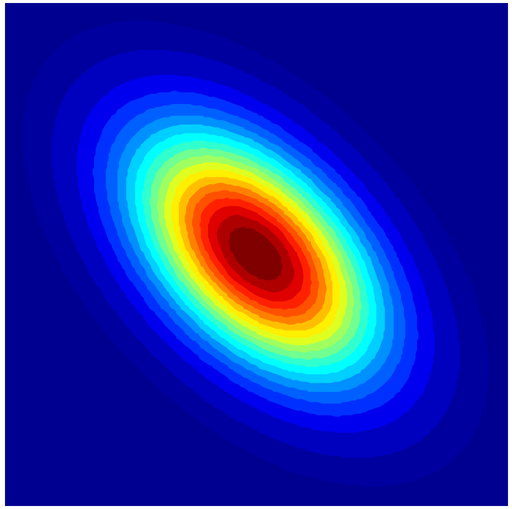


Normal
Marginal
CDFs



Copula

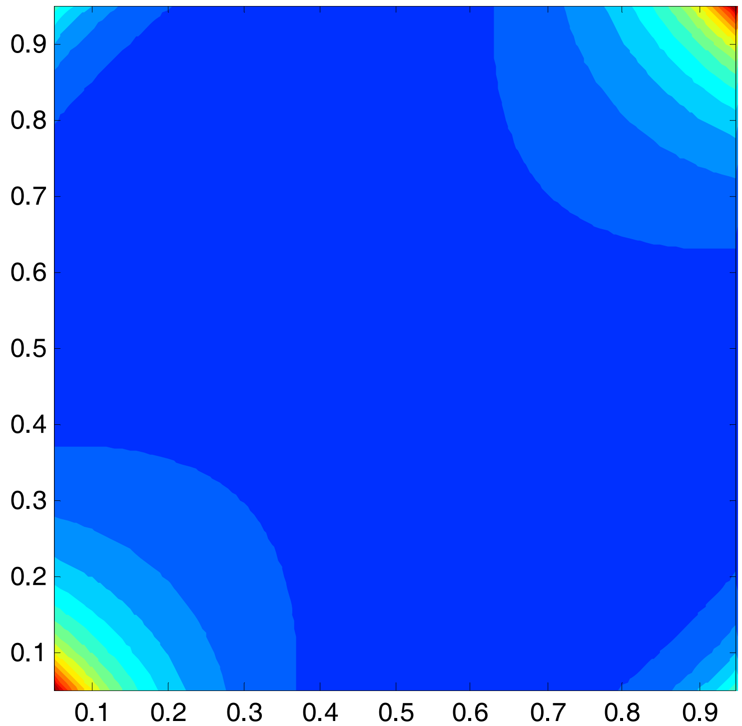




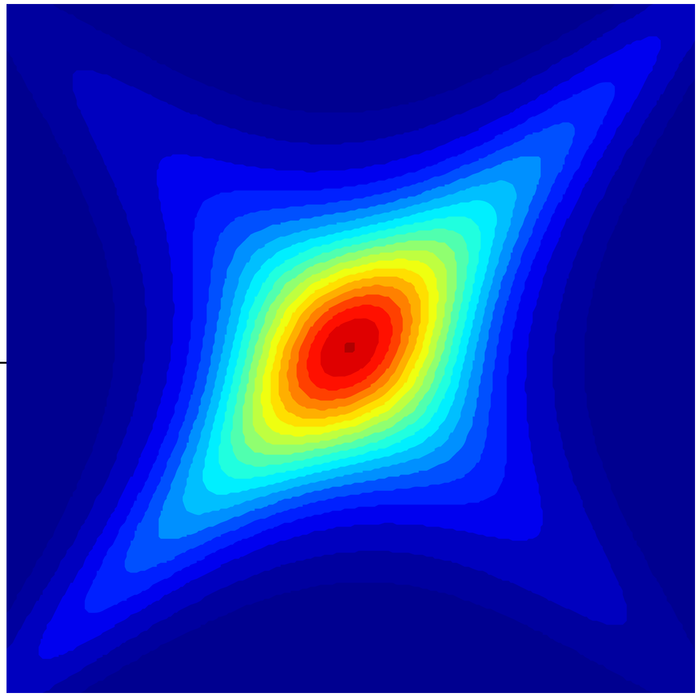
Bivariate Distribution

Example 3: Student *t*-Copula with 3 DoF (r = 0.5)

Copula

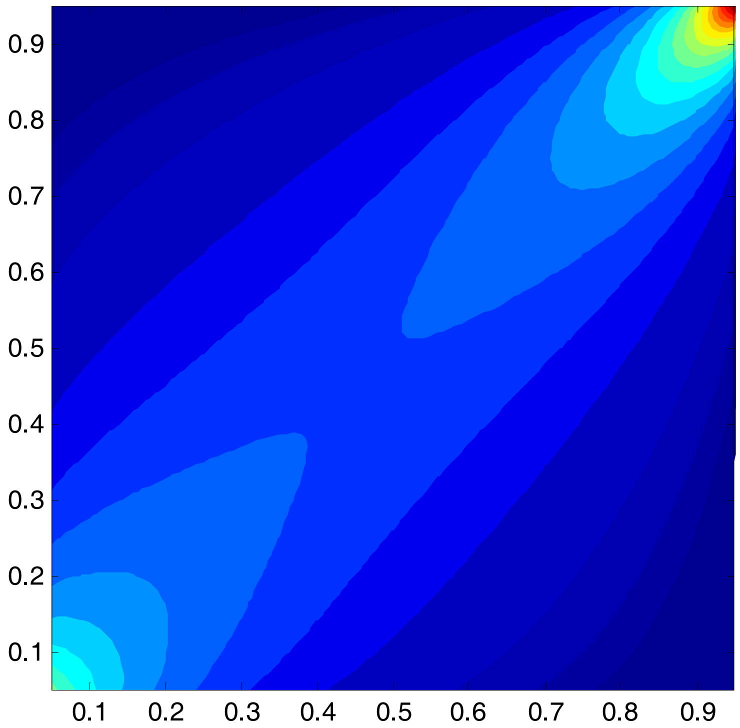


Distribution

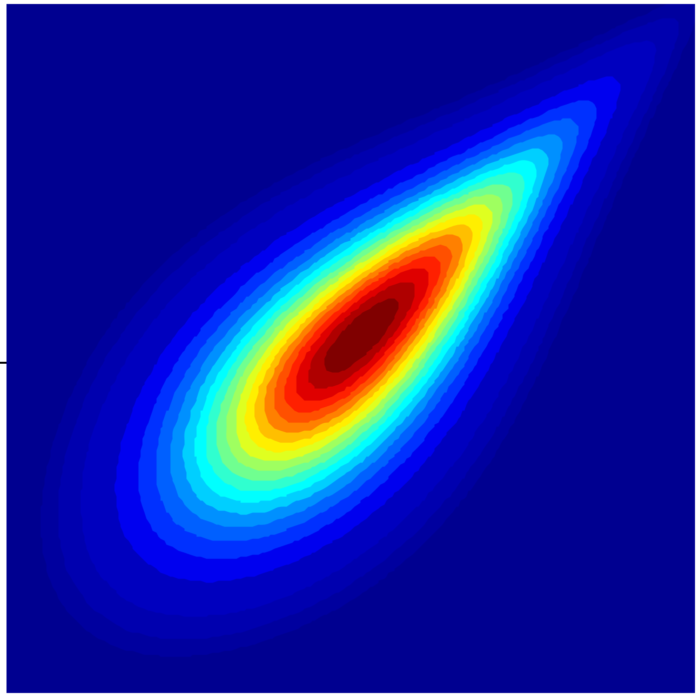


Example 4: Gumbel copula (p = 1)

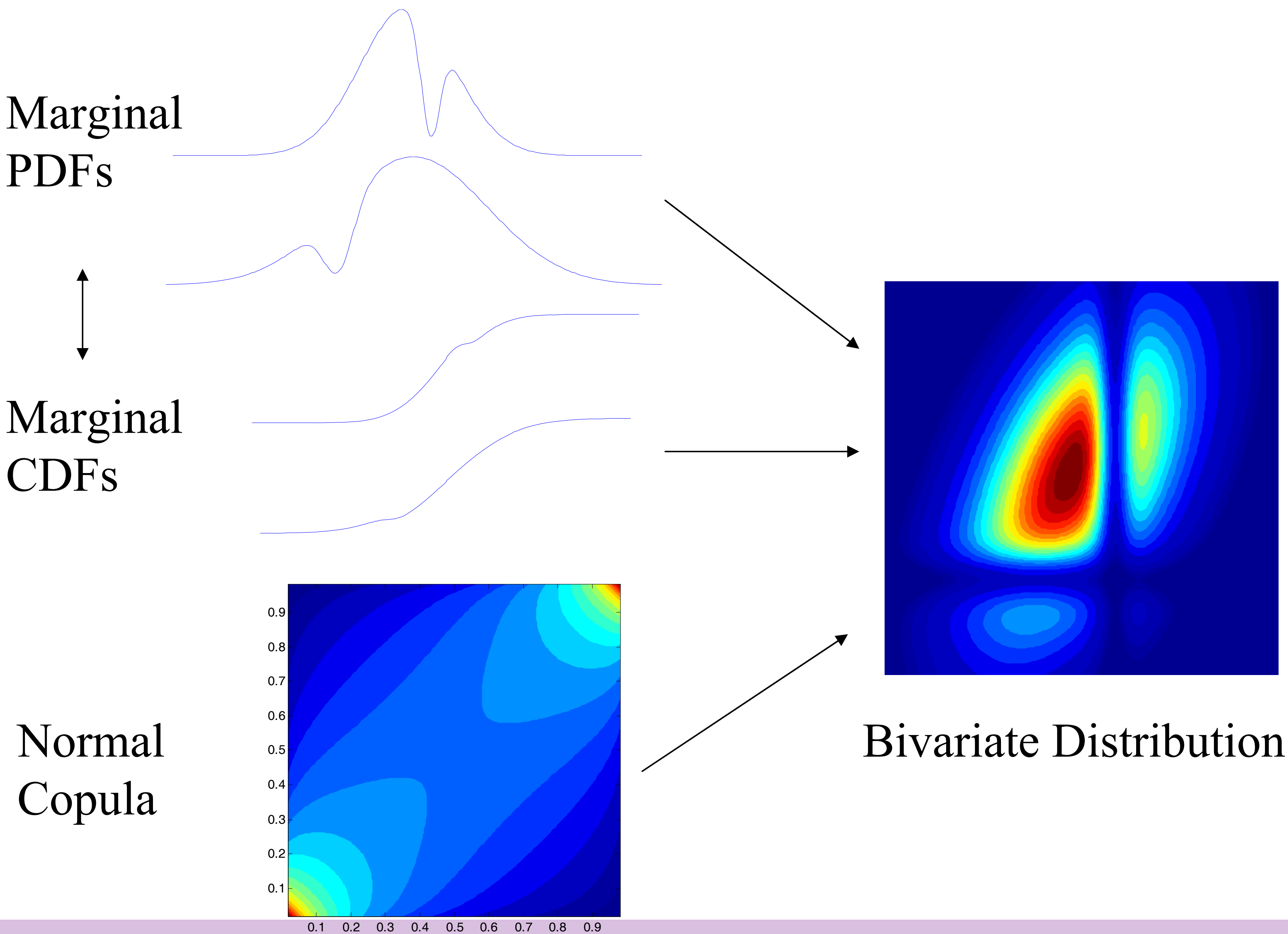
Copula



Distribution



Example 5: Normal copula with arbitrary margins



Properties of Copulae:

- Completely describe dependence structure between 2 (or more) random variables
- Invariant to increasing transformations

  eg: variables in logs have the same copula as variables in levels
- Bounded by *minimum copula* (perfectly negative association) and *maximum copula* (perfectly positive association)

How to choose the right copula?

- Large number of parametric + non-parametric copulae
- Maximum Likelihood
  - estimate margins and copula jointly
  - 2-step: estimate margins and then copula- misspecification issues
- AIC, Goodness of fit, Formal Significance tests (Guo and Salmon 2006)
- Robust Tests for Normality and Ellipticity (Chu and Salmon 2006) , L moments- Stein approximations- GMLM tests

Exchange rates: Need to ensure triangular no-arbitrage condition

Previous work

- Bikos (2000) and Taylor and Wang (2004): Use parametric copula to fit implied correlation coefficient from risk neutral margins-estimates one parameter with one observation.
- Rosenberg (2003 JD): Uses historical data to estimate copula parameter- doesn't obey risk neutrality.
- Bennett and Kennedy (2004 JD): Use semi-parametric copula – spline adjusted Gaussian-to fit several contracts on the third bilateral- uses more but still very limited information- 5 observations to estimate one parameter.

$$C_\phi(u,v) = \phi^{-1}(C_N(\phi(u),\phi(v)))$$

Our Main Objectives

- Fit a copula that is consistent with all three risk neutral marginal distributions (GBPEUR, GBPUSD and EURUSD) – obeys triangular arbitrage and employs a consistent numeraire.
- Efficient estimation by using information on full range of distributions and efficient and consistent inference.
- Need to derive the relationship between the univariate pdf of $z^{a,b}$ under risk neutral measure $\mathbb{Q}_a$ and the bivariate pdf of $z^{c,a}$ and $z^{c,b}$ under the risk neutral measure $\mathbb{Q}_c$ recognising the triangular arbitrage condition that $z^{a,b} = z^{c,b} - z^{c,a}$

First Step: Find relationship between the three margins under different risk-neutral measures

$$\boxed{f_z^{\mathbb{Q}_\mathrm{E}}(s) = \int_{-\infty}^{\infty} f_{xy}^{\mathbb{Q}_\mathrm{S}}(u, u-s)e^{-u}du}$$

where

x = EURGBP

y = USDGBP

z = EURUSD   ($x = y + z$)

Second Step: Use Sklar's theorem to construct the bivariate distribution between EURGBP and USDGBP

$$\boxed{f_{xy}^{\mathbb{Q}_S}(u,v) = c\left(F_x^{\mathbb{Q}_S}(u), F_y^{\mathbb{Q}_S}(v)\right) f_x^{\mathbb{Q}_S}(u) f_y^{\mathbb{Q}_S}(v)}$$

where c is a copula density.

Third Step: Estimate the copula by minimising the squared distance between the actual and the fitted third bilateral (*z*)

$$\hat{\theta} = \operatorname{arginf}_{\theta} \left[\int_{-\infty}^{\infty} \left(f_z^{\mathbb{Q}_{\mathrm{E}}}(s) - \hat{f}_z^{\mathbb{Q}_{\mathrm{E}}}(s;\theta) \right)^2 ds \right]^{\frac{1}{2}}$$

where

$$\hat{f}_z^{\mathbb{Q}_{\mathrm{E}}}(s,\theta) = \int_{-\infty}^{\infty} \hat{c}\left(F_x^{\mathbb{Q}_{\mathrm{S}}}(u), F_y^{\mathbb{Q}_{\mathrm{S}}}(u-s);\theta\right) f_x^{\mathbb{Q}_{\mathrm{S}}}(u) f_y^{\mathbb{Q}_{\mathrm{S}}}(u-s) e^{-u} du$$

Estimation

Following Basu et al. Biometrika 1998 this min density power divergence estimator lies within a general class of estimators (that includes KLIC (MLE)) where the divergence is given by ;

$$d_\alpha(g,f) = \int \{ f^{1+\alpha}(z) - (1+\tfrac{1}{\alpha}) g(z) f^\alpha(z) + \tfrac{1}{\alpha} g^{1+\alpha}(z) \} dz$$

L2 is given by taking $\alpha=1$. The KLIC minimiser follows by choosing $\alpha=0$

$$d_0(g,f) = \int g(z) \log\left\{ \frac{g(z)}{f(z)} \right\} dz$$

- Within this class between the L2 and KLIC estimators we trade off robustness for efficiency but under standard regularity conditions we obtain weak consistency in the divergence metric, asymptotic normality and a sandwich covariance estimator in either case.

- Alternatively in a local asymptotic geometry the embedding space of distributions is flat and L2 norm provides the geodesic distance minimiser.

Fourth Step: Extract the distribution of the SERI

ERI in log deviation:

$$\boxed{\xi = \omega_E x + \omega_D y}$$

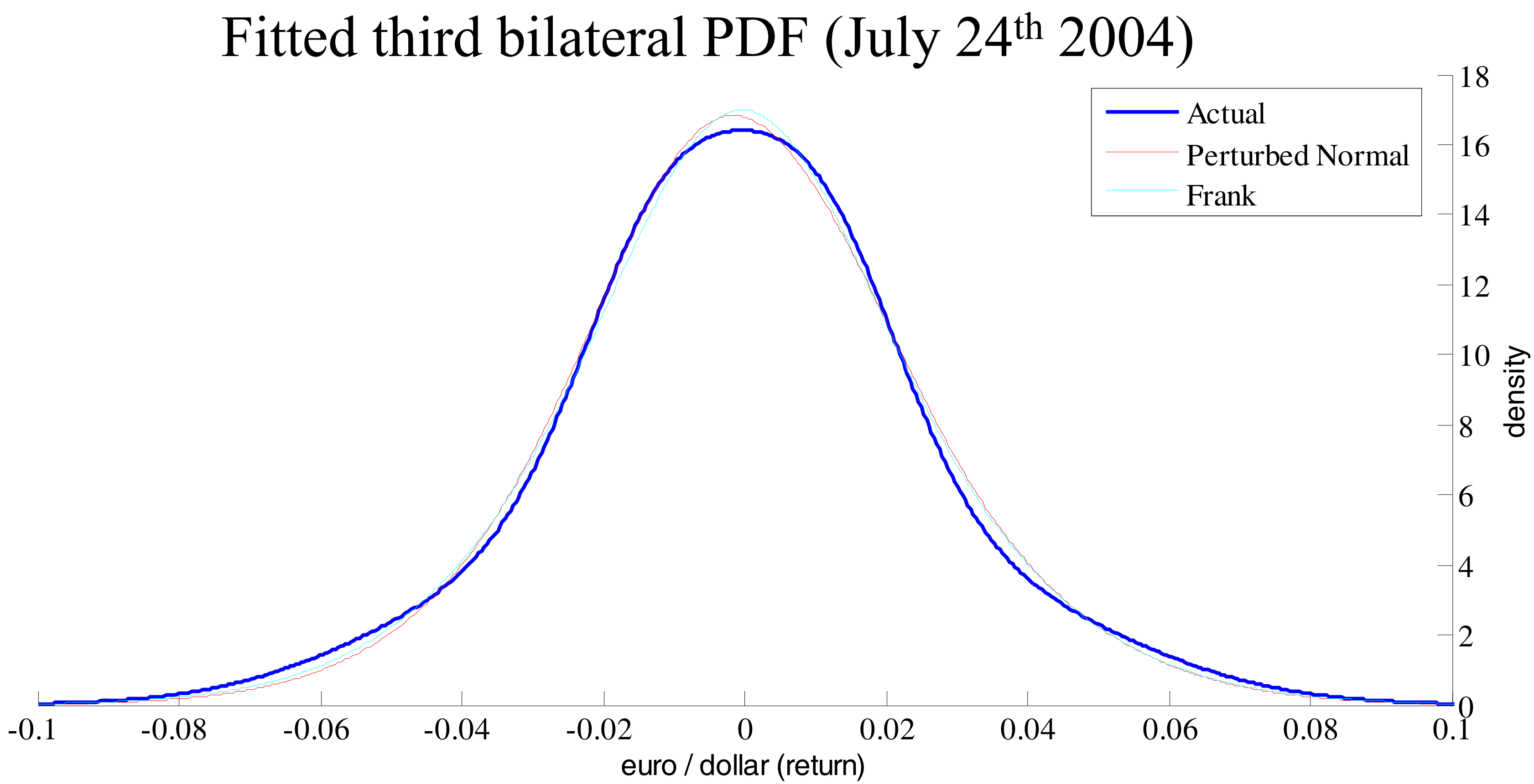
Integrate out:

$$\boxed{f_\xi(s) = \int_{-\infty}^{\infty} f_{xy}^{\mathbb{Q}_S}\left(u, \frac{s - \omega_E u}{\omega_D}\right) \frac{1}{\omega_D} du}$$

Summary

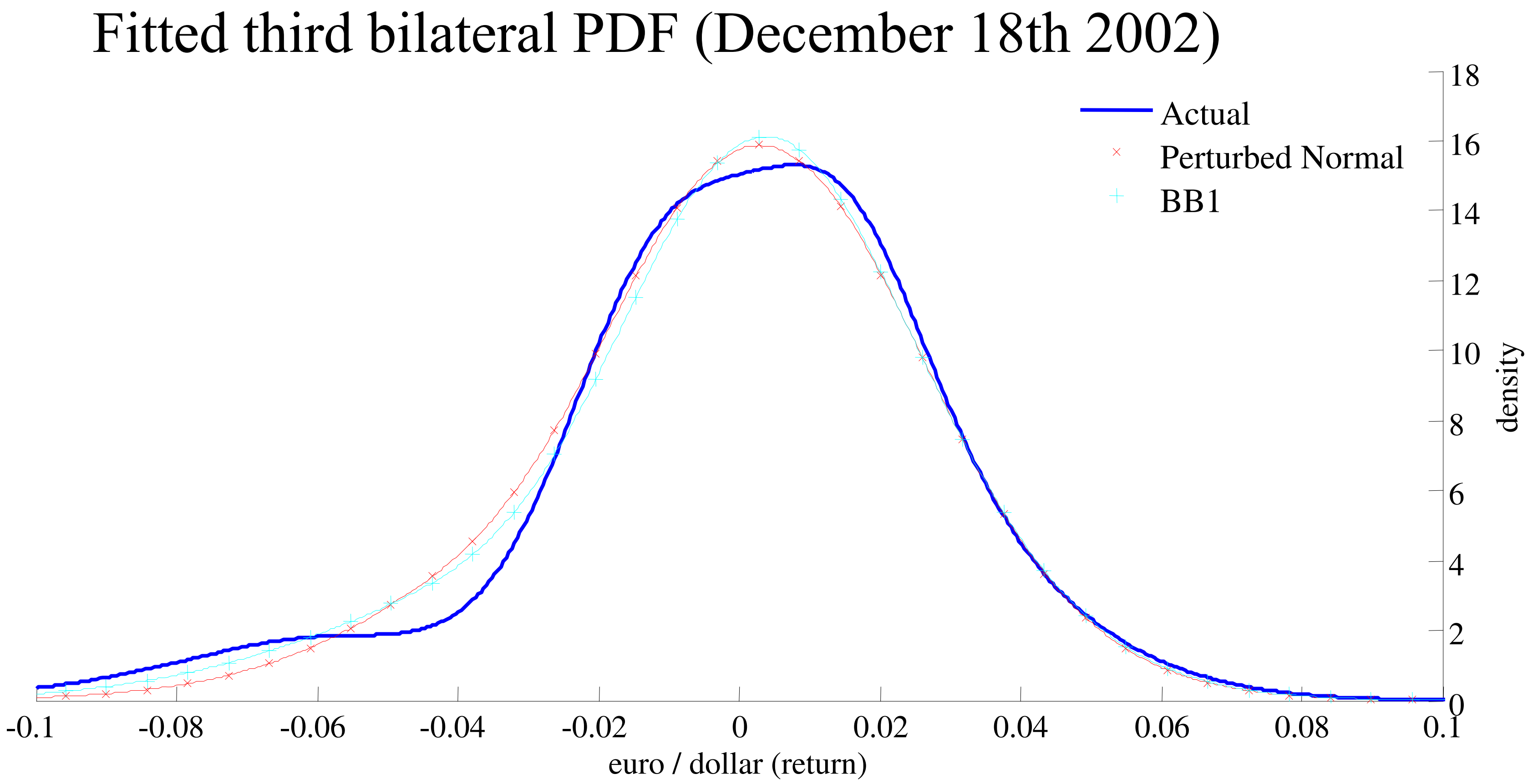
1. Establish a connection between a bivariate distribution of GBPEUR, GBPUSD and EURUSD under the respective RN measures
2. Express the bivariate distribution using a copula
3. Fit the copula to match the third bilateral (EURUSD)
4. Integrate out the SERI (or multivariate currency option payoff)

Comparing different copulas



ok when symmetric …

Comparing different copulas



… but bad fit when PDF is asymmetric

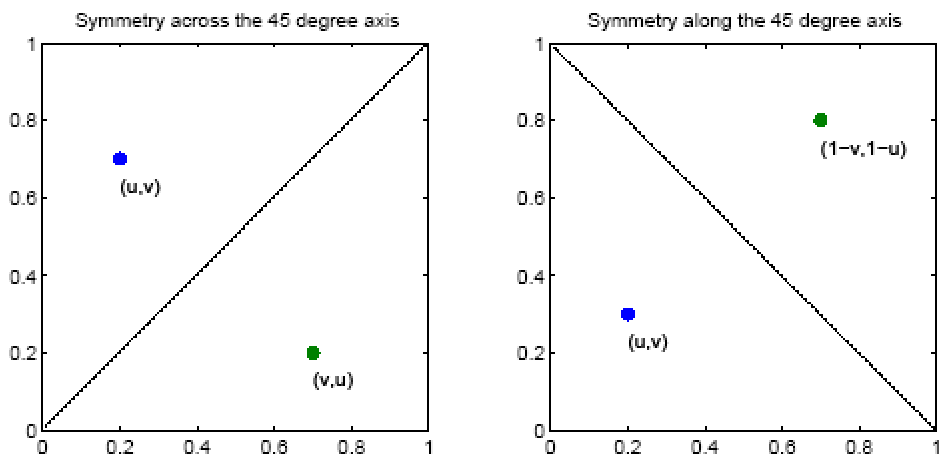
Asymmetric Dependence

- Asymmetry along the 45 degree line

$$c(u,v) \neq c(1-u,1-v)$$

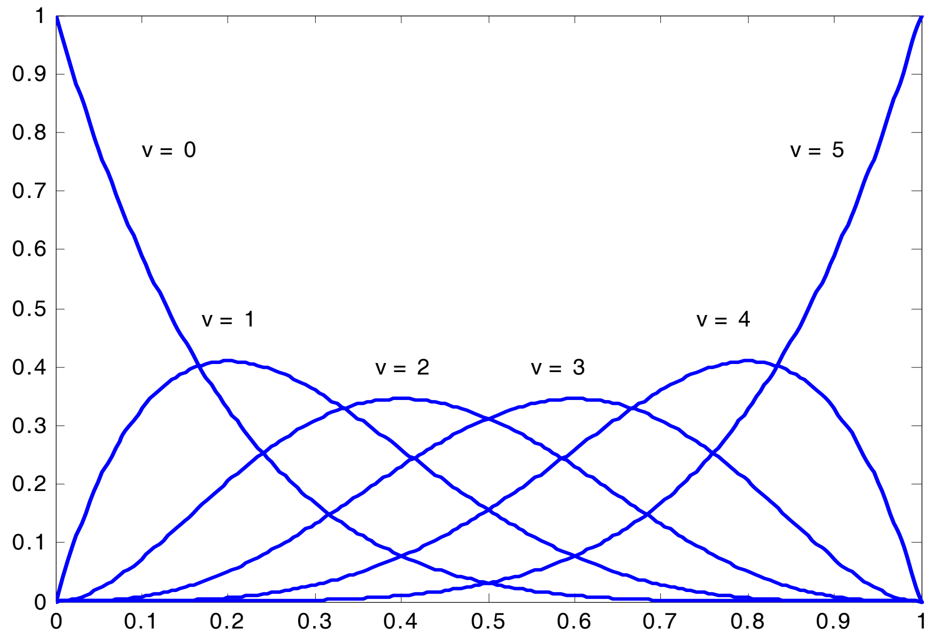
- Asymmetry across the 45 degree line

$$c(u,v) \neq c(v,u)$$

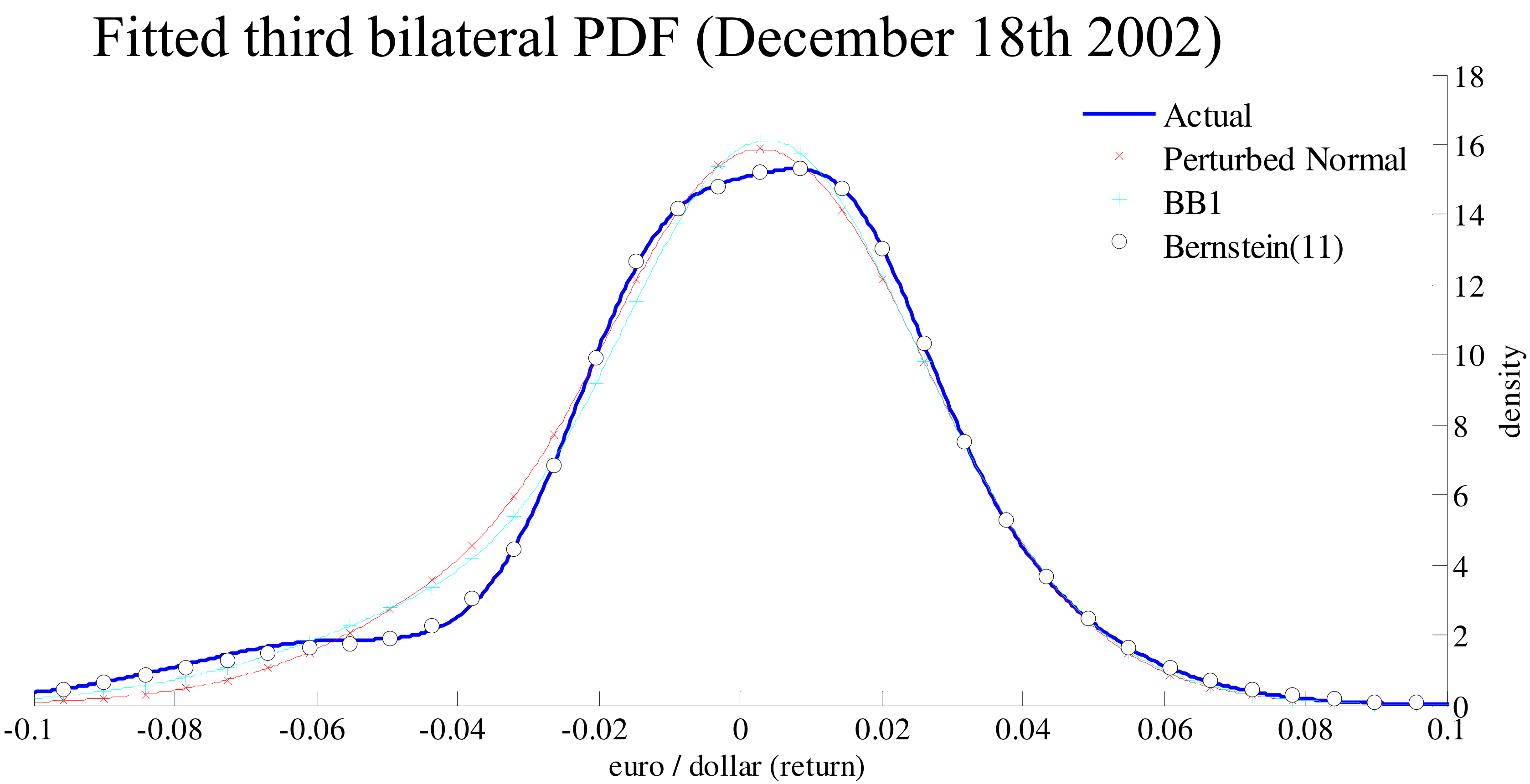


The Bernstein Copula

- Based on Bernstein polynomials (Sancetta and Satchell, 2004)
- Can approximate the shape of <u>any</u> possible copula



Comparing different copulas (cont'd)

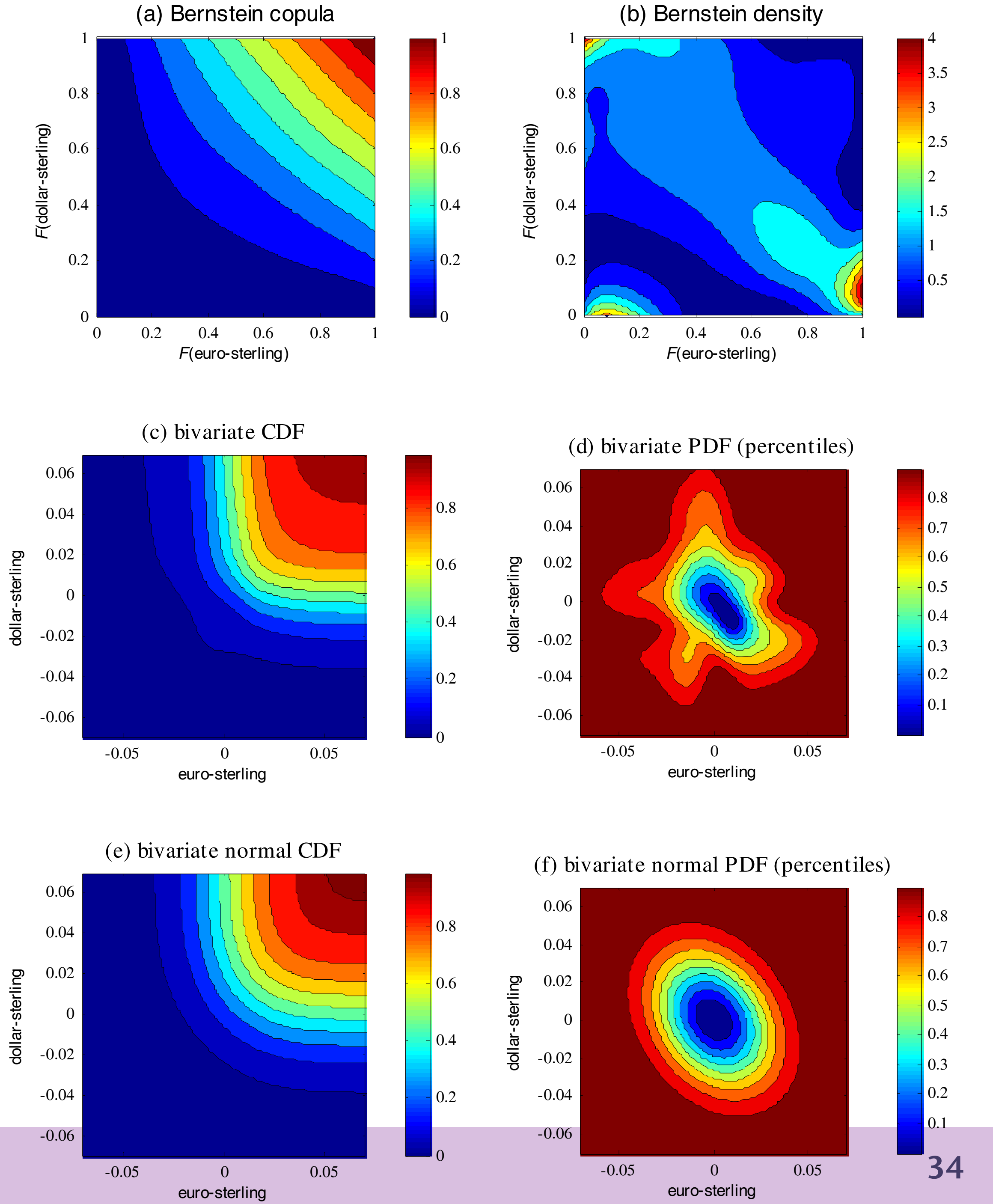


Comparing different copulas (cont'd)

Table 2: Comparison of Fit (18 Dec 2002)

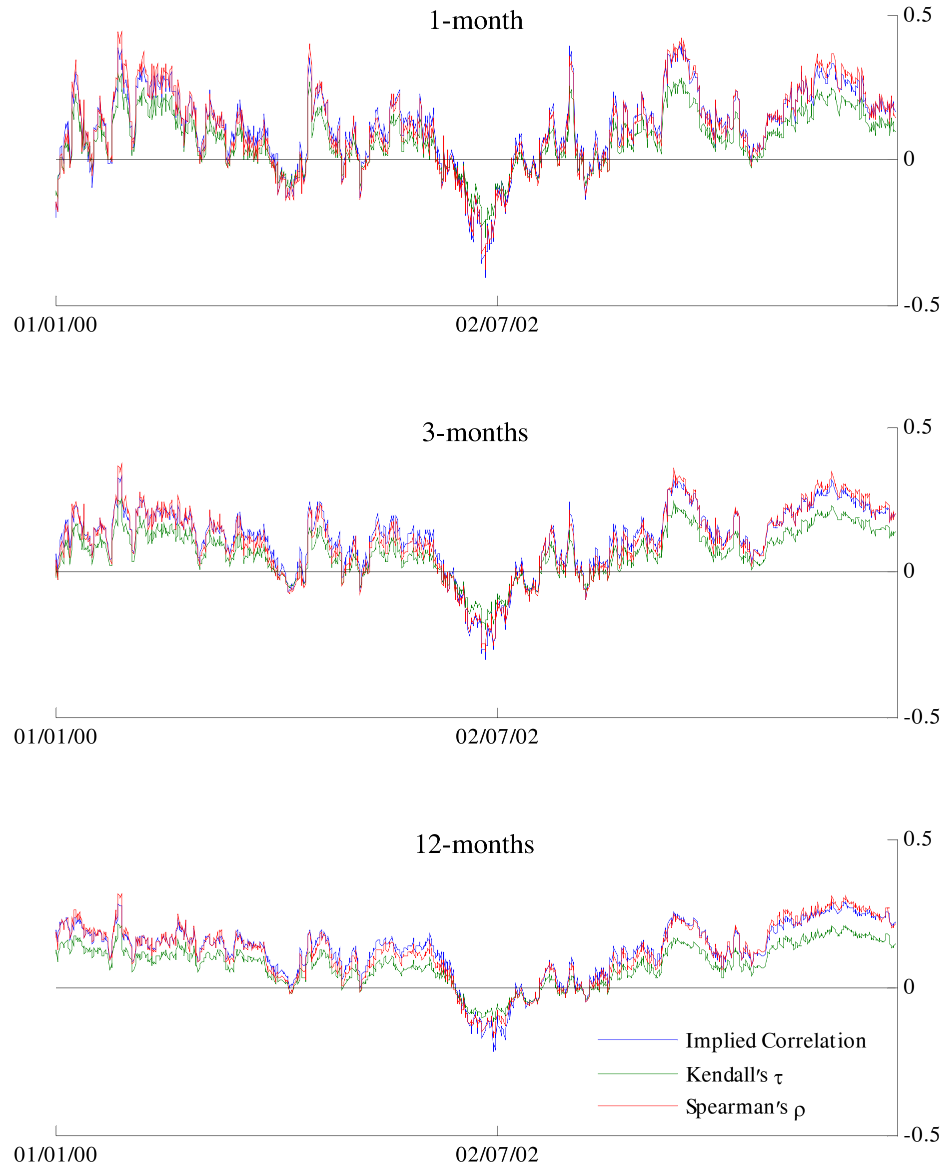
PDF	L^2-dist (%)	mean	std	skew	kurt
Actual	–	**-0.000**	**0.0297**	**-0.682**	**4.248**
Normal	29.95	-0.000	0.0273	-0.294	3.456
Frank	29.79	-0.000	0.0278	-0.310	3.614
Plackett	29.87	-0.000	0.0279	-0.311	3.621
BB1	23.96	-0.001	0.0284	-0.498	3.721
BB7	28.24	-0.000	0.0273	-0.339	3.570
Asy. Gumbel	27.31	-0.000	0.0277	-0.440	3.645
Pert. Normal	27.38	-0.000	0.0271	-0.319	3.462
Bernstein(2)	29.52	-0.000	0.0277	-0.307	3.587
Bernstein(5)	19.19	-0.000	0.0283	-0.529	4.038
Bernstein(7)	9.91	0.000	0.0295	-0.649	4.418
Bernstein(9)	5.55	0.000	0.0299	-0.685	4.511
Bernstein(11)	3.59	0.000	0.0297	-0.629	4.340
Bernstein(13)	4.89	0.000	0.0297	-0.659	4.420

Example of fitted copula and the implied bivariate distribution between EURGBP and USDGBP

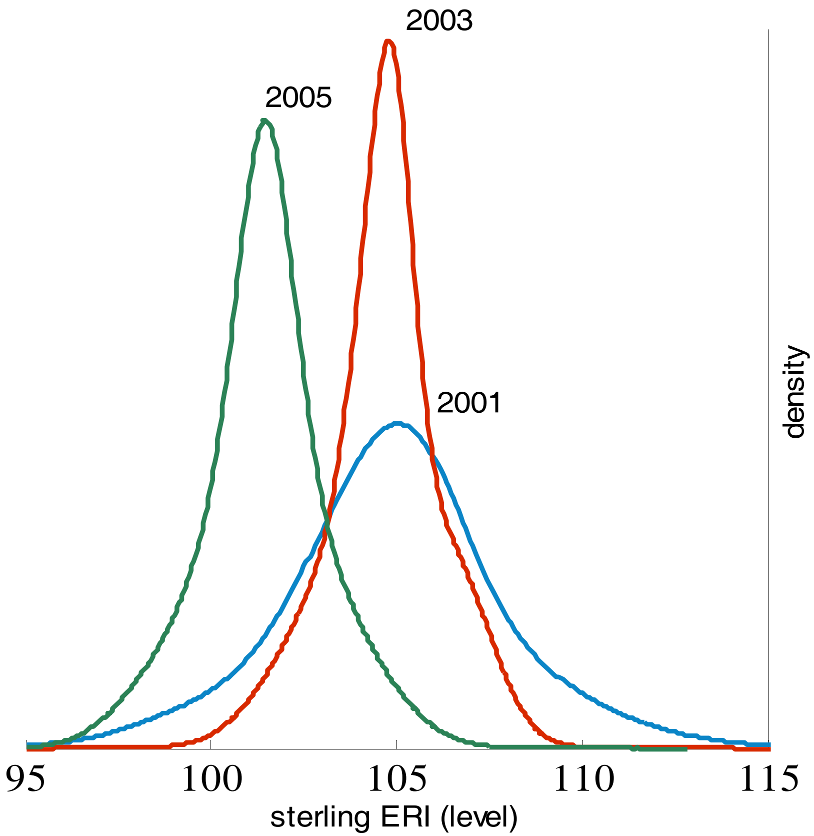


First four moments of the estimated SERI distributions

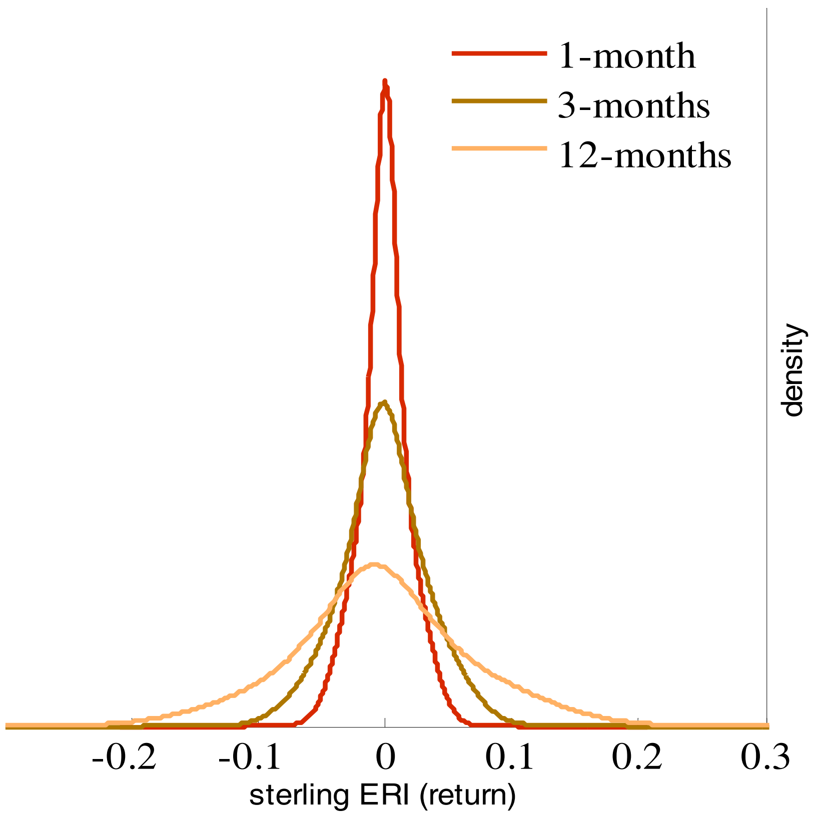
Dependence measures



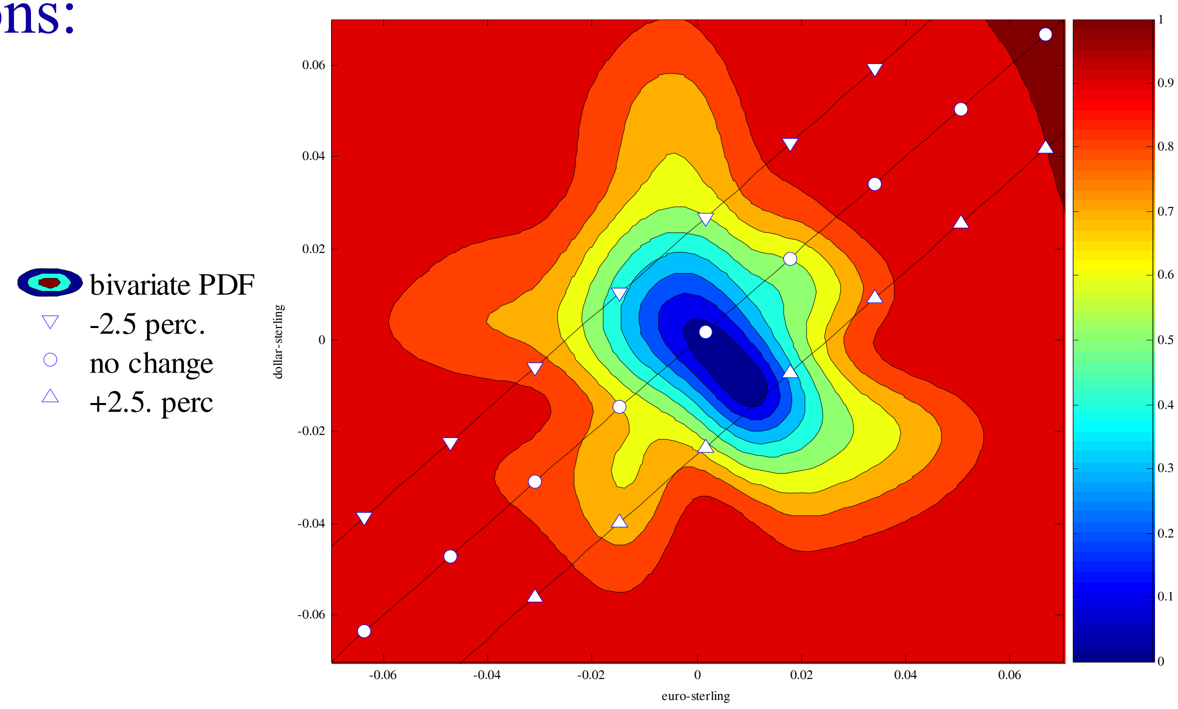
1-month level PDFs

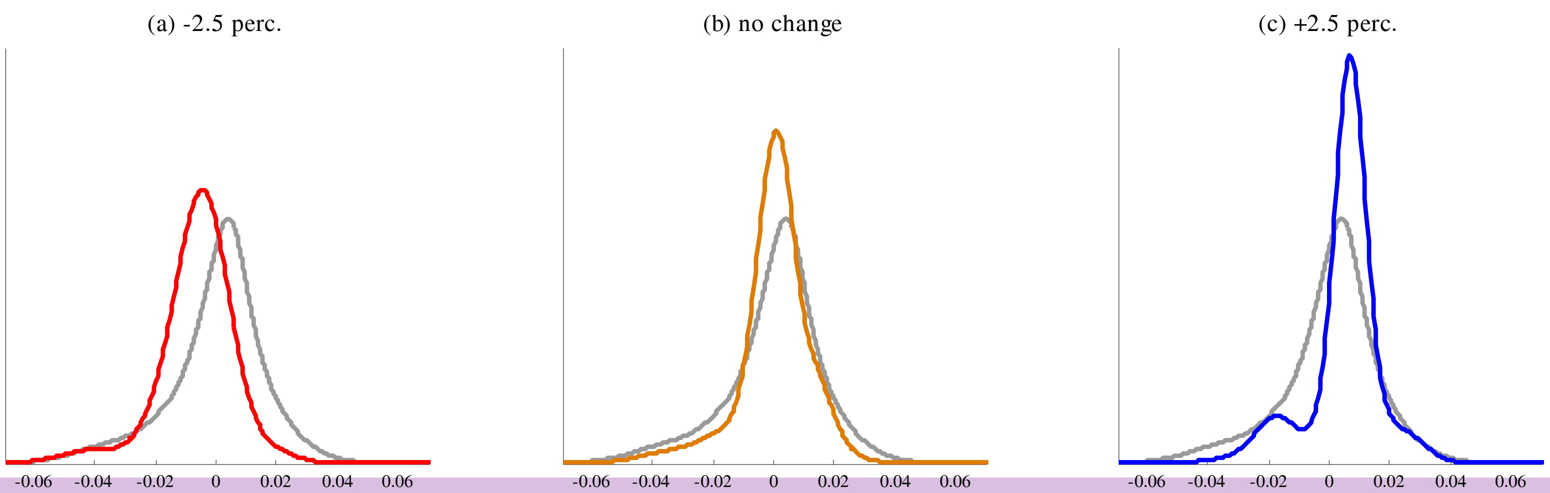


return PDFs (June 24th 2004)



Conditional distributions:





Pricing Index Options – general multivariate contingent claims

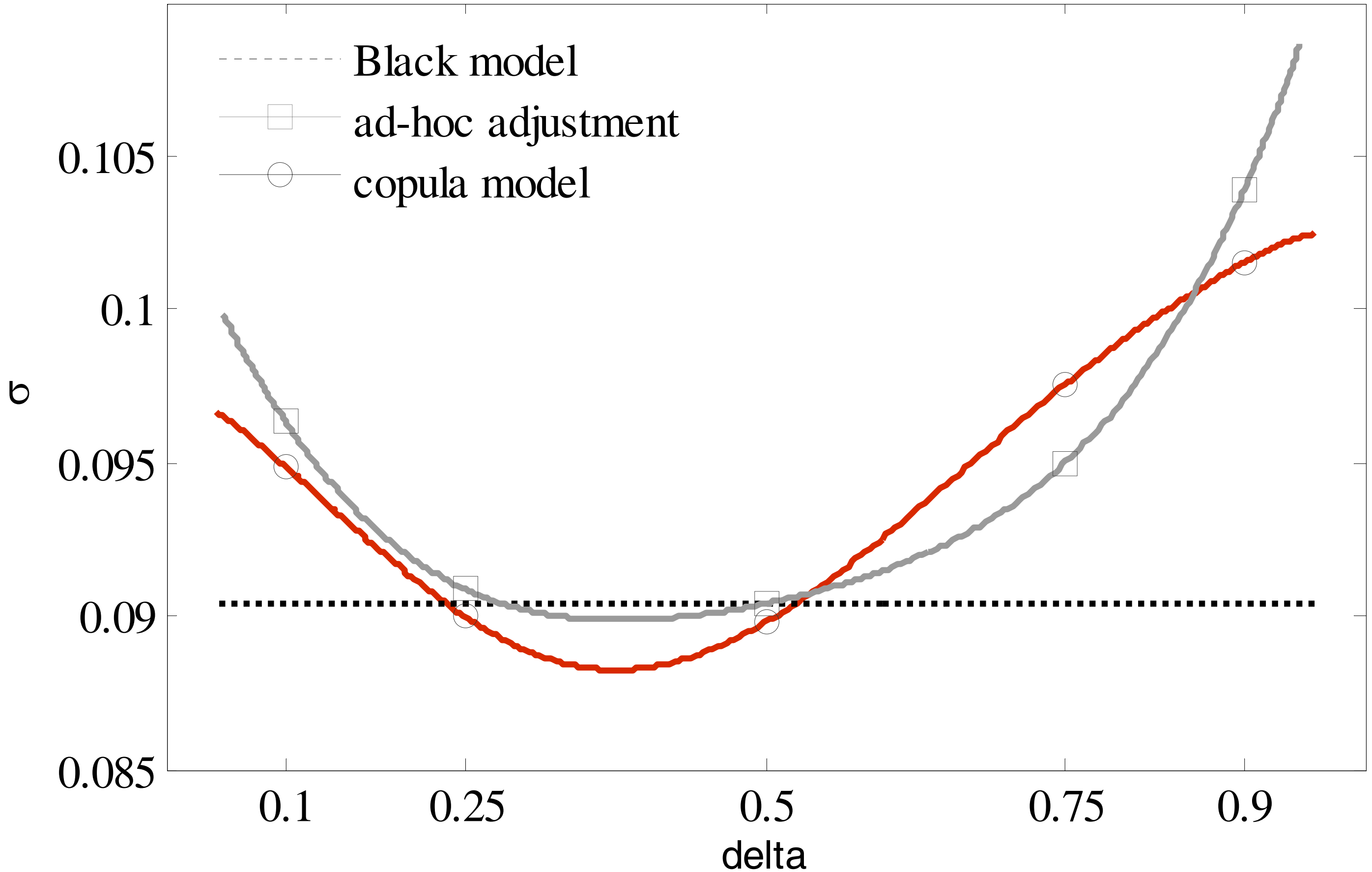
- Examples: CME (CME$INDEX™) and NYBOT (USDX©)
- Closely related problem to effective exchange rates:

$$\text{payoff} = \max\left(\left(\frac{S_T^{c,a}}{M_{t,T}^{c,a}}\right)^{\omega_a}\left(\frac{S_T^{c,b}}{M_{t,T}^{c,b}}\right)^{(1-\omega_a)} - K, 0\right)$$

- General pricing formula: Multivariate Feynman-Kaç:

$$V_t\left(G(S_t^{c,a}, S_t^{c,b})\right) = e^{-r_c(T-t)}\int_0^\infty\int_0^\infty G(u,v)\tilde{f}_{S_T^{c,a},S_T^{c,b},t}(u,v)\,du\,dv$$

Volatility smile for 50:50 euro-yen vs dollar: 1-month contracts June 24th 2004:



Pricing errors relative to alternative models:

Table 3: Prices of index option-contracts for June 24 2004 in vols

	straddle	risk rev.$_{25\Delta}$	risk rev.$_{10\Delta}$	butterfly$_{25\Delta}$	butterfly$_{10\Delta}$
Black model	9.04	0.00	0.00	0.00	0.00
ad-hoc adj.	9.04	-0.41	-0.76	0.25	0.97
copula model	8.99	-0.75	-0.67	0.38	0.83
$error_{\mathrm{Black}}$	*-1%*	*100%*	*100%*	*100%*	*100%*
$error_{\mathrm{ad-hoc}}$	*-1%*	*45%*	*-14%*	*34%*	*-17%*

Summary

- Described a new approach for constructing bivariate risk-neutral distributions implied by option prices
- Model the dependence between set of 3 currencies using the copula function that satisfies a triangular non arbitrage condition such that the bivariate distribution of any two bilaterals is consistent with the univariate distribution of the third currency pair.
- Enables us to derive arbitrage consistent risk neutral distribution of ERI under a consistent numeraire
- Enables valid pricing of multivariate contingent claims.
- Method exploits all the available information on the set of option implied density functions
- Parametric Copulae are not able to capture the asymmetry across the 45 degree line
- Method is independent of estimation of marginal PDFs