# AMM Designs Beyond Constant Functions

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# Automated Market Makers

# Constant Function Markets

## Constant Function Markets

- A pool with assets $X$ and $Y$
- Available liquidity or reserves: $x$ and $y$
- Deterministic trading function $f(x, y)$

  $\Longrightarrow$ defines the state of the pool before and after a trade
- Liquidity providers (LPs) deposit assets in the pool.

  Liquidity takers (LTs) trade with the pool.

## Liquidity Takers

- LTs send a quantity $\Delta y$ of $Y$. They receive a quantity $\Delta x$ of $X$ given by the trading function

$$\boxed{\underbrace{f(x,y) = f(x-\Delta x, y+\Delta y)}_{\text{LT trading condition}} = \underbrace{\kappa^2}_{\text{Depth}}}$$

- Level function

$$f(x,y) = \kappa^2 \iff x = \varphi(y)$$

- Execution and marginal exchange rates

$$\frac{\Delta x}{\Delta y} \xrightarrow{\Delta y \longrightarrow 0} \underbrace{-\varphi'(y) \equiv Z}_{\text{Instantaneous rate}}$$

## Liquidity Providers

- LPs change the depth:

$$f\,(x+\Delta x, y+\Delta y) = K^2 > f\,(x,y) = \kappa^2\,.$$

- LPs do not change the rate:

$$Z = \underbrace{-\varphi^{\kappa\prime}(y) = -\varphi^{K\prime}(y+\Delta y)}_{\text{LP trading condition}}\,.$$

- LPs hold a portion of the pool and earn fees.

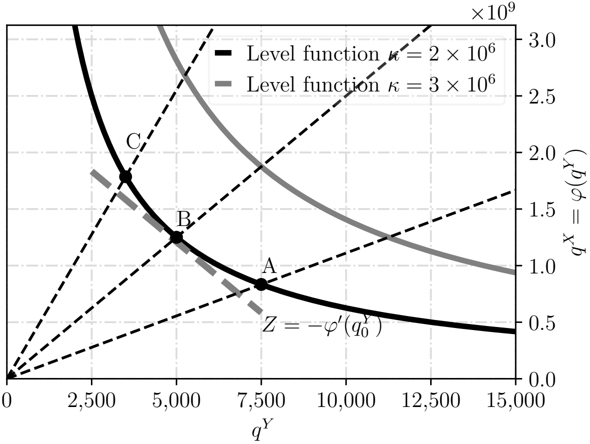


Figure: Geometry of CFMs: level function $\varphi\left(q^Y\right) = q^X$ for two values of the pool depth.

## In CPMs (Uniswap)

- LT trading condition:

$$f(x, y) = x \times y \quad \text{and} \quad Z = x/y.$$

- LP trading condition:

$$\frac{x + \Delta x}{y + \Delta y} = \frac{x}{y}$$

- Depth variations

$$K^2 = (x + \Delta x)(y + \Delta y) > \kappa = x\, y$$

# Automated Market Makers Designs Beyond Constant Functions

**This talk:** Arithmetic liquidity pool (ALP).
**For more:** see the paper where we study the geometric liquidity pool (GLP) too.

## Generalising CFMs: ALP

- Reserves: quantities $x$ and $y$ of assets $X$ and $Y$.

- Liquidity taking:

  LT sends buy and sell orders with (minimum) size $\zeta$ of asset $Y$.

- Liquidity provision:

  The LP chooses the shifts $\delta_t^b$ and $\delta_t^a$ such that :
  - $Z_t - \delta_t^b$ is the price to sell a constant amount $\zeta > 0$.
  - $Z_t + \delta_t^a$ is the price to buy a constant amount $\zeta > 0$.

- Marginal rate:

  The marginal rate is impacted by buy/sell orders following impact function $\eta^a$ and $\eta^b$ .

## ALP: the dynamics

- The ALP receives orders with size $\zeta$ throughout a trading window $[0, T]$.
- $\left(N_t^b\right)_{t\in[0,T]}$ and $\left(N_t^a\right)_{t\in[0,T]}$ are counting processes for the number of sell and buy orders filled by the LP.
- The dynamics of the ALP reserves:

$$\begin{aligned} \mathrm{d}y_t &= \zeta \, \mathrm{d}N_t^b - \zeta \, \mathrm{d}N_t^a \,, \\ \mathrm{d}x_t &= -\zeta \, \left(Z_{t-} - \delta_t^b\right) \mathrm{d}N_t^b + \zeta \left(Z_{t-} + \delta_t^a\right) \mathrm{d}N_t^a \,. \end{aligned}$$

- The dynamics of the marginal rate

$$\mathrm{d}Z_t = -\,\eta^b(y_{t-})\,\mathrm{d}N_t^b + \eta^a(y_{t-})\,\mathrm{d}N_t^a \,,$$

for impact functions $\eta^a(\cdot)$ and $\eta^b(\cdot)$.

- The reserves take finitely many values $\{\underline{y}, \underline{y} + \zeta, \ldots, \overline{y}\}$.

**Theorem: CFM $\subset$ ALP**

Let $\varphi(\cdot)$ be the level function of a CFM. Assume one chooses the impact functions

$$\eta^a(y) = \varphi'(y) - \varphi'(y-\zeta)\,, \qquad \eta^b(y) = -\varphi'(y) + \varphi'(y+\zeta)\,,$$

and chooses the quotes

$$\delta_t^a = \frac{\varphi(y_{t^-} - \zeta) - \varphi(y_{t^-})}{\zeta} + \varphi'(y_{t^-}) - \underbrace{\mathfrak{f}\,\zeta\,\varphi'(y_t^-)}_{\text{If fees} \neq 0}\,,$$

$$\delta_t^b = \frac{\varphi(y_{t^-} + \zeta) - \varphi(y_{t^-})}{\zeta} - \varphi'(y_{t^-}) - \underbrace{\mathfrak{f}\,\zeta\,\varphi'(y_t^-)}_{\text{If fees} \neq 0}\,.$$

Then ALP $\equiv$ CFM !

**Idea of the proof**

The dynamics of the reserves and the marginal rate $Z^{\text{CFM}}$ in the CFM pool are given by

$$
\begin{aligned}
\mathrm{d}y_t^{\text{CFM}} &= \zeta\,\mathrm{d}N_t^b - \zeta\,\mathrm{d}N_t^a\,,\\
\mathrm{d}x_t^{\text{CFM}} &= \left(\varphi\left(y_{t^-}^{\text{CFM}} + \zeta\right) - \varphi\left(y_{t^-}^{\text{CFM}}\right)\right)\,\mathrm{d}N_t^b\\
&\qquad + \left(\varphi\left(y_{t^-}^{\text{CFM}} - \zeta\right) - \varphi\left(y_{t^-}^{\text{CFM}}\right)\right)\,\mathrm{d}N_t^a\,,\\
\mathrm{d}Z_t^{\text{CFM}} &= \left(-\varphi'\left(y_{t^-}^{\text{CFM}} + \zeta\right) + \varphi'\left(y_{t^-}^{\text{CFM}}\right)\right)\,\mathrm{d}N_t^b\\
&\qquad + \left(-\varphi'\left(y_{t^-}^{\text{CFM}} - \zeta\right) + \varphi'\left(y_{t^-}^{\text{CFM}}\right)\right)\,\mathrm{d}N_t^a\,.
\end{aligned}
$$

## Arbitrage in the ALP

Round-trip sequence $=$ any sequence of trades $\{\epsilon_1, \ldots, \epsilon_\mathfrak{m}\}$, where $\epsilon_k = \pm 1$ (buy/sell) for $k \in \{1, \ldots, \mathfrak{m}\}$ and $\sum_{k=1}^{\mathfrak{m}} \epsilon_k = 0$.

## Theorem: no-arbitrage

Under reasonable conditions on the impact functions $\eta^a$ and $\eta^b$ (see the paper), there is no round-trip sequence of trades to arbitrage the ALP.

## Example of "reasonable" conditions

The impact functions $\eta^a(\cdot)$ and $\eta^b(\cdot)$ are bounded above by functions we give in the paper.

$\iff$

The bid after a buy trade is lower than the ask before the trade.
The ask after a sell trade is higher than the bid before the trade.

**Proposition: no price manipulation**

The marginal rate $Z$ takes only the ordered finitely many values

$$\mathcal{Z} = \{\mathfrak{z}_1, \dots, \mathfrak{z}_N\}$$

with the property that $Z_0 \in \mathcal{Z}$ and for $i \in \{1, \dots, N-1\}$

$$\mathfrak{z}_{i+1} - \eta^b(\mathfrak{y}_{N-i}) = \mathfrak{z}_i \qquad \text{and} \qquad \mathfrak{z}_i + \eta^a(\mathfrak{y}_{N-i} + \zeta) = \mathfrak{z}_{i+1}\,,$$

if and only if $\eta^a(\cdot)$ and $\eta^b(\cdot)$ are such that

$$\eta^b(\mathfrak{y}_i) = \eta^a(\mathfrak{y}_i + \zeta)\,.$$

# The optimal price of liquidity in the ALP

So far we have only discussed the mechanics/microstructure of the ALP, which is general enough to have CFMs as a subset.

Let's write a model to underpin the new design.

## Assumptions of the strategy

- The LP models the intensity of order arrivals as:

$$\begin{cases} \lambda^b_t\left(\delta^b_t\right) = c^b\, e^{-\kappa\, \delta^b_t}\, \mathbb{1}^b\left(y_{t-}\right)\,, \\ \lambda^a_t\left(\delta^a_t\right) = c^a\, e^{-\kappa\, \delta^a_t}\, \mathbb{1}^a\left(y_{t-}\right)\,, \end{cases}$$

- $c^a \geq 0$ and $c^b \geq 0$: capture the baseline selling and buying pressure.

- Inventory limits (concentrated liquidity): the ALP stops using the LP's liquidity upon reaching her inventory limits $\underline{y}, \overline{y}$

$$\mathbb{1}^b(y) = \mathbb{1}_{\{y+\zeta\leq\overline{y}\}} \quad \text{and} \quad \mathbb{1}^a(y) = \mathbb{1}_{\{y-\zeta\geq\underline{y}\}}\,,$$

**Admissible strategies**

For $t \in [0, T]$, we define the set $\mathcal{A}_t$ of admissible shifts

$$\mathcal{A}_t = \Big\{ \delta_s = (\delta^b_s, \delta^a_s)_{s\in[t,T]},\ \mathbb{R}^2\text{-valued},\ \mathbb{F}\text{-adapted},$$
$$\text{square-integrable, and bounded from below by } \underline{\delta} \Big\},$$

where $\underline{\delta} \in \mathbb{R}$ is given and write $\mathcal{A} := \mathcal{A}_0$.

## The performance criterion of the LP

- The LP chooses the impact functions $\eta^b$ and $\eta^a$, the inventory limits $\underline{y}$ and $\overline{y}$.
- The LP estimates (or predicts) the strategy parameters $c^b,\ c^a,\ \kappa$.
- The performance criterion using the price of liquidity $\delta = \left(\delta^b, \delta^a\right)$ is the function $w^\delta$:

$$w^\delta(t,x,y,z) = \mathbb{E}_{t,x,y,z}\left[x_T + y_T\, Z_T - \alpha\,(y_T - \hat{y})^2 - \phi \int_t^T (y_s - \hat{y})^2\, \mathrm{d}s\right] .$$

- The LP wishes to find $\delta^* = \arg\max_\delta w^\delta(0,x,y,z)$

**Proposition: the problem is well-posed**

There is $C \in \mathbb{R}$ such that for all $(\delta_s)_{s\in[t,T]} \in \mathcal{A}_t$, the performance criterion of the LP satisfies

$$w^{\delta}(t,x,y,z) \leq C < \infty\,,$$

so the value function $w$ is well defined.

**Results**

- Closed-form solution !
- In our design: CFMs are suboptimal.

Let us go through these claims in a little more detail.

# Closed-form solution

## Closed-form solution

The admissible optimal Markovian control $(\delta^\star_s)_{s\in[t,T]} = \left(\delta^{b\star}_s, \delta^{a\star}_s\right)_{s\in[t,T]} \in \mathcal{A}_t$ is given by

$$\delta^{b\star}(t, y_{t-}) = \frac{1}{\kappa} - \frac{\theta\,(t, y_{t-} + \zeta) - \theta\,(t, y_{t-})}{\zeta} - \frac{(y_{t-} + \zeta)\,\eta^b(y_{t-})}{\zeta}\,,$$
$$\delta^{a\star}(t, y_{t-}) = \frac{1}{\kappa} - \frac{\theta\,(t, y_{t-} - \zeta) - \theta\,(t, y_{t-})}{\zeta} + \frac{(y_{t-} - \zeta)\,\eta^a(y_{t-})}{\zeta}\,,$$

where $\theta$ is in the paper.

## No arbitrage

$$\eta^a(\mathfrak{y}_i) \leq \frac{1}{\kappa}\,, \quad \text{and} \qquad \eta^b(\mathfrak{y}_i) \leq \frac{1}{\kappa}\,.$$

# CFMs are suboptimal

## Proposition: CFMs are suboptimal

- Let $\varphi$ be the level function of a CFM. Consider an LP who deposits her initial wealth $(x_0, y_0)$ in the CFM and whose performance criterion is

$$J^{\text{CFM}} = \mathbb{E}\left[x_T^{\text{CFM}} + y_T^{\text{CFM}}\, Z_T^{\text{CFM}} - \alpha\,(y_T^{\text{CFM}} - \hat{y})^2 - \phi \int_0^T (y_s^{\text{CFM}} - \hat{y})^2\,\mathrm{d}s\right] .$$

- Consider an LP in a ALP with the same initial wealth $(x_0, y_0)$ and with impact functions $\eta^a(\,\cdot\,)$ and $\eta^b(\,\cdot\,)$ that match the dynamics of a CFM.

- Let $\delta_t^{CFM} = \left(\delta_t^{a,\text{CFM}}, \delta_t^{b,\text{CFM}}\right)$ be the price of liquidity that matches that in a CFM.

- Then

$$J^{\text{CFM}} = J\left(\delta^{CFM}\right) \qquad \text{and} \qquad J^{\text{CFM}} \leq J\left(\delta^{\star}\right) .$$

# The ALP in practice & numerical examples

## Some practicalities in the ALP

Our theorem states what price of liquidity $\delta^*$ is once $\eta^a(\cdot), \eta^b(\cdot)$ and model parameters (e.g. $\alpha, \phi, \hat{y}$) are specified.

The ALP asks that LPs specify their impact functions and model parameters and the "venue" plays by the rules imposed by the dynamics and the optimal strategy.

Implementation on-chain

- With hooks for impact functions.
- Computationally efficient & closed-form $\iff$ low gas fees, low storage burden.

## Numerical examples: Impact functions and strategy parameters

We assume

- Buy/Sell pressure: $c^a = c^b = c > 0$.

- The inventory risk constraint is $y \in \{\underline{y}, \ldots, \overline{y}\}$ where $\underline{y} \geq \zeta$.

- We employ the following impact functions:

$$\eta^b(y) = \frac{\zeta}{2\,y + \zeta}\,L \quad \text{and} \quad \eta^a(y) = \frac{\zeta}{2\,y - \zeta}\,L\,,$$

  where $L > 0$ is the impact parameter.

- No price manipulation: $\eta^b(y) = \eta^a(y + \zeta)$

- No arbitrage: we choose $L < \frac{1}{\kappa}$.

# Numerical examples: price of liquidity

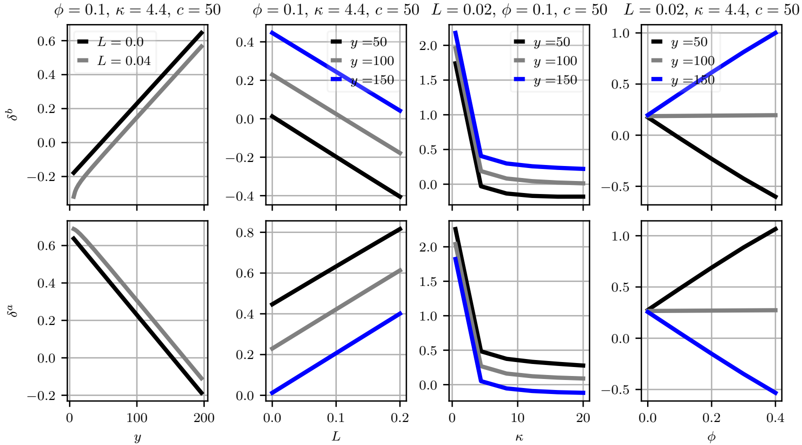


Figure: ALP: Optimal shifts as a function of model parameters, where $\hat{y} = 100$ ETH, $[\underline{y}, \overline{y}] = [\zeta, 200]$, and $\alpha = 0\,\text{USDC} \cdot \text{ETH}^{-2}$.

## Numerical examples: fighting arbitrageurs

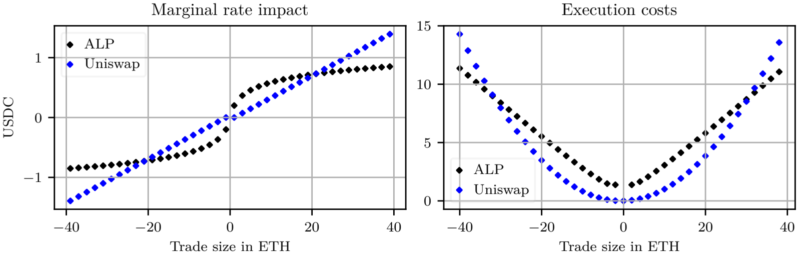


Figure: Marginal rate impact and execution costs in the ALP as a function of the size of the trade.

## Numerical examples: fighting arbitrageurs

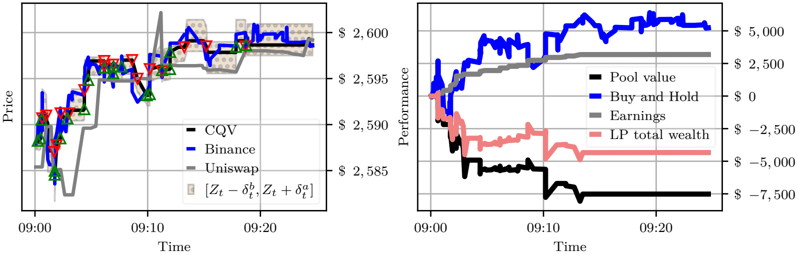


Figure: LP wealth when arbitrageurs trade in the ALP and Binance. **Left**: Exchange rates from ALP, Binance, and Uniswap v3. **Right**: *Pool value* is computed as $x_t + y_t Z_t$ , *Buy and Hold* is computed as the wealth from holding the LP's inventory outside the ALP, i.e., $y_t Z_t$, *Earnings* are the revenue from the quotes, and *LP total wealth* is the total LP's wealth.

## Numerical examples: fighting arbitrageurs

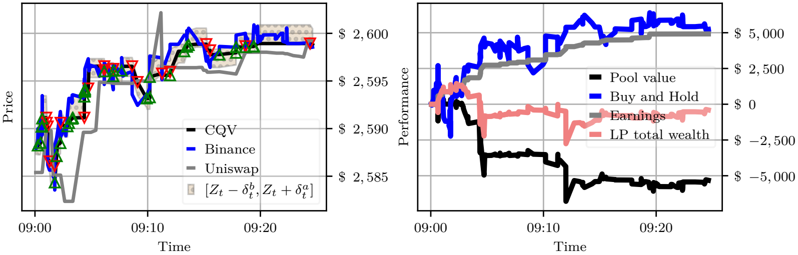


Figure: LP wealth when only an arbitrageur interacts in the ALP and with an increased value of the penalty parameter $\phi$.

## Numerical simulations: Uniswap vs ALP

| | Average | Standard deviation |
|---|---|---|
| ALP (scenario I) | $-0.004\%$ | 0.719% |
| ALP (scenario II) | 0.717% | 2.584% |
| Buy and Hold | 0.001% | 0.741% |
| Uniswap v3 | $-1.485\%$ | 7.812% |

Table: Average and standard deviation of 30-minutes performance of LPs in the ALP for both simulation scenarios, LPs in Uniswap, and buy-and-hold.

Thank you

# The geometric liquidity pool (GLP) design.

# The GLP design

Let $\zeta^{\mathrm{b}} \in (0,1)$ and $\zeta^{\mathrm{a}} \in (0,1)$ be two constants, and let the impact functions at the bid and the ask be $y \mapsto \eta^b(y) \in (0,1)$ and $y \mapsto \eta^a(y) \in (0,\infty)$, respectively.[1] In the GLP, the LP is ready to buy the quantity $\zeta^{\mathrm{b}}\, y_{t-}$ and to sell the quantity $\zeta^{\mathrm{a}}\, y_{t-}$ of asset $Y$ at any time $t \in [0,T]$. The quantities of assets $X$ and $Y$ in the pool follow the dynamics

$$\begin{aligned} \mathrm{d}y_t &= \zeta^{\mathrm{b}}\, y_{t-}\, \mathrm{d}N^b_t - \zeta^{\mathrm{a}}\, y_{t-} \mathrm{d}N^a_t\,, \\ \mathrm{d}x_t &= -\zeta^{\mathrm{b}}\, y_{t-} Z_{t-} \left(1 - \delta^b_t\right) \mathrm{d}N^b_t + \zeta^{\mathrm{a}}\, y_{t-} Z_{t-} \left(1 + \delta^a_t\right) \mathrm{d}N^a_t\,. \end{aligned}$$

[1] These assumptions are not restrictive because the impact functions in the GLP are relative movements in the marginal rate $Z$, so a value of 1 means a 100% rate innovation.

# The GLP design

The marginal rate in the pool is updated as follows

$$\mathrm{d}Z_t = Z_{t^-} \left(-\eta^b(y_{t^-})\,\mathrm{d}N_t^b + \eta^a(y_{t^-})\,\mathrm{d}N_t^a\right) .$$

From (**??**), we see that the changes in the marginal rate are proportional to the current rate in the pool. Moreover, the process $(Z_s)_{s\in[t,T]}$ is non-negative as long as $Z_t \geq 0$ because $y \mapsto \eta^b(y) \in (0, 1)$.

# The GLP design

Similar to the ALP, the LP in the GLP assumes that the arrival intensity decays exponentially as a function of the shifts $\delta^a$ and $\delta^b$. However, the order size at the ask is smaller than that at the bid by an overall factor equal to $(1+\zeta)^{-1}$, thus the LP assumes that the exponential decay of the liquidity trading flow at the ask is slower by the same fraction, and she writes

$$\begin{cases} \lambda_t^b\left(\delta_t^b\right) = c^b\, e^{-\kappa\, \delta_t^b}\, \mathbb{1}^b\left(y_{t^-}\right)\,, \\ \lambda_t\left(\delta_t^a\right) = c^a\, e^{-\frac{\kappa}{1+\zeta}\delta_t^a}\, \mathbb{1}^a\left(y_{t^-}\right)\,, \end{cases}$$

for some positive constant $\kappa$.

# The GLP design

The LP is continuously updating the shifts $\delta^b_t$ and $\delta^a_t$ until a fixed horizon $T > 0$. The performance criterion of the LP using the strategy $\delta = \left(\delta^b, \delta^a\right) \in \mathcal{A}$, where the admissible set is in (**??**), is a function $w^\delta \colon [0, T] \times \mathbb{R} \times \mathcal{Y} \times \mathbb{R}^+ \to \mathbb{R}$, which is given by

$$\mathbb{E}_{t,x,y,z}\left[x_T + y_T\, Z_T - \alpha\, Z_T\,(y_T - \hat{y})^2 - \phi \int_t^T Z_s\,(y_s - \hat{y})^2\,\mathrm{d}s\right].$$

Note that in contrast to the performance criterion in the ALP, the aversion to inventory deviations from $\hat{y}$ in (30) is proportional to the marginal pool rate.

# The GLP design

We find closed-form solutions (and hence a new design) for when the impact functions are:

$$\eta^b(y) = \frac{\zeta}{1+\zeta} \in (0,1)\,, \quad \eta^a(y) = \zeta \in (0,1)\,.$$

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