Numerical Methods

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The aim of this paper is to review and compare the standard approaches to the numerical evaluation of options. We will pay particular attention to accuracy and efficiency, and to dividends and the early exercise opportunity in the case of American options. This report can be regarded as an introduction to Chapter 6b, and it leans heavily on Geske and Shastri (1985).

We will make the usual assumptions about market behaviour, namely that continuous trading is possible with no transaction costs, and that there is no penalty for selling short, and no taxes. Also, we will assume that the risk-free continuously compounded interest rate is a constant, r. Our standing assumption about the stock price s_t on which the option is written is that it has constant proportional drift (expected rate of return), μ, and constant proportional volatility, σ.

I European options

These values obey the celebrated Black–Scholes equation:

$$\frac{\partial \varphi_t(x)}{\partial t} = r\varphi_t(x) - rx\frac{\partial \varphi_t(x)}{\partial x} - \frac{1}{2}x^2\sigma^2\frac{\partial^2 \varphi_t(x)}{\partial x^2}. \tag{1}$$

where $\varphi_t(x)$ is the option value at time t if the stock price is then x. This is a diffusion equation in reverse time, and its solution is determined by the initial (final!) condition:

$$\varphi_T(x) = \begin{cases} \max\{(c - x), 0\} & \text{for a put} \\ \max\{(x - c), 0\} & \text{for a call} \end{cases} \tag{2}$$

where T is the maturity time of the option, and c is its strike price. The following four subsections all present ways of solving equation (1) numerically.

The analytic formula

This is the most efficient solution of equation (1), and was actually given by Black and Scholes (1973). For a call it is

$$\varphi_t(x) = xN(d_1) - \exp(-r(T-t))\cdot c\cdot N(d_2),$$

where
$$d_1 = \frac{\ln(x/c) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = d_1 - \sigma\sqrt{T-t},$$

and $N(\cdot)$ is the cumulative normal function (with variance equal to 1). The disadvantage of this formula is that it is not easily adapted to dealing with options on stocks which pay dividends or where exercise before maturity is possible. These issues are discussed below.

The explicit finite difference method

For this method it is appropriate first to make a logarithmic transformation of equation (1), so that it becomes:

$$\frac{\partial\psi_t(x)}{\partial t} = r\psi_t(x) - \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial\psi_t(x)}{\partial x} - \frac{1}{2}\sigma^2\frac{\partial^2\psi_t(x)}{\partial x^2} \qquad (3)$$

where $\psi_t(x)$ is the option value at time t if the stock price is $\exp(x)$.

The explicit finite difference method then proceeds as follows. First, replace the space axis for equation (3) by the grid $\{x_{-m}, x_{-m+1}, \ldots, x_0, \ldots, x_m\}$ such that the distance between neighbouring points is a constant h, and $\exp(x_0) = c$ (strike price). (We discuss the choice of h and m below.) Also replace the time axis by the grid (labelled backwards for convenience) $\{\ldots, t_2, t_1, t_0\}$ such that the distance between neighbouring points is a constant k, and $t_0 = T$.

Then write $\psi_{t_i}(x_j)$ as w_j^i for each i and j, so that the option value at any time t_i (as a function of the logarithmic stock price) is represented by the vector of values $(w_{-m}^i, \ldots, w_{+m}^i)$. (This vector represents the values of the option for prices $\exp(x_{-m}), \ldots, \exp(x_m)$.)

Next, replace the derivative expressions in equation (3) using the following finite difference approximations:

$$\frac{\partial\psi_{t_i}(x_j)}{\partial t} = (w_j^i - w_j^{i+1})/k + O(k) \qquad (4a)$$

$$\frac{\partial\psi_{t_i}(x_j)}{\partial x} = (w_{j+1}^i - w_{j-1}^i)/2h + O(h^2) \qquad (4b)$$

$$\frac{\partial^2\psi_{t_i}(x_j)}{\partial x^2} = (w_{j+1}^i - 2w_j^i + w_{j-1}^i)/2h + O(h^2) \qquad (4c)$$

to yield the approximation to equation (3):

$$w_j^{i+1} = p^- w_{j-1}^i + p w_j^i + p^+ w_{j+1}^i \tag{5}$$

where

$$p^- = k(\sigma^2/2h^2 - (r - \tfrac{1}{2}\sigma^2)/2h)$$
$$p = 1 - k\sigma^2/h^2 - rk$$
$$p^+ = k(\sigma^2/2h^2 + (r - \tfrac{1}{2}\sigma^2)/2h)$$

The solution is obtained by evolving backwards in time: if the solution at time t_i is known (that is, the vector $(w_{-m}^i, \ldots, w_{+m}^i)$), then we calculate $(w_{-m+1}^{i+1}, \ldots, w_{m-1}^{i+1})$ explicitly using equation (5); and we calculate the side values w_{-m}^{i+1} and w_{+m}^{i+1} using the equations

$$\left.\begin{aligned} w_{-m+1}^{i+1} - w_{-m}^{i+1} &= 0 \\ w_{+m}^{i+1} - w_{m-1}^{i+1} &= \exp(x_m) - \exp(x_{m-1}) \end{aligned}\right\} \text{for a call,}$$
$$\left.\begin{aligned} w_{-m+1}^{i+1} - w_{-m}^{i+1} &= \exp(x_{-m}) - \exp(x_{-m+1}) \\ w_m^{i+1} - w_{m-1}^{i+1} &= 0 \end{aligned}\right\} \text{for a put.} \tag{6}$$

(These equations for $w_{\pm m}^{i+1}$ come from the derivative conditions

$$\frac{\partial \phi_t(x)}{\partial x} = \begin{cases} 1 \text{ if } x \text{ is large} \\ 0 \text{ if } x \text{ is small} \end{cases}$$

for a call, and

$$\frac{\partial \phi_t(x)}{\partial x} = \begin{cases} 0 \text{ if } x \text{ is large} \\ -1 \text{ if } x \text{ is small} \end{cases}$$

for a put.)

The numerical stability of this method is ensured by the condition $|p^-| + |p| + |p^+| \leqslant 1$; without numerical stability the errors associated with the method may become amplified as it evolves through time, so that the solution becomes useless. Note that since $p^- + p + p^+ = 1 - kr$, which is very close to 1, the stability condition is almost equivalent to the condition that p^-, p, p^+ are all non-negative. In fact this latter condition also ensures stability. Having chosen the space step h, this stability condition dictates that the time step k cannot be too big, and in fact that $k = O(h^2)$.

Here we explain why the condition $|p^-| + |p| + |p^+| \leqslant 1$ ensures numerical stability. Suppose that at time t_i the numerical procedure introduces the error $(\varepsilon_{-m}^i, \ldots, \varepsilon_m^i)$ to the solution $(w_{-m}^i, \ldots, w_m^i)$. As time evolves, this error is perpetrated through the system via equation (5) itself, and so its effect at time t_{i+1} is $(\tilde{\varepsilon}_{-m}^i, \ldots, \tilde{\varepsilon}_m^i)$, where

$$\tilde{\varepsilon}_j^i = p^- \varepsilon_{j-1}^i + p\varepsilon_j^i + p^+ \varepsilon_{j+1}^i \leq (|p^-| + |p| + |p^+|) \max\{\varepsilon_{-m}^i, \ldots, \varepsilon_m^i\}.$$

From this we see that if the condition is satisfied, then $|\tilde{\varepsilon}_j^i| \leq \max\{|\varepsilon_{-m}^i|, \ldots, |\varepsilon_m^i|\}$, and so the error is not amplified.

To implement this method, first choose h to give a sufficiently detailed table of results (a value of 0.01 or 0.02 might be appropriate — this makes the result sensitive to a 1 or 2 per cent change in stock price). Also choose m to give a sufficiently wide spread of results — it is appropriate to go perhaps two standard deviations beyond any stock price for which the option value is required, in order to minimise the effect of the approximation equations (6). Then choose k to give numerical stability; for h = 0.02 (and r = 0.7, σ = 0.3, stability will be achieved for k = 1/300 years. With these values, and for time to a maturity of 0.5 years, the method will solve equation (1) to within 0.1 per cent.

The explicit method is simplest among finite difference methods, and Geske and Shastri (1985) conclude that for our problem it is the most efficient.

The implicit finite difference method

For this method it is also sensible to work in logarithmic transformation, that is with equation (3) rather than equation (1) (see Geske and Shastri, 1985). The essential difference from the explicit method is that equation (4a) is replaced by

$$\frac{\partial \psi_{t_i}(x_j)}{\partial t} = (w_j^{i-1} - w_j^i)/k + O(k),$$

to yield the following approximation to equation (3):

$$p^- w_{j-1}^i + p w_j^i + p^+ w_{j+1}^i = w_j^{i-1} \tag{7}$$

where now

$$p^- = -k(\sigma^2/2h^2 - (r - \tfrac{1}{2}\sigma^2)/2h),$$

$$p = 1 + k\sigma^2/h^2 + rk,$$

$$p^+ = -k(\sigma^2/2h^2 + (r - \tfrac{1}{2}\sigma^2)/2h).$$

From the option values $(w_{-m}^i, \ldots, w_{+m}^i)$ we can calculate the values $(w_{-m}^{i+1}, \ldots, w_{+m}^{i+1})$ as before but using equation (7) (with i replaced by $i + 1$), and the side conditions in equations (6) above. However, equations (7) and (6) together now do not give each answer explicitly, they give us a simultaneous equation for the values $(w_{-m}^{i+1}, \ldots, w_{+m}^{i+1})$. In matrix notation this equation is

$$\begin{bmatrix} p^- + p, & p^+ & & & \\ & p^-, & p, & p^+ & \\ & & \ddots & \ddots & \ddots \\ & & & \ddots & \ddots & \ddots \\ & & & & \ddots & \ddots & \ddots \\ & & & & & \ddots & \ddots & \ddots \\ & & & & & p^-, & p, & p^+ \\ & & & & & & p^-, & p + p^+ \end{bmatrix} \begin{bmatrix} w^{i+1}_{-m+1} \\ \\ \\ \\ \\ \\ w^{i+1}_{m-1} \end{bmatrix} = \begin{bmatrix} w^{i}_{-m+1} \\ \\ \\ \\ \\ \\ w^{i}_{m-1} \end{bmatrix} - \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \exp x_m - \exp x_{m-1} \end{bmatrix}$$

for a call, and the same for a put, but with the last term replaced by

$$\begin{bmatrix} \exp x_{-m} - \exp x_{-m+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

(Having solved this, the side values w^{i+1}_{-m}, w^{i+1}_{m} are obtained using equations (6).) This matrix equation is tridiagonal and so it is quite easy to solve numerically — it can be solved in two passes through the matrix, one to eliminate the lower diagonal, and one to back-substitute for the answers.

To implement this method, choose h and m as before, and choose k to ensure the accuracy of the method. Note that there are now no constraints imposed on k by numerical stability.

The implicit method is more difficult to understand and to implement than the explicit, and it requires the solution of a tridiagonal matrix equation at each time step. On the other hand a longer time step (and hence fewer iterations) is allowed than for the explicit method.

The binomial method

In this method the stock price process is approximated by a random walk with discrete time steps, and with the same drift and variance. (Actually we work with the logarithm of the stock price, and the drift and variance are required to match only in the limit of small time steps of the random walk.) Then the option value is calculated simply as the mean (expected) payoff $\varphi_T(x)$ (given by equation (2)) at time T (and discounted back to the present time), given by this random walk. The binomial method is very elegant, but the reasons for its success are rather subtle. In fact we must pretend that the drift of the stock price is r rather than μ for the method to succeed (see Chapter 6b; Smith, 1979).

Let us denote the parameters of the random walk as follows: the discrete time step is k, and as time increases by k the log-price might jump up by h^+ or down by h^-, with probabilities p^+ and p^-, respectively. There is some leeway in choosing these parameters: having chosen k, then common choices for the others are the following:

	$h^{\pm}$	$p^{\pm}$
Cox, Ross and Rubinstein	$\pm\sigma\sqrt{k}$	$\frac{1}{2} \pm (r - \frac{1}{2}\sigma^2)\sqrt{k}/2\sigma$
Jarrow and Rudd	$(r - \frac{1}{2}\sigma^2)k \pm \sigma\sqrt{k}$	$\frac{1}{2}$

(See Omberg, 1987.)

To implement the binomial method, suppose that the current logarithm of the stock price is x, and that there are n time steps to go until the maturity of the option. After k time steps the logarithmic stock price process will be at the point $x + ih^{+} - (k - i)h^{-}$, where i is the number of up steps performed by the process. This point is represented as b_i^k in Figure 6a.1.

To obtain the option value for the current logarithmic stock price x (i.e.

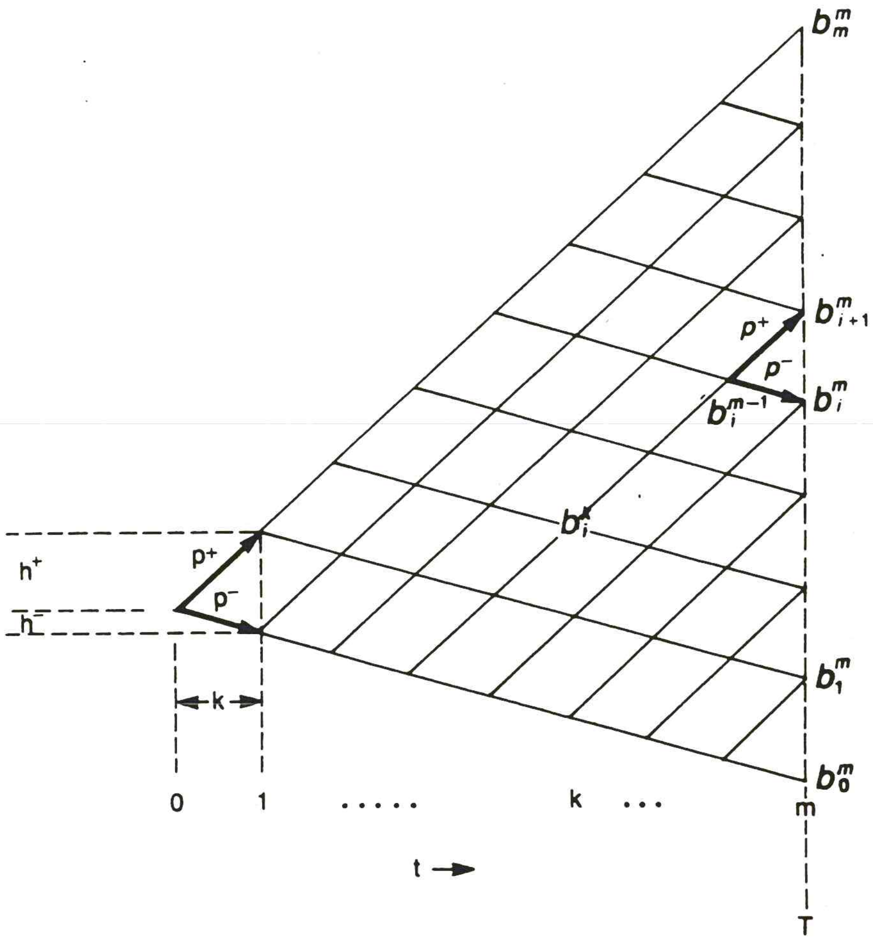

Figure 6a.1

the point b_0^0 on the lattice) we work recursively back from time T, as we did for the finite difference methods: the values at $b_0^n, \ldots, b_n^n$ are known from the final condition (2); to obtain the values $b_0^{n-1}, \ldots, b_{n-1}^{n-1}$ we simply use the explicit equations

$$b_i^{n-1} = \exp(-rk)(p^- b_i^n + p^+ b_{i+1}^n)$$

for $i = 0, \ldots, n-1$, then continue inductively backwards.

This method is accurate to within 0.1 per cent if we take $k = 1/100$ years, and the Jarrow–Rudd parameters from the table above (which seem preferable to the other parameters). For this value of k, $h^+ + h^-$ is about 0.07. This helps explain why the binomial method is much more efficient than the explicit finite difference method, although the procedures look similar: the binomial method can work with a much coarser space mesh. Geske and Shastri (1985) report that for a single evaluation the binomial method is more efficient by a factor of about 10.

The main disadvantages of the binomial method over the finite difference method are that it must be implemented separately for each option evaluation, and that it is not easily adapted to dealing with a stock which pays a constant dividend (see below).

II American options

The value of an American option is theoretically rather difficult to derive and to express; it is the solution to a free boundary problem associated with equation (1) (see Chapter 6b; Karatzas, 1988). However, it is not difficult to adapt the iterative procedures in Subsections I.2, I.3 and I.4 above to dealing with the early exercise opportunity; just replace the calculated value $\varphi_t(x)$ at each step by $\max\{\varphi_t(x), \chi(x)\}$, where $\chi(x)$ is the payoff from immediate exercise (thus $\chi(x)$ is given by equation (2)). If we have $\varphi_t(x) < \chi(x)$ and x is the current stock price, then exercise the option immediately. This adaptation is in standard use, and it is theoretically justified In Chapter 6b.

The analytic formula (Subsection I.1) is more difficult to adapt, but it has been done by Geske (1979), Geske and Johnson (1979) and Selby and Hodges (1987), and a description of this adaptation is given in Chapter 6b. The adaptation evaluates the option with exercise opportunities restricted to a small collection of, say, n times (with perhaps $n = 3$ or 4), and thus only gives an approximation to the option value. However, this approximation is surprisingly accurate, perhaps within 2 per cent of the strike price; this accuracy is justified in Chapter 6b. The procedure requires the evaluation of cumulative normal distributions of dimension n, which makes it much more time-consuming than that of Subsection I.1.

III Dividends

To value an option when the underlying stock pays dividends it is necessary to assume in advance the terms of the dividends, and how they affect the stock-price process. It is usual to assume that the dividends are fixed and constant (say, equal to d), and are paid at fixed times. This is usually how the firm issuing the stock is expected to behave. On payment of the dividend the price of the stock falls by d; it is usual to assume that the proportional drift, μ, and volatility, σ, of the stock remain constant across dividend dates.

In principle, it is easy to adapt the recursive procedures of Subsections I.2, I.3 and I.4 to deal with dividends under these assumptions, simply by altering the value function φ_t as we evolve backwards through the dividend date. This alteration is just given by

$$\varphi_{t^-}(x) = \begin{cases} \max\{\varphi_t + (x-d), (x-c)\} & \text{for a call} \\ \varphi_{t^+}(x-d) & \text{for a put.} \end{cases}$$

(Exercise the call immediately antedividend if $(x - c) > \varphi_{t^+}(x - d)$; never exercise a put immediately antedividend.) However, this procedure is a little awkward when applied to the finite difference methods because the option value function, φ_t, is stored in the computer as a table of its values at the grid points $\{\exp(x_{-m}), \ldots, \exp(x_{+m})\}$; to find $\varphi_{t^-}(\exp(x_i))$ we must estimate $\varphi_{t^+}(\exp(x_i) - d)$ by interpolation because $\exp(x_i) - d$ is in general not a grid point. Moving the grid to the points $\{\exp(x_{-m}) + d, \ldots, \exp(x_{+m}) + d\}$ leads to other difficulties because this grid is not equally spaced when it is logarithmically transformed.

To adapt the binomial method to dealing with dividends with the above assumptions is also difficult for the same reasons, and the difficulty cannot be overcome by interpolation because the binomial methods uses a much coarser mesh than the finite difference method in order to gain an efficiency advantage over it. Geske and Shastri (1985) explain the problem in terms of the triangular lattice of Figure 6a.1 failing to fit together when the grid is moved to $\{\exp(x_{-m}) + d, \ldots, \exp(x_{+m}) + d\}$ — they refer to it as the 'exploding tree problem'.

Dividends are easier to handle if we make the simplifying assumption that the amount of the dividend is a constant proportion (say, $\bar{d}$) of the stock price. This allows us to move the grid to $(\exp(x_{-m}) \cdot (1 + \bar{d}), \ldots, \exp(x_{+m}) \cdot (1 + \bar{d}))$, which has the same regular spacing as the original grid when it is logarithmically transformed. Thus we can continue backwards from the new grid with the same parameters in our finite difference or binomial procedure as before, and we can calculate $\varphi_{t^-}(\exp(x_i)(1 + \bar{d}))$ as $\varphi_{t^-}(\exp x_i)$ for a put or $\max\{\varphi_{t^-}(\exp(x_i)), (\exp(x_i) - c)\}$ for a call.

A further simplifying assumption is that the dividend is a constan. ˈroportion of the share price and is paid continuously at a rate of, say, d^*s_t. To model this, one would use the following equation by Garman and Kolhagen (1983), rather than our equation (1):

$$\frac{\partial \varphi_t(x)}{\partial t} = r\varphi_t(x) - (r - d^*)x\frac{\partial \varphi_t}{\partial x} - \frac{1}{2}x^2\sigma^2\frac{\partial^2 \varphi_t(x)}{\partial x^2}.$$

The Garman–Kolhagen equation was originally developed to value foreign exchange options, and for such an application d^* is the foreign interest rate. Note that an American call will not be exercised between dividend dates if these are discrete, but for a continuous dividend the option might be exercised prematurely if $d^* > r$.

IV Summary and conclusions

We have described in detail and compared some standard methods for valuing stock options, namely the analytic formula, the explicit and implicit finite difference methods, and the binomial method.

The analytic formula is most efficient for the basic valuation, but it is very inflexible, and cannot easily be adapted to dealing with early exercise or stocks which pay dividends. The finite difference methods are very flexible and can deal with these factors, though they are rather cumbersome and inefficient. The explicit method is simpler and more efficient than the implicit. The binomial method is readily adapted to dealing with early exercise, and is much more efficient than the explicit finite difference method. However, it is only easily adapted to constant proportional dividends and not the more usual constant absolute dividends.

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