Valuation and Hedging:

Theoretical Issues

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I Introduction

This paper reviews the main theoretical issues related to valuing and hedging options. We first consider the assumptions made by valuation models, and the general form of these models. Option valuations can only be made conditional on inputs which describe the probability distribution of future values on the underlying asset. Our ability to value contingent claims is mostly limited by our ignorance of these distributions. The effectiveness of extensions of Black and Scholes (for example, to take account of dividends, jump-diffusion or CEV processes, or of stochastic volatility) depends on how much they improve the modelling of the distributional characteristics of the underlying asset. Issues concerning valuation and model choice are discussed in the light of this perspective.

Valuations which may be accurate in an equilibrium sense do not necessarily enable price discrepancies to be turned into profitable arbitrage if transactions costs must be incurred or markets are incomplete. The paper reviews various sources of risk in hedged positions and discusses their relative importance. Recent work on hedging in imperfect markets (subject to incompleteness or transaction costs) is reviewed. The difficulties of exactly replicating options positions suggest that we should put different bounds around the value of an instrument, depending on the market conditions, the assumptions, and the securities that would be used to arbitrage any mispricing.

The paper concludes with a brief discussion of a number of techniques for controlling risk in option portfolios.

II Common elements of valuation models

It is easy to overlook the fact that at the heart of any option model is the set of assumptions that it makes concerning the stochastic process followed by the value of the underlying asset, and which define the probability distribution of the asset's value at the expiry date. The Cox and Ross (1976) 'risk-neutral' valuation insight makes it clear that this distribution is the critical determinant of option values. The technique can be applied provided we can construct a zero risk hedge portfolio without it depending on investors' risk preferences. This applies to a large class of option pricing models, in fact to the majority of them. After a minor transformation to the probabilities of the process to make its expected return equal the risk-free interest rate, the option payoffs can be evaluated by discounting their expected value at the risk-free rate.

The value of an option is thus in essence obtained simply by integrating the appropriate part of the probability distribution of the underlying asset. It is fairly easy to see this for European options. For American options, where early exercise may occur, it is a little more complicated and the path taken by the asset value also matters. There it is necessary to find the optional stopping rule for when to exercise the option to maximise its value. Once the 'free boundary' on which the option should be exercised has been determined, the framework is the same as before. The payoffs along the boundary are integrated with their probabilities (under the transformed risk-neutral distribution).

Finally, in some models the Cox–Ross approach does not quite apply. For example, in the Merton (1976) jump-diffusion model, the jump component cannot be hedged. It is assumed that the jump risk is diversifiable and not priced in the market. Thus again it is simply the shape of the assumed distribution that matters.

Similarly, in some other models, risk aversion parameters are necessary in order to complete the model (as, for example, in Brennan and Schwartz, 1979). In models of this kind we may again regard the valuation as consisting of integrating the assumed probability distributions, but now after some (relatively minor) risk adjustments have been made.

Given this general introduction, we can now state the general form of assumptions made by most (continuous-time) valuation models:

1. *Frictionless markets.* There are no transaction costs, and no taxes or costly margin requirements.
2. *Well-defined processes.* There are suitable well-defined processes for
   (a) the risk-free rate
   (b) dividends
   (c) the underlying asset.
   Common assumptions would be a constant (or at least certain) interest rate, no dividends and a log-normal process for the underlying asset. However, all of these can be generalised considerably.
3. *Elimination of priced risk.* The processes assumed must enable
   (a) the option to be duplicated by a dynamically adjusted self-financing portfolio, or

Table 3.1 Some generalisations of the Black and Scholes model

Interest rate		
Stochastic interest rate	Merton (1973)	1a
Dividends		
Known proportional dividends	Merton (1973)	1a
Known fixed dividends	Whaley (1982)	1a
Compound options	Geske (1979)	1a
	Selby and Hodges (1987)	1a
Asset process		
Displaced diffusion	Rubinstein (1983)	1a
Constant elasticity of variance	Beckers (1981)	1a
Jump	Cox and Ross (1976)	1a
Jump-diffusion	Merton (1976)	1a, 1b
Stochastic volatility	Hull and White (1987)	1a, 1b
	Wiggins (1989)	1c
Brownian bridge	Ball and Torous (1983)	1c
General (for European options)	Jarrow & Rudd (1982)	2

Notes:
1. Letters refer to type of assumption under Assumption 3:
   a. Riskless arbitrage
   b. Unpriced risk, $\lambda = 0$
   c. Priced risk, $\lambda \neq 0$
2. Jarrow and Rudd dispense with a continuous-time argument and directly assume that the Cox-Ross approach can be used.

(b) if some risks cannot be hedged they may be regarded as diversifiable and not commanding any risk premium, or

(c) one or more risk aversion parameters may be retained within the model.

Table 3.1 provides some elaboration of these assumptions, by giving examples of the kinds of generalisation made under Assumption 2 by some standard models. The table also indicates the type of argument used under Assumption 3 to obtain a pricing relationship.

Finally, for completeness, we illustrate in Table 3.2, the nature of the transformation of probabilities used in the valuation. It shows for a standard binomial tree how the transformed probabilities, p^*, are calculated, which give the values for pricing. The relationship between p^* and the objective probability, p, is given by

Table 3.2 Option valuation in complete markets

The stochastic process enables us to calculate prices for each future state.

Binomial example			*Stage two*	
			Probabilities	*State prices*
		Su^2	p^2	$p^{*2} \cdot r^{-2}$
	Su			
S		Sud	$2pq$	$2p^*q^* \cdot r^{-2}$
	Sd			
		Sd^2	q^2	$q^{*2} \cdot r^{-2}$
B	Br	Br^2	1	

Using either a *replication* argument or a *risk-neutrality* one,
the *state prices* are as given above,

with $p^* = \dfrac{r - d}{u - d}$ and $q^* = 1 - p^*$

Value of any instrument is given by summing payoffs times state prices.

$$p^* = p - \frac{k}{u - d}$$

where k is the expected risk premium on the underlying asset.

III Issues concerning valuation and model choice

Although the role of the assumed probability distribution is complicated by the kind of transformation described above, it is nevertheless the main driving force within any valuation model. It is not surprising that the estimation of volatility plays such a key role in option valuation, nor that a number of systematic deviations from the Black and Scholes (1973) model have been interpreted as consistent with non-normality. In some cases this stems from stochastic volatility (see Galai, 1983; Jarrow, 1987; and Geske and Torous, 1987). The timing and size of dividends can also have a significant effect on the distribution of the asset's subsequent value. It is well known that the valuation of long-term warrants and convertibles is particularly sensitive to assumptions about the company's dividend policy. Similarly, uncertainty about the exact timing of future dividends can be a real hazard in pricing American put options.

In a world in which the assumptions of the Black–Scholes model are strictly adhered to, asset volatility is constant, and it can be estimated to an arbitrary degree of precision from observable data. In practice, there is strong evidence that volatilities vary through time, and we see increasing use made of Autoregressive Conditional Heteroscedasticity (ARCH) and Generalised Autoregressive Conditional Heteroscedasticity (GARCH) models for price processes. It is clear that the information contained in a historical volatility estimate is quite limited, and it is not surprising that implied volatilities are often more useful.

However, it is only in the unlikely case of stationary log-normal distributions with constant parameters that there will be an absence of exercise price and maturity effects. The claims of Jarrow (1987) for the ability of implied variance to take account of a wide range of model misspecifications seem rather optimistic. Finally, while implied volatility indicates a market consensus, it is always possible that the consensus may be wrong.

If we are able to forecast the distributional properties of assets better than the market consensus, we shall be able to make money. Statistical analyses clearly play a role in this, but we need also to bear in mind that all assets are different from one another, and that their properties also change through time. This implies that statistical analysis is a potentially misleading tool unless it is combined with elements of fundamental analysis.

The remarks we have just made also throw some light on the difficulties of performing empirical tests on option valuation models. It is notoriously difficult to test the fit of particular distributions, particularly when observations are limited. We are unlikely to be able to choose between competing models on the basis of empirical evidence on the appropriateness of their assumptions. Some ambiguity also surrounds the interpretations that can be placed on a model's ability to explain market prices, and on the performance of hedged (arbitrage) positions constructed using it. A better fit may be obtained because of additional degrees of freedom, or because market participants are using a similar model, rather than because the intrinsic valuations are in any sense better. Similarly, if participants happen to use a slightly inappropriate model, this model may be the best way to find short-term arbitrage opportunities, even though its valuations are poor. The ideal test would be to hedge returns right out to expiry, but there are problems here caused by changes in parameters such as interest rates and volatility. Galai (1983) provides some further discussion of these and related issues.

We have so far ignored the possibility of interactions between derivative securities and the underlying. If markets are perfect, no such interactions should exist. On the other hand, it can be argued that, from a micro-market perspective, transactions on the two kinds of instrument are to some extent independent and either one could lead the other. There is also some concern at the potential for manipulation between the two markets. Some recent papers have tested the integration between the markets for assets and those for their derivative securities. Nabar and Park (1988) found evidence suggesting that after the inception of options trading, the volatility of the underlying stock decreased on average by 4–8 per cent. The study was based on 390 new listings between 1973 and 1985. Stoll and Whaley (1987) found some evidence that the volatility of price changes of S & P 500 stocks was significantly higher on expiration days of the index futures contracts. They also found a significant average expiration-day price effect of −0.4 per cent. Another study by Snelling (1987) analysed price behaviour on stocks and their associated options at the times of 135 earnings announcements made during August to November 1979. The results of the study suggest that in anticipating and responding to these announcements, the option prices led the underlying stocks — but by minutes rather than hours.

We conclude this section with a brief discussion of a number of issues affecting the choice of a valuation model. Four obvious considerations are the following: appropriateness of the assumptions; convenience of the inputs; the degree of fit with market data; and the practicality of the calculations. In general some trade-offs must be made between these criteria. We may illustrate these by means of some examples. A variety of models are available which treat volatility as stochastic. Theoretically, one might expect this risk to be priced in the market and its price of risk to require the estimation of a parameter. Although this more general model will give a better fit to current market data, a simpler one might be preferred as being more robust and easier to compute and understand.

The recent models by Ho and Lee (1986), and by Schaefer and Schwartz (1987) are appealing for quite different reasons. The Ho and Lee model is designed to start exactly from an entire current term structure and to pursue its binomial evolution through time. The Schaefer and Schwartz approach is also designed to enjoy the advantages of a single state variable, but does so in a way which is closer in spirit to the Constant Elasticity of Variance (CEV) model, by exploiting the relationship between volatility and duration.

Computational and pedagogic virtues are also important, and the binomial method enjoys much of its current popularity because it is so easy to understand, to code and to modify to new situations. Two generalisations have appeared recently, by Boyle (1988) to two state variables and by Madan and Milne (1987) to a multinomial distribution with many assets. It will be interesting to see how quickly these methods are taken up.

IV **Hedging**

The problems of hedging the risks of options positions are every bit as important as those of valuing options. Some aspects are easier to deal with:

it is easier to judge whether hedging has been successful than to judge whether a valuation was accurate. Other aspects such as the best hedging strategies under transactions costs are very difficult indeed. Under the idealised conditions assumed by the continuous-time models derived from the no-arbitrage argument (Assumption 3a), exact hedging is not a problem. Provided the assumptions of the models hold, delta-hedging (with continuous revision of the hedge ratio) is sufficient to entirely eliminate all risk.

In practice there are at least five reasons why risk cannot be eliminated completely.

Incomplete markets

The prices of securities move by discrete amounts rather than continuously. At best we face a trinomial distribution at each instant. This suggests that in all probability real markets are incomplete, and risk can never be entirely eliminated.

Hakansson (1979) and others have noted the paradox that we can only obtain exact valuations for securities which are redundant (because they can be replicated by combinations of others).

The hedging errors which arise with delta-hedges which are revised at discrete intervals have been studied by Boyle and Emanuel (1980). Given a market which is incomplete, we may want to find the risk-minimising strategy. Work in this vein has been done by Follmer and Sonderman (1986) (see also Sonderman, 1987). Other work on this topic seems likely, and there are various ways in which incompleteness can occur and be modelled.

Transactions costs

The continuous price process models unhappily imply infinite amounts of turnover. With any positive level of transactions costs this means going bust in no time at all. Transactions costs therefore prohibit the ideal from of continuous rebalancing, and other methods of rebalancing imply residual risks.

The analysis of option pricing under transactions costs is an important and growing area of study. Leland (1985) showed that option returns can be replicated in the presence of transactions costs, if the delta-hedging portfolio is revised at discrete time intervals using a Black–Scholes formula with the variance term σ^2 adjusted to

$$\sigma^2(1 + \sqrt{2/\pi}\, k/\sigma\sqrt{\Delta t})$$

where k is the round trip cost and Δt is the period between revisions.

In a similar vein, Neuhaus (1988) has analysed the properties of related strategies which rebalance depending on the magnitudes of price movements rather than time intervals.

A simulation study by Figlewski (1987) examines the performance of several alternative strategies, including the Leland one, and rebalancing when the delta changes by a fixed amount. The general conclusions are that a reduction in trading costs can only be bought at some substantial increase in risk. The alternative methods seem to give roughly similar results.

Another paper by Davis (1988) provides an elegant extension of Merton's (1971) 'Optional Consumption and Portfolio Rules in a Continuous Time Model'. The optimal solution puts an asymmetric collar around the usual asset mix proportions, and constrains the mix within this collar. In the context of the conventional hedging problem this suggests new strategies constraining delta not to move too far from the normal figure, but without rebalancing all the way back to it.

Varying parameters

We have already noted that misestimation of volatility is a serious problem in valuing options. Changes in volatility are a significant cause of hedging errors, unless they are themselves hedged against.

Hedging can be used to neutralise $\partial c/\partial \sigma$ (kappa). For Black–Scholes options of the same expiry date this also amounts to neutralising $\partial^2 c/\partial S^2$ (gamma). However, for other models gamma- and kappa-hedging are not necessarily the same thing.

More seriously, both these measures are sensitive to the price of the underlying asset and other model parameters. It is therefore usually desirable to consider what will happen to a portfolio over ranges of values, rather than just evaluate delta, kappa, and so on, at today's prices.

Similarly, if other parameters change, such as interest rates, or expected dividends significant gains or losses may arise.

Misspecified model

It is fairly obvious that if our pricing and hedging model is misspecified, then hedging errors will arise. The risk is similar to that of changes in model parameters, but harder to identify and control.

Market variation

Finally, even if none of the sources of error we have discussed was present and we had a 'perfect' model at our disposal, short-term hedging errors would still arise if market prices contained noise. In the long term our strategy would work and our arbitrages would make money.

However, in a market which prices options poorly there is little to stop cheap instruments getting cheaper and even less to stop dear ones getting dearer.

V Conclusions

We have noted the key role played in valuation by the assumptions that determine the probability distributions of the value of the underlying asset. We have also noted the reasons which make risks difficult to hedge. We must conclude that value is a somewhat ambiguous concept. Although we may be confident in our valuation of an instrument, we cannot necessarily arbitrage deviations from that value. We can instead think in terms of ranges of values, defined by the method and certainty by which arbitrage could be accomplished. However, even with quite pure arbitrages, market prices can worsen before a profit is realised.

Since risk is here to stay, so too are a variety of methods for risk control. Position and book limits related to risk point calculations for particular instruments are easy to administer and have a part to play. So, too, do the traditional hedging methods of options and futures markets, usually involving a handful of instruments at a time (for example, delta-hedging, delta- and gamma-hedging).

We can, however, also expect to see much more development and use of risk-control methods at the portfolio level.

Linear programming is a powerful tool in this respect (see, for instance, Garman, 1976, and Hodges and Schaefer, 1977). Variance minimisation methods are also important, but need to be carefully adapted for use with options, to take account of the particular dependencies and distributions involved.

References

Ball, C. A. and Torous, W. N. (1983), Bond Price Dynamics and Options, *Journal of Financial and Quantitative Analysis*, 18, December, 517–31.

Beckers, S. (1981), Standard Deriations Implied in Options Prices on Predictors of Future Stock Price Variability, *Journal of Banking and Finance*, 5, 363–81.

Black F. (1976), The Pricing of Commodity Contracts, *Journal of Financial Economics*, 3, 167–79.

Black, F. and Scholes M. (1973), The Pricing of Options and Corporate Liabilities, *Journal of Political Economy*, 81, 637–59.

Boyle, P. P. (1988), A Lattice Framework for Option Pricing with Two State Variables, *Journal of Financial and Quantitative Analysis*, 23, 1–12.

Boyle, P. and Emanuel, D. (1980), Discretely Adjusted Option Hedges, *Journal of Financial Economics*, 8, 259–82.

Brennan, M. J. and Schwartz, E. S. (1979), A continuous time approach to the pricing of bonds, *Journal of Banking and Finance*, 3, 133–55.

Brenner, M., Courtadon, G. and Subrahmanyam, M. (1985), Options on the Spot and Options on Futures, *Journal of Finance*, 40, 1303–17.

Cox, J. and Ross, S. A. (1976), The Valuation of Options for Alternative Stochastic Processes, *Journal of financial Economics*, 3, 145–66.

Davis, M. H. A. (1988), Local Time on the Stock Exchange. In J. R. Norris (ed.), *Stochastic Calculus in Application*, Pitman Research Notes in Mathematics, Longman.

Figlewski, S. (1987), Options Arbitrage in Imperfect Markets. Working Paper, New York University.

Follmer, H. and Sonderman, D. (1986), Hedging of Non-redundant Contingent Claims. In W. Hildenbrand and A. Mas-Colell (eds), *Contributions to Mathematical Economics in Honour of Gerard Debreu*, North-Holland, Amsterdam.

Galai, D. (1983), A Survey of Empirical Tests of Option Pricing Models. In M. Brenner (ed.), *Option Pricing*, D. C. Heath, Lexington, Mass.

Garman, M. B. (1976), An Algebra for Evaluating Hedge Portfolios, *Journal of Financial Economics*, 3, 405–27.

Geske, R. (1979), The Valuation of Compound Options, *Journal of Financial Economics*, 7, 63–81.

Geske, R. and Torous, W. (1987), Volatility and Mispricing: Robust Variace Estaimation and Black–Scholes Option Pricing. Working Paper, UCLA.

Hakansson, N. (1979), The Fantastic World of Finance Progress and the Free Lunch, *Journal of Financial and Quantitative Analysis*, 14, 717–34.

Ho, T. S. Y. and Lee, S. (1986), Term Structure Movements and Pricing Interest Rate Contingent Claims, *Journal of Finance*, 41, 1011–29.

Hodges, S. D. and Schaefer, S. M. (1977), A Model for Bond Portfolio Improvements, *Journal of Financial and Qualitative Analysis*, 12, 243–60.

Hull, J. and White, A. (1987), The Pricing of Options on Assets with Stochastic Volatilities, *Journal of Finance*, 42, 281–301.

Jarrow, R. (1987), Option Pricing and Implicit Volatilities: A Review and a New Perspective. Working Paper, Johnson Graduate School of Management, Cornell University.

Jarrow, R. and Rudd, A. (1982), Approximate Option Valuation for Arbitrary Stochastic Processes, *Journal of Financial Economics*, 10, 347–69.

Leland, H. E. (1985), Option Pricing and Replication with Transaction Costs, *Journal of Finance*, 15, 1283–1301.

Madan, D. B. and Milne, F. (1987), The Multinomial Option Pricing Model and its Limits. Working Paper, University of Sydney.

Merton, R. C. (1971), Optional Consumption and Portfolio Rules in a Continuous Time Model, *Journal of Economic Theory*, 3, 373–413.

Merton, R. C. (1973), Theory of Rational Option Pricing, *Bell Journal*, 4, 141–83.

Merton R. C. (1976), Option Pricing When Underlying Stock Returns are Discontinuous, *Journal of Financial Economics*, 3, 125–44.

Nabar, P. G. and Park, S. Y. (1988), Options Trading and Stock Price Volatility. Working Paper, New York University.

Neuhaus, H. (1988), Option Valuation and Hedging under Transactions Costs. Ph.D. thesis, London Business School.

Rubinstein, M. (1983), Displaced Diffusion Option Pricing, *Journal of Finance*, 38, March, 213–17.

Schaefer, S. M. and Schwartz, E. S. (1987), Time-dependent Variance and the Pricing of Bond Options, *Journal of Finance*, 42, 1113–28.

Selby, M. J. P. and Hodges, S. D. (1987), On the Evaluation of Compound Options, *Management Science*, 33, 347–55.

Snelling, P. V. (1987), Earnings Surprises and the Information Content of Option Prices. Ph.D. thesis, University of North Carolina, Chapel Hill.

Sonderman, D. (1987), Currency Options: Hedging and Social Value, *European Economic Review*, 31, 246–56.

Stoll, H. R. and Whaley, R. E. (1987), Program Trading and Expiration-Day Effects, *Financial Analysts Journal*, March–April, 16–28.

Whaley, R. (1982), Valuation of American Call Options on Dividend Paying Stocks: Empirical Tests, *Journal of Financial Economics*, 10, 29–58.

Wiggins, J. (1989), Option Values Under Stochastic Volatility: Theory and Empirical Estimates, *Journal of Financial Economics*, 19, 351–72.