

Liquidity Takers

- LTs send a quantity Δy of Y . They receive a quantity Δx of X given by the **trading function**

$$\underbrace{f(x, y) = f(x - \Delta x, y + \Delta y)}_{\text{LT trading condition}} = \underbrace{\kappa^2}_{\text{Depth}}$$

- Level** function

$$f(x, y) = \kappa^2 \iff x = \varphi(y)$$

- Execution** and **marginal** exchange rates

$$\frac{\Delta x}{\Delta y} \xrightarrow{\Delta y \rightarrow 0} \underbrace{-\varphi'(y)}_{\text{Instantaneous rate}} \equiv Z$$

Liquidity Providers

- LPs change the depth:

$$f(x + \Delta x, y + \Delta y) = K^2 > f(x, y) = \kappa^2.$$

- LPs do not change the rate:

$$Z = \underbrace{-\varphi^{\kappa'}(y) = -\varphi^{K'}(y + \Delta y)}_{\text{LP trading condition}}.$$

- LPs hold a portion of the pool and **earn fees**.

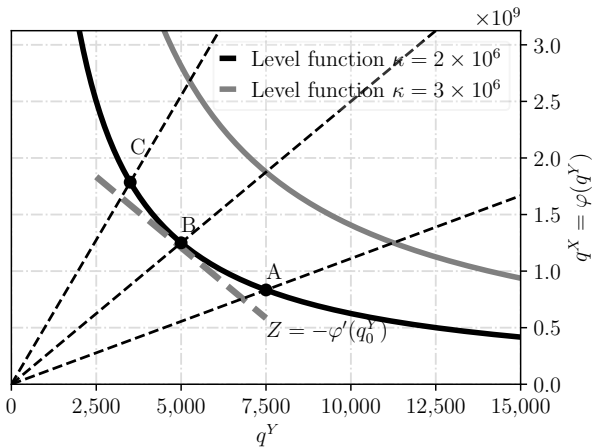


Figure: Geometry of CFMs: level function $\varphi(q^Y) = q^X$ for two values of the pool depth.

In CPMs (Uniswap)

- LT trading condition:

$$f(x, y) = x \times y \quad \text{and} \quad Z = x/y.$$

- LP trading condition:

$$\frac{x + \Delta x}{y + \Delta y} = \frac{x}{y}$$

- Depth variations

$$K^2 = (x + \Delta x)(y + \Delta y) > \kappa = x y$$

Automated Market Makers Designs Beyond Constant Functions

This talk: Arithmetic liquidity pool (ALP).

For more: see the paper where we study the geometric liquidity pool (GLP) too.

Generalising CFMs: ALP

- **Reserves:** quantities x and y of assets X and Y .

- **Liquidity taking:**

LT sends buy and sell orders with (minimum) size ζ of asset Y .

- **Liquidity provision:**

The LP chooses the shifts δ_t^b and δ_t^a such that :

- $Z_t - \delta_t^b$ is the price to sell a constant amount $\zeta > 0$.
- $Z_t + \delta_t^a$ is the price to buy a constant amount $\zeta > 0$.

- **Marginal rate:**

The marginal rate is impacted by buy/sell orders following **impact function** η^a and η^b .

ALP: the dynamics

- The ALP receives orders with size ζ throughout a trading window $[0, T]$.
- $(N_t^b)_{t \in [0, T]}$ and $(N_t^a)_{t \in [0, T]}$ are counting processes for the number of sell and buy orders filled by the LP.

- The **dynamics** of the **ALP reserves**:

$$\begin{aligned} dy_t &= \zeta dN_t^b - \zeta dN_t^a, \\ dx_t &= -\zeta (Z_{t-} - \delta_t^b) dN_t^b + \zeta (Z_{t-} + \delta_t^a) dN_t^a. \end{aligned}$$

- The **dynamics** of the **marginal rate**

$$dZ_t = -\eta^b(y_{t-}) dN_t^b + \eta^a(y_{t-}) dN_t^a,$$

for **impact functions** $\eta^a(\cdot)$ and $\eta^b(\cdot)$.

- The reserves take **finitely many values** $\{\underline{y}, \underline{y} + \zeta, \dots, \bar{y}\}$.

Theorem: CFM \subset ALP

Let $\varphi(\cdot)$ be the level function of a CFM. Assume one chooses the impact functions

$$\eta^a(y) = \varphi'(y) - \varphi'(y - \zeta), \quad \eta^b(y) = -\varphi'(y) + \varphi'(y + \zeta),$$

and chooses the quotes

$$\delta_t^a = \frac{\varphi(y_{t-} - \zeta) - \varphi(y_{t-})}{\zeta} + \varphi'(y_{t-}) - \underbrace{f \zeta \varphi'(y_{t-}^-)}_{\text{If fees } \neq 0},$$

$$\delta_t^b = \frac{\varphi(y_{t-} + \zeta) - \varphi(y_{t-})}{\zeta} - \varphi'(y_{t-}) - \underbrace{f \zeta \varphi'(y_{t-}^-)}_{\text{If fees } \neq 0}.$$

Then **ALP \equiv CFM !**

Idea of the proof

The dynamics of the reserves and the marginal rate Z^{CFM} in the CFM pool are given by

$$\begin{aligned}dy_t^{\text{CFM}} &= \zeta dN_t^b - \zeta dN_t^a, \\dx_t^{\text{CFM}} &= \left(\varphi \left(y_{t^-}^{\text{CFM}} + \zeta \right) - \varphi \left(y_{t^-}^{\text{CFM}} \right) \right) dN_t^b \\&\quad + \left(\varphi \left(y_{t^-}^{\text{CFM}} - \zeta \right) - \varphi \left(y_{t^-}^{\text{CFM}} \right) \right) dN_t^a, \\dZ_t^{\text{CFM}} &= \left(-\varphi' \left(y_{t^-}^{\text{CFM}} + \zeta \right) + \varphi' \left(y_{t^-}^{\text{CFM}} \right) \right) dN_t^b \\&\quad + \left(-\varphi' \left(y_{t^-}^{\text{CFM}} - \zeta \right) + \varphi' \left(y_{t^-}^{\text{CFM}} \right) \right) dN_t^a.\end{aligned}$$

Arbitrage in the ALP

Round-trip sequence = any sequence of trades $\{\epsilon_1, \dots, \epsilon_m\}$, where $\epsilon_k = \pm 1$ (buy/sell) for $k \in \{1, \dots, m\}$ and $\sum_{k=1}^m \epsilon_k = 0$.

Theorem: no-arbitrage

Under reasonable conditions on the impact functions η^a and η^b (see the paper), there is **no round-trip sequence of trades to arbitrage the ALP**.

Example of “reasonable” conditions

The impact functions $\eta^a(\cdot)$ and $\eta^b(\cdot)$ are bounded above by functions we give in the paper.



The bid after a buy trade is lower than the ask before the trade.
The ask after a sell trade is higher than the bid before the trade.

Proposition: no price manipulation

The marginal rate Z takes only the **ordered finitely many values**

$$\mathcal{Z} = \{\mathfrak{z}_1, \dots, \mathfrak{z}_N\}$$

with the property that $Z_0 \in \mathcal{Z}$ and for $i \in \{1, \dots, N-1\}$

$$\mathfrak{z}_{i+1} - \eta^b(\eta_{N-i}) = \mathfrak{z}_i \quad \text{and} \quad \mathfrak{z}_i + \eta^a(\eta_{N-i} + \zeta) = \mathfrak{z}_{i+1},$$

if and only if $\eta^a(\cdot)$ and $\eta^b(\cdot)$ are such that

$$\eta^b(\eta_i) = \eta^a(\eta_i + \zeta).$$

Assumptions of the strategy

- The LP models the **intensity** of **order arrivals** as:

$$\begin{cases} \lambda_t^b(\delta_t^b) = c^b e^{-\kappa \delta_t^b} \mathbb{1}^b(y_{t-}), \\ \lambda_t^a(\delta_t^a) = c^a e^{-\kappa \delta_t^a} \mathbb{1}^a(y_{t-}), \end{cases}$$

- $c^a \geq 0$ and $c^b \geq 0$: capture the baseline selling and buying pressure.
- **Inventory limits** (concentrated liquidity): the ALP stops using the LP's liquidity upon reaching her inventory limits \underline{y}, \bar{y}

$$\mathbb{1}^b(y) = \mathbb{1}_{\{y+\zeta \leq \bar{y}\}} \quad \text{and} \quad \mathbb{1}^a(y) = \mathbb{1}_{\{y-\zeta \geq \underline{y}\}},$$

Admissible strategies

For $t \in [0, T]$, we define the set \mathcal{A}_t of admissible shifts

$$\mathcal{A}_t = \left\{ \delta_s = (\delta_s^b, \delta_s^a)_{s \in [t, T]}, \mathbb{R}^2\text{-valued, } \mathbb{F}\text{-adapted,} \right. \\ \left. \text{square-integrable, and bounded from below by } \underline{\delta} \right\},$$

where $\underline{\delta} \in \mathbb{R}$ is given and write $\mathcal{A} := \mathcal{A}_0$.

The performance criterion of the LP

- The LP **chooses** the **impact functions** η^b and η^a , the inventory limits \underline{y} and \bar{y} .
- The LP **estimates** (or predicts) the **strategy parameters** c^b , c^a , κ .
- The performance criterion using the price of liquidity $\delta = (\delta^b, \delta^a)$ is the function w^δ :

$$w^\delta(t, x, y, z) = \mathbb{E}_{t,x,y,z} \left[x_T + y_T Z_T - \alpha (y_T - \hat{y})^2 - \phi \int_t^T (y_s - \hat{y})^2 ds \right].$$

- The LP wishes to find $\delta^* = \arg \max_\delta w^\delta(0, x, y, z)$

Proposition: the problem is well-posed

There is $C \in \mathbb{R}$ such that for all $(\delta_s)_{s \in [t, T]} \in \mathcal{A}_t$, the performance criterion of the LP satisfies

$$w^\delta(t, x, y, z) \leq C < \infty,$$

so the value function w is well defined.

Results

- Closed-form solution !
- In our design: **CFMs are suboptimal.**

Let us go through these **claims** in a little more detail.

Closed-form solution

Closed-form solution

The **admissible** optimal Markovian control $(\delta_s^*)_{s \in [t, T]} = (\delta_s^{b*}, \delta_s^{a*})_{s \in [t, T]} \in \mathcal{A}_t$ is given by

$$\delta^{b*}(t, y_{t-}) = \frac{1}{\kappa} - \frac{\theta(t, y_{t-} + \zeta) - \theta(t, y_{t-})}{\zeta} - \frac{(y_{t-} + \zeta) \eta^b(y_{t-})}{\zeta},$$

$$\delta^{a*}(t, y_{t-}) = \frac{1}{\kappa} - \frac{\theta(t, y_{t-} - \zeta) - \theta(t, y_{t-})}{\zeta} + \frac{(y_{t-} - \zeta) \eta^a(y_{t-})}{\zeta},$$

where θ is in the paper.

No arbitrage

$$\eta^a(\eta_i) \leq \frac{1}{\kappa}, \quad \text{and} \quad \eta^b(\eta_i) \leq \frac{1}{\kappa}.$$

CFMs are suboptimal

Proposition: CFMs are suboptimal

- Let φ be the level function of a CFM. Consider an LP who deposits her initial wealth (x_0, y_0) in the CFM and whose performance criterion is

$$J^{\text{CFM}} = \mathbb{E} \left[x_T^{\text{CFM}} + y_T^{\text{CFM}} Z_T^{\text{CFM}} - \alpha (y_T^{\text{CFM}} - \hat{y})^2 - \phi \int_0^T (y_s^{\text{CFM}} - \hat{y})^2 ds \right].$$

- Consider an LP in a ALP with **the same initial wealth** (x_0, y_0) and with impact functions $\eta^a(\cdot)$ and $\eta^b(\cdot)$ that **match the dynamics** of a CFM.
- Let $\delta_t^{\text{CFM}} = (\delta_t^{a,\text{CFM}}, \delta_t^{b,\text{CFM}})$ be the price of liquidity that **matches** that in a CFM.
- Then

$$J^{\text{CFM}} = J(\delta^{\text{CFM}}) \quad \text{and} \quad J^{\text{CFM}} \leq J(\delta^*).$$

The ALP in practice & numerical examples

Some practicalities in the ALP

Our theorem states what price of liquidity δ^* is **once** $\eta^a(\cdot), \eta^b(\cdot)$ and model parameters (e.g. α, ϕ, \hat{y}) are **specified**.

The ALP asks that LPs specify their impact functions and model parameters and the “venue” plays by the rules imposed by the dynamics and the optimal strategy.

Implementation **on-chain**

- With hooks for impact functions.
- Computationally efficient & closed-form \iff low gas fees, low storage burden.

Numerical examples: Impact functions and strategy parameters

We assume

- Buy/Sell pressure: $c^a = c^b = c > 0$.
- The inventory risk constraint is $y \in \{\underline{y}, \dots, \bar{y}\}$ where $\underline{y} \geq \zeta$.
- We employ the following impact functions:

$$\eta^b(y) = \frac{\zeta}{2y + \zeta} L \quad \text{and} \quad \eta^a(y) = \frac{\zeta}{2y - \zeta} L,$$

where $L > 0$ is the **impact parameter**.

- **No price manipulation**: $\eta^b(y) = \eta^a(y + \zeta)$
- **No arbitrage**: we choose $L < \frac{1}{\kappa}$.

Numerical examples: price of liquidity

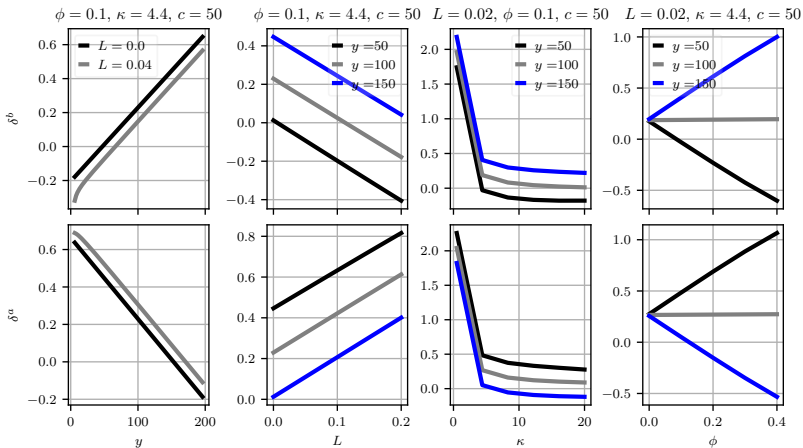


Figure: ALP: Optimal shifts as a function of model parameters, where $\hat{y} = 100 \text{ ETH}$, $[\underline{y}, \bar{y}] = [\zeta, 200]$, and $\alpha = 0 \text{ USDC} \cdot \text{ETH}^{-2}$.

Numerical examples: fighting arbitrageurs

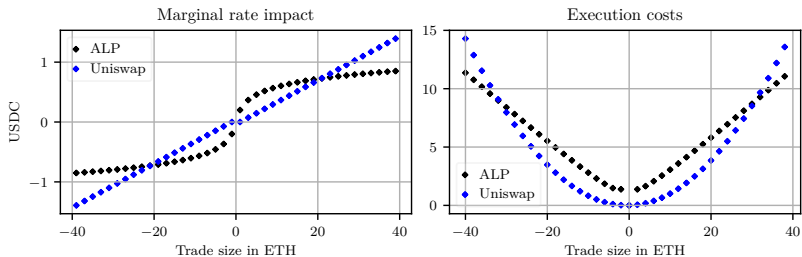


Figure: Marginal rate impact and execution costs in the ALP as a function of the size of the trade.

Numerical examples: fighting arbitrageurs

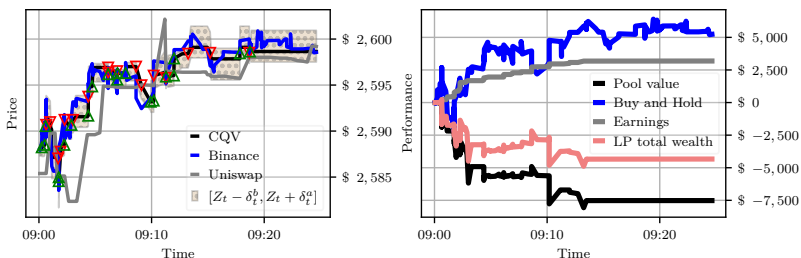


Figure: LP wealth when arbitrageurs trade in the ALP and Binance. **Left:** Exchange rates from ALP, Binance, and Uniswap v3. **Right:** *Pool value* is computed as $x_t + y_t Z_t$, *Buy and Hold* is computed as the wealth from holding the LP's inventory outside the ALP, i.e., $y_t Z_t$, *Earnings* are the revenue from the quotes, and *LP total wealth* is the total LP's wealth.

Numerical examples: fighting arbitrageurs

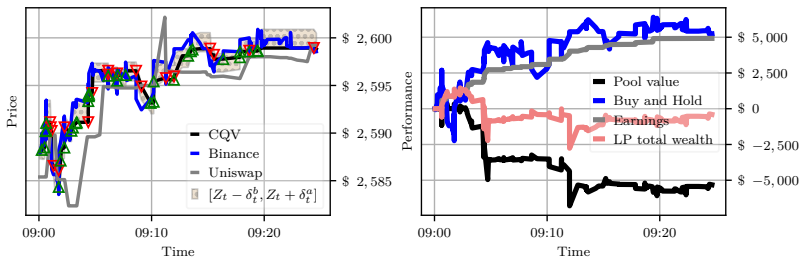


Figure: LP wealth when only an arbitrageur interacts in the ALP and with an **increased value** of the **penalty** parameter ϕ .

Numerical simulations: Uniswap vs ALP

| | Average | Standard deviation |
|-------------------|---------|--------------------|
| ALP (scenario I) | -0.004% | 0.719% |
| ALP (scenario II) | 0.717% | 2.584% |
| Buy and Hold | 0.001% | 0.741% |
| Uniswap v3 | -1.485% | 7.812% |

Table: Average and standard deviation of 30-minutes performance of LPs in the ALP for both simulation scenarios, LPs in Uniswap, and buy-and-hold.

Thank you

The geometric liquidity pool (GLP) design.

The GLP design

Let $\zeta^b \in (0, 1)$ and $\zeta^a \in (0, 1)$ be two constants, and let the impact functions at the bid and the ask be $y \mapsto \eta^b(y) \in (0, 1)$ and $y \mapsto \eta^a(y) \in (0, \infty)$, respectively.¹ In the GLP, the LP is ready to buy the quantity $\zeta^b y_{t-}$ and to sell the quantity $\zeta^a y_{t-}$ of asset Y at any time $t \in [0, T]$. The quantities of assets X and Y in the pool follow the dynamics

$$dy_t = \zeta^b y_{t-} dN_t^b - \zeta^a y_{t-} dN_t^a,$$

$$dx_t = -\zeta^b y_{t-} Z_{t-} (1 - \delta_t^b) dN_t^b + \zeta^a y_{t-} Z_{t-} (1 + \delta_t^a) dN_t^a.$$

¹These assumptions are not restrictive because the impact functions in the GLP are relative movements in the marginal rate Z , so a value of 1 means a 100% rate innovation.

The GLP design

The marginal rate in the pool is updated as follows

$$dZ_t = Z_{t-} (-\eta^b(y_{t-}) dN_t^b + \eta^a(y_{t-}) dN_t^a) .$$

From (??), we see that the changes in the marginal rate are proportional to the current rate in the pool. Moreover, the process $(Z_s)_{s \in [t, T]}$ is non-negative as long as $Z_t \geq 0$ because $y \mapsto \eta^b(y) \in (0, 1)$.

The GLP design

Similar to the ALP, the LP in the GLP assumes that the arrival intensity decays exponentially as a function of the shifts δ^a and δ^b . However, the order size at the ask is smaller than that at the bid by an overall factor equal to $(1 + \zeta)^{-1}$, thus the LP assumes that the exponential decay of the liquidity trading flow at the ask is slower by the same fraction, and she writes

$$\begin{cases} \lambda_t^b(\delta_t^b) = c^b e^{-\kappa \delta_t^b} \mathbb{1}^b(y_{t-}) , \\ \lambda_t^a(\delta_t^a) = c^a e^{-\frac{\kappa}{1+\zeta} \delta_t^a} \mathbb{1}^a(y_{t-}) , \end{cases}$$

for some positive constant κ .

The GLP design

The LP is continuously updating the shifts δ_t^b and δ_t^a until a fixed horizon $T > 0$. The performance criterion of the LP using the strategy $\delta = (\delta^b, \delta^a) \in \mathcal{A}$, where the admissible set is in (??), is a function $w^\delta: [0, T] \times \mathbb{R} \times \mathcal{Y} \times \mathbb{R}^+ \rightarrow \mathbb{R}$, which is given by

$$\mathbb{E}_{t,x,y,z} \left[x_T + y_T Z_T - \alpha Z_T (y_T - \hat{y})^2 - \phi \int_t^T Z_s (y_s - \hat{y})^2 ds \right].$$

Note that in contrast to the performance criterion in the ALP, the aversion to inventory deviations from \hat{y} in (30) is proportional to the marginal pool rate.

The GLP design

We find closed-form solutions (and hence a new design) for when the impact functions are:

$$\eta^b(y) = \frac{\zeta}{1 + \zeta} \in (0, 1), \quad \eta^a(y) = \zeta \in (0, 1).$$

[Download This Paper](#)[Open PDF in Browser](#)[Add Paper to My Library](#)[Revise my Submission](#)

Automated Market Makers Designs Beyond Constant Functions

40 Pages • Posted: 25 May 2023 • Last revised: 16 Aug 2023

[Álvaro Cartea](#)

University of Oxford; University of Oxford - Oxford-Man Institute of Quantitative Finance

[Fayçal Drissi](#)

University of Oxford - Oxford-Man Institute of Quantitative Finance

[Leandro Sánchez-Betancourt](#)

Mathematical Institute, University of Oxford; University of Oxford - Oxford-Man Institute of Quantitative Finance

[David Siska](#)

University of Edinburgh - School of Mathematics; Vega

[Lukasz Szpruch](#)

University of Edinburgh - School of Mathematics; The Alan Turing Institute; Simtopia

Date Written: May 25, 2023